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Attribute Reduction in Students Selection for a Merit cum Means Based Scholarship Scheme using Basis in Nano Topological Spaces

Kalarani M[*]
Nithyakala R[†]

Abstract

The objective of this paper is to identify the key attributes that have a significant impact on the merit cum means-based scholarship selection process, ensuring that the allocation of resources is fair and efficient. Scholarships aid students with limited resources in their pursuit of a college education. The complexity of selecting suitable candidates with multiple characteristics has made it harder to determine the essential factors for deciding whether or not an applicant is eligible. To reduce attributes, we use the concepts of basis, lower and upper approximations of Nano Topological Spaces to identify the most relevant traits that affect scholarship eligibility. By using a systematic approach, decision-making is minimized and resources are allocated more effectively. Administrators can ensure that scholarships reach deserving students in need by focusing on essential attributes. The research will help to improve scholarship awarding methods and increase inclusion and accessibility in higher education.

Keywords: Upper Approximations, Lower Approximations, Nano Topology, Basis, Core, Attributes.

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1. Introduction

The concept of nano topology was originally introduced by Lellis Thivagar et al. [4] and since then it has proven to be a valuable tool in various applications. Lellis Thivagar used nano topological basis to identify the key factors for adult dietary needs, Jayalakshmi [1] and Aggour [2] used it in a medical diagnosis problem. In this study, we adapted and extended this concept to the area of scholarship selection to help university administrators identify the most deserving students who are truly in need of financial aid. A Nano topological basis provides a sophisticated framework for analyzing subsets of the given universe U. This concept uses lower and upper approximations and boundary regions defined by an equivalent relation to represent and study data. In reducing attributes, as well as identifying the main factors for selecting eligible students on a given attribute basis, the concepts of basis in Nano Topological Spaces are applied.

The merit-based and means-based scholarship program aims to provide financial assistance to deserving students from economically disadvantaged backgrounds. However, selecting the most deserving candidates from a pool of applicants can be difficult due to the multitude of attributes that must be taken into account. By leveraging this fundamental concept in the nano topological space, we aim to streamline and optimize this selection process.

Providing financial support to deserving students pursuing higher education has always been an important aspect of promoting an inclusive and equitable education system. The merit cum means-based scholarship program is one such initiative to support deserving students from economically disadvantaged backgrounds. In recent years, the process of selecting the most suitable students for scholarships has become increasingly complicated. College administrators face the challenge of screening a large pool of applicants, each with their own unique backgrounds and qualifications. The traditional approach of considering a few standard criteria often falls short of accurately assessing the true needs and merit of individual students.

The merit cum means-based scholarship scheme for meritorious students is usually awarded to a needy student with a good educational background. The scheme aims to provide financial assistance to poor, deserving students enrolled in college. To select a suitable student for this purpose, it is necessary to take into account a separate set of parameters that may be applied when granting such scholarships. The parameters are based on

(i) The eligibility criteria for availing of the scholarship is the minimum aggregate marks greater than 50% in the last examination or in the qualifying exam, which is specially conducted for scholarship selection.

(ii) Family background, which includes family annual income, parents' education, occupation of parents (government/private), number of siblings, and whether the family owns any agricultural property.

(iii) Medium of previous study, which can be any regional language or English.

(iv) Residential area, that is, students belonging to rural areas or from the city side, or the student is a hosteller or day scholar.
(v) Involvement in extra-curricular/co-curricular activities and
(vi) Awarded any other scholarship or the student is a first-generation learner.

This paper presents a promising methodology that leverages the concept of Nano Topological Spaces to enhance the scholarship selection process. By considering a broader range of parameters and incorporating individual circumstances, this approach will aid college administrators in identifying the most suitable candidates for the scholarship, thus promoting greater inclusivity and social mobility within the education system.

2. Preliminaries

Definition 2.1[4]**.** Given X is a nonempty finite set of objects and R_X is an equivalence relation on X called the indiscernibility relation. The X is divided into disjoint equivalence classes. The elements of the same equivalence class are said to be indiscernible with each other. The pair (X, R_X) is called an approximation space. Let $A \subseteq X$, then
(i) The lower approximation of A with respect to R_X is $L_{R_X}(A) = \bigcup_{x \in X}\{R_X(x)/ R_X(x) \subseteq A\}$, where $R_X(x)$ is the equivalence class determined by $L_{R_X}(A)$.
(ii) The upper approximation of A with respect to R_X is $U_{R_X}(A) = \bigcup_{x \in X}\{R_X(x)/ R_X(x) \cap A \neq \phi\}$.
(iii) The boundary region of A is $B_{R_X}(A) = U_{R_X}(A) - L_{R_X}(A)$.

Definition 2.2[4]**.** Let X be the universe, R_X is an equivalence relation on X and $A \subseteq X$, $\tau_{R_X}(A) = \{\phi, X, L_{R_X}(A), U_{R_X}(A), B_{R_X}(A)\}$. Then $\tau_{R_X}(A)$ satisfies the axioms
(i) X and ϕ are in $\tau_{R_X}(A)$.
(ii) The arbitrary union of the elements of $\tau_{R_X}(A)$ is in $\tau_{R_X}(A)$.
(iii) The finite intersection of any finite sub collection of $\tau_{R_X}(A)$ is in $\tau_{R_X}(A)$.
Here $\tau_{R_X}(A)$is a topology on X called the nano topology on X with respect to A and $(X, \tau_{R_X}(A))$ is called a nano topological space. The elements of $\tau_{R_X}(A)$ are called the nano open sets and the complement of the nano open sets are called nano closed sets.

Definition 2.3[3]**.** Let $\mathcal{M} = \{U, A, V, f\}$ is a set valued information, where U is a non-empty finite set of objects, A is a finite set of attributes, $V = \cup V_a$ is the domain of attribute 'a' and $f: U \times A \to \wp(V)$ is a function such that for every $x \in U$ and $a \in A$, $f(x, a) \subseteq V_a$. The attribute set is divided into two subsets a set C of condition attributes and a decision attribute d, where $C \cap \{d\} = \emptyset$

Definition 2.4[3]**.** The relation $R_A = \{(x, y) \in U \times U: f(x, a) \supseteq f(y, a)$ for every $a \in A\}$ is called the dominance relation on U.

Definition 2.5[3]**.** The dominance class of $x \in U$ is defined as $[x]_A^R = \{y \in U: (y, x) \in R_A\} = \{y \in U: f(y, a) \supseteq f(x, a)$ for every $a \in A$. Here $E_A^R = \{[x]_A^R : x \in U\}$ is the family of dominance classes on U.

Definition 2.6[3]**.** If $\mathcal{M} = \{U, A, V, f\}$ is a set valued information and a subset X of U, the upper approximation of X is defined as $U_{R_X}(X) = \{x \in U: [x]_A^R \cap X \neq \emptyset\}$ and the lower approximation is defined as $L_{R_X}(X) = \{x \in U: [x]_A^R \subseteq X\}$. The boundary region of X is denoted by $B_{R_X}(X) = U_{R_X}(X) - L_{R_X}(X)$.

Definition 2.7[5]**.** Given a set valued ordered information system $\mathcal{M}$, a subset B of A is said to be a criterion reduction of S if $R_A = R_B$ and $R_M \neq R_A$ for any $\mathcal{M} \subset B$. Criterion reduction is a minimal attribute set B such that $R_A = R_B$.

Definition 2.8[3]**.** If $\tau_R(X)$ is a nano topology on U with respect to X, then the set $\beta_R(X) = \{U, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.

Definition 2.9[5]**.** The criterion reduction of a set valued information system is a minimal attribute subset B of the attribute set A such that $\beta_B = \beta_A$.

Definition 2.10[5]**.** The core of the attribute set A is $\mathrm{CORE(A)} = \{\mathrm{a} \in \mathrm{A}: \beta_{\mathrm{A}} \neq \beta_{\mathrm{A}-\{\mathrm{a}\}}\}$. That is CORE(A) is the intersection of the criterion reduction.

3. Attribute Reduction Using Nano Topological Basis

Consider the following information table, which provides information about eight students. The eligibility criterion for availing of the merit-based and means-based scholarship schemes includes the minimum requirement of aggregate marks greater than 50% in the last examination, in the qualifying exam, or in the first semester.

Suppose a batch of students who joined a higher education institution in a particular year are examined for awarding the scholarship. In the selection process, the following attributes are taken into consideration: the students' mark, which is Honor or Medium; the family income, which is less than 3 lakhs or more than 3 lakhs; the medium of previous study of the student, which is either English or any other regional language; the family residence, which is either rural or urban; the student's involvement in extracurricular activities like sports or not involved in sports; and other scholarships availed by the students.

Students	**Group-I**	**Group-II**	**Group-III**	**Group-IV**	**Group-V**	**Group-VI**	**Decision (D)**
	(MA)	**(FI)**	**(ME)**	**(RE)**	**(EC)**	**(OS)**	
T_1	H	< 3L	E	Ur	NS	Not Availed	Selected
T_2	H	< 3L	R	Ru	S	Availed	Selected
T_3	H	> 3L	R	Ru	S	Availed	Selected
T_4	M	> 3L	R	Ur	NS	Not Availed	Not Selected
T_5	M	< 3L	E	Ru	NS	Not Availed	Selected
T_6	H	> 3L	R	Ru	S	Availed	Not Selected
T_7	H	> 3L	R	Ur	NS	Not Availed	Selected
T_8	M	< 3L	E	Ru	NS	Not Availed	Not Selected

Table.3.1

In table.3.1,

Group-I: Marks (MA) → Honor (H), Medium (M)
Group-II: Family Income (FI) → Income < 3 lakhs(<3L), Income > 3 lakhs(>3L)
Group-III: Medium (ME) → English (E), Regional Language (R)
Group-IV: Residence (RE)→ Rural (Ru), Urban (Ur)
Group-V: Extra-Curricular (EC)→ Sports (S) / Non-Sports (NS)
Group-VI: Other Scholarship (OS)→ Availed / Not Availed

Given $\mathcal{M} = \{U, A, V, f\}$ is the set valued information system, $U = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ is the non-empty finite set of objects, and $A = \{MA, FI, ME, RE, EC, OS, D\}$ is the finite set of attributes, $V = \cup V_a$ is the domain of attribute 'a'.
Here, $V = \{H, M, < 3L, > 3L, E, R, Ru, Ur, S, NS, Availed, Not\ Availed\}$. The function f is defined by $f: U \times A \to \wp(V)$ is a function such that for every $x \in U$ and $a \in A$, $f(x, a) \subseteq V_a$, where $\wp(V)$ is the power set of V. Here $f(S_1, MA) = H$, $f(S_3, RE) = Ru$ and so on. The attribute set is divided into two subsets a set C condition attributes and a decision attribute D, where $C \cap D = \emptyset$ and the decision attribute D describes whether the student is selected for the scholarship or not. The concept of elimination of attributes in Nano Topology is applied to identify the key factors for selecting eligible students to get the scholarship.

The family of dominance classes is given by

$$U_C/R = \{\{T_1\},\{T_2\},\{T_3,T_6\},\{T_4\},\{T_5,T_8\},\{T_7\}\}$$

Case-1: Students who received the Scholarship

Assume $X = \{T_1, T_2, T_3, T_5, T_7\}$, the set of students selected for scholarship. The lower and upper approximations are $L_C(X) = \{T_1, T_2, T_7\}$, $U_C(X) = \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}$ and the boundary region is $B_C(X) = \{T_3, T_5, T_6, T_8\}$. The nano topology on U is given by $\tau_C(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}$ and the basis for $\tau_C(X)$ is $\beta_C(X) = \{U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}\}$.

Step-1:

Let $B_1 = C - \{MA\}$. Then $U_{B1}/R = \{\{T_1\},\{T_2\},\{T_3,T_6\},\{T_4,T_7\},\{T_5,T_8\}\}$ and the lower and upper approximations are $L_{B1}(X) = \{T_1, T_2\}$, $U_{B1}(X) = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$. The corresponding nano topology and the basis are given by $\tau_{B1}(X) = \{\emptyset, U, \{T_1, T_2\}, \{T_3, T_4, T_5, T_6, T_7, T_8\}\}$, $\beta_C(X) = \{U, \{T_1, T_2\}, \{T_3, T_4, T_5, T_6, T_7, T_8\}\}$. $\beta_{B1}(X) \neq \beta_C(X)$.

Let $B_2 = C - \{FI\}$. Then $U_{B2}/R = \{\{T_1\},\{T_2,T_3,T_6\},\{T_4\},\{T_5,T_8\},\{T_7\}\}$ and the corresponding nano topology and basis are $\tau_{B2}(X) = \{\emptyset, U, \{T_1, T_7\}, \{T_2, T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\}$ and $\beta_{B2}(X) = \{U, \{T_1, T_7\}, \{T_2, T_3, T_5, T_6, T_8\}$. $\beta_{B2}(X) \neq \beta_C(X)$.

If $B_3 = C - \{ME\}$. Then $U_{B3}/R = \{\{T_1\},\{T_2\},\{T_3,T_6\},\{T_4\},\{T_5,T_8\},\{T_7\}\}$ and $\tau_{B3}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} = \tau_C(X)$. $\beta_{B3}(X) = \{U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}\} = \beta_C(X)$.

If $B_4 = C - \{RE\}$. Then $U_{B4}/R = \{\{T_1\},\{T_2\},\{T_3,T_6\},\{T_4\},\{T_5,T_8\},\{T_7\}\}$ and $\tau_{B4}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} = \tau_C(X)$. Thus, $\beta_{B4}(X) = \beta_C(X)$.

If $B_5 = C - \{EC\}$. Then $U_{B5}/R = \{\{T_1\},\{T_2\},\{T_3,T_6\},\{T_4\},\{T_5,T_8\},\{T_7\}\}$ and $\tau_{B5}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} = \tau_C(X)$. $\beta_{B5}(X) = \beta_C(X)$.

If $B_6 = C - \{OS\}$. Then $U_{B6}/R = \{\{T_1\},\{T_2\},\{T_3,T_6\},\{T_4\},\{T_5,T_8\},\{T_7\}\}$ and $\tau_{B6}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} = \tau_C(X)$. Then, $\beta_{B6}(X) = \beta_C(X)$.

Hence B_3, B_4, B_5, B_6 are the criterion reducts for the student's selection.

Step-2: Consider $B_3 = C - \{ME\} = \{MA, FI, RE, EC, OS\}$.

If $S_1 = B_3 - \{MA\} = \{FI, RE, EC, OS\}$, then $U_{S1}/R = \{\{T_1\},\{T_2\},\{T_3,T_6\},\{T_4,T_7\},\{T_5,T_8\}\}$

and the lower and upper approximations are $L_{S1}(X) = \{T_1, T_2\}$, $U_{S1}(X) = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$. The corresponding nano topology and the basis are $\tau_{S1}(X) = \{\emptyset, U, \{T_1, T_2\}, \{T_3, T_4, T_5, T_6, T_7, T_8\}\}$, $\beta_{S1}(X) = \{U, \{T_1, T_2\}, \{T_3, T_4, T_5, T_6, T_7, T_8\}\}$. So, $\beta_{S1}(X) \neq \beta_C(X)$.

If $S_2 = B_3 - \{FI\} = \{MA, RE, EC, OS\}$, then $U_{S2}/R = \{\{T_1, T_7\}, \{T_2, T_3, T_6\}, \{T_4\}, \{T_5, T_8\}\}$ and the corresponding nano topology is $\tau_{S2}(X) = \{\emptyset, U, \{T_1, T_7\}, \{T_2, T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} \neq \tau_C(X)$. Therefore, $\beta_{S2}(X) \neq \beta_C(X)$.

If $S_3 = B_3 - \{RE\} = \{MA, FI, EC, OS\}$, then $U_{S3}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and the corresponding nano topology is $\tau_{S3}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} = \tau_C(X)$. Thus, $\beta_{S3}(X) = \beta_C(X)$.

If $S_4 = B_3 - \{EC\} = \{MA, FI, RE, OS\}$, then $U_{S4}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and $\tau_{S4}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} = \tau_C(X)$. $\beta_{S4}(X) = \beta_C(X)$.

Similarly, if $S_5 = B_3 - \{OS\} = \{MA, FI, RE, EC\}$. Then $U_{S5}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and the corresponding nano topology is $\tau_{S5}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} = \tau_C(X)$. So, $\beta_{S5}(X) = \beta_C(X)$.

Hence S_3, S_4, S_5 are the criterion reducts for the information system.

CORE(A)=$S_3 \cap S_4 \cap S_5$

$= \{MA, FI, EC, OS\} \cap \{MA, FI, RE, OS\} \cap \{MA, FI, RE, EC\} = \{MA, FI\}$.

CORE={Mark, Family Income}

Step-3: $B_4 = C - \{RE\} = \{MA, FI, ME, EC, OS\}$.

If $V_1 = B_4 - \{MA\} = \{FI, ME, EC, OS\}$. Then $U_{V1}/R = \{\{T_1, T_5, T_8\}\}, \{T_2\}, \{T_3, T_6\}, \{T_4, T_7\}\}$. The nano topology and the basis are $\tau_{V1}(X) = \{\emptyset, U, \{T_2\}, \{T_1, T_3, T_4, T_5, T_6, T_7, T_8\}\}$, $\beta_{V1}(X) = \{U, \{T_2\}, \{T_1, T_3, T_4, T_5, T_6, T_7, T_8\}\}$. This implies that $\beta_{V1}(X) \neq \beta_C(X)$.

If $V_2 = B_4 - \{FI\} = \{MA, ME, EC, OS\}$, then $U_{V2}/R = \{\{T_1\}, \{T_2, T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and the nano topology is $\tau_{V2}(X) = \{\emptyset, U, \{T_1, T_7\}, \{T_2, T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} \neq \tau_C(X)$. Thus, $\beta_{V2}(X) = \{U, \{T_1, T_7\}, \{T_2, T_3, T_5, T_6, T_8\}\} \neq \beta_C(X)$.

If $V_3 = B_4 - \{ME\} = \{MA, FI, EC, OS\}$. Then $U_{V3}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and $\tau_{V3}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} = \tau_C(X)$. Therefore, $\beta_{V3}(X) = \beta_C(X)$.

If $V_4 = B_4 - \{EC\} = \{MA, FI, ME, OS\}$, then $U_{V4}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and $\tau_{V4}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\}$ $= \tau_C(X)$. $\beta_{V4}(X) = \beta_C(X)$.

If $V_5 = B_4 - \{OS\} = \{MA, FI, ME, EC\}$, then $U_{V5}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and $\tau_{V5}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\}$ $= \tau_C(X)$. $\beta_{V5}(X) = \beta_C(X)$.
Therefore V_3, V_4, V_5 are the criterion reducts.
CORE(A) = $V_3 \cap V_4 \cap V_5 = \{MA, FI, EC, OS\} \cap \{MA, FI, ME, OS\} \cap \{MA, FI, ME, EC\}$
$= \{MA, FI\}$.
So, CORE={Mark, Family Income}

Step-4: $B_5 = C - \{EC\} = \{MA, FI, ME, RE, OS\}$.
If $W_1 = B_5 - \{MA\} = \{FI, ME, RE, OS\}$. Then $U_{W1}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4, T_7\}, \{T_5, T_8\}\}$ and $\tau_{W1}(X) = \{\emptyset, U, \{T_1, T_2\}, \{T_3, T_4, T_5, T_6, T_7, T_8\}\}$, $\beta_{W1}(X) = \{U, \{T_1, T_2\}, \{T_3, T_4, T_5, T_6, T_7, T_8\}\} \neq \beta_C(X)$.

If $W_2 = B_5 - \{FI\} = \{MA, ME, RE, OS\}$. Then $U_{W2}/R = \{\{T_1, T_7\}, \{T_2, T_3, T_6\}, \{T_4\}, \{T_5, T_8\}\}$ and the corresponding nano topology is given by $\tau_{W2}(X) = \{\emptyset, U, \{T_1, T_7\}, \{T_2, T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} \neq \tau_C(X)$. Therefore, $\beta_{W2}(X) \neq \beta_C(X)$.

If $W_3 = B_5 - \{ME\} = \{MA, FI, RE, OS\}$. Then $U_{W3}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and $\tau_{W3}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\}$ $= \tau_C(X)$. So, $\beta_{W3}(X) = \beta_C(X)$.

If $W_4 = B_5 - \{RE\} = \{MA, FI, ME, OS\}$. Then $U_{W4}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and $\tau_{W4}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\}$ $= \tau_C(X)$. $\beta_{W4}(X) = \beta_C(X)$.
Similarly, if $W_5 = B_5 - \{OS\} = \{MA, FI, ME, RE\}$. Then $U_{W5}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and the corresponding nano topology is given by $\tau_{W5}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} = \tau_C(X)$. Thus, $\beta_{W5}(X) = \beta_C(X)$.

Hence W_3, W_4, W_5 are the criterion reducts and CORE(A)=$W_3 \cap W_4 \cap W_5$
$= \{MA, FI, RE, OS\} \cap \{MA, FI, ME, OS\} \cap \{MA, FI, ME, RE\} = \{MA, FI\}$
That is CORE={Mark, Family Income}

Step-5: $B_6 = C - \{OS\} = \{MA, FI, ME, RE, EC\}$.
If $Y_1 = B_6 - \{MA\} = \{FI, ME, RE, EC\}$. Then $U_{Y1}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4, T_7\}, \{T_5, T_8\}\}$ and $\tau_{Y1}(X) = \{\emptyset, U, \{T_1, T_2\}, \{T_3, T_4, T_5, T_6, T_7, T_8\}\}$. The basis is $\beta_{Y1}(X) = \{U, \{T_1, T_2\}, \{T_3, T_4, T_5, T_6, T_7, T_8\}\} \neq \beta_C(X)$

If $Y_2 = B_6 - \{FI\} = \{MA, ME, RE, EC\}$. Then $U_{Y2}/R = \{\{T_1\}, \{T_2, T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and the nano topology is $\tau_{Y2}(X) = \{\emptyset, U, \{T_1, T_7\}, \{T_2, T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} \neq \tau_C(X)$. Therefore, $\beta_{Y2}(X) \neq \beta_C(X)$.

Similarly, for $Y_3 = B_6 - \{ME\} = \{MA, FI, RE, EC\}$, $Y_4 = B_6 - \{RE\} = \{MA, FI, ME, EC\}$ and $Y_5 = B_6 - \{EC\} = \{MA, FI, ME, RE\}$, $U_{Y3}/R = U_{Y4}/R = U_{Y5}/R = \{\emptyset, \{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4\}, \{T_5, T_8\}, \{T_7\}\}$ and $\tau_{Y3}(X) = \{\emptyset, U, \{T_1, T_2, T_7\}, \{T_3, T_5, T_6, T_8\}, \{T_1, T_2, T_3, T_5, T_6, T_7, T_8\}\} = \tau_{Y4}(X) = \tau_{Y5}(X) = \tau_C(X)$. Thus, $\beta_{Y3}(X) = \beta_{Y4}(X) = \beta_{Y5}(X) = \beta_C(X)$.

Hence Y_3, Y_4, Y_5 are the criterion reducts and CORE(A)=$Y_3 \cap Y_4 \cap Y_5$
$=\{MA, FI, RE, EC\} \cap \{MA, FI, ME, EC\} \cap \{MA, FI, ME, RE\} = \{MA, FI\}$
Therefore, CORE={Mark, Family Income}

<u>Case-2</u>: Students who didn't received the Scholarship

Assume $X = \{T_4, T_6, T_8\}$, the set of students not selected for the scholarship. Then, $U_C/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4, T_7\}, \{T_5, T_8\}\}$ and the lower and upper approximations are $L_C(X) = \{T_4\}$,$U_C(X) = \{T_3, T_4, T_5, T_6, T_8\}$. The boundary region is $B_C(X) = \{T_3, T_5, T_6, T_8\}$. The nano topology on U is $\tau_C(X) = \{\emptyset, U, \{T_4\}, \{T_3, T_5, T_6, T_8\}, \{T_3, T_4, T_5, T_6, T_8\}\}$ and the basis for $\tau_C(X)$ is $\beta_C(X) = \{U, \{T_4\}, \{T_3, T_4, T_5, T_6, T_8\}\}$.

Step-1:

Let $B_1 = C - \{MA\}$. Then $U_{B1}/R = \{\{T_1\}, \{T_2\}, \{T_3, T_6\}, \{T_4, T_7\}, \{T_5, T_8\}\}$ and the lower and upper approximations are $L_{B1}(X) = \emptyset$,$U_{B1}(X) = \{T_3, T_4, T_5, T_6, T_7, T_8\}$ corresponding nano topology and the basis are $\tau_{B1}(X) = \{\emptyset, U, \{T_3, T_4, T_5, T_6, T_7, T_8\}\}$, $\beta_{B1}(X) = \{U, \emptyset, \{T_3, T_4, T_5, T_6, T_7, T_8\}\}$. This implies that $\beta_{B1}(X) \neq \beta_C(X)$.

If $B_2 = C - \{FI\}$. Then $\tau_{B2}(X) = \{\emptyset, U, \{T_4\}, \{T_2, T_3, T_5, T_6, T_8\}, \{T_2, T_3, T_4, T_5, T_6, T_8\}\}$, $\beta_{B2}(X) = \{U, \{T_4\}, \{T_2, T_3, T_5, T_6, T_8\}\}$. Thus $\beta_{B2}(X) \neq \beta_C(X)$.

If $B_3 = C - \{ME\}$. Then $\tau_{B3}(X) = \{\emptyset, U, \{T_4\}, \{T_3, T_5, T_6, T_8\}, \{T_3, T_4, T_5, T_6, T_8\}\}$, $\beta_{B3}(X) = \{U, \emptyset, \{T_3, T_5, T_6, T_8\}\} = \beta_C(X)$.

Similarly, if $B_4 = C - \{RE\}$, $B_5 = C - \{EC\}$, $B_6 = C - \{OS\}$, then $\tau_{B4}(X) = \{\emptyset, U, \{T_4\}, \{T_3, T_5, T_6, T_8\}, \{T_3, T_4, T_5, T_6, T_8\}\} = \tau_{B5}(X) = \tau_{B6}(X)$,
Thus $\beta_{B4}(X) = \beta_{B5}(X) = \beta_{B6}(X) = \beta_C(X)$.

Hence B_3, B_4, B_5, B_6 are the criterion reducts for student's selection and CORE=$B_3 \cap B_4 \cap B_5 \cap B_6$= {Marks, Family income}

4. Discussion

In this article, attribute reduction using the basis of nanotopological spaces is introduced and explored to improve the selection process of students eligible for the merit-based scholarship program. Applying this new approach, we sought to identify the key factors that play a central role in determining which deserving students will receive a scholarship. Our analysis revealed that the main attributes that influence the selection process are the students' marks and family income.

The academic performance of the students, as represented by their marks, serves as a critical indicator of their potential and dedication to their studies. On the other hand, family income is a key determinant of the financial needs of students. By considering both of these attributes, administrators can ensure that the scholarship reaches those students who retain the necessary academic merit and financial constraints to truly benefit from the financial assistance.

Applying the nano topological basis concepts in attribute reduction provides an efficient and systematic way to identify the most relevant attributes for stock market selection, identifying suitable crops for a particular season, etc. By focusing on key attributes, administrators can avoid unnecessary complexity and redundancy in the decision-making process, which facilitates efficient and fair resource allocation. Our findings contribute to the field of scholarship awarding and education policy as they provide an objective, data-driven method for identifying the most deserving students. This approach ensures that scholarships are awarded to those who need them and who have demonstrated their academic potential.

In conclusion, the concept of attribute reduction using the basis in Nano Topological Spaces proves to be a valuable tool in the scholarship selection process. By identifying key attributes of student grades and family income, we can ensure a more efficient and effective allocation of the scholarship program based on merit and means. This research serves as a foundation for further exploration and optimization of scholarship awarding methods, ultimately leading to a more inclusive and equitable education system.

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Inventory model with preservation technology and exponential holding cost in fuzzy scenario

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Abstract

Inventories are ubiquitous in the business sector. Since inventory is most frequently incurring expense, stock control is critical for an organization and it must be scrimping and saving in contemplation of function the merchandising fruitfully. In this paper, an inventory model for a deteriorating item under exponential holding cost with collaborative preservation technology investment under carbon policy is considered. Also, this study is developed in a fuzzy scenario by employing triangular fuzzy numbers. Signed distance method is utilized to enhance decision making and optimization. Further the convexity of the total cost function for both the crisp and the fuzzy case is established. The objective is to determine the optimal investment in preservation technology and the optimal cycle length so as to minimize the total cost. Moreover, some managerial results are obtained by using sensitivity analysis and graphical representation is also carried out. The applications of the proposed model is used in the fields of constructing machinery or heavy duty construction equipment, specific chemicals and processed food.

Keywords: carbon emission; preservation technology; signed distance method; stock dependent demand; triangular fuzzy number.
2010 AMS subject classification: 94D05, 35Q93.[‡]

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1. Introduction

The displayed inventory level is a promotional device in today's globalized technology to boost income. According to numerous researches, having a significant number of products on display may attract more consumers. This implies a positive relationship between demand and stock levels. As a result, demand in this model is considered as stock-dependent, which is more practical. Bardhan et al. [1] investigated an optimal replenishment policy and preservation technology investment for a non-instantaneous deteriorating item with stock-dependent demand. Preservation technology is a critical component in reducing deterioration. Giri et al. [7] have explored a supply chain model for time-dependent deteriorating item with preservation technology investment. For a single-vendor multi-buyer model, Setiawan and Endrayanto [11] implemented a coordination strategy and synchronization in the production flow, including adjustable lead time. Khanna et al. [9] adopted an optimizing preservation strategy for deteriorating items with time-varying holding cost and stock dependent demand.

Global warming poses a significant hazard to our planet. The world's attention is currently focused on reducing carbon emissions. Dye and Yang [6] anticipated that ordering and storing inventory causes carbon emissions. They looked at sustainability in the context of a collaborative trade credit arrangement, where demand is tied to the credit period. Daryanto and Wee [4] considered a production lot size decision of a manufacturer incorporating environmental impact of carbon emission. Tao and Xu [14] developed an inventory model concerning emission-regulation policies with consumer's low carbon awareness, providing decision support. Shen et al. [13] developed a production inventory model for deteriorating items with collaborative preservation technology investment under carbon tax. Yu et al. [3] presented an inventory model of a deteriorating product considering carbon emissions. Patel et al. [5] decided optimal order quantity for the industries especially chemical industries with trended demand under trade credit with existence of cap and trade structure to reduce carbon emissions. Tripathi and Mishra [12] investigated an EOQ model with linear time dependent demand and different holding cost functions.

In the above analysis, it is presumed that all parameters are precisely known. But in real world, parameters are imprecise in nature and one has to deal with approximation of numbers that are close to real numbers. Fuzzy number provides a way to model this epistemic uncertainty and its propagation. Bjork [2] analysed an EOQ model in a fuzzy environment. Alrefaei and Tuffaha [10] studied an intuitionistic polygonal fuzzy numbers. Hemalatha and Annadurai [8] proposed an integrated production-distribution inventory system for deteriorating products in fuzzy environment by ensuring extra investment thereby reducing setup cost.

We develop the model including some points which highlight the novelty of our model. In our model, the deterioration effect of the product is considered and preservation technology is addressed to regulate the deterioration rate. Demand is cogitated as stock-dependent and the holding cost is ruminated as an exponential. Under

carbon emission regulations, the goal is to resolute the optimal investment in preservation technology and cycle time (Dye & Yang, [6]).

As the path of developing EOQ models with uncertainty expressed as fuzzy numbers are quite profitable, the fuzzy model is discussed in this study. The cost parameters are considered as a triangular fuzzy numbers. The total cost function is defuzzified and proven to be convex using the signed distance method.

Following an introduction, the remainder of the article is organized as follows: The second section is devoted to the related preliminary definitions. In Section 3, notations and assumptions are shown. In Section 4, a mathematical model for the crisp model is developed and another mathematical model for a fuzzy model is developed in Section 5. Numerical example is provided to illustrate the crisp and fuzzy models in Section 6. In Section 7, sensitivity analysis and managerial insights are provided to validate the concept. Comparative study is given in Section 8. Finally, conclusion and future research direction are given in the last section.

2. Preliminaries

The following definitions of fuzzy sets are relevant to the method used in the proposed model.

Definition 2.1 Triangular Fuzzy Numbers: Let $\tilde{D} = (p_1, p_2, p_3), p_1 < p_2 < p_3$ be a triangular fuzzy number with membership function:

$$\mu_{\tilde{B}}(x) = \begin{cases} \dfrac{x - p_1}{p_2 - p_1}, & p_1 \le x \le p_2 \\ \dfrac{p_3 - x}{p_3 - p_2}, & p_2 \le x \le p_3 \\ 0, & otherwise\ . \end{cases}$$

Definition 2.2 Signed distance method (Bjork [2]): The signed distance of $[L_\alpha, R_\alpha]$ measured from 0 is $\mathrm{d}_0\left([L_\alpha, R_\alpha], \tilde{0}\right) = \dfrac{\tilde{D}_L(\alpha) + \tilde{D}_R(\alpha)}{2}$. For the triangular fuzzy number $\tilde{B} \in R^-$, the distance from $\tilde{D}$ to 0 is written as $d(\tilde{D}, \tilde{0}) = \dfrac{1}{4}(p_1 + 2p_2 + p_3)$.

3. Notations and assumptions

To develop the proposed model, we adopt the following notations and assumptions.

Notations:

The notations used in our model are listed as follows:

T_e	Length of cycle
τ_e	Investment in preservation technology per unit time
A_{ece}	Fixed carbon emission per order
c_{ece}	Carbon emission per unit per order
C_{eoc}	Cost of ordering (per order)

C_{edc}	Unit cost due to deterioration
C_{ehc}	Holding cost per unit time t $(C_{ehc}(t) = h_{ehc}\, e^{rt})$
C_{epc}	Purchasing cost per unit
h_{ece}	Carbon emission for inventory per unit time
$I_{eh}(t)$	Level of inventory at time $t, 0 \le t \le T_e$
Q	Size of order
$TC_{ep}(T_e, \tau_e)$	Total cost of the system
y_0	Rate of deterioration in the absence of preservation technologies
$y(\tau)$	Investment in preservation technologies reduces the rate of deterioration

Assumptions:

The following assumptions are made in the model:

1) The rate of demand is directly proportional to the stock level. i.e.,
$$D(I_{eh}(t)) = a + bI_{eh}(t), \quad a > 0,\ 0 < b < 1.$$
2) The time horizon is infinite with negligible lead time.
3) Shortages are not allowed.
4) Preservation technology investment reduces the rate of deterioration gradually. The reduced deterioration rate $y(\tau_e)$, is a function of preservation technology cost τ_e such that $y(\tau_e) = y_0 e^{-u\tau_e}$, which satisfies the conditions $\partial TC_{ep}/\partial \tau_e < 0,\ \partial^2 TC_{ep}/\partial \tau_e^{\,2} > 0$ and $y(0) = y_0$, where u is sensitivity parameter of investment $0 < u < 1$.
5) The holding cost is considered to be dependent on time as $C_{ehc}(t) = h_{ehc}\, e^{rt},\ 0 < r < 1.$
6) The total amount of carbon emissions includes emissions from ordering, holding and purchasing inventory.

4. Mathematical model

In this section, a mathematical model is developed to determine the cycle time and optimal investment in preservation technology. Deteriorating items are likewise regarded with low carbon emission cost and preservation technology in our inventory system. The equation governing the inventory level (Khanna et al. [9]) can be expressed as follows:

$$\frac{dI_{eh}(t)}{dt} + y(\tau) I_{eh}(t) = -(a + bI_{eh}(t)), \quad 0 \le t \le T_e. \tag{1}$$

The solution of equation (1) using the boundary condition $I_{eh}(T_e) = 0,$ is given by

$$I_{eh}(t) = a\left[(T_e - t) + \left((T_e - t)^2 (y(\tau_e) + b)/2\right)\right], \tag{2}$$

and the initial inventory level is

$$Q = I_{eh}(0) = a\left[T_e + \left(\left(T_e^2 (y(\tau_e) + b)\right)/2\right)\right]. \tag{3}$$

The total cost of the system is calculated by adding the following costs.

Ordering cost: C_{eoc},

Preservation technology investment: $PT = \tau_e T_e$,

Holding cost: $EHC = \int_0^{T_e} h_{ehc} e^{rt} I_{eh}(t)dt$

$$= ah_{ehc}\left[\left(-T_e / r\right)+\left(e^{rT_e} / r^2\right)-\left(1/r^2\right)+\left(\left(y(\tau_e)+b\right)/2\right)\left(\left(-T_e^2 / r\right)-\left(2T_e / r^2\right)+\left(2e^{-rT_e} / r^3\right)+\left(2/r^3\right)\right)\right],$$

Deterioration cost: $EDC = C_{edc}\left(Q - \int_0^{T_e} D\left(I_{eh}(t)\right)dt\right)$

$$= C_{edc}\left[\left(T_e^2 y(\tau_e)\left(3a - abT_e\right) - ab^2 T_e^3\right)/6\right],$$

and Purchasing cost: $EPC = C_{epc} Q = aC_{epc}\left[T_e + \left(\left(T_e^2\left(y(\tau_e)+b\right)\right)/2\right)\right].$

Then total carbon emissions TEC in a finite time horizon T (Dye & Yang, [6]) is

$$TC_{ep}\left(T_e, \tau_e\right) = A_{ece} + c_{ece} Q + h_{ece}\int_0^{T_e} I_{eh}(t)dt$$

$$= A_{ece} + ac_{ece}\left(T_e + \left(T_e^2\left(y(\tau_e)+b\right)/2\right)\right) + ah_{ece}\left(\left(T_e^2/2\right)+\left(T_e^3\left(y(\tau_e)+b\right)/6\right)\right).$$

i.e.,

$$TC_{ep}\left(T_e, \tau_e\right) = \left(C_{eoc}/T_e\right)+\left(ah_{ehc}/T\right)\left[\left(-T_e/r\right)+\left(e^{rT_e}/r^2\right)-\left(1/r^2\right)+\left(\left(y(\tau_e)+b\right)/2\right)\right.$$
$$\left.\left(\left(-T_e^2/r\right)-\left(2T_e/r^2\right)+\left(2e^{-rT_e}/r^3\right)+\left(2/r^3\right)\right)\right]+\left(aC_{epc}/T\right)\left[T_e+\left(\left(T_e^2\left(y(\tau_e)+b\right)\right)/2\right)\right]$$
$$+\left(C_{edc}/T_e\right)\left[\left(T_e^2 y(\tau_e)\left(3a-abT_e\right)-ab^2T_e^3\right)/6\right]+\left(\tau_e T_e/T_e\right)$$
$$+\frac{1}{T_e}\left[A_{ece}+ac_{ece}\left(T_e+\left(T_e^2\left(y(\tau_e)+b\right)/2\right)\right)+ah_{ece}\left(\left(T_e^2/2\right)+\left(T_e^3\left(y(\tau_e)+b\right)/6\right)\right)\right]. \quad (4)$$

The objective is to minimize the total cost by jointly optimizing the cycle time T_e and the investment in preservation technology τ_e. To establish optimality, taking the first order partial derivative and equate it into zero, we get

$$\partial TC_{ep}/\partial T_e = 0 \text{ and } \partial TC_{ep}/\partial \tau_e = 0, \quad (5)$$

That is

$$\left(-C_{eoc}/T_e^2\right)+\left(ah_{ehc}e^{rT_e}/r^3T_e^2\right)\left(r+\left(y_0e^{-u\tau_e}+b\right)\right)\left(rT_e-1\right)+\left(ah_{ehc}/r^2T_e^2\right)+\left(y_0e^{-u\tau_e}+b\right)$$
$$\left(\left(-ah_{ehc}/2r\right)-\left(ah_{ehc}/r^3T_e^2\right)\right)+\left(aC_{epc}/2\right)\left(y_0e^{-u\tau_e}+b\right)+\left(C_{edc}/6\right)\left(3ay_0e^{-u\tau_e}-2abT_ey_0e^{-u\tau_e}-2ab^2T_e\right) \quad (6)$$
$$+\left(ac_{ece}/2\right)\left(y_0e^{-u\tau_e}+b\right)-\left(A_{ece}/T_e^2\right)+\left(ah_{ece}/6\right)\left[3+2T_e\left(y_0e^{-u\tau_e}+b\right)\right]=0,$$

and

$$1-\left(h_{ehc}auy_0e^{-(u\tau_e-rT_e)}/r^3T_e\right)+uy_0e^{-u\tau_e}\left(\left(ah_{ehc}T_e/2r\right)+\left(ah_{ehc}/r^2\right)-\left(ah_{ehc}/r^3T_e\right)\right)-\left(C_{dc}uy_0e^{-u\tau_e}T_e(3a-abT_e)/6\right) \quad (7)$$
$$-\left(ah_{ece}T_e^2uy_0e^{-u\tau_e}/6\right)-\left(a\left(C_{epc}+c_{ece}\right)T_euy_0e^{-u\tau_e}/2\right)=0.$$

We now derive the optimal values of T_e and τ_e as T_e^* and τ_e^* by simultaneously solving equations (6) and (7). We derive the total cost of the system by replacement these values into equation (4). Equation (3) is used to determine the optimal order quantity. By examining the second order sufficient conditions, for the total cost equation (4) to be minimum are

$$\frac{\partial^2 TC_{ep}}{\partial T_e^{\,2}} = \frac{2C_{eoc}}{T_e^{\,3}} + \frac{ah_{ehc}e^{-rT_e}}{r^3T_e^{\,3}}\left(r-\left(y_0e^{-u\tau_e}+b\right)\right)\left(2rT_e+r^2T_e^{\,2}+2\right)-\frac{2ah_{ehc}}{r^2T_e^{\,3}}$$
$$+\frac{2ah_{ehc}}{r^3T_e^{\,3}}\left(y_0e^{-u\tau_e}+b\right)-\frac{C_{edc}ab}{3}\left(y_0e^{-u\tau_e}+b\right)+\frac{2A_{ece}}{T_e^{\,3}}+\frac{2ah_{ece}\left(y_0e^{-u\tau_e}+b\right)}{3}>0,$$

$$\frac{\partial^2 TC_{ep}}{\partial \tau_e^{\,2}} = -\frac{h_{ehc}au^2y_0e^{-(u\tau_e+rT_e)}}{r^3T_e}+u^2y_0e^{-u\tau_e}\left(\frac{ah_{ehc}T_e}{2r}-\frac{ah_{ehc}}{r^2}+\frac{ah_{ehc}}{r^3T}\right)$$
$$+\frac{C_{edc}au^2y_0e^{-u\tau_e}T_e(3-bT_e)}{6}+\frac{ah_{ece}T_eu^2y_0e^{-u\tau_e}}{3}+\frac{a\left(C_{epc}+c_{ece}\right)T_eu^2y_0e^{-u\tau_e}}{2}>0,$$

and

$$\frac{\partial^2 TC_{ep}}{\partial \tau_e \partial T_e} = \frac{\partial^2 TC_{ep}}{\partial T_e \partial \tau_e} = -\frac{h_{hc}auy_0e^{-(u\tau_e+rT_e)}}{r^3T_e^{\,3}}\left(rT_e+1\right)-ah_{hc}uy_0e^{-u\tau_e}\left(\frac{1}{2r}-\frac{1}{r^3T_e^{\,2}}\right)$$
$$-\frac{C_{edc}auy_0e^{-u\tau_e}(3-2bT_e)}{6}-\frac{a\left(C_{epc}+c_{ece}\right)uy_0e^{-u\tau_e}}{2}-\frac{2ah_{ece}T_euy_0e^{-u\tau_e}}{3}.$$

Then, we have

$$\left(\left(\partial^2 TC_{ep}(T_e,\tau_e)/\partial T_e^{\,2}\right).\left(\partial^2 TC_{ep}(T_e,\tau_e)/\partial \tau_e^{\,2}\right)\right)-\left(\left(\partial^2 TC_{ep}(T_e,\tau_e)/\partial T_e\partial \tau_e\right).\left(\partial^2 TC_{ep}(T_e,\tau_e)/\partial T_e\partial \tau_e\right)\right)>0. \quad (8)$$

Since all the second order derivatives are highly nonlinear, the optimality is determined graphically (Figure 1).

5. Fuzzy model

The fuzzy inventory model, including the fuzzification and defuzzification processes, is described in this section..

The method of fuzzification involves transforming crisp parameters into fuzzy parameters. Fuzzy variables can be represented by the membership function given in the preliminary section for triangular fuzzy numbers. Here, we consider the ordering cost, holding cost, holding cost component, demand parameters, and purchasing cost as uncertain. They are represented as triangular fuzzy numbers as follows:

$$\begin{aligned}
\tilde{C}_{eoc} &= (C_{eoc}-\Delta_1,\ C_{eoc},\ C_{eoc}+\Delta_2), && 0<\Delta_1<C_{eoc},\ \Delta_2>0,\\
\tilde{C}_{edc} &= (C_{edc}-\Delta_3,\ C_{edc},\ C_{edc}+\Delta_4), && 0<\Delta_3<C_{edc},\ \Delta_4>0,\\
\tilde{a} &= (a-\Delta_5,\ a,\ a+\Delta_6), && 0<\Delta_5<a,\ \Delta_6>0,\\
\tilde{b} &= (b-\Delta_7,\ b,\ b+\Delta_8), && 0<\Delta_7<b,\ \Delta_8>0,\\
\tilde{h}_{ehc} &= (h_{ehc}-\Delta_9,\ h_{ehc},\ h_{ehc}+\Delta_{10}), && 0<\Delta_9<h_{ehc},\ \Delta_{10}>0,\\
\tilde{r}_{hc} &= (\mathrm{r}_{hc}-\Delta_{11},\ r_{hc},\ r_{hc}+\Delta_{12}), && 0<\Delta_{11}<r_{hc},\ \Delta_{12}>0,\\
\tilde{C}_{epc} &= (C_{epc}-\Delta_{13},\ C_{epc},\ C_{epc}+\Delta_{14}), && 0<\Delta_{13}<C_{epc},\ \Delta_{14}>0.
\end{aligned} \quad (9)$$

Then the left and right α cuts of the various parameters C_{eoc}, C_{edc}, h_{ehc}, r, a, b and C_{epc} are given by

$$\tilde{C}_{eoc_L}(\alpha)=C_{eoc}-\Delta_1+\alpha\Delta_1>0; \qquad \tilde{C}_{eoc_R}(\alpha)=C_{eoc}-\Delta_2+\alpha\Delta_2>0,$$

$$\begin{aligned}
&\tilde{C}_{edc_L}(\alpha)=C_{edc}-\Delta_3+\alpha\Delta_3>0; && \tilde{C}_{edc_R}(\alpha)=C_{edc}-\Delta_4+\alpha\Delta_4>0,\\
&\tilde{a}_L(\alpha)=a-\Delta_5+\alpha\Delta_5>0; && \tilde{a}_R(\alpha)=a-\Delta_6+\alpha\Delta_6>0,\\
&\tilde{b}_L(\alpha)=b-\Delta_7+\alpha\Delta_7>0; && \tilde{b}_R(\alpha)=b-\Delta_8+\alpha\Delta_8>0,\\
&\tilde{h}_{ehc_L}(\alpha)=h_{ehc}-\Delta_9+\alpha\Delta_9>0; && \tilde{h}_{ehc_R}(\alpha)=h_{ehc}-\Delta_{10}+\alpha\Delta_{10}>0,\\
&\tilde{r}_L(\alpha)=r-\Delta_{11}+\alpha\Delta_{11}>0; && \tilde{r}_R(\alpha)=r-\Delta_{12}+\alpha\Delta_{12}>0,\\
&\tilde{C}_{epc_L}(\alpha)=C_{epc}-\Delta_{13}+\alpha\Delta_{13}>0; && \tilde{C}_{epc_R}(\alpha)=C_{epc}-\Delta_{14}+\alpha\Delta_{14}>0.
\end{aligned} \tag{10}$$

Hence, when the costs C_{eoc}, C_{edc}, h_{ehc}, r, a, b and C_{epc} in equation (4) are fuzzified with triangular fuzzy numbers $\tilde{C}_{eoc}$, $\tilde{C}_{edc}$, $\tilde{h}_{ehc}$, $\tilde{r}$, $\tilde{a}$, $\tilde{b}$ and $\tilde{C}_{epc}$ as expressed in equation (10). Thus, the total cost is obtained in fuzzy sense is given by

$$\begin{aligned}
\tilde{\prod} TC_{ep}(T_e,\tau_e)=&\frac{\tilde{C}_{eoc}}{T_e}+\frac{\tilde{a}\tilde{h}_{ehc}}{T_e}\left[\frac{-T_e}{\tilde{r}}+\frac{e^{\tilde{r}T_e}}{\tilde{r}^2}-\frac{1}{\tilde{r}^2}+\frac{\left(y(\tau_e)+\tilde{b}\right)}{2}\left(\frac{-T_e^{\,2}}{\tilde{r}}-\frac{2T_e}{\tilde{r}^2}+\frac{2e^{-\tilde{r}T_e}}{\tilde{r}^3}+\frac{2}{\tilde{r}^3}\right)\right]\\
&+\frac{\tilde{a}\tilde{C}_{epc}}{T_e}\left[T_e+\frac{T_e^{\,2}\left(y(\tau_e)+\tilde{b}\right)}{2}\right]+\frac{\tilde{C}_{edc}}{T_e}\left[\frac{T_e^{\,2}y(\tau_e)\left(3\tilde{a}-\tilde{a}\tilde{b}T_e\right)-\tilde{a}\tilde{b}^2T_e^{\,3}}{6}\right]+\frac{\tau_eT_e}{T_e}\\
&+\frac{1}{T_e}\left[A_{ece}+\tilde{a}c_{ece}\left(T_e+\frac{T_e^{\,2}\left(y(\tau_e)+\tilde{b}\right)}{2}\right)+\tilde{a}h_{ece}\left(\frac{T_e^{\,2}}{2}+\frac{T_e^{\,3}\left(y(\tau_e)+\tilde{b}\right)}{6}\right)\right].
\end{aligned} \tag{11}$$

From equation (11), we obtain the left and right α cuts of $\tilde{\prod} TC_{ep}(T_e,\tau_e)$ is as follows:

$$\begin{aligned}
\tilde{\prod} TC_{ep_L}(T_e,\tau_e)=&\frac{\tilde{C}_{eoc_L}(\alpha)}{T_e}+\frac{\tilde{a}_L(\alpha)\tilde{h}_{ehc_L}(\alpha)}{T_e}\left[\frac{-T_e}{\tilde{r}_L(\alpha)}+\frac{e^{\tilde{r}_L(\alpha)T_e}}{\tilde{r}_L(\alpha)^2}-\frac{1}{\tilde{r}_L(\alpha)^2}+\frac{\left(y(\tau_e)+\tilde{b}_L(\alpha)\right)}{2}\right.\\
&\left.\left(\frac{-T_e^{\,2}}{\tilde{r}_L(\alpha)}-\frac{2T_e}{\tilde{r}_L(\alpha)^2}+\frac{2e^{-\tilde{r}_L(\alpha)T_e}}{\tilde{r}_L(\alpha)^3}+\frac{2}{\tilde{r}_L(\alpha)^3}\right)\right]+\frac{\tilde{a}_L(\alpha)\tilde{C}_{epc_L}(\alpha)}{T_e}\left[T_e+\frac{T_e^{\,2}\left(y(\tau_e)+\tilde{b}_L(\alpha)\right)}{2}\right]\\
&+\frac{\tilde{C}_{edc_L}(\alpha)}{T_e}\left[\frac{T_e^{\,2}y(\tau_e)\left(3\tilde{a}_L(\alpha)-\tilde{a}_L(\alpha)\tilde{b}_L(\alpha)T_e\right)-\tilde{a}_L(\alpha)\tilde{b}_L(\alpha)^2T_e^{\,3}}{6}\right]+\frac{\tau_eT_e}{T_e}\\
&+\frac{1}{T_e}\left[A_{ece}+\tilde{a}_L(\alpha)c_{ece}\left(T_e+\frac{T_e^{\,2}\left(y(\tau_e)+\tilde{b}_L(\alpha)\right)}{2}\right)+\tilde{a}_L(\alpha)h_{ece}\left(\frac{T_e^{\,2}}{2}+\frac{T_e^{\,3}\left(y(\tau_e)+\tilde{b}_L(\alpha)\right)}{6}\right)\right],
\end{aligned}$$

$$\tilde{\prod} TC_{ep_R}(T_e, \tau_e) = \frac{\tilde{C}_{eoc_R}(\alpha)}{T_e} + \frac{\tilde{a}_R(\alpha)\tilde{h}_{ehc_R}(\alpha)}{T_e}\left[\frac{-T_e}{\tilde{r}_R(\alpha)} + \frac{e^{-\tilde{r}_R(\alpha)T_e}}{\tilde{r}_R(\alpha)^2} - \frac{1}{\tilde{r}_R(\alpha)^2} + \frac{\left(y(\tau_e)+\tilde{b}_R(\alpha)\right)}{2}\right.$$
$$\left.\left(\frac{-T_e^2}{\tilde{r}_R(\alpha)} - \frac{2T_e}{\tilde{r}_R(\alpha)^2} + \frac{2e^{-\tilde{r}_R(\alpha)T_e}}{\tilde{r}_R(\alpha)^3} + \frac{2}{\tilde{r}_R(\alpha)^3}\right)\right] \frac{\tilde{a}_R(\alpha)\tilde{C}_{epc_R}(\alpha)}{T_e}\left[T_e + \frac{T_e^2\left(y(\tau_e)+\tilde{b}_R(\alpha)\right)}{2}\right]$$
$$+ \frac{\tilde{C}_{edc_R}(\alpha)}{T_e}\left[\frac{T_e^2 y(\tau_e)\left(3\tilde{a}_R(\alpha) - \tilde{a}_R(\alpha)\tilde{b}_R(\alpha)T_e\right) - \tilde{a}_R(\alpha)\tilde{b}_R(\alpha)^2 T_e^3}{6}\right] + \frac{\tau_e T_e}{T_e}$$
$$+ \frac{1}{T_e}\left[A_{ece} + \tilde{a}_R(\alpha)c_{ece}\left(T_e + \frac{T_e^2\left(y(\tau_e)+\tilde{b}_R(\alpha)\right)}{2}\right) + \tilde{a}_R(\alpha)h_{ece}\left(\frac{T_e^2}{2} + \frac{T_e^3\left(y(\tau_e)+\tilde{b}_R(\alpha)\right)}{6}\right)\right].$$

The process of defuzzifying involves turning the fuzzified results into quantifiable quantities. Hence, the fuzzified total cost equation (11) narrated with triangular fuzzy number is transformed into the crisp function by utilizing signed distance formula. Then the defuzzified total cost is calculated and is given by

$$d(\tilde{\prod} TC_{ep}(T_e, \tau_e), \tilde{0}) = \frac{H_1}{T_e} + \frac{H_3 H_5}{T_e}\left[\frac{-T_e}{H_6} + \frac{e^{-H_6 T_e}}{H_6^2} - \frac{1}{H_6^2} + \frac{\left(y(\tau_e)+H_4\right)}{2}\left(\frac{-T_e^2}{H_6} - \frac{2T_e}{H_6^2} + \frac{2e^{-H_6 T_e}}{H_6^3} + \frac{2}{H_6^3}\right)\right]$$
$$+ \frac{H_3 H_7}{T_e}\left[T_e + \frac{T_e^2\left(y(\tau_e)+H_4\right)}{2}\right] + \frac{H_2}{T_e}\left[\frac{T_e^2 y(\tau_e)\left(3H_3 - H_3 H_4 T_e\right) - H_3 H_4^2 T_e^3}{6}\right] + \frac{\tau_e T_e}{T_e} \quad (12)$$
$$+ \frac{1}{T_e}\left[A_{ece} + H_3 c_{ece}\left(T_e + \frac{T_e^2\left(y(\tau_e)+H_4\right)}{2}\right) + H_3 h_{ece}\left(\frac{T_e^2}{2} + \frac{T_e^3\left(y(\tau_e)+H_4\right)}{6}\right)\right].$$

Where,

$$H_1 = C_{eoc} + \frac{\Delta_2 - \Delta_1}{4} > 0,\ H_2 = C_{edc} + \frac{1}{4}(\Delta_4 - \Delta_3),\ H_3 = a + \frac{1}{4}(\Delta_6 - \Delta_5),$$
$$H_4 = b + \frac{1}{4}(\Delta_8 - \Delta_7),\ H_5 = h_{ehc} + \frac{1}{4}(\Delta_{10} - \Delta_9),\ H_6 = r_{hc} + \frac{1}{4}(\Delta_{12} - \Delta_{11}),$$

and $H_7 = C_{epc} + \frac{1}{4}(\Delta_{14} - \Delta_{13})$

Since all the second order derivatives are highly nonlinear, the optimality is determined graphically (Figure 2).

6. Numerical example

In this section, a suitable example is given to illustrate the model. Here, we consider an inventory system with the same data as in Khanna et al. [9] and Yu et al. [3]. C_{eoc} = 40/order, C_{edc} = 50/year, u = 0.05, y_0 = 0.09, h_{ehc} = 0.7 per unit per year, r = 0.5, a = 25, b = 0.07, C_{epc} = 90 per unit, $A_{ece} = 0.02$, $c_{ece} = 0.1$, $h_{ece} = 0.1$. Moreover, we summarize the input parameters as fuzzy triangular values and defuzzified values in Table 1. The total cost for the crisp model is $TC_{ep}(T_e, t_e)$ = 2460,

the cycle time is T_e = 0.8 year and investment in preservation technology t_e = 40.52. The total cost of the fuzzy model is $\tilde{\prod} TC_{ep}(T_e, \tau_e)$ = 1631, the cycle time is $\tilde{T_e}$ = 0.92 year and investment in preservation technology $\tilde{t_e}$ =37.59.

7. Sensitivity analysis

We inspect the effects of variations in the system variables C_{eoc}, a, b, h_{ehc}, r, and C_{epc} on the optimal ordering quantity Q the cycle time is T_e and investment in preservation technology τ_e with minimum total expected cost. The optimal values of Q, T_e, τ_e and $TC_{ep}(T_e, t_e)$ are derived, when one of the parameters changes (increases or decreases) by 25% and all other parameters remain unchanged. The results of sensitivity analysis are presented for both the cases in Table 2 and are graphically shown in Figures 3 – 8. On the basis of the results of Table 2 and Figures 3 – 8, we see that fuzzy model provides best optimal solution as compared to crisp model.

7. 1 Managerial insights

In this section, we study the effect of changes in the cost components of the system on the optimal length of the cycle T_e^*, the optimal ordering quantity Q^*, the optimal investment in preservation technology τ_e^* and the minimum total cost for crisp model as $\prod TC_{ep}$ and for fuzzy model as $\tilde{\prod} TC_{ep}$. A sensitivity analysis is carried out by considering the same numerical example and computed results are shown in Table 2. Based on the computational results, we obtain the following managerial insights.

(1) It's interesting to note that increasing the holding costs components r_{hc} has a positive effect. This will lead to a decrease in Q, T_e, τ_e and $TC_{ep}(T_e, t_e)$. But increase in the values of the holding costs components h_{hc} will lead to an increase in Q, T_e, τ_e and $TC_{ep}(T_e, t_e)$.

(2) The optimal solution for several values of D, increase in demand parameter a results increase in Q and τ_e. This result has implication on the holding cost, ordering cost as well as delivery cost. Therefore, an increase in a will lead to an increase of $TC_{ep}(T_e, t_e)$ and decrease in T_e.

(3) From Table 2, the values of Q, T_e, and τ_e decrease with decrease in the values of parameter b but increases of $TC_{ep}(T_e, t_e)$

(4) It is foreseeable that if the buyer's ordering cost C_{eoc} rises, $TC_{ep}(T_e, t_e)$ and Q will increases. This is because, for high values of ordering cost, departing from the

optimal solution has a substantial effect on T_e and τ_e respectively. As a result, an increase in C_{eoc} will result in an increase in T_e and τ_e in both circumstances.

(5) In Table 2, with an increase in purchasing cost C_{epc}, τ_e and $TC_{ep}(T_e, t_e)$ increases but decrease of T_e and Q .

Fuzzy Input parameters	Triangular fuzzy numbers	Defuzzified values
$\tilde{C}_{eoc}$	(30, 40, 50)	37.5
$\tilde{C}_{edc}$	(40, 50, 60)	45
$\tilde{a}$	(20, 25, 30)	22.5
$\tilde{b}$	(0.06, 0.07, 0.08)	0.06
$\tilde{h}_{ehc}$	(0.6, 0.7, 0.8)	0.6
$\tilde{r}$	(0.4, 0.5, 0.6)	0.45
$\tilde{C}_{epc}$	(80, 90, 100)	65

Table 1. Fuzzy input parameters as triangular values

8. Comparative study

From Table 2, it is observed that triangular fuzzy number gives the best optimum solution. The fuzzy model with triangular fuzzy numbers generates a better result than the crisp model with the total cost with 33.70% savings. In this paper, it is shown that the knowledge of the crisp model is gradually improved to a fuzzy model with triangular fuzzy and fine-tuned our model into more specific knowledge with minimum total cost. The main reason for this situation is the low carbon emission cost and exponential holding cost which helps to increase the sales and a positive impact on customer preference.

The triangular fuzzy model finds lower values of τ_e = 37.59 and total cost $\tilde{\prod} TC_{ep}(T_e, \tau_e)$ = 1631 (better) at each performance criterion than the crisp model t_e = 40.52 and total cost $TC_{ep}(T_e, t_e)$ = 2460 indicating that total cost is higher than the fuzzy model with triangular fuzzy numbers. Thus fuzzy model gives a better result than the crisp model. Hence, fuzzy model gives the advantages of the application of fuzzy in real-world environment on Supply Chain management.

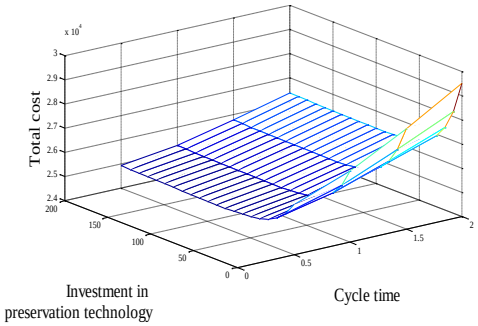

Figure 1. Graphical representation for crisp model

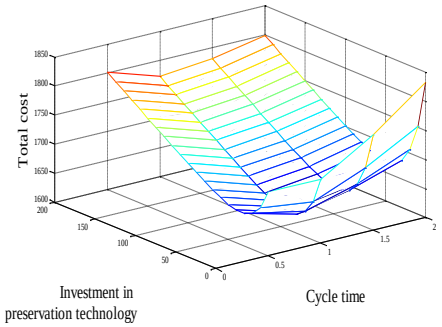

Figure 2. Graphical representation for fuzzy model

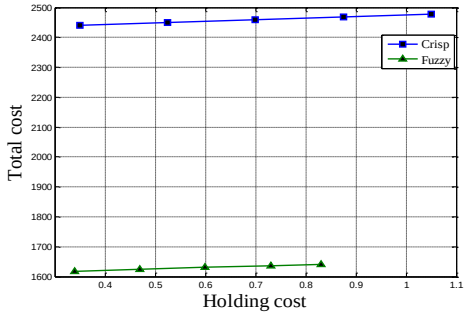

Figure 3. Effect of holding cost h_{ehc}

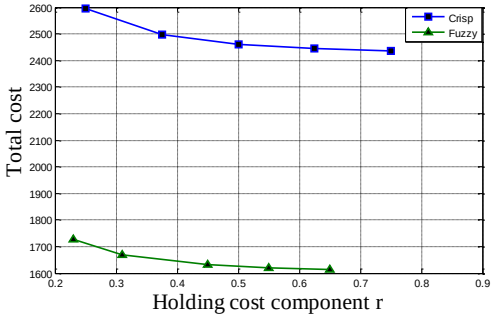

Figure 4. Effect of holding cost component r

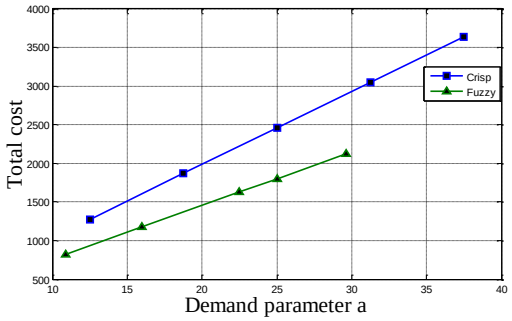

Figure 5. Effect of demand parameter a

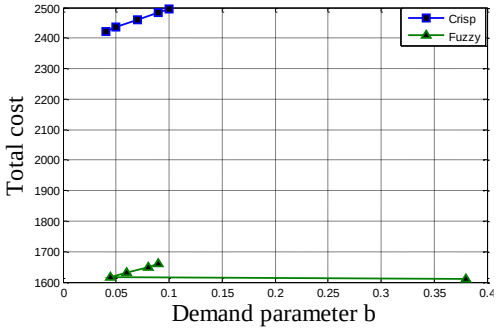

Figure 6. Effect of demand parameter b

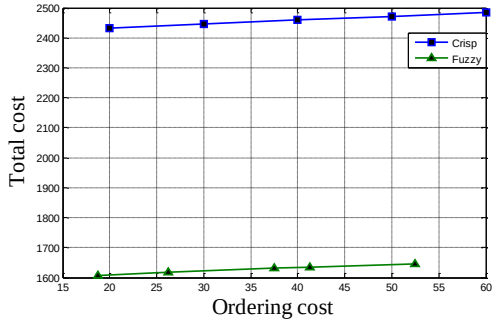

Figure 7. Effect of ordering cost C_{eoc}

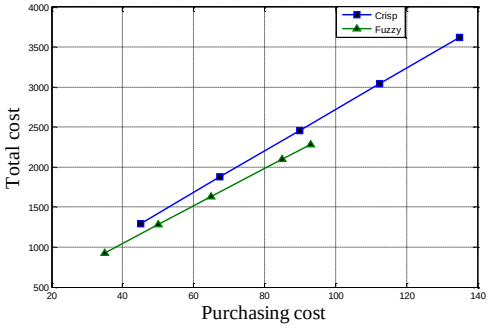

Figure 8. Effect of purchasing cost C_{epc}

Parameter	%	Crisp Parameter value	Fuzzy Parameter value	Crisp Optimal values				Fuzzy Optimal values			
				T_e^*	τ_e^*	Q^*	$\prod TC_{ep}$	T_e^*	τ_e^*	Q^*	$\tilde{\prod} TC_{ep}$
C_{eoc}	-50	20	18.75	0.6	38.63	15.37	2431	0.76	35.72	17.59	1608
	-25	30	26.25	0.7	39.59	18.01	2446	0.83	36.48	19.25	1618
	0	40	37.50	0.75	40.52	19.33	2460	0.92	37.59	21.40	1631
	+25	50	41.25	0.81	41.40	20.92	2472	0.95	37.95	22.12	1635
	+50	60	52.50	0.87	42.24	22.52	2484	1.04	38.98	24.29	1646
a	-50	12.5	10.9	0.91	28.99	11.85	1271	1.10	25.29	12.55	824.8
	-25	18.75	16	0.81	35.65	15.71	1867	1.00	31.74	16.63	1181
	0	25	22.5	0.75	40.52	19.33	2460	0.92	37.59	21.40	1631
	+25	31.25	25	0.71	44.38	22.82	3050	0.90	39.43	23.23	1803
	+50	37.5	29.63	0.68	47.59	26.18	3640	0.86	42.42	26.26	2122
b	-50	0.04	0.038	0.84	41.85	21.45	2423	0.99	38.45	22.84	1609
	-25	0.05	0.045	0.8	41.30	20.49	2435	0.96	38.13	22.21	1616
	0	0.07	0.06	0.75	40.52	19.33	2460	0.92	37.59	21.40	1631
	+25	0.09	0.08	0.72	39.98	18.66	2483	0.88	37.08	20.62	1650
	+50	0.1	0.09	0.70	39.77	18.19	2495	0.87	36.89	20.46	1660
h_{ehc}	-50	0.35	0.34	0.69	36.87	17.75	2440	0.86	34.53	19.98	1617
	-25	0.525	0.47	0.72	38.86	18.54	2450	0.89	36.19	20.69	1624
	0	0.70	0.60	0.75	40.52	19.33	2460	0.92	37.59	21.40	1631
	+25	0.875	0.73	0.78	41.93	20.12	2469	0.95	38.82	22.12	1637
	+50	1.05	0.83	0.80	43.17	20.64	2478	0.97	39.66	22.59	1642
r	-50	0.25	0.23	1.46	58.14	38.50	2595	1.76	54.65	41.89	1728
	-25	0.375	0.31	0.94	47.10	24.37	2496	1.25	46.43	29.33	1670
	0	0.5	0.45	0.75	40.52	19.33	2460	0.92	37.59	21.40	1631
	+25	0.625	0.55	0.67	36.58	17.22	2444	0.82	33.91	19.03	1619
	+50	0.75	0.65	0.63	34.19	16.18	2436	0.77	31.53	17.85	1613
C_{epc}	-50	45	35	0.97	37.24	25.24	1295	1.13	34.83	26.51	930.1
	-25	67.5	50	0.84	38.99	21.73	1879	1.01	36.30	23.58	1281
	0	90	65	0.75	40.52	19.33	2460	0.92	37.59	21.40	1631
	+25	112.5	85	0.69	41.85	17.73	3039	0.83	39.11	19.24	2095
	+50	135	93	0.64	43.04	16.41	3616	0.81	39.66	18.76	2281

Table 2. Effects of parameters on optimal solution

9. Conclusions

In this study, we examined the impacts of cycle time and investment in preservation technology on an inventory model with exponential holding cost in a fuzzy scenario. Our research revealed that various integrated inventory models would be beneficial for both the seller and buyer in cases where the cost parameters take the form of a triangular fuzzy number. We obtained more information about the cost parameters in relation to the decision variables and total fuzzy profit from the managerial insights. The model's viability is investigated using numerical analysis and sensitivity analysis. The present study can be extended with the trade credit financing policy, seasonal and expiry products, inflation and multi items.

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Topology via Graph Ideals

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Abstract

The study of ideal topological space has started since 1933 and till date it is being developed by several mathematicians. Various classes of open sets, different types of operators and exploration of elementary topological results in ideal topological spaces have been discussed in various research papers. Methods of generating topologies using various relations have been explored by many researchers. Many researchers explored the methods of inducing topologies via graphs. This paper, introduces the notions of graph ideals, graph local function and characterizes some of their properties. It also describes a method of generating a new graph topology on the vertex set of a graph from the graph adjacency topology using Kuratowski closure operator and depicts the nature of open sets with respect to the new topology. Further, it explores the condition for compatibility of the graph adjacency topology with graph ideal.

Keywords: graph ideal, open neighbourhood function, graph local function, graph adjacency topological space
2020 AMS subject classifications: 54H99, 57M15, 54A10, 54A05 [1]

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1 Introduction

Topology is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending. For a very long time, it was believed that abstract topological structures have limited applications in the generalization of real line and complex plane or some connections to Algebra and other branches of Mathematics. Further, it seems that there is a large gap between these structures and real life applications. Generating topologies by relations and the representation of topological concepts through binary relations will narrow the gap between Topology and its applications. A relation on graph represents a key for bridging Graph Theory and topological structures. The relation induces new types of topological structures to the graph. In 1967, J.W.Evans et al.[5] showed that there is a one-to-one correspondence between the labelled transitive directed graph with n points and the labelled topologies on n points. In 1967, S.S.Anderson and G.Chartrand[1] investigated the lattice-graph of the topologies of transitive directed graphs presented by J.W.Evans et al.[5]. In 2010, C.Marijuan[13] studied the relation between directed graphs and finite topologies. In 2013, S.M.Amiri et al.[2] induced a topology on vertex set of undirectd graph. In 2018, A.Kilicman and K.Abdulkalex[8] associated an incidence topology with vertex set of simple graphs without isolated vertices. In 2018, Shokry Nada et al.[15] generated topologies using the post class relations on the vertex set of graphs and discussed some of its applications in the biomedical field. K.Lalithambigai and P.Gnanachandra in [12], described a method of generating topology using adjacency, incidence relations on vertex set of graphs and studied the properties of closure and interior of vertex set of subgraphs in the graph adjacency topological space. Further, K.Lalithambigai and P.Gnanachandra in [11], introduced graph grills and explored the properties of topologies induced by graph grills on vertex set of graphs. The topic of ideals in general topological spaces is treated in the classic text by Kuratowski[9],[10] and also by R.Vaidyanathaswamy in [16],[17]. The properties which a topological space possess locally and the conditions through which those properties become a global one was studied by O.Njastad in 1966 [14]. Generation of new topologies from the old one via ideals in general topological space was studied by D.Jankovic and T.R.Hamlett[7]. Ideal on a topological space can be used to study the similarity of two structures. Till now, scholars have mostly contributed to the study of ideals in general topological space. It can be noticed that most of the real life problems can be modelled as a graph and can be solved using the graph theory concepts. The aim of this paper is to bridge the gap between abstract and concrete concepts in ideal topological spaces, by defining graph ideals and graph local functions in graph adjacency topological space. This paper explores the basic facts of graph local function and describes the method of generating a

new topology from the older via graph ideals. Further, this paper characterizes the nature of the open sets of the new topology in terms of the closure operator and investigates the compatibility of graph adjacency topology with the graph ideal.

2 Preliminaries

Fundamental definitions and preliminaries of graph theory and topology may be found in the sources [3],[4],[6].
A graph G consists of a pair (V(G),X(G)), where V(G) is a nonempty finite set whose elements are called vertices and X(G) is a set of unordered pairs of distinct elements of V(G). The elements of X(G) are called edges of the graph G. An edge joining a vertex to itself is called a loop. Edges joining the same vertices are called multiple edges. A graph without loops and multiple edges is called a simple graph. A graph G is called a bipartite graph if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 to a vertex of V_2. (V_1,V_2) is called a bipartition of G. If further G contains every edge joining the vertex of V_1 to the vertex of V_2, then G is called a complete bipartite graph. If V_1 contains m vertices and V_2 contains n vertices then the complete bipartite graph G is denoted by $K_{m,n}$. It should be noticed that $K_{1,m}$ is called a star for m$\geqslant$1. The degree of a vertex v in a graph G is the number of edges incident with v and it is denoted by deg(v). A graph in which degree of every vertex are the same is called a regular graph. A vertex of degree 0 is called an isolated vertex.
The concept of ideals in topological spaces has been studied by Kuratowski[9] and Vaidyanathaswamy[14].
An ideal on a topological space (X, τ) is a non empty collection I of subsets of X satisfying the following two conditions: (i) If $A \in I$ and $B \subset A$, then $B \in I$
(ii) If $A \in I$ and $B \in I$, then $A \cup B \in I$
An ideal topological space is a topological space (X, τ) with an ideal I on X, and is denoted by (X, τ, I)

Throughout the paper, the graphs under discussion are the simple undirected graphs which are not star graphs.

3 Graph Ideal and Graph Local Function

In this section, graph ideal and graph local function in a graph adjacency topological space are defined with illustrations.

Definition 3.1. *Let G = (V(G),X(G)) be a graph. For v ∈ V(G), the neighbourhood set N_v of v is defined as* $N_v = \{u \in V(G) : uv \in X(G)\}$.

Definition 3.2. *Let G = (V(G),X(G)) be a graph without isolated vertices. Define* S_N *as the family of* N_v *for all v ∈ V(G), i.e.,* $S_N = \{N_v : v \in V(G)\}$. *Then* S_N *forms a subbase for a topology* $\mathcal{T}_A$ *on V(G) and the pair (V(G),*$\mathcal{T}_A$*) is called graph adjacency topological space. If W is a vertex induced subgraph of G, then the closure of V(W) is defined by cl(V(W)) = V(W)* $\cup\{v \in V(G) : N_v \cap V(W) \neq \phi\}$ *and the interior of V(W) is defined by int(V(W)) =* $\{v \in V(G) : N_v \subseteq V(G)\}$.

Definition 3.3. *Let G = (V(G),X(G)) be a graph for which* $P(V)$ *and* $P(X)$ *are the power sets of V(G) and X(G) respectively. The set* $\mathcal{I} = \{\mathcal{G}^{'} : \mathcal{G}^{'} = (V^{'}, X^{'})$, *where* $V^{'} \subseteq V, X^{'} \subseteq X\}$ *is said to be a graph ideal on a graph adjacency topological space (V(G),*$\mathcal{T}_A$*) if it satisfies the following two conditions*
i) if $\mathcal{G}^{'}$ *and* $\mathcal{G}^{''} \in \mathcal{I}$, *then* $\mathcal{G}^{'} \cup \mathcal{G}^{''} \in \mathcal{I}$
ii) if $\mathcal{G}^{'}$ *is a subgraph of* $\mathcal{G}^{''}$ *and* $\mathcal{G}^{''} \in \mathcal{I}$, *then* $\mathcal{G}^{'} \in \mathcal{I}$.

Example 3.1. *Consider the following graph*

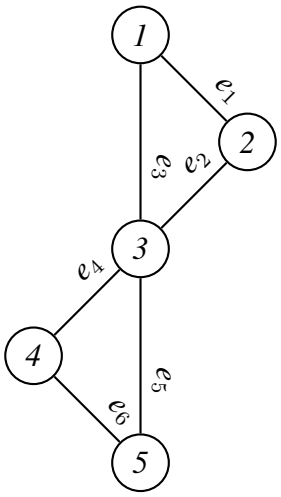


Figure. 3.1

For the graph in Figure 3.1,
$\mathcal{I} = \{(\{1,2\},\{e_1\}),(\{1\},\phi),(\{2\},\phi),(\{1,2\},\phi)\}$ *is a graph ideal.*
$\mathcal{I} = \{(\{1,2,3\},\{e_2\}),(\{4,5\},\{e_6\}),(\{1,2,3,4,5\},\{e_2,e_6\}),(\{1,2\},\{e_1\})$ *is not a graph ideal, since,* $(\{1\},\phi) \notin \mathcal{I},(\{2\},\phi) \notin \mathcal{I},(\{3\},\phi) \notin \mathcal{I},(\{4\},\phi) \notin \mathcal{I},(\{5\},\phi) \notin \mathcal{I}$.

Definition 3.4. *Let* $(V(G),\mathcal{T}_A)$ *be a graph adjacency topological space such that v ∈ V(G). The open neighbourhood system at v denoted by N(v) is defined as N(v) =* $\{U \in \mathcal{T}_A : v \in U\}$.

Definition 3.5. *Let* $(V(G),\mathcal{T}_A)$ *be a graph adjacency topological space with a graph ideal* $\mathcal{I}$. *Let W be a subgraph of G. Then* $(V(W))^*(\mathcal{I},\mathcal{T}_A) = \{v \in V(G) :$ *for every* $U \in N(v), V(W) \cap U \neq V(\mathcal{G}^{'})$ *for any* $\mathcal{G}^{'} \in \mathcal{I}\}$ *is called the graph local function of V(W) with respect to* $\mathcal{I}$ *and* $\mathcal{T}_A$.

Example 3.2. *Consider the following graph*

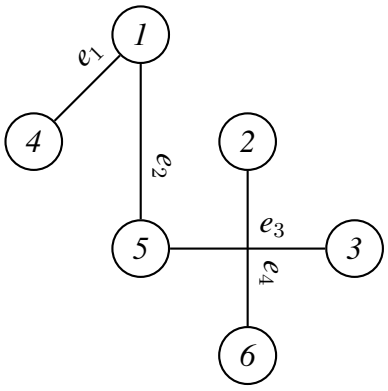


Figure 3.2

Graph local function of a subgraph of the graph in Figure 3.2 is illustrated below:
$S_N = \{\{4,5\},\{6\},\{5\},\{1\},\{1,3\},\{2\}\}$.
B = $\{\phi,\{4,5\},\{6\},\{5\},\{1\},\{1,3\},\{2\}\}$.
$\mathcal{T}_A = \{\phi,\{4,5\},\{6\},\{5\},\{1\},\{1,3\},\{2\},\{4,5,6\},\{4,5,1\},\{1,3,4,5\},\{2,4,5\},\{5,6\},\{1,6\},$
$\{1,3,6\},\{2,6\},\{1,5\},\{1,3,5\},\{2,5\},\{1,2\},\{1,5,6\},\{1,4,5\},\{1,4,5,6\},\{1,5,6\},\{1,3,5,6\},$
$\{1,3,4,5,6\},\{2,4,5,6\},\{2,4,5,1\},\{1,2,3,4,5\},\{2,5,6\},\{1,2,6\},\{1,2,3,6\},\{1,2,5\},$
$\{1,2,3,5\},\{1,2,5,6\},\{1,2,4,5\},\{1,2,4,5,6\},\{1,2,5,6\},\{1,2,3,5,6\},\{1,2,3,4,5,6\}\}$.
Let $\mathcal{I} = \{(\{1\},\phi),(\{5\},\phi),(\{1,5\},\phi),(\{1,5\},e_2)\}$ *be a graph ideal.*
Let $W = (\{1,2,3\},\phi)$ *be a subgraph of the given graph. Then* $(V(\mathcal{W}))^*(\mathcal{I},\mathcal{T}_A) = \{2,3\}$.

Note 3.6.
1. *Let G be a graph with n vertices. For each i = 1,2, ...n,* $\{(\{v_i\},\phi)\}$ *is a graph ideal. In this case, for any subgraph* W, $(V(\mathcal{W}))^*(\mathcal{I},\mathcal{T}_A) = cl(V(\mathcal{W}))$.
2. *Let* $P(G)$ *be the collection of all subgraphs of a graph* G. *Clearly* $P(G)$ *is a graph ideal. In this case, for any subgraph* W, $(V(W))^*(\mathcal{I},\mathcal{T}_A) = \phi$.

4 Facts Concerning Graph Local Function

This section describes some basic facts concerning the graph location function which are useful in the generation of new topology from the old one.

Theorem 4.1. *Let* $(V(G),\mathcal{T}_A)$ *be a graph adjacency topological space with graph ideals* $\mathcal{I}$ *and* $\mathcal{J}$. *Let* H *and* W *be subgraphs of* G. *Then*

1. *V(H)* $\subseteq$ *V(W)* $\Rightarrow$ *(V(H))** $\subseteq (V(W))^*$
2. $\mathcal{I} \subseteq \mathcal{J} \Rightarrow (V(W))^*(\mathcal{J}) \subseteq (V(W))^*(\mathcal{I})$

3. *(V(W))*$^* \subseteq$ *cl(V(W))*
4. *((V(W))*$^*)^* \subseteq (V(W))^*$
5. *(V(*$H \cup W))^* = (V(H))^* \cup (V(W))^*$
6. $(V(H))^* - (V(W))^* \subseteq (V(H) - V(W))^*$
7. $U \in \mathcal{T}_A \Rightarrow U \cap (V(W))^* = U \cap (U \cap V(W))^* \subseteq (U \cap V(W))^*$
8. *(V(W))*$^*(\mathcal{I}) \cup (V(W))^*(\mathcal{J}) = (V(W))^*(\mathcal{I} \cap \mathcal{J})$.
9. $(V(H) \cap V(W))^* \subseteq (V(H))^* \cap (V(W))^*$.
10. *I = V(*$G^{'}$*) for some* $G^{'} \in \mathcal{I} \Rightarrow (V(H) \cup I)^* = (V(H))^* = (V(H) - I)^*$.
11. *(V(H))** *-* *((V(H))*$^*)^*$ $\subseteq (V(H) - (V(H))^*)^*$.

Proof. 1. Let $v \notin (V(W))^*$.
Hence there exists $U \in N(v)$ such that $V(W) \cap U = V(\mathcal{G}^{'})$, for some $\mathcal{G}^{'} \in \mathcal{I}$.
Since $V(H) \subseteq V(W), V(H) \cap U \subseteq V(W) \cap U$.
Hence there exists $U \in N(v)$ such that $V(H) \cap U = V(\mathcal{G}^{'})$, for some $\mathcal{G}^{'} \in \mathcal{I}$.
So $v \notin (V(H))^*$. Hence $(V(H))^* \subseteq (V(W))^*$.
2. Let $v \in (V(W))^*(\mathcal{J})$.
Hence, for every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{J}$.
Since $\mathcal{I} \subseteq \mathcal{J}, V(W) \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{I}$.
Hence for every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{I}$.
Hence $(V(W))^*(\mathcal{J}) \subseteq (V(W))^*(\mathcal{I})$
3. Let $v \in (V(W))^*$.
Hence for every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{I}$.
Thus $V(W) \cap N_v \neq \phi$ and $v \in cl(V(W))$.
Hence $(V(W))^* \subseteq cl(V(W))$.
4. Let $v \in ((V(W))^*)^*$.
Hence for every $U \in N(v), (V(W))^* \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{I}$.
For every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{I}$.
So $v \in (V(W))^*$ and $((V(W))^*)^* \subseteq (V(W))^*$.
5. Let $v \notin (V(H))^* \cup (V(W))^*$.
Then $v \notin (V(H))^*$ and $v \notin (V(W))^*$.
So there exists $U_1 \in N(v)$ such that $V(H) \cap U_1 = V(\mathcal{G}^{'})$, for some $\mathcal{G}^{'} \in \mathcal{I}$ and $U_2 \in N(v)$ such that $V(W) \cap U_2 = V(\mathcal{G}^{''})$, for some $\mathcal{G}^{''} \in \mathcal{I}$.
Hence $(V(H) \cup V(W)) \cap U = V(\mathcal{G}^{'''})$, where $U = U_1 \cup U_2$ and $V(\mathcal{G}^{'''}) = V(\mathcal{G}^{'}) \cup V(\mathcal{G}^{''})$.
So there exists $U \in N(v)$ such that $(V(H) \cup V(W)) \cap U = V(\mathcal{G}^{'''})$, for some $\mathcal{G}^{'''} \in \mathcal{I}$.

Hence $v \notin (V(H \cup W))^*$ and $(V(H \cup W))^* \subseteq (V(H))^* \cup (V(W))^*$.
Thus $V(H) \subseteq V(H \cup W)$ and $V(W) \subseteq V(H \cup W)$.
By (1), $(V(H))^* \subseteq (V(H \cup W))^*$ and $(V(W))^* \subseteq (V(H \cup W))^*$.
Therefore $(V(H))^* \cup (V(W))^* \subseteq (V(H \cup W))^*$.
Hence $(V(H))^* \cup (V(W))^* = (V(H \cup W))^*$.
6. Let $v \notin (V(H) - V(W))^*$.
Hence there exists $U \in N(v)$ such that $(V(H) - V(W)) \cap U = V(\mathcal{G}')$, for some $\mathcal{G}' \in \mathcal{I}$.
So $(V(H) \cap U) - (V(W) \cap U) = V(\mathcal{G}')$, for some $\mathcal{G}' \in \mathcal{I}$.
Thus $(V(H) \cap U) = V(\mathcal{G}')$ and $V(W) \cap U \neq V(\mathcal{G}')$, for some $\mathcal{G}' \in \mathcal{I}$.
So $v \notin (V(H))^*$ and $v \in (V(W))^*$.
Thus $v \notin (V(H))^* - (V(W))^*$.
Hence $(V(H))^* - (V(W))^* \subseteq (V(H) - V(W))^*$.
7. Let $U \in \mathcal{T}_A$ and $v \in U \cap (V(W))^*$.
$v \in U$ and $v \in (V(W))^*$.
$v \in U$ and $U \in \mathcal{T}_A \Rightarrow U \in N(v)$.
$v \in (V(W))^* \Rightarrow$ for every $U_1 \in N(v), V(W) \cap U_1 \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$.
$U \cap (V(W) \cap U_1) \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$.
$(U \cap V(W)) \cap U_1 \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$.
Hence $v \in (U \cap V(W))^*$.
So $v \in U \cap (U \cap V(W))^*$.
$U \cap (V(W))^* \subseteq U \cap (U \cap V(W))^*$.
By reserving the above steps, it follows that $U \cap (U \cap V(W))^* \subseteq U \cap (V(W))^*$.
So $U \cap (V(W))^* = U \cap (U \cap V(W))^*$.
Moreover, $U \cap (U \cap V(W))^* \subseteq (U \cap V(W))^*$.
8. Since $\mathcal{I} \cap \mathcal{J} \subseteq \mathcal{I}$ and $\mathcal{I} \cap \mathcal{J} \subseteq \mathcal{J}, (V(W))^*(\mathcal{I}) \subseteq (V(W))^*(\mathcal{I} \cap \mathcal{J})$ and $(V(W))^*(\mathcal{J}) \subseteq (V(W))^*(\mathcal{I} \cap \mathcal{J})$.
Hence $(V(W))^*(\mathcal{I}) \cup (V(W))^*(\mathcal{J}) \subseteq (V(W))^*(\mathcal{I} \cap \mathcal{J})$.
Let $v \in (V(W))^*(\mathcal{I} \cap \mathcal{J})$. So, for every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I} \cap \mathcal{J}$.
For every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$ or $V(W) \cap U \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{J}$.
So $v \in (V(W))^*(\mathcal{I})$ or $v \in (V(W))^*(\mathcal{J})$ and so $v \in (V(W))^*(\mathcal{I}) \cup (V(W))^*(\mathcal{J})$.
$(V(W))^*(\mathcal{I} \cap \mathcal{J}) \subseteq (V(W))^*(\mathcal{I}) \cup (V(W))^*(\mathcal{J})$.
So $(V(W))^*(\mathcal{I} \cap \mathcal{J}) = (V(W))^*(\mathcal{I}) \cup (V(W))^*(\mathcal{J})$.
9. Since $V(H) \cap V(W) \subseteq V(H)$ and $V(H) \cap V(W) \subseteq V(W)$, by (1),
$(V(H) \cap V(W))^* \subseteq (V(H))^*$ and $(V(H) \cap V(W))^* \subseteq (V(W))^*$.
Hence $(V(H) \cap V(W))^* \subseteq (V(H))^* \cap (V(W))^*$.
10. Since $V(H) \subseteq V(H) \cup I$ and $V(H) - I \subseteq V(H)$, by (1),
$(V(H))^* \subseteq (V(H) \cup I)^*$ and $(V(H) - I)^* \subseteq (V(H))^*$.
Now, $v \notin (V(H) - I)^*$

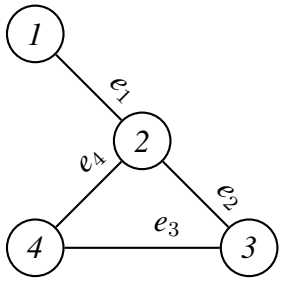
$\Rightarrow$ there exists $U \in N(v)$ such that $U \cap (V(H) - I) = V(\mathcal{G}')$, for some $\mathcal{G}' \in \mathcal{I}$.
$\Rightarrow$ there exists $U \in N(v)$ such that $U \cap (V(H) - V(\mathcal{G}'')) = V(\mathcal{G}')$, for some $\mathcal{G}', \mathcal{G}'' \in \mathcal{I}$.
So there exists $U \in N(v)$ such that $(U \cap V(H)) - (U \cap V(\mathcal{G}'')) = V(\mathcal{G}')$, for some $\mathcal{G}', \mathcal{G}'' \in \mathcal{I}$.
This implies that there exist $U \in N(v)$ such that $U \cap V(H) = V(\mathcal{G}')$, for some $\mathcal{G}' \in \mathcal{I}$.
$\Rightarrow v \notin (V(H))^*$.
So $(V(H))^* \subseteq (V(H) - I)^*$.
Let $v \in (V(H) \cup I)^*$.
For every $U \in N(v), U \cap (V(H) \cup I) \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$.
For every $U \in N(v), (U \cap V(H)) \cup (U \cap I) \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$.
But $I = V(\mathcal{G}'')$, for some $\mathcal{G}'' \in \mathcal{I}$.
So $U \cap V(H) \neq V(\mathcal{G}''')$, for any $\mathcal{G}''' \in \mathcal{I}$.
So $v \in (V(H))^*$.
Hence $(V(H) \cup I)^* \subseteq (V(H))^*$.
So $(V(H) \cup I)^* = (V(H))^* = (V(H) - I)^*$.
11. $v \in (V(H))^* - ((V(H))^*)^*$.
So $v \in (V(H))^*$ and $v \notin ((V(H))^*)^*$.
For every $U \in N(v), U \cap V(H) \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$ and there exists $U_1 \in N(v)$ such that $U_1 \cap (V(H))^* = V(\mathcal{G}'')$, for some $\mathcal{G}'' \in \mathcal{I}$.
For every $U \in N(v), (U \cap V(H)) - (U \cap (V(H))^*) \neq V(\mathcal{G}''')$, for any $\mathcal{G}''' \in \mathcal{I}$.
For every $U \in N(v), U \cap (V(H) - (V(H))^*) \neq V(\mathcal{G}''')$, for any $\mathcal{G}''' \in \mathcal{I}$.
$v \in (V(H) - (V(H))^*)^*$.
So $(V(H)^* - ((V(H))^*)^* \subseteq (V(H) - (V(H))^*)^*$. □

5 Topology via Graph Ideals

This section describes a method of generating a new topology from the old one using Kuratowski closure operator cl^*.

Definition 5.1. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. Let W be a subgraph of G. Define $cl^*(V(W)) = V(W) \cup (V(W))^*$ and $\mathcal{T}_A^*(\mathcal{I}) = \{V(H) \subseteq V(G) : cl^*(V(G) - V(H)) = V(G) - V(H)\}$. $\mathcal{T}_A^*(\mathcal{I})$ is called the graph adjacency topology generated by cl^*. When there is no ambiguity, it is denoted as $\mathcal{T}_A^*$.*

Example 5.1. *Consider the following graph*



$S_N = \{\{2\}, \{1,3,4\}, \{2,4\}, \{2,3\}\}$
$\mathsf{B} = \{\phi, \{2\}, \{1,3,4\}, \{2,4\}, \{2,3\}, \{4\}, \{3\}\}$
$\mathcal{T}_A = \{\phi, \{2\}, \{1,3,4\}, \{2,4\}, \{2,3\}, \{4\}, \{3\}, \{3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$
$\mathcal{I} = \{(\{1,2\}, \{e_1\}), (\{1\}, \phi), (\{2\}, \phi), (\{1,2\}, \phi)\}$
$N(1) = \{\{1,3,4\}, \{1,2,3,4\}\}, N(2) = \{\{2\}, \{2,4\}, \{2,3\}, \{2,3,4\}, \{1,2,3,4\}\},$
$N(3) = \{\{1,3,4\}, \{2,3\}, \{3\}, \{3,4\}, \{2,3,4\}, \{1,2,3,4\}\},$
$N(4) = \{\{1,3,4\}, \{2,4\}, \{4\}, \{3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$
$\mathcal{T}_A^* = \{\phi, \{3\}, \{4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}.$

Observation 5.2. 1. *If* $\mathcal{I} = \{(\{v_i\}, \phi)\}$ *is a graph ideal, then* $(V(W))^*(\mathcal{I}, \mathcal{T}_A) = cl(V(W))$ *for every* $V(W) \subseteq V(G)$.
Hence $cl^*(V(W)) = cl(V(W))$.
$\mathcal{T}_A^* = \{V(H) \subseteq V(G) : cl^*(V(G) - V(H)) = V(G) - V(H)\}$
= $\{V(H) \subseteq V(G) : cl(V(G) - V(H)) = V(G) - V(H)\}$
= $\mathcal{T}_A$.

2. *If* $P(G)$ *is the collection of all subgraphs of a graph* G. *Then* $P(G)$ *is a graph ideal. Hence for any subgraph W,* $(V(W))^*(\mathcal{I}, \mathcal{T}_A) = \phi$ *and* $cl^*(V(W)) = V(W)$.
$\mathcal{T}_A^* = \{V(H) \subseteq V(G) : cl^*(V(G) - V(H)) = V(G) - V(H)\}$
= $\{V(H) \subseteq V(G) : V(G) - V(H) = V(G) - V(H)\}$
= $P(G)$
Hence $\mathcal{T}_A^*$ *is a discrete topology.*

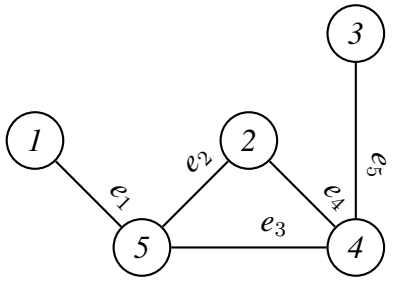
3. *For every graph ideal* $\mathcal{I}, \{(\{v_i\}, \phi)\} \subseteq \mathcal{I} \subseteq P(G)$. *Hence* $\mathcal{T}_A \subseteq \mathcal{T}_A^* \subseteq$ *discrete topology.*

Proposition 5.1. *If* $\mathcal{I}$ *and* $\mathcal{J}$ *are graph ideals such that* $\mathcal{I} \subseteq \mathcal{J}$, *then* $\mathcal{T}_A^*(\mathcal{I}) \subseteq \mathcal{T}_A^*(\mathcal{J})$.

Proof. $\mathcal{I} \subseteq \mathcal{J} \Rightarrow (V(H))^*(\mathcal{J}) \subseteq (V(H))^*(\mathcal{I})$. Thus $\{V(H) \cup (V(H))^*(\mathcal{J})\} \subseteq \{V(H) \cup (V(H))^*(\mathcal{I})\}$. Therefore $cl^*(V(H))(\mathcal{J}) \subseteq cl^*(V(H))(\mathcal{I})$. This means $cl^*(V(G) - V(H))(\mathcal{I}) \subseteq cl^*(V(G) - V(H))(\mathcal{J})$. Hence $\mathcal{T}_A^*(\mathcal{I}) \subseteq \mathcal{T}_A^*(\mathcal{J})$.
□

Definition 5.3. *The derived set of V(H) in V(G), $\mathcal{T}_A$) denoted by $(V(H))^d$ is defined by v $\in (V(H))^d$ if and only if $(U - \{v\}) \cap V(H) \neq \phi$ for every U ∈ N(v). The derived set of V(H) in V(G), $\mathcal{T}_A^*$) is denoted by $(V(H))^{d*}$.*

Example 5.2. *Consider the following graph*



$S_N = \{\{1\}, \{5,4\}, \{4\}, \{3,2,5\}, \{1,2,4\}\}$
$\mathsf{B} = \{\phi, \{1\}, \{5,4\}, \{4\}, \{3,2,5\}, \{1,2,4\}, \{2\}\}$
$\mathcal{T}_A = \{\phi, \{1\}, \{5,4\}, \{4\}, \{3,2,5\}, \{1,2,4\}, \{2\}, \{1,2,4,5\}, \{2,5\}, \{2,3,4,5\}, \{2,4,5\}, \{2,4\}, \{1,2,3,4,5\}\}$
Let $V(H) = \{3,5\}$. *Then* $(V(H))^d = \{3\}$.

Note 5.4. 1. $v \in (V(H))^{d*} \Leftrightarrow v \in cl^*(V(H) - \{v\})$
$\Leftrightarrow v \in (V(H) - \{v\}) \cup (V(H) - \{v\})^*$
$\Leftrightarrow v \in (V(H) - \{v\})^*$
$\Leftrightarrow$ *for every* $U \in N(v), (V(H) - \{v\}) \cap U \neq V(G')$, *for any* $G' \in \mathcal{I}$
Hence $v \in (V(H))^{d*} \Leftrightarrow$ *for every* $U \in N(v), (V(H) - \{v\}) \cap U \neq V(G')$, *for any* $G' \in \mathcal{I}$.

2. $(V(H))^{d*} \subseteq (V(H))^*$

Definition 5.5. *A graph which has either an infinite number of vertices or edges or both is called an infinite graph.*

Definition 5.6. *Let G be an infinite graph. Let $\mathcal{I}_f$ be the graph ideal of finite subgraphs of G and H be a subgraph of G. A vertex v is said to be an ω-accumulation point of V(H) if and only if $U \cap V(H)$ is infinite for every $U \in N(v)$. The set of all ω-accumulation points of V(H) is denoted by $(V(H))^\omega$.*

Observation 5.7. $(V(H))^\omega = (V(H))^*(\mathcal{I}_f)$.

Definition 5.8. *Let G be an infinite graph. Let $\mathcal{I}_c$ be the graph ideal of countable subgraphs of G and H be a subgraph of G. A vertex v is said to be a condensation point of V(H) if U ∩ V(H) is uncountable for every U ∈ N(v). The set of all condensation points of V(H) is denoted by $(V(H))^*(\mathcal{I}_c)$.*

Definition 5.9. *A graph adjacency topological space $(V(G), \mathcal{T}_A)$ is said to be a T_1 space if for each pair of distinct vertices v_i and v_j, v_i belong to every $U \in N(v_i)$ for which v_i does not belong to any $U \in N(v_j)$ and v_j belongs to every $U \in N(v_j)$ and v_j does not belong to any $U \in N(v_i)$.*

Remark 5.1.

1. *Let $\mathcal{I}_f$ be a graph ideal of finite subgraphs of G. Since $(\{v_i\}, \phi) \in \mathcal{I}_f$ for each $v_i \in V(G)$, $(V(H))^* = (V(H))^{d*}$.*
   Also $(V(H))^* = (V(H))^{\omega}$. Hence ω-accumulation points of V(H) in $(V(G), \mathcal{T}_A)$ are precisely the limit points of V(H) in $(V(G), \mathcal{T}_A^*(\mathcal{I}_f))$. Since $(\{v_i\}, \phi) \in \mathcal{I}_f$, $(\{v_i\}, \phi)^* = \phi$. So $cl^*(\{v_i\}) = \{v_i\}$.Hence $(V(G), \mathcal{T}_A^*)$ is T_1.
   In T_1 spaces, ω-accumulation point and limit point coincides. Hence the set of ω-accumulation points of V(H) in $(V(G), \mathcal{T}_A)$ and $(V(G), \mathcal{T}_A^*)$ coincide.
   $(V(H))^*(\mathcal{I}_f) = (V(H))^d$ iff $\mathcal{T}_A = \mathcal{T}_A^*(\mathcal{I}_f))$ iff $(V(G), \mathcal{T}_A)$ is T_1.
2. *Let $\mathcal{I}_c$ be a graph ideal of countable subgraphs of G. Since $(V(H))^*$ is the set of condensation points of V(H) and $(V(H))^* = (V(H))^{d*}$, the condensation points of V(H) in (V(G), $\mathcal{T}_A$) are precisely the limit points of V(H) in $(V(G), \mathcal{T}_A^*(\mathcal{I}_c))$.*

Definition 5.10. *Let H be a subgraph of a graph G. We say that V(H) is closed and discrete in $(V(G), \mathcal{T}_A)$ if and only if $(V(H))^d = \phi$.*

Lemma 5.1. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. If $I \in \mathcal{I}$, then I is closed and discrete in $(V(G), \mathcal{T}_A^*)$.*

Proof. $\mathrm{I} \in \mathcal{I} \Rightarrow I^* = \phi \Rightarrow I^{d*} = \phi$. □

Remark 5.2. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space. Let $\mathcal{I}_{cd} = \{H : H$ is a subgraph of G and $(V(H))^d = \phi\}$.*
Let us prove $\mathcal{I}_{cd}$ is a graph ideal.
$H_1, H_2 \in \mathcal{I}_{cd} \Rightarrow (V(H_1))^d = \phi$ and $(V(H_2))^d = \phi$.
Hence for every $v \in V(G), (U - \{v\}) \cap V(H_1) = \phi$ and $(U - \{v\}) \cap V(H_2) = \phi$.
So for every $v \in V(G), (U - \{v\}) \cap V(H_1 \cup H_2) = \phi$.
So $H_1 \cup H_2 \in \mathcal{I}_{cd}$.
Let $H_1 \in \mathcal{I}_{cd}$ and H_2 be a subgraph of H_1.
$H_1 \in \mathcal{I}_{cd} \Rightarrow (V(H_1))^d = \phi$.
Hence for every $v \in V(G), (U - \{v\}) \cap V(H_1) = \phi$ and so for every $v \in V(G), (U - \{v\}) \cap V(H_2) = \phi$.
So $(V(H_2))^d = \phi$ which implies $H_2 \in \mathcal{I}_{cd}$.
$\mathcal{I}_{cd}$ is a graph ideal. Also $(V(H))^d \subseteq (V(H))^*$. Hence $\mathcal{T}_A^* = \mathcal{T}_A$.
Lemma 5.1 implies that $\mathcal{I}_{cd}$ is the largest graph ideal with the property that $\mathcal{T}_A^* = \mathcal{T}_A$.
Finally, $(V(H))^* = (V(H))^d$ iff $(V(G), \mathcal{T}_A)$ is T_1.

Example 5.3. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space. Let $\mathcal{I} = \{G' : G' \text{ is a subgraph of G and } int(cl(V(G'))) = \phi\}$, i.e., $\mathcal{I}$ is the collection of nowhere dense subgraphs of $(V(G), \mathcal{T}_A)$.*
Let $G', G'' \in \mathcal{I}$. Hence $int(cl(V(G'))) = \phi$ and $int(cl(V(G''))) = \phi$.
Now, $int(cl(V(G' \cup G''))) = int(cl(V(G') \cup cl(V(G''))) \supseteq int(cl(V(G')) \cup int(cl(V(G'')) = \phi$.
Hence $int(cl(V(G' \cup G''))) \supseteq \phi$.
So $\mathcal{I}$ is not a graph ideal.

6 The Open sets of $\mathcal{T}_A^*$

Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. Let H be a subgraph of G. Then V(H) is said to be $\mathcal{T}_A^*$-closed if and only if $((V(H))^* \subseteq V(H)$. Hence V(H) is $\mathcal{T}_A^*$-open if and only if V(G) - V(H) is $\mathcal{T}_A^*$-closed, i.e., if and only if $(V(G) - V(H))^* \subseteq V(G) - V(H)$, i.e., if and only if $V(H) \subseteq V(G) - (V(G) - V(H))^*$.
Hence V(H) is $\mathcal{T}_A^*$-open if and only if $v \in V(H) \Rightarrow v \notin (V(G) - V(H))^*$. So V(H) is $\mathcal{T}_A^*$-open if and only if $v \in V(H) \Rightarrow$ there exists $U \in N(v)$ such that $U \cap (V(G) - V(H)) = V(G')$, for some $G' \in \mathcal{I}$.
Let $W = U \cap (V(G) - V(H))$. V(H) is $\mathcal{T}_A^*$-open if and only if, for $v \in V(H)$ there exists $U \in N(v)$ such that $W = V(G')$, for some $G' \in \mathcal{I}$.
If $v \in V(H)$ then $v \notin V(G) - V(H)$. So $v \notin W$. Also $v \in U$. Hence $v \in U - W$.
V(H) is $\mathcal{T}_A^*$-open if and only if, for $v \in V(H)$, there exists $U \in N(v)$ such that $v \in U - W$ and $W = V(G')$ for some $G' \in \mathcal{I}$.
Let $\beta(\mathcal{I}, \mathcal{T}_A^*) = \{U - W : U \in \mathcal{T}_A, W = V(G') \text{ for some } G' \in \mathcal{I}\}$. $\beta(\mathcal{I}, \mathcal{T}_A^*)$ is a basis for $\mathcal{T}_A^*$.

Note 6.1. *If $\mathcal{I}$ and $\mathcal{J}$ are graph ideals then $\mathcal{I} \vee \mathcal{J} = \{I \cup J : I \in \mathcal{I}, J \in \mathcal{J}\}$ is also a graph ideal.*

Theorem 6.1. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideals $\mathcal{I}$ and $\mathcal{J}$. Let H be a subgraph of G. Then $(V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A) = (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J})) \cap (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I}))$.*

Proof. Let $v \notin (V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A)$.
Hence there exists $U \in N(v)$ such that $U \cap V(H) = V(G')$, for some $G' \in \mathcal{I} \vee \mathcal{J}$.
Let $G'' \in \mathcal{I}$ and $G''' \in \mathcal{J}$ such that $U \cap V(H) = V(G'''')$ where $G'''' = G'' \cup G'''$.
Assume that $V(G'') \cap V(G''') = \phi$.
Hence $(U \cap V(H)) - V(G'') = V(G''')$ and $(U \cap V(H)) - V(G''') = V(G'')$.
So $(U - V(G'')) \cap V(H) = V(G''')$, where $G''' \in \mathcal{J}$ and $(U - V(G''')) \cap V(H) = V(G'')$, where $G'' \in \mathcal{I}$.

$\Rightarrow v \notin (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I}))$ or $v \notin (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J}))$.
$\Rightarrow v \notin (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I})) \cap (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J}))$.
Hence $(V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J})) \cap (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I})) \subseteq (V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A)$.
$v \notin (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J}))$ implies that there exists $U \in N(v)$ such that $(U - V(G^{''})) \cap V(H) = V(G^{'})$ for some $G^{'} \in \mathcal{I}, G^{''} \in \mathcal{J}$.
Assume that $V(G^{''}) \subseteq V(H)$. Hence $U \cap V(H) = V(G^{'}) \cup V(G^{''}) = V(G^{'''})$, where $G^{'''} \in \mathcal{I} \vee \mathcal{J}$. So $v \notin (V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A)$.
$(V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A) \subseteq (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J}))$.
Similarly, $(V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A) \subseteq (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I}))$.
$(V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A) \subseteq (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J})) \cap (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I}))$.
Hence $(V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A) = (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J})) \cap (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I}))$. □

Remark 6.1. *Given a graph adjacency topological space $(V(G), \mathcal{T}_A)$ and a graph ideal $\mathcal{I}$, $\mathcal{T}_A^{**} = (\mathcal{T}_A^*(\mathcal{I}))^*(\mathcal{I})$ is a topology on $V(G)$ and $\mathcal{T}_A^{**}$ is finer than $\mathcal{T}_A^*$.*

Corollary 6.1. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. Then $(V(H))^*(\mathcal{I}, \mathcal{T}_A) = (V(H))^*(\mathcal{I}, \mathcal{T}_A^*)$ and hence $\mathcal{T}_A^* = \mathcal{T}_A^{**}$.*

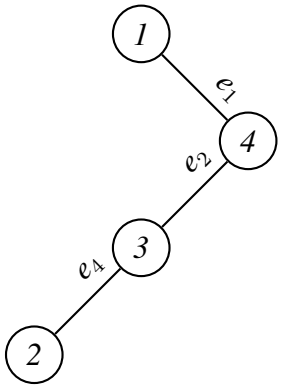
Proof. $(V(H))^*(\mathcal{I}, \mathcal{T}_A) =$
$(V(H))^*(\mathcal{I} \vee \mathcal{I}, \mathcal{T}_A) = (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{I})) \cap (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{I}))$
= $(V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{I}))$
$\mathcal{T}_A^{**} = \mathcal{T}_A^*(\mathcal{I}))^*(\mathcal{I}) = \mathcal{T}_A^*$. □

7 Compatibility of $\mathcal{T}_A$ with $\mathcal{I}$

This section introduces the definition of compatibility of $\mathcal{T}_A$ with the graph ideal $\mathcal{I}$ and explores its significance.

Definition 7.1. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. The graph adjacency topology $\mathcal{T}_A$ is said to be compatible with the graph ideal $\mathcal{I}$, denoted $\mathcal{T}_A \backsim \mathcal{I}$, if the following holds for every vertex induced subgraph H of G :*
if for every $v \in V(H)$, there exists a $U \in N(v)$ such that $U \cap V(H) = V(G^{'})$, for some $G^{'} \in \mathcal{I}$, then $V(H) = V(G^{''})$, for some $G^{''} \in \mathcal{I}$.

Example 7.1. *Consider the following graph*



$S_N = \{\{4\}, \{3\}, \{2,4\}, \{1\}\}$.
$\mathsf{B} = \{\phi, \{4\}, \{3\}, \{2,4\}, \{1\}\}$.
$\mathcal{T}_A = \{\phi, \{4\}, \{3\}, \{2,4\}, \{1\}, \{3,4\}, \{1,4\}, \{2,3,4\}, \{1,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\}$.
$N(1) = \{\{1\}, \{1,4\}, \{1,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\}$.
$N(2) = \{\{2,4\}, \{2,3,4\}, \{1,2,4\}, \{1,2,3,4\}\}$.
$N(3) = \{\{3\}, \{3,4\}, \{2,3,4\}, \{1,3\}, \{1,3,4\}, \{1,2,3,4\}\}$.
$N(4) = \{\{4\}, \{2,4\}, \{3,4\}, \{1,4\}, \{2,3,4\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\}$.
$\mathcal{T}_A$ is compatible with the graph ideal $\mathcal{I} = \{(\{1\}, \phi), (\{2\}, \phi), (\{1,2\}, \phi)\}$ and not compatible with the graph ideal $\mathcal{I} = \{(\{1\}, \phi), (\{4\}, \phi), (\{1,4\}, \phi)\}$.

Observation 7.2. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. Then the following are implied by $\mathcal{T}_A \backsim \mathcal{I}$:*

(a) For every vertex induced subgraph H of G, $V(H) \cap (V(H))^* = \phi \Rightarrow (V(H))^* = \phi$.

(b) For every vertex induced subgraph H of G, $(V(H) - (V(H))^*)^* = \phi$.

(c) For every vertex induced subgraph H of G, $(V(H) \cap (V(H))^*)^* = (V(H))^*$.

(d) $\beta(\mathcal{I}, \mathcal{T}_A^*) = \{U - W : U \in \mathcal{T}_A, W = V(G'), \text{for some } G' \in \mathcal{I}\} = \mathcal{T}_A^*$.

Theorem 7.1. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. Then the following are equivalent:*
(1) $\mathcal{T}_A \backsim \mathcal{I}$
(2) For every vertex induced subgraphs H of G, $V(H) \cap (V(H))^* = \phi \Rightarrow V(H) = V(G'), \text{for some } G' \in \mathcal{I}$.
(3) For every vertex induced subgraphs H of G, $V(H) - (V(H))^* = \phi$ or $V(H) - (V(H))^* = V(G'), \text{for some } G' \in \mathcal{I}$.
(4) For every $\mathcal{T}_A^*$ closed subgraph H, $V(H) - (V(H))^* = V(G')$ for some $G' \in \mathcal{I}$.
(5) For every vertex induced subgraphs H of G, if V(H) contains no induced subgraph W with $V(W) \subseteq (V(W))^*$, then $V(H) = V(G')$ for some $G' \in \mathcal{I}$.

Proof. (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (3): For every vertex induced subgraphs H of G, $V(H) - (V(H))^* = \phi$ or $(V(H) - (V(H))^*) \cap (V(H) - (V(H))^*)^* = \phi$. So by (2), $V(H) - (V(H))^* = \phi$ or $V(H) - (V(H))^* = V(G^{'})$, for some $G^{'} \in \mathcal{I}$.
(3) $\Rightarrow$ (4) is straightforward.
(4) $\Rightarrow$ (1): Let H be a vertex induced subgraph of G and assume that for every v $\in$ V(H), there exists U $\in$ N(v) such that $U \cap V(H) = V(G^{'})$ for some $G^{'} \in \mathcal{I}$.
Then $V(H) \cap (V(H))^* = \phi$.
Since $(V(H) \cup (V(H))^*)^* = (V(H))^* \cup ((V(H))^*)^* \subseteq (V(H))^* \cup V(H), V(H) \cup (V(H))^*$ is $\mathcal{T}_A^*$ closed.
So by (4), $(V(H) \cup (V(H))^*) - (V(H) \cup (V(H))^*)^* = V(G^{'})$ for some $G^{'} \in \mathcal{I}$.
But $(V(H) \cup (V(H))^*) - (V(H) \cup (V(H))^*)^* = (V(H) \cup (V(H))^*) - ((V(H))^* \cup (V(H))^{**}) = (V(H) \cup (V(H))^*) - (V(H))^* = V(H)$.
Hence $V(H) = V(G^{'})$, for some $G^{'} \in \mathcal{I}$.
So $\mathcal{T}_A \backsim \mathcal{I}$
(3) $\Rightarrow$ (5): Let H be a vertex induced subgraph of G and assume that V(H) contains no induced subgraph W with $V(W) \subseteq (V(W))^*$.
Since $V(H) - (V(H))^* = V(G^{'})$, for some $G^{'} \in \mathcal{I}, V(H) \cap (V(H))^* \subseteq (V(H) \cap (V(H))^*)^*, V(H) \cap (V(H))^* = \phi$.
So $V(H) = V(H) - (V(H))^*$ and $V(H) = V(G^{'})$, for some $G^{'} \in \mathcal{I}$.
(5) $\Rightarrow$ (3) : Since $(V(H) - (V(H))^*) \cap (V(H) - (V(H))^*)^* = \phi, V(H) - (V(H))^*$ contains no vertex induced subgraph W such that $V(W) \subseteq (V(W))^*$.
So by (5), $V(H) - (V(H))^* = V(G^{'})$ for some $G^{'} \in \mathcal{I}$.
□

Theorem 7.2. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$ and $\mathcal{T}_A \backsim \mathcal{I}$. Let H be a vertex induced subgraph of G. If V(H) is $\mathcal{T}_A^*$ closed then it is the union of V(W), where $(V(W))^* \subseteq cl(V(W))$, and $V(G^{'})$ for some $G^{'} \in \mathcal{I}$.*

Proof. Let V(H) be $\mathcal{T}_A^*$ closed. Then $(V(H))^* \subseteq V(H)$. So $V(H) = (V(H) - (V(H))^*) \cup (V(H))^*$.
By theorem 7.1(3), $V(H) - (V(H))^* = V(G^{'})$ for some $G^{'} \in \mathcal{I}$.
Also $(V(H))^* \subseteq cl(V(H))$.
So V(H)is the union of V(W), where $(V(W))^* \subseteq cl(V(W))$, and $V(G^{'})$ for some $G^{'} \in \mathcal{I}$. □

8 Conclusions

Graph ideal and graph local function in a graph adjacency topological space have been defined and the basic facts concerning graph local function have been

proved. The method of generating a new topology from the older one using Kuratowski closure operator via graph ideal has been discussed. Further, the characteristics of the open sets of the new topology in terms of the closure operator have been analyzed. New topologies generated by two different ideals have been compared. The feature of graph location function based on the union of two ideals has been studied. The compatibility of the graph adjacency topology with graph ideal has been defined and the equivalent conditions for compatibility of the graph adjacency topology with graph ideal have been investigated. Further, the results in this paper are useful in the study of some new sets and topologies in graph adjacency topological space with graph ideal. These concepts can be studied further by stating graph local function of induced subgraphs of a graph with respect to graph adjacency topology using graph prime ideal and graph principal ideal. The open sets of topologies generated via prime ideals and principal ideals can be compared and investigated in future studies.

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W_8 - Curvature tensor in generalized Sasakian-space-forms

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Abstract

The generalized Sasakian-space-forms and their properties have been examined by various researchers such as Alegre and Carriazo [2008], Prakasha [2012], Sarkar and Akbar [2014], Shanmukha et al. [2018], Sarkar and Sen [2012], Rajan and Singh [2020] and Sarkar and Sen [2012]. Motivated by the results of these works, we have proposed the idea of the W_8-curvature tensor in generalized Sasakian-space-forms. The main goal of this paper is to investigate the curvature properties of generalized Sasakian-space-forms that satisfy the conditions $\xi - W_8-$ flatness, $\phi - W_8-$semi-symmetric, $W_8 \cdot Q = 0$, $W_8 \cdot R = 0$ and to prove some interesting results.

Keywords: Sasakian-space-form, generalized Sasakian-space-form, $\phi-$recurrent, $\phi-$symmetric, $\phi-$semi-symmetric, W_8- curvature tensor, Einstein manifold, $\eta-$Einstein manifold.
2020 AMS subject classifications:53C15, 53C25, 53D15 [1]

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1 Introduction

The nature of a Riemannian manifold depends on the curvature tensor R of the manifold. It is well known that the sectional curvatures of a manifold determine its curvature tensor completely. Real space-forms are Riemannian manifolds with constant sectional curvature c, and their curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

Representation for these spaces are hyperbolic spaces $(c < 0)$, spheres $(c > 0)$ and Euclidean spaces $(c = 0)$.
The Sasakian manifold determines the ϕ-sectional curvature of a Sasakian-space-form, and it has a specific form for its curvature tensor. The Kenmotsu and cosymplectic space-forms use the same notation. Alegre et al. [2004] developed and researched generalized Sasakian-space-forms in an effort to generalize such space -forms in a shared frame. A generalized Sasakian-space-form is an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, whose curvature tensor is given by

$$\begin{aligned} R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y \\ - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned}$$

The Riemannian curvature tensor of a generalized Sasakian-space-form $\mathrm{M}^{2n+1}(f_1, f_2, f_3)$ is simply given

$$R = f_1R_1 + f_2R_2 + f_3R_3,$$

where f_1, f_2, f_3 are differential functions on $M^{2n+1}(f_1, f_2, f_3)$ and

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{1}$$

$$\begin{aligned} R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ +2g(X,\phi Y)\phi Z, \ and \end{aligned} \tag{2}$$

$$\begin{aligned} R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ +g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \end{aligned} \tag{3}$$

where $f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$. Here c denotes the constant ϕ-sectional curvature. Numerous geometers, including Alegre and Carriazo [2008], De and Majhi [2015], De and Sarkar [2010], Kim [2006], Prakasha [2012], Sarkar and Akbar [2014], Sarkar and Sen [2012], Shanmukha et al. [2018], Singh [2016],

have explored the characteristics of the generalized Sasakian-space-form. Numerous writers have addressed the idea of local symmetry of a Riemannian manifold in various ways and to different extents. Takahashi introduced the Sasakian manifold's locally ϕ-symmetry in Toshio [1977]. This is extended by De, Shaikh, and Sudipta to the notation of ϕ-symmetry in De et al. [2003], after which they introduce the notation of ϕ-recurrent Sasakian manifold. On the Kenmotsu manifold De et al. [2009], LP-Sasakian manifold Venkatesha [2008], W_8- Curvature Tensor in the Lorentzian Sasakian manifold Rajan and Singh [2020] and $(LCS)_n$-manifold Shaikh et al. [2008], the ϕ-recurrent condition was further studied.

In Tripathi and Gupta [2012] have define the W_8-curvature tensor, given by

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[S(X,Y)Z - S(Y,Z)X],$$

where R and S are curvature tensor and Ricci tensor of the manifold respectively.

A new class of almost contact Riemann manifold was presented by K. Kenmotsu [1972], sometimes referred to as a Kenmotsu manifold. Kenmotsu studied at the underlying characteristics of these manifolds local structure. Kenmotsu manifolds have a one-dimensional basis, a Kahler fibre, and are locally isometric to warped product spaces. According to Kenmotsu's research, a Kenmotsu manifold has a negative curvature of -1 if R(X,Y)Z = 0, where R is the Riemannian curvature tensor and R(X,Y)Z is the derivative of the tensor algebra at each point of the tangent space. Because odd dimensions hyperbolic spaces cannot admit Sasakian structures, unlike odd dimensional spheres, which are well known to do so, odd dimensional hyperbolic Kenmotsu structure is permitted in spaces. Normal Kenmotsu manifolds the almost contact Riemannian manifolds. Several properties of Kenmotsu manifold have been studied by many authors like Bagewadi et al. [2007], Blair [1976], Chaubey and Ojha [2010], DE [2008], Ingalahalli and Bagewadi [2012], Hui and Chakraborty [2017], Baishya and Chowdhury [2016], Nagaraja et al. [2018], Özgür [2006], Prakasha and Balachandra [2018], Ali Shaikh and Kumar Hui [2009], Sinha and Srivastava [1991].

These concepts served as our inspiration as we made an effort to research the characteristics of generalized Sasakian-space-form. The structure of the current paper is as follows. In section 2, we review some preliminary results. In section 3, we study $\xi - W_8 -$flat generalized Sasakian-space-forms. Section 4, deals with the $\phi - W_8 -$semi-symmetric condition in generalized Sasakian-space-form and found to be Einstein manifold. In section 5, we discuss generalized Sasakian-space-form satisfying $W_8 \cdot Q$ = 0 and also found to be Einstein manifold. Finally in the last

section, we discuss the generalized Sasakian-space-form satisfying $W_8 \cdot R = 0$ and found to be $\eta-$Einstein Manifolds.

2 Generalized Sasakian-space-forms

The Riemannian manifold M^{2n+1} is called an almost contact metric manifold if the following result holds Blair [1976, 2002]:

$$\phi^2 X = -X + \eta(X)\xi, \tag{4}$$
$$\eta(\xi) = 1, \phi\xi = 0, \eta(\phi X) = 0, g(X,\xi) = \eta(X), \tag{5}$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{6}$$
$$g(\phi X, Y) = -g(X, \phi Y), g(\phi X, X) = 0, \tag{7}$$
$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y), \forall X, Y \in (T_p M). \tag{8}$$

A almost contact metric manifold is said to be Sasakian if and only if Blair [1976], Sasaki [1965]

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{9}$$
$$\nabla_X \xi = -\phi X. \tag{10}$$

Again we know that Alegre et al. [2004] in (2n+1)-dimensional generalized Sasakian-space -form:

$$\begin{aligned} S(X,Y) &= (2nf_1 + 3f_2 - f_3)g(X,Y) \\ &\quad - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \end{aligned} \tag{11}$$
$$S(\phi X, \phi Y) = S(X,Y) + 2n(f_1 - f_3)\eta(X)\eta(Y), \tag{12}$$
$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{13}$$
$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{14}$$
$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \tag{15}$$
$$R(\xi,X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\}, \tag{16}$$
$$\eta(R(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{17}$$
$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{18}$$
$$Q\xi = 2n(f_1 - f_3)\xi, \tag{19}$$

for any vector fields X,Y,Z where R,S,Q and r are the Riemannian curvature tensor, Ricci tensor, Ricci operator $g(QX,Y) = S(X,Y)$ and scalar curvature tensor

of generalized Sasakian-space-forms in that order.

3 $\xi - W_8-$flat generalized Sasakian-space-form

In this section, we study $\xi - W_8-$flat in generalized Sasakian-space-form:

Definition 3.1. *A generalized Sasakian-space-form is said to be $\xi - W_8-$flat if*

$$W_8(X,Y)\xi = 0, \tag{20}$$

for any vector fields X, Y on M.

W_8-curvature tensor Tripathi and Gupta [2012] is defined as

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[S(X,Y)Z - S(Y,Z)X], \tag{21}$$

where R and S are curvature tensor and Ricci tensor of the manifold respectively. Replacing Z by ξ in (21), we get

$$W_8(X,Y)\xi = R(X,Y)\xi + \frac{1}{(n-1)}[S(X,Y)\xi - S(Y,\xi)X]. \tag{22}$$

By using (20) in (22), we get

$$R(X,Y)\xi + \frac{1}{(n-1)}[S(X,Y)\xi - S(Y,\xi)X] = 0. \tag{23}$$

By virtue of (15), (18) in (23) and on simplification, we obtained

$$(f_1-f_3)\{\eta(Y)X-\eta(X)Y\}+\frac{1}{(n-1)}[S(X,Y)\xi-2n(f_1-f_3)\eta(Y)X] = 0. \tag{24}$$

By taking inner product with ξ in (24) and on simplification, we have

$$S(X,Y) = 2n(f_1 - f_3)\eta(Y)\eta(X). \tag{25}$$

Hence above discussion, we state the following theorem:

Theorem 3.1. *If a generalized Sasakian-space-forms satisfying $\xi - W_8-$flat condition then the generalized Sasakian-space-form is a special type of $\eta-$Einstein manifolds.*

4 $\phi - W_8-$semi-symmetric condition in generalized Sasakian-space-form

In this section, we study $\phi - W_8-$semi-symmetric condition in generalized Sasakian-space-form:

Definition 4.1. *A generalized Sasakian-space-form is said to be $\phi - W_8-$semi-symmetric if*

$$W_8(X,Y)\cdot\phi = 0, \tag{26}$$

for any vector fields X,Y on M.

Now, (26) turns into

$$(W_8(X,Y)\cdot\phi)Z = W_8(X,Y)\phi Z - \phi W_8(X,Y)Z = 0. \tag{27}$$

From equation (21), we get

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[S(X,Y)Z - S(Y,Z)X]. \tag{28}$$

Replace Z by ϕZ in (28), we obtain

$$W_8(X,Y)\phi Z = R(X,Y)\phi Z + \frac{1}{(n-1)}[S(X,Y)\phi Z - S(Y,\phi Z)X]. \tag{29}$$

Making use of (28) and (29) in (27) and on simplification, we get

$$R(X,Y)\phi Z - \phi R(X,Y)Z + \frac{1}{(n-1)}[S(Y,Z)\phi X - S(Y,\phi Z)X] = 0. \tag{30}$$

Putting $X = \xi$ in (30) and by virtue of (16) and on simplification, we obtain

$$(f_1 - f_3)g(Y,\phi Z)\xi - \frac{1}{(n-1)}S(Y,\phi Z)\xi = 0. \tag{31}$$

Replace ϕZ by Z in (31) and on simplification, we get

$$S(Y,Z)\xi = (n-1)(f_1 - f_3)g(Y,Z)\xi. \tag{32}$$

By taking inner product with ξ in (32), we get

$$S(Y,Z) = (n-1)(f_1 - f_3)g(Y,Z). \tag{33}$$

Hence, we state the following theorem:

Theorem 4.1. *If a generalized Sasakian-space-form satisfying $\phi - W_8-$semi-symmetric condition then the generalized Sasakian-space-form is an Einstein manifolds.*

5 Generalized Sasakian-space-form satisfying $W_8 \cdot Q = 0$

In this section, we study generalized Sasakian-space-form satisfying $W_8 \cdot Q = 0$. Then we have

$$W_8(X,Y)QZ - Q(W_8(X,Y)Z) = 0. \tag{34}$$

Putting Y = ξ in (34), we obtain

$$W_8(X,\xi)QZ - Q(W_8(X,\xi)Z) = 0. \tag{35}$$

By virtue of (21) in (35), we get

$$\begin{aligned} R(X,\xi)QZ + \frac{1}{(n-1)}[S(X,\xi)QZ - S(\xi,QZ)X] - Q\{R(X,\xi)Z \\ + \frac{1}{(n-1)}[S(X,\xi)Z - S(\xi,Z)X]\} = 0. \end{aligned} \tag{36}$$

By using (16), (18) in (36), we obtain

$$\begin{aligned} -(f_1-f_3)[g(X,QZ)\xi - \eta(QZ)X] \\ + \frac{1}{(n-1)}[2n(f_1-f_3)\eta(X)QZ - 2n(f_1-f_3)\eta(QZ)X] \\ -Q[-(f_1-f_3)g(X,Z)\xi - \eta(Z)X] \\ + \frac{1}{(n-1)}[2n(f_1-f_3)\eta(X)Z - 2n(f_1-f_3)\eta(Z)X] = 0, \end{aligned} \tag{37}$$

$$\begin{aligned} -(f_1-f_3)S(X,Z)\xi + (f_1-f_3)Q\eta(Z)X \\ + \frac{2n}{(n-1)}(f_1-f_3)\eta(X)QZ - \frac{2n}{(n-1)}(f_1-f_3)Q\eta(Z)X \\ +(f_1-f_3)g(X,Z)Q\xi - (f_1-f_3)\eta(Z)QX \\ - \frac{2n}{(n-1)}(f_1-f_3)\eta(X)QZ + \frac{2n}{(n-1)}(f_1-f_3)Q\eta(Z)X = 0. \end{aligned} \tag{38}$$

Using (19) and simplifying (38), we have

$$S(X,Z)\xi = 2n(f_1 - f_3)g(X,Z)\xi. \tag{39}$$

Taking inner product with ξ in (39) and on simplifaction, we have

$$S(X,Z) = 2n(f_1 - f_3)g(X,Z). \tag{40}$$

Hence, we state the following theorem:

Theorem 5.1. *A generalized Sasakian-space-form satisfying $W_8 \cdot Q = 0$ is an Einstein manifolds.*

6 Generalized Sasakian-space-form satisfying $W_8 \cdot R = 0$

In this section, we study generalized Sasakian-space-form satisfying $W_8 \cdot R = 0$. Then we have

$$\begin{aligned} W_8(\xi,U)R(X,Y)Z - R(W_8(\xi,U)X,Y)Z \\ -R(X,W_8(\xi,U)Y)Z - R(X,Y)W_8(\xi,U)Z = 0. \end{aligned} \tag{41}$$

Putting Z = ξ in (41), we have

$$\begin{aligned} W_8(\xi,U)R(X,Y)\xi - R(W_8(\xi,U)X,Y)\xi \\ -R(X,W_8(\xi,U)Y)\xi - R(X,Y)W_8(\xi,U)\xi = 0. \end{aligned} \tag{42}$$

By using (15) in (42) and on simplification, we get

$$\begin{aligned} (f_1 - f_3)\eta(W_8(\xi,U)X)Y - (f_1 - f_3)\eta(W_8(\xi,U)Y)X \\ -R(X,Y)W_8(\xi,U)\xi = 0. \end{aligned} \tag{43}$$

By using (21) in (43), we get

$$\begin{aligned} (f_1 - f_3)\eta[R(\xi,U)X + \frac{1}{(n-1)}\{S(\xi,U)X - S(U,X)\xi\}]Y \\ -(f_1 - f_3)\eta[R(\xi,U)Y + \frac{1}{(n-1)}\{S(\xi,U)Y - S(U,Y)\xi\}]X \\ -R(X,Y)[R(\xi,U)\xi + \frac{1}{(n-1)}\{S(\xi,U)\xi - S(U,\xi)\xi\}] = 0. \end{aligned} \tag{44}$$

By using (16), (17), (18) in (44) and on simplification, we get

$$(f_1 - f_3)\{g(U,X)Y - g(U,Y)X\} + \frac{2n}{(n-1)}(f_1 - f_3)\eta(U)\eta(X)Y$$
$$-\frac{2n}{(n-1)}(f_1 - f_3)\eta(U)\eta(Y)X + \frac{1}{(n-1)}\{S(U,Y)X - S(U,X)Y\}$$
$$+R(X,Y)U = 0. \quad (45)$$

Putting Y = ξ in (45), we get

$$(f_1 - f_3)\{g(U,X)\xi - g(U,\xi)X\} + \frac{2n}{(n-1)}(f_1 - f_3)\eta(U)\eta(X)\xi$$
$$-\frac{2n}{(n-1)}(f_1 - f_3)\eta(U)\eta(\xi)X + \frac{1}{(n-1)}\{S(U,\xi)X - S(U,X)\xi\}$$
$$+R(X,\xi)U = 0. \quad (46)$$

By using (5), (16), (18) in (46) and on simplification, we get

$$S(X,U)\xi = 2n(f_1 - f_3)\eta(U)\eta(X)\xi. \quad (47)$$

By taking inner product with ξ in (47), we have

$$S(X,U) = 2n(f_1 - f_3)\eta(U)\eta(X). \quad (48)$$

Hence, we state the following theorem:

Theorem 6.1. *If a generalized Sasakian-space-form satisfies $W_8 \cdot R$ = 0, then the manifold is a special type of $\eta-$ Einstein manifolds.*

7 Conclusions

In this paper, we proposed the notion of the W_8-curvature tensor in generalized Sasakian-space-forms drawing inspiration from the generalized Sasakian-space-form and the W_8-curvature tensor. Several definitions are provided to support this new mechanism. The concept of generalized Sasakian-space-forms has been extensively studied by various authors,including Alegre et al. [2004], Prakasha [2012], Sarkar and Akbar [2014] and Shanmukha et al. [2018]. Their research has shed light on the properties and characteristics of these space-forms.

The main objective of this study is to investigate the curvature properties of the generalized Sasakian-space-form. The analysis begins by examining the $\xi - W_8-$ flat generalized Sasakian-space-forms which are revealed to be a specific type of Einstein manifolds. Furthermore, the paper explores the $\phi - W_8-$ semi-symmetric condition in the generalized Sasakian-space-form establishing it as an Einstein manifold. Additionally, the generalized Sasakian-space-form satisfying $W_8 \cdot Q = 0$ is discussed demonstrating its status as an Einstein manifold. Lastly, the paper presents the discovery of generalized Sasakian-space- forms that satisfy $W_8 \cdot R = 0$ and are identified as $\eta-$ Einstein manifolds.

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Approximation and moduli of continuity for a function belonging to Hölder's class $H^{\alpha}[0,1)$ and solving Lane-Emden differential equation by Boubaker wavelet technique

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Abstract

In this paper, Boubaker wavelet is considered. The Boubaker wavelets are orthonormal. The series of this wavelet is verified for the function $f(t) = t \ \ \forall\, t \in$[0,1). The convergence analysis of solution function of Lane-Emden differential equation has been studied. New Boubaker wavelet estimator $E_{2^k,M}(f)$ for the approximation of solution function f belong to Hölder's class $H^{\alpha}[0,1)$ of order $0 < \alpha \leq 1$, has been developed. Furthermore, the moduli of continuity of $\big(f - S_{2^k,M}(f)\big)$ of solution function f of Lane-Emden differential equation has been introduced and it has been estimated for solution function f$\in H^{\alpha}[0,1)$ class. These estimator and moduli of continuity are new and best possible in wavelet analysis. Boubaker wavelet collocation method has been proposed to solve Lane-Emden differential equations with unknown Boubaker coefficients. In this process, Lane-Emden differential equations are reduced into a system of algebraic equations and these equations are solved by collocation method. Three Lane-Emden type equations are solved to demonstrate the applicability of the proposed method. The solutions obtained by the proposed method are compared with their exact solutions. The

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absolute errors are negligible. Thus, this shows that the method described in this paper is applicable and accurate.
Keywords: Boubaker wavelet, Boubaker polynomial, Boubaker wavelet approximation, Moduli of continuity, convergence analysis, Collocation method, Lane-Emden differential equations.
2020 AMS subject classifications:42C40, 65T60, 45G10, 45B05.[1]

1 Introduction

Wavelet theory is a newly emerging area of research in a mathematical sciences. It has applications in engineering disciplines; such as signal analysis for wave representation and segmentation etc. Wavelets allow the accurate representation of a variety of signals and operators. Wavelets are assumed as a basis function $\{\psi_{n,m}(\cdot)\}$ continuously in time domain. Special feature of wavelets basis is that all functions $\{\psi_{n,m}(\cdot)\}$ are constructed from a single mother wavelet $\psi(\cdot)$ which is small pulse. Many practical and physical problems in the field of science and engineering are formulated as intial and boundry value problems. Approximation of functions by the wavelet method has been discussed by many researchers like Devorce[2], Morlet[5], Meyer[4], Debnath[3], Lal and Satish[11]. The wavelet functions have been applied for finding approximate solutions for some problems arising in numerous branches of science and engineering.

In this paper, Boubaker wavelet has been studied. This wavelet is defined by the orthogonal Boubaker polynomials. It has several interesting and useful properties.
The main aims of present paper are as follows:
(i) To define Boubaker wavelet and to verify Boubaker wavelet series by examples.
(ii) To study the properties of Boubaker wavelet coefficient in expansion of characteristic function.
(iii) To estimate the approximation of solution function f of Lane-Emden differential equations belonging to Hölder's class $H^{\alpha}[0,1)$ by the Boubaker wavelet series.
(iv) To estimate the moduli of continuity of $\left(f - S_{2^k,M}(f)\right)$ of solution function f of Lane-Emden differential equations belonging to Hölder's class $H^{\alpha}[0,1)$ by the Boubaker wavelet series.
(v) To solve Lane-Emden differential equation by Boubaker wavelet series by

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collocation method.
The paper is organized as follows: In Section-2, Boubaker wavelet,Boubaker wavelet approximation and moduli of continuity of Function of Hölder's class $H^{\alpha}[0,1)$ are defined. The Boubaker wavelet series is verified by example. In Section-3 ,theorem concerning the convergence analysis of Boubaker wavelet has been discussed. In Section-4, theorem concerning Boubaker wavelet coefficient in the expansion of characteristic function and approximation in $H^{\alpha}[0,1)$ have been obtained. In Section-5, theorem concerning the moduli of continuity of $\big(f - S_{2^k,M}(f)\big)$ have been determined. In Section-6, Boubaker wavelet method for solution of differential equation has been discussed . In Section-7, Lane-Emden differential equations have been solved using Boubaker wavelet series by collocation method. Finally, the main conclusions are summarized in Section-8.

2 Definitions and Preliminaries

2.1 Boubaker wavelet

Wavelet functions are constructed from dilation and translation of a definite function, named mother wavelet ψ . $\psi_{b,a}$ may be defined as

$$\psi_{b,a}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right) \ a, b \in \mathbb{R}, a \neq 0 \qquad \text{(Daubechies[6])} \qquad (1)$$

where a and b are dilation and translation parametres respectively while t is normalized time. By taking $a = \frac{1}{2^k}$, $b = \frac{n}{2^k}$ and $\psi(t) = \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}B_m(t)$, where $B_m(t)$ is Boubaker polynomial of degree m, in equation (1), it reduces into form

$$\psi^{(B)}_{n,m}(t) = \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}2^{\frac{k}{2}}B_m(2^k t - n).$$

In precise,

$$\psi^{(B)}_{n,m}(t) = \begin{cases} \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}2^{\frac{k}{2}}B_m(2^k t - n), & if \ \frac{n}{2^k} \leqslant t < \frac{n+1}{2^k}, \\ 0, & otherwise. \end{cases}$$

where $n = 0, 1, 2..., 2^k - 1,\ k = 0, 1, 2, 3, ...,$ while m represents the order of orthogonal Boubaker polynomial (shiralashetti et al.[8]). It has four parameters m, n, k & t. The orthogonal Boubaker polynomial $B_m(t)$ of order m satisfies the

following conditions:

$$B_0(t) = 1,\ B_1(t) = \frac{1}{2}(2t-1), B_2(t) = \frac{1}{6}(6t^2 - 6t + 1)\ ;$$
$$B_3(t) = \frac{1}{20}(20t^3 - 30t^2 + 12t - 1),\ \ B_4(t) = \frac{1}{70}(70t^4 - 140t^3 + 90t^2 - 20t + 1)\ ;$$
$$B_5(t) = \frac{1}{252}(252t^5 - 630t^4 + 560t^3 - 210t^2 + 30t - 1);$$
$$B_6(t) = \frac{1}{924}(924t^6 - 2772t^5 + 3150t^4 - 1680t^3 + 420t^2 - 42t + 1);$$

2.2 Boubaker Wavelet Approximation

A function $f \in L^2[0,1)$ may be expanded in Boubaker wavelet series as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t), \tag{2}$$

where the coefficients $c_{n,m}$ are given by

$$c_{n,m} =< f(t), \psi_{n,m}^{(B)}(t) > . \tag{3}$$

If the series (2) is truncated, then

$$(S_{2^k,M}f)(t) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t) = C^T\Psi(t) \tag{4}$$

where $(S_{2^k,M}f)$ is the $(2^k, M)^{th}$ partial sum of series (2) and C, $\Psi(t)$ are $2^kM \times 1$ matrices given by

$$C = [c_{0,0}, c_{0,1}, ..., c_{0,}M-1, c_{1,0}, ..., c_{1,M-1}, ..., c_{2^k-1,0}, ..., c_{2^k-1,M-1}]^T \tag{5}$$

and

$$\Psi(t) = [\psi_{0,0}(t), \psi_{0,1}(t), ..., \psi_{0,M-1}(t), \psi_{1,0}(t), \psi_{1,M-1}(t), ..., \psi_{2^k-1,0}(t), ..., \psi_{2^k-1,M-1}]^T. \tag{6}$$

The Boubaker wavelet approximation of f by $(S_{2^k,M}f)$ under norm $||\ ||_2$, denoted by $E_{2^k,M}(f)$, is defined by

$$E_{2^k,M}(f) = \min||f - (S_{2^k,M})||_2 \qquad \text{(Zygmund)[10])}.$$

If $E_{2^k,M}(f) \to 0$ as k $\to \infty$, M$\to\infty$,then $E_{2^k,M}(f)$ is best approximation of f order $(2^k, M)$ (Zygmund[10]).

2.3 Moduli of continuity

The moduli of continuity of a function $f \in L^2[0,1)$ is defined as

$$\begin{aligned} W(f,\delta) &= \sup_{0\le h\le\delta} ||f(\cdot+h)-f(\cdot)||_2 \\ &= \sup_{0\le h\le\delta}\left(\int_0^1 |f(t+h)-f(t)|^2 dt\right)^{\frac{1}{2}} \end{aligned}$$

It is remarkable to note that $W(f,\delta)$ is a non-decreasing function of δ and $W(f,\delta)\to 0$ as $\delta\to 0$, (Chui [1]).

2.4 Function of Hölder's class $H^\alpha[0,1)$

A function $f \in H^\alpha[0,1)$ if f is continuous and satisfies the inequality

$f(x)-f(y)=O(|x-y|^\alpha), \forall x,y\in[0,1)$ and $0<\alpha\le 1$ (Das)[9]).

2.5 Example

The example of this section illustrates the validity of the Boubaker wavelet series as follows:
Consider the function $f:[0,1)\to\mathbb{R}$ defined by $f(t)=t \ \ \forall\, t\in$[0,1).
Let

$$f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t). \tag{7}$$

$$\begin{aligned} c_{n,m} &= <f(t),\psi_{n,m}^{(B)}(t)> = \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)\psi_{n,m}^{(B)}(t)dt \\ &= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} t\sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}2^{\frac{k}{2}}B_m(2^kt-n)\,dt \\ &= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}2^{\frac{k}{2}}\int_0^1 \frac{v+n}{2^k}B_m(v)\frac{dv}{2^k}, \ \ 2^kt-n=v \\ &= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}\int_0^1 (v+n)B_m(v)dv. \end{aligned}$$

By above expansion ,taking $m=0$,

$$c_{n,0} = \frac{1}{2^{\frac{3k}{2}}}\int_0^1 (v+n)B_0(v)dv = \frac{1}{2^{\frac{3k}{2}}}\int_0^1 (v+n)dv = \frac{2n+1}{2^{\frac{3k+2}{2}}}. \tag{8}$$

Next

$$\begin{aligned} c_{n,1} &= \frac{2\sqrt{3}}{2^{\frac{3k}{2}}}\int_0^1 (v+n)B_1(v)dv = \frac{\sqrt{3}}{2^{\frac{3k}{2}}}\int_0^1 (v+n)(2v-1)dv \\ &= \frac{\sqrt{3}}{3.2^{\frac{3k+2}{2}}}. \qquad (9) \\ c_{n,2} &= \frac{6\sqrt{5}}{2^{\frac{3k}{2}}}\int_0^1 (v+n)B_2(v)dv = \frac{\sqrt{5}}{2^{\frac{3k}{2}}}\int_0^1 (v+n)(6v^2-6v+1)dv = 0 \end{aligned}$$

For, $m \geq 2$

$$\begin{aligned} c_{n,m} &= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}\int_0^1 (v+n)B_m(v)dv \\ &= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}\bigg(\int_0^1 vB_m(v)dv + \int_0^1 nB_m(v)dv\bigg) \\ &= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}\bigg(\int_0^1 \frac{1}{2}(2v-1)B_m(v)dv \\ &+ \int_0^1 \frac{1}{2}B_m(v)dv + \int_0^1 nB_m(v)dv\bigg) \\ &= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}\bigg(\int_0^1 B_1(v)B_m(v)dv \\ &+ \frac{1}{2}\int_0^1 B_0(v)B_m(v)dv + n\int_0^1 B_0(v)B_m(v)dv\bigg) \\ &= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}(0+0+0) = 0 \text{ , } B_m(v) \text{ is orthogonal .} \\ c_{n,m} &= 0 \ \forall\, n \geq 2^k, \text{ by defintion of } \psi_{n,m}^{(B)}. \end{aligned}$$

$$\text{Then , } \quad f(t) = \sum_{n=0}^{2^k-1} c_{n,0}\psi_{n,0}^{(B)}(t) + \sum_{n=0}^{2^k-1} c_{n,1}\psi_{n,1}^{(B)}(t) + \sum_{n=2^k}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t), \qquad (10)$$

Next,

$$\begin{aligned} ||f||_2^2 &= < f, f > \\ &= < \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t), \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t) > \\ &= \bigg\langle \sum_{n=0}^{2^k-1} c_{n,0}\psi_{n,0}^{(B)}(t) + \sum_{n=0}^{2^k-1} c_{n,1}\psi_{n,1}^{(B)}(t) + \sum_{n=2^k}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t), \\ &\quad \sum_{n=0}^{2^k-1} c_{n,0}\psi_{n,0}^{(B)}(t) + \sum_{n=0}^{2^k-1} c_{n,1}\psi_{n,1}^{(B)}(t) + \sum_{n=2^k}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t)\bigg\rangle \\ &= \sum_{n=0}^{2^k-1} c_{n,0}^2||\psi_{n,0}^{(B)}||_2^2 + \sum_{n=0}^{2^k-1} c_{n,1}^2||\psi_{n,1}^{(B)}||_2^2 + \sum_{n=2^k}^{\infty} c_{n,m}^2||\psi_{n,m}^{(B)}||_2^2, \end{aligned}$$

$$= \sum_{n=0}^{2^k-1} c_{n,0}^2 + \sum_{n=0}^{2^k-1} c_{n,1}^2 + 0, \{\psi_{n,m}\}_{n,m\in\mathbb{Z}} \text{ being orthonormal}$$

$$= \sum_{n=0}^{2^k-1} \frac{(2n+1)^2}{2^{(3k+2)}} + \sum_{n=0}^{2^k-1} \frac{1}{3.2^{3k+2}} = \frac{1}{3}, \text{ by eqns (8) and (9).}$$

$$\text{Also, } ||f||_2^2 = < f, f >= \int_0^1 |f(t)|^2 dt = \int_0^1 t^2 dt = \frac{1}{3}.$$

Hence, the Boubaker wavelet expansion (7) is verified for $f(t) = t$.

3 Convergence analysis

In this section, the convergence analysis of solution of Lane-Emden differential equation has been studied.

3.1 Theorem

If f be a exact solution of of Lane-Emden differential equation and its Boubaker wavelet series is

$$f(\cdot) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(\cdot) \tag{11}$$

then its $(2^k, M)^{th}$ partial sums $(S_{2^k,M}f)(\cdot) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(B)}(\cdot)$ converges to $f(\cdot)$ as M$\to\infty$, k$\to\infty$.

Proof of Theorem 3.1

$$\text{Now, } < f, f > = \left\langle \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}, \sum_{n'=0}^{\infty}\sum_{m'=0}^{\infty} c_{n',m'}\psi_{n',m'}^{(B)} \right\rangle$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{m'=0}^{\infty} c_{n,m}\overline{c_{n',m'}} < \psi_{n,m}^{(B)}, \psi_{n',m'}^{(B)} >$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\overline{c_{n,m}} < \psi_{n,m}^{(B)}, \psi_{n,m}^{(B)} >$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |c_{n,m}|^2 ||\psi_{n,m}^{(B)}||^2$$

$$
\begin{aligned}
&= \sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty}|c_{n,m}|^2||\psi_{n,m}^{(B)}||^2 + \sum_{n=2^k}^{\infty}\sum_{m=0}^{\infty}|c_{n,m}|^2||\psi_{n,m}^{(B)}||^2 \\
&= \sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty}|c_{n,m}|^2 + 0, \text{ by defintion of } \{\psi_{n,m}\}_{n,m\in\mathbb{Z}} \\
\sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty}|c_{n,m}|^2 &= ||f||_2^2 < \infty, f \in L^2[0,1). \qquad (12)
\end{aligned}
$$

For $M > N$, using eqn (12),

$$
\begin{aligned}
||(S_{2^k,M}f) - (S_{2^k,N}f)||_2^2 &= \left|\left|\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}^{(B)}(t) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{N-1}c_{n,m}\psi_{n,m}^{(B)}(t)\right|\right|_2^2 \\
&= \left|\left|\sum_{n=0}^{2^k-1}\sum_{m=N}^{M-1}c_{n,m}\psi_{n,m}^{(B)}(t)\right|\right|_2^2 \\
&= \left\langle \sum_{n=0}^{2^k-1}\sum_{m=N}^{M-1}c_{n,m}\psi_{n,m}^{(B)}(t), \sum_{n=0}^{2^k-1}\sum_{m=N}^{M-1}c_{n,m}\psi_{n,m}^{(B)}(t)\right\rangle \\
&= \sum_{n=0}^{2^k-1}\sum_{m=N}^{M-1}c_{n,m}\overline{c_{n,m}} < \psi_{n,m}^{(B)}(t), \psi_{n,m}^{(B)}(t) > \\
&= \sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty}|c_{n,m}|^2||\psi_{n,m}^{(B)}(t)||^2 \\
&= \sum_{n=0}^{2^k-1}\sum_{m=N}^{M-1}|c_{n,m}|^2 \to 0 \text{ as } M\to\infty, N\to\infty.
\end{aligned}
$$

Hence, $\left\{(S_{2^k,M}f)\right\}_{M\in\mathbb{N}}$ is a Cauchy sequence in $L^2[0,1)$, $L^2[0,1)$ is a Banach space and hence $\left\{(S_{2^k,M}f)\right\}_{M\in\mathbb{N}}$ converges to a function $g(t) \in L^2[0,1)$.
Now we need to prove that $g(t) = f(t)$.
For this

$$
\begin{aligned}
< g(t) - f(t), \psi_{n_0,m_0}^{(B)}(t) > &= < g(t), \psi_{n_0,m_0}^{(B)}(t) > - < f(t), \psi_{n_0,m_0}^{(B)}(t) > \\
&= < \lim_{M\to\infty}(S_{2^k,M}f)(t), , \psi_{n_0,m_0}^{(B)}(t) > - c_{n_0,m_0} \\
&= \lim_{M\to\infty}\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1}c_{n,m} < \psi_{n,m}^{(B)}(t), \psi_{n_0,m_0}^{(B)}(t) > - c_{n_0,m_0} \\
&= c_{n_0,m_0} < \psi_{n_0,m_0}(t), \psi_{n_0,m_0}(t) > - c_{n_0,m_0} \\
&= c_{n_0,m_0} - c_{n_0,m_0} \quad = 0.
\end{aligned}
$$

Thus $< g(t) - f(t), \psi_{n,m}^{(B)}(t) > = \ 0 \ \forall \quad n \geqslant n_0, m \geqslant m_0$.
Then $g(t) = f(t)$.
Hence, $\sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(B)}(t)$ converges to $f(t)$ as $k \to \infty, M \to \infty$.

4 Approximation analysis

In this Section, approximation f by $(S_{2^k,M} f)$ is estimated as follows.

4.1 Theorem

Let a function $f = \chi_{\left[\frac{n_0}{2^k}, \frac{n_0+1}{2^k}\right)}$, where n_0 is positive integer less than equal to 2^k and Boubaker wavelet expansion of f is

$$f(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}^{(B)}(t) \tag{13}$$

then the coefficients $c_{n,m}$ satisfy

$$c_{n,m} = \begin{cases} O\left(\frac{((2m)! \sqrt{(2m+1)}\ 2^{-k/2})}{(m!)^2} \right), & if \ \ n = n_0, \\ 0, & n \neq n_0, \end{cases}$$

4.2 Theorem

Let the solution function f of Lane-Emden differential equation be a uniformly continuous defined in $[0, 1)$ such that

$$|f(t_1) - f(t_2)| \leq |t_1 - t_2|^{\alpha} \frac{1}{2^{m-1} m^{\frac{3}{2}}}, \ \ \forall t_1, t_2 \ \ \in [0, 1), m \geq 1 \tag{14}$$

and its Boubaker wavelet series

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}^{(B)}(t) \tag{15}$$

having $(2^k, M)^{th}$ partial sums

$$(S_{2^k,M} f)(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(B)}(t) \tag{16}$$

then Boubaker wavelet approximation $E_{2^k,M}(f)$ satisfies

$$E_{2^k,M}(f) = \min||f - (S_{2^k,M}f)||_2 \quad = O\left(\frac{1}{2^{k\alpha}\sqrt{M}}\right).$$

Proof of theorem 4.1 For

$$f = \chi_{[\frac{n_0}{2^k}, \frac{n_0+1}{2^k})},$$

and

$$\begin{aligned}
c_{n_0,m} &= < f(t), \psi^{(B)}_{n_0,m}(t) > \\
&= \int_{\frac{n_0}{2^k}}^{\frac{n_0+1}{2^k}} f(t)\psi^{(B)}_{n_0,m}(t)dt \\
&= \int_{\frac{n_0}{2^k}}^{\frac{n_0+1}{2^k}} \chi_{[\frac{n_0}{2^k}, \frac{n_0+1}{2^k})}(t)\sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{\frac{k}{2}}B_m(2^kt - n_0)\, dt \\
&= \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{\frac{k}{2}}\int_{\frac{n_0}{2^k}}^{\frac{n_0+1}{2^k}} B_m(2^kt - n_0)\, dt \\
&= \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{\frac{k}{2}}\int_0^1 B_m(v)\frac{dv}{2^k}, \;\; 2^kt - n_0 = v \\
&= \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}\frac{1}{2^{\frac{k}{2}}}\int_0^1 B_m(v)\, dv \\
|c_{n_0,m}| &\leq \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{-\frac{k}{2}}\int_0^1 |B_m(v)|\, dv \\
&\leq \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{-\frac{k}{2}}, \quad \int_0^1 |B_m(v)|dv \leq 1. \qquad (17)
\end{aligned}$$

Then

$$c_{n,m} = \begin{cases} O\left(\frac{((2m)!\sqrt{(2m+1)}\ 2^{-k/2})}{(m!)^2}\right), & if \;\; n = n_0, \\ 0, & n \neq n_0, \end{cases}$$

Thus, theorem 4.1 is completely established.

Proof of theorem 4.2

$$\begin{aligned}
f(t)-(S_{2^k,M}f)(t) &= \sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t)-\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(B)}(t)\\
&= \sum_{n=0}^{2^k-1}(\sum_{m=0}^{M-1}+\sum_{m=M}^{\infty})c_{n,m}\psi_{n,m}^{(B)}(t)-\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(B)}(t)\\
&= \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(B)}(t)+\sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t)\\
&- \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(B)}(t)\\
&= \sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t).
\end{aligned}$$

Next

$$\begin{aligned}
c_{n,m} &= < f(t),\psi_{n,m}^{(B)}(t) >\\
&= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)\psi_{n,m}^{(B)}(t)dt\\
&= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \{f(t)-f(\frac{n}{2^k})\}\psi_{n,m}^{(B)}(t)dt+\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(\frac{n}{2^k})\psi_{n,m}^{(B)}(t)dt\\
&= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \{f(t)-f(\frac{n}{2^k})\}\psi_{n,m}^{(B)}(t)dt+f(\frac{n}{2^k})\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \psi_{n,m}^{(B)}(t)dt\\
&= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \{f(t)-f(\frac{n}{2^k})\}\psi_{n,m}^{(B)}(t)dt\ ,\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \psi_{n,m}^{(B)}(t)dt=0, m\geq 1.
\end{aligned}$$

Then

$$\begin{aligned}
|c_{n,m}| &\leq \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} |f(t)-f^(\frac{n}{2^k})|\ |\psi_{n,m}^{(B)}(t)|dt\\
&= \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{\frac{k}{2}}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} |f(t)-f^(\frac{n}{2^k})|\ |B_m(2^kt-n)|dt\\
&= \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}\frac{1}{2^{\frac{k}{2}}}\int_0^1 |f(\frac{u+n}{2^k})-f^(\frac{n}{2^k})||B_m(u)|du, 2^kt-n=u\\
&\leq \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}\frac{1}{2^{\frac{k}{2}}}\frac{1}{2^{k\alpha}}\frac{1}{2^{(m-1)}m^{\frac{3}{2}}}\int_0^1 |B_m(u)|du
\end{aligned}$$

$$\begin{aligned}
&\leq \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}\frac{1}{2^{k\alpha+\frac{k}{2}}}\frac{1}{2^{(m-1)}m^{\frac{3}{2}}}, \int_0^1 |B_m(u)|du \leq 1 \\
&\leq \frac{1}{2^{k(\alpha+\frac{1}{2})}m}, \qquad \frac{(m!)^2}{(2m)!} \leqslant \frac{1}{2^{m-1}}. \qquad (18)
\end{aligned}$$

Next,

$$\begin{aligned}
||f-(S_{2^k,M}f)||_2^2 &= \int_0^1 |f(t)-(S_{2^k,M}f)(t)|^2 dt \\
&= \int_0^1 \left(\sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t)\right)^2 dt \\
&= \int_0^1 \Bigg(\sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} c_{n,m}^2(\psi_{n,m}^{(B)}(t))^2 \\
&\quad + \sum_{0\leqslant n\neq n'\leq 2^k-1}\sum \sum_{M\leqslant m\neq m'\leq\infty}\sum c_{n,m}c_{n',m'}\psi_{n,m}^{(B)}(t)\psi_{n',m'}^{(B)}(t)\Bigg)dt \\
&= \sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} c_{n,m}^2\int_0^1 (\psi_{n,m}^{(B)}(t))^2\,dt \\
&\quad + \sum_{0\leqslant n\neq n'\leq 2^k-1}\sum \sum_{M\leqslant m\neq m'\leq\infty}\sum c_{n,m}c_{n',m'}\int_0^1 \psi_{n,m}^{(B)}(t)\psi_{n',m'}^{(B)}(t)\,dt \\
&= \sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} |c_{n,m}|^2||\psi_{n,m}^{(B)}(t)||_2^2 \\
&= \sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} |c_{n,m}|^2\ ,\ \{\psi_{n,m}\}_{n,m\in\mathbb{Z}} \text{ being orthonormal in [0,1)} \\
&\leq \sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} \frac{1}{2^{k(2\alpha+1)}m^2},\ \text{ by eqn (18).} \\
&= \frac{2^k}{2^{k(2\alpha+1)}}\sum_{m=M}^{\infty}\frac{1}{m^2} \\
&\leq \frac{2^k}{2^{k(2\alpha+1)}}(\frac{1}{M^2}+\int_M^{\infty}\frac{dm}{m^2}), \text{by Cauchy's intergal test} \\
&= \frac{1}{2^{2k\alpha}}(\frac{1}{M^2}+\frac{1}{M}) \\
&\leq \frac{2}{2^{2k\alpha}M}
\end{aligned}$$

$$\text{Then } E_{2^k,M}(f) = \min||f-(S_{2^k,M}f)||_2 \leqslant \frac{\sqrt{2}}{2^{k\alpha}\sqrt{M}} = O(\frac{1}{2^{k\alpha}\sqrt{M}})$$

Thus, theorem 4.2 is completely established.

5 Moduli of continuity

The moduli of continuity of $\big(f-(S_{2^k,M}f)\big)$ have been determined in this section as follows :

5.1 Theorem

If the solution function f of Lane-Emden differential equation satisfies eqns (14), (15) & (16), then moduli of continuity of $\big(f-(S_{2^k,M}f)\big)$ is given by

$$\begin{aligned} W\bigg(\big(f-(S_{2^k,M}f)\big),\frac{1}{2^k}\bigg) &= \sup_{0\le h\le\frac{1}{2^k}} ||\big(f-(S_{2^k,M}f)\big)(\cdot+h)-\big(f-(S_{2^k,M}f)\big)(\cdot)||_2 \\ &= O\bigg(\frac{1}{2^{k\alpha}\sqrt{M}}\bigg) \end{aligned}$$

Proof of theorem (5.1)
Following the proof of theorem (4.2) ,

$$||f-(S_{2^k,M}f)||_2 \;= O\bigg(\tfrac{1}{2^{k\alpha}\sqrt{M}}\bigg).$$

Then

$$\begin{aligned} W\bigg(\big(f-(S_{2^k,M}f)\big),\frac{1}{2^k}\bigg) &= \sup_{0\le h\le\frac{1}{2^k}} ||\big(f-(S_{2^k,M}f)\big)(t+h)-\big(f-(S_{2^k,M}f)\big)(t)||_2 \\ &\le ||\big(f-(S_{2^k,M}f)\big)||_2+||\big(f-(S_{2^k,M}f)\big)||_2 \\ &= 2||\big(f-(S_{2^k,M}f)\big)||_2 \\ &= 2.O\bigg(\frac{1}{2^{k\alpha}\sqrt{M}}\bigg). \\ W\bigg(\big(f-(S_{2^k,M}f)\big),\frac{1}{2^k}\bigg) &= O\bigg(\frac{1}{2^{k\alpha}\sqrt{M}}\bigg) \end{aligned}$$

6 Boubaker wavelet method for solution of differential equations

In this Section, the solution of Lane-Emden differential equations are obtained by applying Boubaker wavelet collocation method.

Consider the Lane-Emden differential of the form

$$f''(t) + \frac{\alpha}{t} f'(t) + f(t) = h(t), \text{ where } t \in [0,1) \qquad (\text{Wazwaz})[7]). \tag{19}$$

$$f(0) = a, f'(0) = b \tag{20}$$

The solution of any differential equation can be expanded as Boubaker wavelet series as follows

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}^{(B)}(t)$$

Now $f(t)$ is approximated by truncated series

$$(S_{2^k,M} f)(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(B)}(t) \tag{21}$$

then the following residual is obtained by substituting $(S_{2^k,M} f)$ from eqn(21) into eqn(19)

$$\begin{aligned} R(t) &= t \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{''(B)}(t) + \alpha \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{'(B)}(t) + t \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(B)}(t) \\ &- t \times h(t). \end{aligned}$$

The collocation method yields

$$R(t_i) = 0, \ i = 1, 2, 3, ..., 2^k M - 2.$$

Moreover using the intial conditions eqn (20),

$$\sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(B)}(0) = a, \quad \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{'(B)}(0) = b. \tag{22}$$

Hence $2^k M$ system of equations are derived in the unknown coefficients $c_{n,m}$ which can be computed. This procedure is applied for differential equations of heigher order.

7 Illustrated Examples

In this Section, three Lane-Emden differential equations have been solved by using the procedure discussed in previous section-6. Illustrated examples are as follows:

Example (1)

Consider the following Lane-Emden differential equation

$$f''(t)+\frac{2}{t}f'(t)+f(t)=1+12t+t^3,\ f(0)=1,\ f'(0)=0,\ 0\le t<1 \quad (23)$$

The exact solution of eqn (23) is $f(t)=t^3+1$.
Now the differential equation has been solved by applying the procedure described in Section-6, using Boubaker wavelet method by taking $M=5, k=0$. Consider

$$\begin{aligned} f(t) &= \sum_{m=0}^{4} c_{0,m}\psi_{0,m}^{(B)}(t) \\ &= c_{0,0}\psi_{0,0}^{(B)}+c_{0,1}\psi_{0,1}^{(B)}+c_{0,2}\psi_{0,2}^{(B)}c_{0,3}\psi_{0,3}^{(B)}+c_{0,4}\psi_{0,4}^{(B)} \quad (24)\\ f(t) &= c_{0,0}+c_{0,1}\sqrt{3}(2t-1)+c_{0,2}\sqrt{5}(6t^2-6t+1) \\ &+ c_{0,3}\sqrt{7}(20t^3-30t^2+12t-1) \\ &+ c_{0,4}3(70t^4-140t^3+90t^2-20t+1) \quad (25)\end{aligned}$$

Differentiate eqn (25) with respect to t,

$$\begin{aligned} f'(t) &= c_{0,1}(2\sqrt{3})+c_{0,2}\sqrt{5}(12t-6)+c_{0,3}\sqrt{7}(60t^2-60t+12) \\ &+ c_{0,4}3(280t^3-420t^3+180t-20) \quad (26)\end{aligned}$$

$$f''(t)=c_{0,2}(12\sqrt{5})+c_{0,3}\sqrt{7}(120t-60)+c_{0,4}3(840t^2-840t+180)$$

Substitute these values of $f(t)$, $f'(t)$ and $f''(t)$ in given differential eqn (23)

$$\begin{aligned} &c_{0,0}+c_{0,1}\big[\frac{4\sqrt{3}}{t}+\sqrt{3}(2t-1)\big]+c_{0,2}\big[12\sqrt{5}+\frac{2\sqrt{5}}{t}(12t-6) \\ &+\sqrt{5}(6t^2-6t+1)\big]+c_{0,3}\big[\sqrt{7}(120t-60)+\frac{2\sqrt{7}}{t}(60t^2-60t+12) \\ &+\sqrt{7}(20t^3-30t^2+12t-1)\big]+c_{0,4}\big[3(840t^2-840t+180) \\ &+\frac{6}{t}(280t^3-420t^2+180t-20)+3(70t^4-140t^3+90t^2-20t)+1\big] \\ &=1+12t+t^3 \quad (27)\end{aligned}$$

Using intial condition, $f(0)=1$ and $f'(0)=0$ in eqns (25) and (26)

$$c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}-\sqrt{7}c_{0,3}+3c_{0,4}=1 \quad (28)$$

$$2\sqrt{3}c_{0,1}-6\sqrt{5}c_{0,2}+12\sqrt{7}c_{0,3}-60c_{0,4}=0 \quad (29)$$

Now collocate the eqn (27) at $t_1 = 0.5$,$t_2 = 0.7$ and $t_3 = 0.9$, which are obtained by $x_i = \frac{i-\frac{1}{2}}{2^k M} = \frac{i-\frac{1}{2}}{5}$,$i = 2, 4, 5$ respectively.
A system of three linear equations are derived.

$$c_{0,0} + 13.8564c_{0,1} + 25.7147c_{0,2} - 31.7490c_{0,3} - 88.875c_{0,4} = 7.125 \quad (30)$$

$$c_{0,0} + 10.5902c_{0,1} + 41.5844c_{0,2} + 57.79832c_{0,3} - 21.7675c_{0,4} = 9.743 \quad (31)$$

$$c_{0,0} + 9.0836c_{0,1} + 51.7127c_{0,2} + 166.0120c_{0,3} + 351.9676c_{0,4} = 12.529 \quad (32)$$

Solving these eqns (30),(31) and (32) with (28) and (29)
$c_{0,0} = 1.2499999999$, $c_{0,1} = 0.2598076211$, $c_{0,2} = 0.1118033988$
$c_{0,3} = 0.0188982236$, $c_{0,4} = -0.0000000000$.
Substitute all these values of $c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}, c_{0,4}$ in eqn (25)

$$\begin{aligned} f(t) &= 1.2499999999 + 0.2598076211\sqrt{3}(2t-1) \\ &+ 0.1118033988\sqrt{5}(6t^2 - 6t + 1) \\ &+ 0.0188982236\sqrt{7}(20t^3 - 30t^2 + 12t - 1) \\ &- 0.0000000000(70t^4 - 140t^3 + 90t^2 - 20t + 1) \end{aligned} \quad (33)$$

Comparison of exact and Boubaker wavelet solutions are given in table (1) for $k = 0, M = 5$.

Table (1)

t	Exact solution	Approximate solution	Absolute error($\times 10^{-15}$)
0.1	1.001000000000000	1.001000000000000	0
0.2	1.008000000000000	1.0080000000000000	0
0.3	1.027000000000000	1.0270000000000000	0
0.4	1.064000000000000	1.0640000000000000	0
0.5	1.125000000000000	1.1250000000000000	0
0.6	1.216000000000000	1.2160000000000000	0
0.7	1.343000000000000	1.3430000000000000	0
0.8	1.512000000000000	1.5120000000000000	0
0.9	1.729000000000000	1.7290000000000000	0.222044604925031

Table(1):Comparison table of exact and Boubaker wavelet solutions.

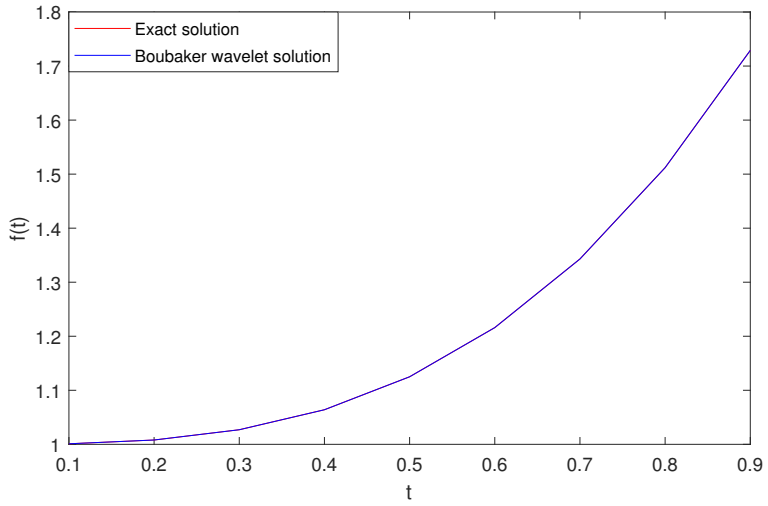

Fig.(1):The graphs of Boubaker wavelet and Exact solutions.

Example (2)

Consider the following Lane-Emden differential equation

$$f''(t)+\frac{2}{t}f'(t)+f(t)=6+12t+t^2+t^3,\ y(0)=0,\ y'(0)=0,\ 0\leq t<1 \quad (34)$$

The exact solution of eqn (34) is $f(t)=t^3+t^2$.
Following the procedure adopted in Example (1);

$$\begin{aligned}
f(t) &= \sum_{m=0}^{4} c_{0,m}\psi_{0,m}^{(B)}(t) \\
&= c_{0,0}\psi_{0,0}^{(B)}+c_{0,1}\psi_{0,1}^{(B)}+c_{0,2}\psi_{0,2}^{(B)}+c_{0,3}\psi_{0,3}^{(B)}+c_{0,4}\psi_{0,4}^{(B)} \\
f(t) &= c_{0,0}+c_{0,1}\sqrt{3}(2t-1)+c_{0,2}\sqrt{5}(6t^2-6t+1) \\
&+ c_{0,3}\sqrt{7}(20t^3-30t^2 12t-1) \\
&+ c_{0,4}3(70t^4-140t^3+90t^2-20t+1) \quad (35) \\
f'(t) &= c_{0,1}(2\sqrt{3})+c_{0,2}\sqrt{5}(12t-6)+c_{0,3}\sqrt{7}(60t^2-60t+12) \\
&+ c_{0,4}3(280t^3-420t^3+180t-20) \quad (36) \\
f''(t) &= c_{0,2}(12\sqrt{5})+c_{0,3}\sqrt{7}(120t-60)+c_{0,4}3(840t^2-840t+180)
\end{aligned}$$

Substitute these values of $f(t)$, $f^{'}(t)$ and $f^{''}(t)$ in given differential eqn (34)

$$c_{0,0} + c_{0,1}\big[\frac{4\sqrt{3}}{t} + \sqrt{3}(2t-1)\big] + c_{0,2}\big[12\sqrt{5} + \frac{2\sqrt{5}}{t}(12t-6)$$
$$\sqrt{5}(6t^2-6t+1)\big] + c_{0,3}\big[\sqrt{7}(120t-60) + \frac{2\sqrt{7}}{t}(60t^2-60t+12)$$
$$\sqrt{7}(20t^3-30t^2+12t-1)\big] + c_{0,4}\big[3(840t^2-840t+180)$$
$$+ \frac{6}{t}(280t^3-420t^2+180t-20) + 3(70t^4-140t^3+90t^2-20t)+1\big]$$
$$= 6 + 12t + t^2 + t^3 \tag{37}$$

Using intial condition, $f(0) = 0$ and $f^{'}(0) = 0$ in eqns (35) and (36)

$$c_{0,0} - \sqrt{3}c_{0,1} + \sqrt{5}c_{0,2} - \sqrt{7}c_{0,3} + 3c_{0,4} = 0 \tag{38}$$

$$2\sqrt{3}c_{0,1} - 6\sqrt{5}c_{0,2} + 12\sqrt{7}c_{0,3} - 60c_{0,4} = 0 \tag{39}$$

Now collocating the equations (37) at $t_1 = 0.5$,$t_2 = 0.7$ and $t_3 = 0.9$,

$$c_{0,0} + 13.856405c_{0,1} + 25.71478c_{0,2} - 31.74901c_{0,3} - 88.875c_{0,4} = 12.375 \tag{40}$$

$$c_{0,0} + 10.59025c_{0,1} + 41.58447c_{0,2} + 57.79832c_{0,3} - 21.76757c_{0,4} = 15.233 \tag{41}$$

$$c_{0,0} + 9.08364c_{0,1} + 51.71279c_{0,2} + 166.01207c_{0,3} + 351.96766c_{0,4} = 18.339 \tag{42}$$

Solving these eqns (40),(41) and (42) with (38) and (39) ,
$c_{0,0} = 0.5833333333$, $c_{0,1} = 0.5484827557$, $c_{0,2} = 0.1863389981$
$c_{0,3} = 0.0188982236$, $c_{0,4} = -0.0000000000$.
Substitute all these values of $c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}, c_{0,4}$ in eqn (35).

$$\begin{aligned} f(t) &= 0.5833333333 + 0.5484827557\sqrt{3}(2t-1) \\ &+ 0.1863389981\sqrt{5}(6t^2-6t+1) \\ &+ 0.0188982236\sqrt{7}(20t^3-30t^2+12t-1) \\ &- 0.0000000000(70t^4-140t^3+90t^2-20t+1) \end{aligned} \tag{43}$$

Comparison of exact and Boubaker wavelet solutions are given in table (2) for $k = 0, M = 5$.

Table (2)

t	Exact solution	Approximate solution	Absolute error ($\times 10^{-15}$)
0.1	0.011000000000000	0.011000000000000	0.017347234759768
0.2	0.048000000000000	0.048000000000000	0.020816681711722
0.3	0.117000000000000	0.117000000000000	0.027755575615629
0.4	0.224000000000000	0.224000000000000	0
0.5	0.375000000000000	0.375000000000000	0
0.6	0.576000000000000	0.576000000000000	0.222044604925031
0.7	0.833000000000000	0.833000000000000	0.111022302462516
0.8	1.152000000000000	1.152000000000000	-0.222044604925031
0.9	1.539000000000000	1.539000000000000	-0.222044604925031

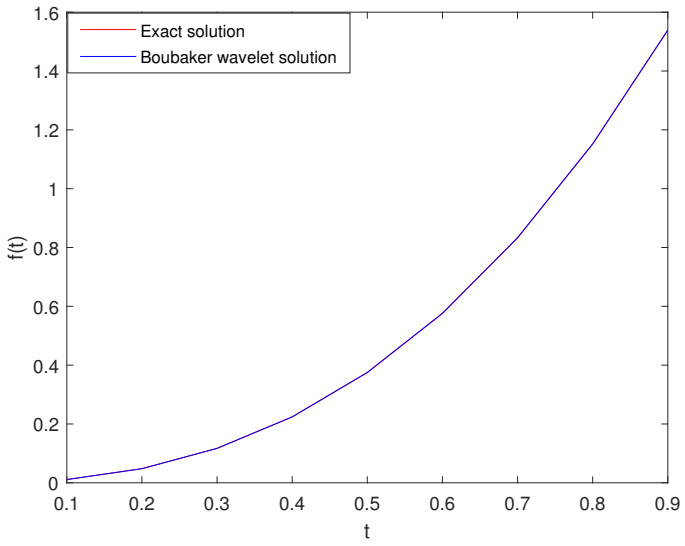

Fig.(2):The graphs of Boubaker wavelet and Exact solutions.

Example 3

Consider the following Lane-Emden differential equation

$$f''(t)+\frac{2}{t}f'(t)=2(2t^2+3)f(t),\ f(0)=1,\ f'(0)=0,\ 0\leq t<1 \tag{44}$$

The exact solution of eqn (44) is $f(t)=e^{t^2}$.
Following the procedure adopted in Example(1);

$$\begin{aligned} f(t) &= \sum_{m=0}^{4} c_{0,m}\psi_{0,m}^{(B)}(t) \\ &= c_{0,0}\psi_{0,0}^{(B)}+c_{0,1}\psi_{0,1}^{(B)}+c_{0,2}\psi_{0,2}^{(B)}+c_{0,3}\psi_{0,3}^{(B)}+c_{0,4}\psi_{0,4}^{(B)} \end{aligned}$$

$$\begin{aligned} f(t) &= c_{0,0}+c_{0,1}\sqrt{3}(2t-1)+c_{0,2}\sqrt{5}(6t^2-6t+1) \\ &+ c_{0,3}\sqrt{7}(20t^3-30t^2+12t-1) \\ &+ c_{0,4}3(70t^4-140t^3+90t^2-20t+1) \qquad (45) \\ f'(t) &= c_{0,1}(2\sqrt{3})+c_{0,2}\sqrt{5}(12t-6)+c_{0,3}\sqrt{7}(60t^2-60t+12) \\ &+ c_{0,4}3(280t^3-420t^3+180t-20) \qquad (46) \\ f''(t) &= c_{0,2}(12\sqrt{5})+c_{0,3}\sqrt{7}(120t-60)+c_{0,4}3(840t^2-840t+180) \end{aligned}$$

Substitute these values of $f(t)$, $f^{'}(t)$ and $f^{''}(t)$ in given differential eqn (44)

$$
\begin{aligned}
&- 2(2t^2+3)c_{0,0} + c_{0,1}\big[\frac{4\sqrt{3}}{t} - 2(2t^2+3)\sqrt{3}(2t-1)\big] \\
&+ c_{0,2}\big[12\sqrt{5} + \frac{2\sqrt{5}}{t}(12t-6) - 2(2t^2+3)\sqrt{5}(6t^2-6t+1)\big] \\
&+ c_{0,3}\big[\sqrt{7}(120t-60) + \frac{2\sqrt{7}}{t}(60t^2-60t+12) \\
&+ 2(2t^2+3)\sqrt{7}(20t^3-30t^2+12t-1)\big] \\
&+ c_{0,4}\big[3(840t^2-840t+180) + \frac{6}{t}(280t^3-420t^2+180t-20) \\
&- 6(2t^2+3)(70t^4-140t^3+90t^2-20t)+1\big] = 0
\end{aligned} \tag{47}
$$

Using intial condition, $f(0) = 1$ and $f^{'}(0) = 0$ in eqns (45) and (46)

$$c_{0,0} - \sqrt{3}c_{0,1} + \sqrt{5}c_{0,2} - \sqrt{7}c_{0,3} + 3c_{0,4} = 1 \tag{48}$$

$$2\sqrt{3}c_{0,1} - 6\sqrt{5}c_{0,2} + 12\sqrt{7}c_{0,3} - 60c_{0,4} = 0 \tag{49}$$

Now collocating the equation (47) at $t_1 = 0.5$,$t_2 = 0.7$ and $t_3 = 0.9$,

$$-7c_{0,0} + 13.85640c_{0,1} + 34.65905c_{0,2} - 31.74901c_{0,3} - 97.8750c_{0,4} = 0 \tag{50}$$

$$-7.96c_{0,0} + 4.38258c_{0,1} + 46.79361c_{0,2} + 68.22893c_{0,3} - 18.73013c_{0,4} = 0 \tag{51}$$

$$-9.24c_{0,0} - 5.10531c_{0,1} + 41.18002c_{0,2} + 163.84467 + 02c_{0,3} + 359.12542c_{0,4} = 0 \tag{52}$$

Solving these eqns (50), (51) and (52) with (48) and (49),
$c_{0,0} = 1.38821917722$, $c_{0,1} = 0.388372764863$, $c_{0,2} = 0.158389887721$
$c_{0,3} = 0.02903271870$, $c_{0,4} = 0.002368327161$
Substitute all these values of $c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}, c_{0,4}$ in eqn (45)

$$
\begin{aligned}
f(t) &= 1.3882191 + 0.38837276\sqrt{3}(2t-1) + 0.1583898\sqrt{5}(6t^2-6t+1) \\
&+ 0.02903271870\sqrt{7}(20t^3-30t^2+12t-1) \\
&+ 0.002368327161(70t^4-140t^3+90t^2-20t+1)
\end{aligned} \tag{53}
$$

Comparison of exact and Boubaker wavelet solutions are given in table (3) for $k = 0, M = 5$

Table (3)

t	Exact solution	Approximate solution	Absolute error
0.1	1.010050167084168	1.005192015148452	0.004858151935716
0.2	1.040810774192388	1.023531157694065	0.017279616498324
0.3	1.094174283705210	1.060057300954230	0.034116982750980
0.4	1.173510870991810	1.121003955135684	0.052506915856126
0.5	1.284025416687741	1.213798267334502	0.070227149353240
0.6	1.433329414560340	1.347061021536102	0.086268393024238
0.7	1.632316219955379	1.530606638615246	0.101709581340133
0.8	1.896480879304952	1.775443176336035	0.121037702968916
0.9	2.247907986676472	2.093772329351916	0.154135657324556

Table(3):Comparision table of exact and Boubaker wavelet solutions.
Comparison of exact and Boubaker wavelet solutions are given in table (4) for $k = 0, M = 6$.

Table (4)

t	Exact solution	Approximate solution	Absolute error
0.1	1.010050167084168	1.000547638755526	0.009502528328642
0.2	1.040810774192388	1.011676778238292	0.029133995954096
0.3	1.094174283705210	1.043773805588921	0.050400478116290
0.4	1.173510870991810	1.103730511655493	0.069780359336317
0.5	1.284025416687741	1.196980758433975	0.087044658253766
0.6	1.433329414560340	1.329537146508637	0.103792268051703
0.7	1.632316219955379	1.510027682492480	0.122288537462899
0.8	1.896480879304952	1.751732446467655	0.144748432837297
0.9	2.247907986676472	2.074620259425891	0.173287727250581

Table(4):Comparison table of exact and Boubaker wavelet solutions.

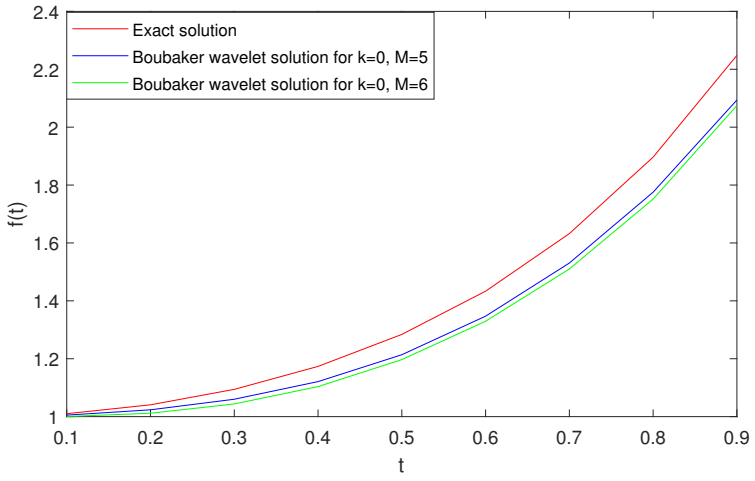

Fig.(3):The graphs of Boubaker wavelet and Exact solutions.

8 Discussion and Conclusion

1. Boubaker wavelets have applications in approximation theory, moduli of continuity and solution of Lane-Emden differential equations. The Boubaker wavelet expansion of solution of Lane-Emden Differential equation is verified and its

convergence analysis has been studied. The estimator $E_{2^k,M}(f)$ of $\big(f - S_{2^k,M}(f)\big)$ has been developed. Furthermore, moduli of continuity

$$W\big((f-(S_{2^k,M}f)),\frac{1}{2^k}\big) \quad \leq \quad 2\,E_{2^k,M}(f)$$

This shows that moduli of continuity $W\bigg(\big(f-(S_{2^k,M}f)\big),\frac{1}{2^k}\bigg)$ is sharper and better than the approximation $E_{2^k,M}(f)$ of $\big(f - S_{2^k,M}(f)\big)$. Hence the moduli of continuity $W\bigg(\big(f-(S_{2^k,M}f)\big),\frac{1}{2^k}\bigg)$ has been also estimated in this research paper.

2. A method has been proposed to solve Lane-Emden differential equation by Boubaker wavelet collocation method. To illustrate the effectiveness and accuracy of the proposed method, three Lane-Emden differential equations have been solved by proposed method, It is observed that the exact solutions of considered differential equations are atmost same to their solutions obtained by proposed method. This is a significant achivement of the research paper in wavelet analysis.

3. Our results are concerned with Boubaker wavelet estimator $E_{2^k,M}(f)$, moduli of continuity $W\bigg(\big(f-(S_{2^k,M}f)\big),\frac{1}{2^k}\bigg)$ and the solutions of Lane-Emden diffential equations by this method.

4. (i) By theorem 4.2,

$$E_{2^k,M}(f) = O\bigg(\frac{1}{2^{k\alpha}\sqrt{M}}\bigg) \to 0 \text{ as } k\to\infty, M\to\infty.$$

.

(ii) As per theorem 5.1,

$$W\bigg(\big(f-S_{2^k,M}(f)\big),\frac{1}{2^k}\bigg) = O\bigg(\frac{1}{2^{k\alpha}\sqrt{M}}\bigg) \to 0 \text{ as } k\to\infty, M\to\infty.$$

Thus $E_{2^k,M}(f)$ and $W\bigg(\big(f-S_{2^k,M}(f)\big),\frac{1}{2^k}\bigg)$ are best possible estimation in wavelet analysis

5. Solution of Lane-Emden differential equation by Boubaker wavelet series by collocation method is approximately same as exact solution of Lane-Emden differential equation. Only a few number of Boubaker wavelet basis is needed to achieve the heigh accuracy. This is significant achivement in wavelet analysis.

6. Limitations and possible future development:

(i) A non-linear Lane-Emden equation can not solved by Boubaker wavelets without using collocation method

(ii) In general, Boubaker wavelets in one variable are ineffective to solve a problem expressed in partial differential equations of two or more variables.

(iii) It is known that $H^\alpha[0,1) \not\subseteq H_2^\alpha[0,1)$.To find the approximate solution of Lane-Emden differential equation in class $H_2^\alpha[0,1)$.

(iv) To define two dimensional Boubaker wavelets and to find the solution of the partial differential equation by this method.

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Group Fuzzy Languages and its Generalizations

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Abstract

In fuzzy language theory, every monoid is the syntactic monoid of some fuzzy language. By using this result the properties of fuzzy language can be studied by the algebraic properties of the syntactic monoids. There are so many methods for studying fuzzy languages. We adopt the above method to analyze different class of fuzzy languages and also characterize certain varieties of fuzzy languages. In this paper we give the variety description of group fuzzy languages and also provide the Eilenberg variety theorem for the class of group fuzzy languages. Moreover we described a fuzzy language whose syntactic monoid is isomorphic to the group $\mathbb{Z}_n$.

Keywords: syntactic congruence; group fuzzy language; pseudovariety.
2020 AMS subject classifications: 20M35; 20M07; 68Q45.[1]

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1 Introduction

The formal study of semigroups has started in the early 20th century. The most relevant and intensively studied classes of semigroups are finite semigroups, regular semigroups and inverse semigroups. The theory of finite semigroups has great importance in the theoretical computer science because of its natural link with finite automata; in the sense that semigroups can arise as syntactic semigroup of regular languages and as transition semigroups of finite automata. This concept is mainly useful to give characterization for regular languages by using the algebraic properties of their corresponding syntactic semigroups. Eilenberg [7] put forward the idea of theory of varieties of rational languages. He establishes a one to one correspondence between the varieties of formal languages and pseudo varieties of finite semigroups through his well known variety theorem.

The notion of fuzzy language was introduced by Lee and Zadeh [9]. It is a genealization of crisp language in the sense that the characteristic function has been replaced by membership function. Since the characteristic function χ_L of a language L is a fuzzy language, every language is a fuzzy language.

The variety of fuzzy languages were introduced by T. Petkovic[11]. A class F of fuzzy language over an alphabet A is called a variety if it is closed under union, intersection, complements, multiplication by constants, quotients, inverse homomorphic images and c–cuts. He proved that there exists a one to one correspondence between variety of fuzzy languages and pseudo variety of monoids. This result is known as Eilenberg variety theorem for fuzzy languages. He also proved that there exists a mutually inverse lattice isomorphism between the lattices of all varieties of fuzzy languages and all varieties of crisp languages. Using this result and Eilenberg variety theorem, he has established a one-to-one correspondence between the lattice of all varieties of fuzzy languages and all pseudo varieties of finite monoids. One of the major result in fuzzy language theory has been considered as every monoid is the syntactic monoid of some fuzzy language[11]. Using this result we study certain classes of fuzzy languages ([2], [3], [4], [5], [6]). The aim of this paper is to provide a variety structure of group fuzzy languages. Also we give the Elienberg variety theorem for the class of group fuzzy languages. Moreover we described a fuzzy language whose syntactic monoid is isomorphic to the group $\mathbb{Z}_n$.

2 Preliminaries

Here we recall the basic definitions and notations that will be used in the sequel. All undefined terms are as in [10]. A non-empty set S with an associative binary operation is called a semigroup. If there is an element $1 \in S$ with

$1s = s = s1$ for all $s \in S$, then S is called a monoid (semigroup with identity).

Definition 2.1 (cf. [7]). *A pseudovariety of monoids (resp. semigroups) is a class* $\mathbf{V}$ *of finite monoids (resp. semigroups) that is closed under finite direct product, morphic image and subobject. That is,*

(i) If $S, T \in \mathbf{V}$, *then* $S \times T \in \mathbf{V}$;

(ii) If $S \in \mathbf{V}$ *and* $\varphi : S \to S'$ *is a homomorphism, then* $\varphi(S) \in \mathbf{V}$;

(iii) If $S \in \mathbf{V}$ *and* S' *is a submonoid (resp. subsemigroup) of* S, *then* $S' \in \mathbf{V}$.

Let A be a non-empty finite set called an alphabet. Elements of A are called letters. A word is a finite sequence of letters from A. The length of a word is the number of letters in it. A word of length zero is called the empty word, it is denoted by 1. A^+ denote the set of all non-empty words. Then, $A^* = A^+ \cup \{1\}(A^+)$ together with the binary operation concatenation is called a free monoid(semigroup) on A.

A fuzzy language λ in $A^*(A^+)$ is a fuzzy subset of $A^*(A^+)$. To each fuzzy language λ we associate a congruence P_λ called syntactic congruence, as follows:

For $u, v \in A^*(A^+)$, $uP_\lambda v$ if and only if $\lambda(xuy) = \lambda(xvy)$ for all $x, y \in A^*(A^+)$. The quotient monoid(semigroup), $\mathrm{Syn}(\lambda) = A^*/P_\lambda (Syn\lambda = A^+/P_\lambda)$ is called the syntactic monoid (semigroup) of λ.

A fuzzy language λ over an alphabet A is recognizable by a monoid (semigroup) S if there is a homomorphism $\phi : A^* \to S (\phi : A^+ \to S)$ and a fuzzy subset π of S such that $\lambda = \pi\phi^{-1}$, where $\pi\phi^{-1}(u) = \pi(\phi(u)$.

For fuzzy languages $\lambda, \lambda_1, \lambda_2$ over an alphabet A, complement, union and intersection are defined respectively by

$$\begin{aligned} \overline{\lambda}(u) &= 1 - \lambda(u), \\ (\lambda_1 \vee \lambda_2)(u) &= \lambda_1(u) \vee \lambda_2(u), \\ (\lambda_1 \wedge \lambda_2)(u) &= \lambda_1(u) \wedge \lambda_2(u). \end{aligned}$$

Further left and right quotients are defined respectively by;

$$(\lambda_1^{-1}\lambda_2)(u) = \bigvee_{v \in A^*} (\lambda_2(vu) \wedge \lambda_1(v)),$$

$$(\lambda_2\lambda_1^{-1})(u) = \bigvee_{v \in A^*} (\lambda_2(uv) \wedge \lambda_1(v)).$$

Let $c \in [0, 1]$ be arbitrary. Then the fuzzy language $c\lambda$ defined by $(c\lambda)(u) = c \cdot \lambda(u)$ is called multiplication by a constant c.

Let A, B be finite alphabets, $\phi : A^* \to B^*(\phi : A^+ \to B^+)$ be a homomorphism and ψ a fuzzy language in $B^*(B^+)$, then the inverse image of ψ (under ϕ) is a fuzzy language $\psi\phi^{-1}$ over A defined by $(\psi\phi^{-1})(u) = \psi(\phi u)$.

For a fuzzy language λ by a c-cut, $c \in [0,1]$, we mean the crisp language λ_c defined by $\lambda_c = \{u \in A^* : \lambda(u) \geq c\}$.

The following theorem gives a characterization for regular fuzzy languages.

Theorem 2.1. *[10] A fuzzy language λ is regular if and only if $Im(\lambda)$ is finite and language λ_c is regular for every $c \in [0,1]$.*

Unless otherwise specified all the fuzzy languages considered here are regular.

Definition 2.2. *A family $\mathscr{F} = \mathscr{F}(A)$ of regular fuzzy languages is a variety of fuzzy languages in $A^*(A^+)$ if it is closed under unions, intersections, complements, multiplication by constants, quotients, inverse homomorphic images and cuts.*

Theorem 2.2. *Let λ be a fuzzy language over an alphabet A. Then we have the following:*

(i) Syn(λ) recognizes λ;

(ii) If M is any other monoid (resp. semigroup) recognizing λ, then Syn(λ) is a homomorphic image of M.

Proof. **(i)** Clearly $\eta_\lambda : A^* \to Syn(\lambda)$(resp. $A^+ \to Syn(\lambda)$) is an onto homomorphism. Define $\pi : Syn(\lambda) \to [0,1]$ by $\pi(\eta_\lambda(u)) = \lambda(u)$. Then $\lambda = \pi\eta_\lambda^{-1}$ and so Syn(λ) recognizes λ.
(ii) Since M recognizes λ, there is an onto homomorphism $\varphi : A^* \to M(A^+ \to M)$ and a function $\pi' : M \to [0,1]$ such that $\lambda = \pi'\varphi^{-1}$. Define $\psi : M \to Syn(\lambda)$ by $\psi(\varphi(u)) = \eta_\lambda(u)$. Then ψ is an onto homomorphism, since η_λ is an onto homomorphism. Thus Syn(λ) is a homomorphic image of M. □

Theorem 2.3. *If the fuzzy languages λ_1 and λ_2 are recognizable by monoids (semigroups), then $\lambda_1 \vee \lambda_2$, $\lambda_1 \wedge \lambda_2$ are recognizable by monoids (semigroups).*

Proof. Let M_1, M_2 be monoids (semigroups), $\varphi_i : A^* \to M_i$ $(A^+ \to M_i), i = 1,2$ be onto homomorphisms and $\pi_i : M_i \to [0,1], i = 1,2$ be mappings such that $\lambda_i = \pi_i\varphi_i^{-1},\ i = 1,2$. Define $\varphi : A^* \to M_1 \times M_2$ $(A^+ \to M_1 \times M_2)$ by

$$\varphi(u) = (\varphi_1(u), \varphi_2(u))$$

and $\pi : M_1 \times M_2 \to [0,1]$ by

$$\pi(\varphi_1(u), \varphi_2(u)) = \pi_1(\varphi_1(u)) \vee \pi_2(\varphi_2(u)).$$

Then φ and π are well defined mappings and φ is a homomorphism.

For all $u \in A^*$ (resp. A^+), we have

$$\begin{aligned}\pi\varphi^{-1}(u) = \pi(\varphi(u)) &= \pi(\varphi_1(u), \varphi_2(u)) \\ &= \pi_1(\varphi_1(u)) \vee \pi_2(\varphi_2(u)) \\ &= \pi_1\varphi_1^{-1}(u) \vee \pi_2\varphi_2^{-1}(u) \\ &= \lambda_1(u) \vee \lambda_2(u) \\ &= (\lambda_1 \vee \lambda_2)(u).\end{aligned}$$

Thus, $\lambda_1 \vee \lambda_2 = \pi\varphi^{-1}$, and hence $\lambda_1 \vee \lambda_2$ is recognized by $M_1 \times M_2$. Now define $\pi' : M_1 \times M_2 \to [0,1]$ by

$$\pi'(\varphi_1(u), \varphi_2(u)) = \pi_1(\varphi_1(u)) \wedge \pi_2(\varphi_2(u)).$$

Then π' is a well defined mapping. For all $u \in A^*$ (resp. A^+), we have

$$\begin{aligned}\pi'\varphi^{-1}(u) = \pi'(\varphi(u)) &= \pi'(\varphi_1(u), \varphi_2(u)) \\ &= \pi_1(\varphi_1(u)) \wedge \pi_2(\varphi_2(u)) \\ &= \pi_1\varphi_1^{-1}(u) \wedge \pi_2\varphi_2^{-1}(u) \\ &= \lambda_1(u) \wedge \lambda_2(u) \\ &= (\lambda_1 \wedge \lambda_2)(u).\end{aligned}$$

Thus $\lambda_1 \wedge \lambda_2 = \pi'\varphi^{-1}$ and hence $\lambda_1 \wedge \lambda_2$ is recognized by $M_1 \times M_2$. □

Theorem 2.4. *Let λ_1, λ_2 be fuzzy languages over over an alphabet A. If a monoid M recognizes λ_1, then $(\lambda_2^{-1}\lambda_1)$ and $(\lambda_1\lambda_2^{-1})$ are recognized by M.*

Proof. Since M recognizes λ, there exists an onto homomorphism $\varphi : A^* \to M$ and a function $\pi : M \to [0,1]$ such that $\lambda_1(u) = \pi(\varphi(u))$ for all $u \in A^*$. Define $\pi_1 : M \to [0,1]$ by

$$\pi_1(m) = \bigvee_{v \in A^*} (\pi(\varphi(vu)) \wedge \lambda_2(v)),$$

where $m = \varphi(u)$. Then $\pi_1(\varphi(u)) = \bigvee_{v \in A^*} (\lambda_1(vu)) \wedge \lambda_2(v)) = \lambda_2^{-1}\lambda_1(u)$. So M recognizes $\lambda_2^{-1}\lambda_1$. Similarly $\lambda_1\lambda_2^{-1}$ is recognized by M. □

Lemma 2.1 (cf. [11], Lemma 2)**.** *Let A and B be alphabets, $\lambda, \lambda_1, \lambda_2$ be fuzzy languages over A, τ be a fuzzy language over B, $\varphi : A^* \to B^*$ be a homomorphism and $c \in [0,1]$ be a constant. Then*

(i) $P_{\overline{\lambda}} = P_{c\lambda} = P_\lambda$, $P_{\lambda_1 \vee \lambda_2}, P_{\lambda_1 \wedge \lambda_2} \supseteq P_{\lambda_1} \cap P_{\lambda_2}$;

(ii) $P_{\lambda_1^{-1}\lambda_2}, P_{\lambda_2\lambda_1^{-1}} \supseteq P_{\lambda_2}$;

(iii) $\varphi \circ P_\tau \circ \varphi^{-1} \subseteq P_\tau\varphi^{-1}$, *where* $\varphi \circ P_\tau \circ \varphi^{-1}$ *is a congruence on* A^* *defined by* $(u, v) \in \varphi \circ P_\tau \circ \varphi^{-1} \Leftrightarrow (u\varphi, v\varphi) \in P_\tau$.

Corolary 2.1. *Let* λ *be a fuzzy language over* A. *Then*

(i) $Syn(\overline{\lambda}) = Syn(\lambda)$ *and*

(ii) $Syn(c\lambda) = Syn(\lambda)$ *for all* $c \in [0, 1]$.

Proof. (i) By Lemma 2.1(i), we have $P_{\overline{\lambda}} = P_\lambda$. So $Syn(\overline{\lambda}) = Syn(\lambda)$.
(ii) Again by Lemma 2.1(i), $P_\lambda = P_{c\lambda}$ and hence $Syn(\lambda) = Syn(c\lambda)$ for all $c \in [0, 1]$. □

For a variety of fuzzy languages $\mathscr{F}$, let $\mathscr{F}^s$ be the family of finite monoids defined by $\mathscr{F}^s = \{\text{Syn}(\lambda) : \lambda \in \mathscr{F}(A), \text{for some} A\}$. For a variety of finite monoids $\mathscr{S}$, let $\mathscr{S}^f = \mathscr{S}^f(A)$ be the family of fuzzy languages defined by $\mathscr{S}^f(A) = \{\lambda$ is a fuzzy language over $A : \text{Syn}(\lambda) \in \mathscr{S}\}$.

Theorem 2.5. *(cf.[11],Theorem 7) The mappings* $\mathscr{F} \to \mathscr{F}^s$ *and* $\mathscr{S} \to \mathscr{S}^f$ *are mutually inverse lattice isomorphisms between the lattices of all varieties of fuzzy languages and all varieties of finite monoids.*

3 Group Fuzzy Languages

A fuzzy language $\lambda \in A^*$ is called group fuzzy language if its syntactic monoid (Syn(λ)) is a group. Also if it is regular, then we call as regular group fuzzy language.

Example 3.1 (cf. [8])**.** *Let* G *be a finite group and* $\varphi : A^* \to G$ *be an onto homomorphism. If* $L = \varphi^{-1}(e)$, *then* χ_L *is a group fuzzy language.*

The class of regular group fuzzy languages in A^* is denoted by **GFL**. By the above example **GFL** $\neq \phi$.
The following lemma shows that **GFL** is closed under the boolean operations.

Lemma 3.1. *Let* $\lambda, \lambda_1, \lambda_2 \in$ ***GFL****. Then* $\overline{\lambda} \in$ ***GFL****,* $\lambda_1 \vee \lambda_2 \in$ ***GFL*** *and* $\lambda_1 \wedge \lambda_2 \in$ ***GFL****.*

Proof. Since Syn(λ)= Syn($\overline{\lambda}$) (Corollary 2.1 (i)) and Syn(λ) is a group, it follows that $\overline{\lambda} \in GFL$. Since Syn($\lambda_1$), Syn($\lambda_2$) are groups and $\lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ are recognized by Syn(λ_1) $\times\, Syn(\lambda_2)$ (Theorem 2.3), we see that $\lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ are in **GFL**. □

Next result shows that **GFL** is closed under the inverse homomorphic images.

Lemma 3.2. *Let A be finite alphabet, λ be a fuzzy language on A^* and $\lambda \in$ **GFL**. If $\varphi : X^* \to A^*$ is a homomorphism, then $\lambda\varphi^{-1} \in$ **GFL**, where $\lambda\varphi^{-1}(u) = \lambda(\varphi(u))$ for all $u \in X^*$.*

Proof. Since Syn(λ) recognizes λ, there is a homomorphism $\psi : A^* \to Syn(\lambda)$ and a mapping $\eta : Syn(\lambda) \to [0,1]$ such that $\lambda = \eta\psi^{-1}$, where $\eta\psi^{-1}(u) = \eta(\psi(u))$. So $\psi \circ \varphi$ is a homomorphism from X^* into the group Syn(λ) and

$$\begin{aligned} \eta((\psi \circ \varphi)^{-1})(u) &= \eta((\psi \circ \varphi)(u)) \\ &= \eta(\psi(\varphi(u))) \\ &= \eta\psi^{-1}(\varphi(u)) \\ &= \lambda(\varphi(u)). \end{aligned}$$

So $\lambda\varphi^{-1}$ is recognized by Syn(λ). Since Syn($\lambda\varphi^{-1}$) is a homomorphic image of Syn(λ), we have $\lambda\varphi^{-1} \in$ **GFL**. □

Lemma 3.3. *Let $\lambda_1, \lambda_2 \in$ **GFL**. Then $\lambda_2^{-1}\lambda_1$ and $\lambda_1\lambda_2^{-1}$ are in **GFL**.*

Proof. By Theorems 2.2 and 2.4, $\lambda_2^{-1}\lambda_1$ and $\lambda_1\lambda_2^{-1}$ are recognized by Syn(λ_1). Since Syn(λ_1) is a group and Syn($\lambda_2^{-1}\lambda_1$), Syn($\lambda_1\lambda_2^{-1}$) are homomorphic images of Syn(λ_1), we have $\lambda_2^{-1}\lambda_1$ and $\lambda_1\lambda_2^{-1}$ are in **GFL**. Hence **GFL** is closed under the quotients. □

Lemma 3.4. *Let $\lambda \in$ **GFL** and $c \in [0,1]$. Then $c\lambda \in$ **GFL**.*

Proof. By Corollary 2.1(i), we have Syn($c\lambda$) = Syn(λ). Since Syn(λ) is a group, we see that $c\lambda \in$ **GFL**. □

Next result shows that **GFL** is closed in c-cuts.

Lemma 3.5. *Let $\lambda \in$ **GFL** and $c \in [0,1]$. Then the syntactic monoid of the language $\lambda_c = \{u \in A^* : \lambda(u) \geq c\}$ is a group.*

Proof. Since $G = Syn(\lambda)$. Then there is an onto homomorphism $\varphi : A^* \to G$ and a function $\pi : G \to [0,1]$ such that $\lambda = \pi\varphi^{-1}$. Let $P = \{m \in G : \pi(m) \geq c\}$. Then

$$\begin{aligned} \varphi^{-1}(P) &= \{u \in A^* : \varphi(u) \in P\} \\ &= \{u \in A^* : \pi(\varphi(u)) \geq c\} \\ &= \{u \in A^* : \pi\varphi^{-1}(u) \geq c\} \\ &= \{u \in A^* : \lambda(u) \geq c\} \\ &= \lambda_c. \end{aligned}$$

So G recognizes λ_c. Since the syntactic monoid of λ_c $(M(\lambda_c))$ is a homomorphic image of G, $M(\lambda_c)$ is a group. □

Theorem 3.1. *(i) **GFL** is a variety of fuzzy languages;*

*(ii) There exists a one-one correspondence between **GFL** and the pseudovariety **G** of finite groups.*

Proof. **(i)** Follows from Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5.
(ii) Let $\lambda \in$ **GFL**. Then by definition $Syn(\lambda) \in$ **G**.

Conversely, assume that $G \in$ **G**. Since every monoid is a syntactic monoid of some fuzzy language (cf. [11], Theorem 4), there is a fuzzy language λ such that $Syn(\lambda) = G$. So $\lambda \in$ **GFL**. □

Next theorem gives an example of a fuzzy language whose syntactic monoid is a group.

Theorem 3.2. *Let $t \geq 1$ and $n > 0$. For $u \in A^*$, define $\lambda_n : A^* \to [0, 1]$ by*

$$\lambda_n(u) = \frac{k}{k+t} \text{ if } |u| \equiv k(\mathrm{mod}\, n).$$

Then Syn(λ_n) is isomorphic to the group $\mathbb{Z}_n$ (under addition modulo n).

Proof. Let $u, v \in A^*$ and let $uP_{\lambda_n}v$. Then $\lambda_n(xuy) = \lambda_n(xvy)$ for all $u, v \in A^*$. If $|xuy| \equiv k_1(\mathrm{mod}\ n)$ and $|xvy| \equiv k_2(\mathrm{mod}\ n)$, then by the definition of λ_n, we have $\frac{k_1}{k_1+t} = \frac{k_2}{k_2+t}$. So $k_1(k_2+t) = k_2(k_1+t)$. That is, $(k_1 - k_2)t = 0$. Since $t \geq 1$, we have $k_1 = k_2$ and so $|xuy| \equiv |xvy|(\mathrm{mod}\ n)$. Since $|xuy| = |x| + |u| + |y|$ and $|xvy| = |x| + |v| + |y|$, we see that $||u| - |v|| \equiv 0(\mathrm{mod}\ n)$. Thus $[u]_{\lambda_n} = \{v \in A^* \mid ||u| - |v|| \equiv 0(\mathrm{mod}\, n)\}$. If $w_0 = \epsilon$ and w_i is a word of length i, then

$$[w_0] = \{\epsilon\} \bigcup \{u \in A^+ : |u| = kn,\ k = 1, 2, \ldots\}$$

and for $i = 1, 2, \ldots, n-1$

$$[w_i] = \{u \in A^+ : |u| = i + kn,\ k = 1, 2, \ldots\}.$$

So $Syn(\lambda_n) = \{[w_0], [w_1], \ldots, [w_{n-1}]\}$. Clearly Syn($\lambda_n$) is a monoid. Since $[w_0][w_i] = [w_i] = [w_i][w_0]$ for all i, it follows that $[w_0]$ is the identity element in Syn(λ_n). If $i + j \equiv 0(\mathrm{mod}\, n)$, then $[w_i][w_j] = [w_i w_j] = [w_0] = [w_j][w_i]$. So $[w_j]$ is the inverse of $[w_i]$. Thus Syn(λ_n) is a group. Define $\varphi : Syn(\lambda_n) \to \mathbb{Z}_n$ by $\varphi([w_i]) = |w_i|$ for all i. Then φ is an isomorphism. □

4 Conclusion

In fuzzy language theory, every monoid is the syntactic monoid of some fuzzy language. With this concept we study different types of fuzzy languages with the algebraic properties of its syntactic monoids. In this paper we give the variety structure of group fuzzy language and also provide the Eleinberg variety theorem for group fuzzy language. We can extend this work into a distributive lattice L, since L-fuzzy languages are generalization of fuzzy languages[1].

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Cactus graphs with cycle blocks and square product labeling

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Abstract

In many real world problems, cactus graphs were considered as models from both algorithmic and theoretical point of view and this graph is a subclass of planar graph and superclass of a tree. In this article, the study has been carried out on some cactus graphs with cycle blocks for obtaining results on square product labeling.
Keywords: Square Product Labeling (SPL) and Cactus Graphs.
2020 AMS subject classifications: 05C05, 05C38, 05C76, 05C78. [1]

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1 Introduction

Graph labeling is one of the field in graph theory and it is the labeling of integers to the vertices or edges, or both under particular conditions which was introduced by A. Rosa [1967] in 1967. Graph labeling has a wide range of applications such as in X-ray crystallography, coding theory, radar, astronomy, circuit design, network theory, communication networks and database management. Nowadays, research in labeling of graph is increasingly expanding by studing more than 300 kinds of labelings. One such labeling is square sum labeling introduced by V. Ajitha et al. [2009] and they explored some results on it. J. B. Babujee and Babitha [2012] also worked on square sum labeling and obtained the results on it. Further, J.Shiama [2012] has worked on square difference labeling and proved some results. K. G. Mirajkar and Sthavarmath [2022] initiated square product labeling and obtained results for some class of graphs, cycle related graphs, Cartesian product of graphs and silicate and oxide networks. Khan et al. [2010a] have studied on cactus graphs for proving the results for $(2,1)-$ total labeling and for $L(2,1)-$ Labeling of cactus graphs in Khan et al. [2010b]. K. Kalaiarasi and Mahalakshmi [2022] shown the applications for cactus fuzzy labeling graphs. Further, S. Philomena, M. Pal, and K. Thirusanga Philomena et al. [2014] investigated some results for square and cube difference labeling on cactus graphs. In this article, the results are obtained for cactus graphs with cycle blocks on square product labeling and applied number theory concepts to establish the results.

All examined graphs here are finite, undirected, simple and connected. For undefined expressions and symbols refer F.Harary [1969], for number theory concepts refer Burton [2006] and for different labeling concepts we refer Gallian [2020].

2 Preliminaries

Definition 2.1. *A graph G is said to be a square product labeling (SPL), if there exists function $f : V(G) \to \{1, 2, 3, \ldots, p\}$ which is bijective, here p is counting of vertices inducing $f^* : E(G) \to N$ which is injective, where $f^*(uv) = f(u)^2 f(v)^2$ and the resulting edges are distinctly labeled.*

Definition 2.2. *Cactus graph Khan et al. [2010a] is a connected graph, in which every block is a cycle or an edge, in other words, no edge belongs to more than one cycle.*

3 Main Results

Theorem 3.1. *For two cycles C_n and C_m of any lengths n and m with a common cutvertex v admits square product labeling.*

Proof. Consider the cycles C_n and C_m of any lengths n and m having a common cutvertex v labeled by 1 with $V(C_n) = \{v_1, v_2, v_3, ..., v_n\}$ and $V(C_m) = \{v_1^{'}, v_2^{'}, v_3^{'}, ..., v_m^{'}\}$, hence the number of vertices are $(n+m)-1$ and edges are $2n$. We consider six cases to prove the result with $f : V(G) \to \{1, 2, 3, \ldots, (n+m)-1\}$

Case 1: For $n < m$

$$\begin{aligned} f(v_i) &= 2i \; for \; 1 \leq i \leq n-1, \; f(v_i^{'}) = f(v_{n-1}) + 2i \; for \; 1 \leq i \leq m-n \\ f(v_{i+1}^{'}) &= 2i+1 \; for \; m-n \leq i \leq m-2 \end{aligned}$$

Labeling of edges acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \; 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v_{m-n+1+(i-1)}^{'} v_{m-n+2+(i-1)}^{'}) : \; m-n \leq i \leq m-2\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_n) &= (f(v_n))^2, \; f^*(vv_1) = 4, \; f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\ f^*(vv_n^{'}) &= (f(v_n^{'}))^2, \; f^*(v_i^{'}v_{i+1}^{'}) = (f(v_n^{'}))^2 (f(v_n^{'}))^2 \; for \; 1 \leq i \leq m-n \end{aligned}$$

Case 2: For $n > m$

$$\begin{aligned} f(v_i) &= 2i \; for \; 1 \leq i \leq n-2, \; f(v_{n-1}) = 3 \\ f(v_i^{'}) &= f(v_{n-1}) + 2i \; for \; 1 \leq i \leq m-1 \end{aligned}$$

Labeling of edges acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \; 1 \leq i \leq n-3\}$ and $(4P^2 + 16P + 15)^2$, $P \geqslant 1$ for $\{f^*(v_i^{'}v_{i+1}^{'}) : \; m-n \leq i \leq m-2\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_n) &= (f(v_n))^2, \; f^*(vv_1) = 4, \; f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\ f^*(v_{n-1}v_n) &= (f(v_{n-1}))^2 \; (f(v_n))^2 \end{aligned}$$

Case 3: For $n = m$

$$\begin{aligned} f(v_i) &= 2i, \; 1 \leq i \leq (n-1), \; f(v_i^{'}) = 2i-1, \; 1 \leq i \leq m-1 \\ f(v_i^{'}) &= f(v_{n-1}) + 2i, \; 1 \leq i \leq m-1 \end{aligned}$$

Labeling of edges acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \; 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$,

$P \geqslant 1$ for $\{f^*(v_i'v_{i+1}') : \ m-n \leq i \leq m-2\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_n) &= (f(v_n))^2,\ f^*(vv_1) = 4,\ f^*(vv_1') = (f(v_1'))^2 \\ f^*(v_m'v) &= (f(v'))^2 \end{aligned}$$

Case 4: For $n = 2P(P+1),\ P \geqslant 1$ and $n < m$

$$\begin{aligned} For\ n = 2P(P+1),\ P = 1 & \\ f(v_1) &= 2,\ f(v_2) = 6,\ f(v_3) = 4,\ f(v_1') = f(v_2) + 2 \\ f(v_{m-1}' - (i-1)) &= 2i+1\ for\ 1 \leq i \leq m-n \\ if\ n \geqslant 4,\ n = 2P(P+1),\ P > 1 & \\ f(v_i) &= 2i\ for\ 1 \leq i \leq n-1,\ f(v_1') = 3 \\ f(v_{i+1}') &= 2i+1\ for\ 2 \leq i \leq m-3 \\ f(v_{i+1}') &= f(v_{n-1}) + 2 + 2(i-1)\ for\ 1 \leq i \leq m-n \end{aligned}$$

Labeling of edges acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \ 1 \leq i \leq n-3\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v'_{(m-1)-(i-1)}v'_{(m-2)-(i-1)}) : \ 1 \leq i \leq m-3\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_n) &= (f(v_n))^2,\ f^*(vv_1) = 4,\ f^*(vv_1') = (f(v_1'))^2,\ f^*(v_m'v) = (f(v'))^2 \\ f^*(v_{n-1}v) &= (f(v_{n-1}))^2,\ f^*(v_1v_2) = f(v_1)^2\ f(v_2)^2 \\ f^*(v_2v_3) &= f(v_2)^2\ f(v_3)^2,\ f^*(v_1'v_2') = (f(v_1'))^2\ (f(v_2'))^2 \end{aligned}$$

Case 5: For $n = 2P^2 + 2P + 1\ m \geqslant (n-1)$

$$\begin{aligned} f(v_i) &= 2i\ for\ 1 \leq i \leq n-3,\ f(v_{n-2}) = (2n-2) \\ f(v_{n-1}) &= (2n-4) \\ f(v_1') &= 3,\ f(v_{i+1}') = f(v_{n-2}) + 2i\ for\ 1 \leq i \leq m-n \\ f(v'_{(m-n+1)+(i-1)}) &= 2i+1\ for\ m-n \leq i \leq m-2 \end{aligned}$$

Labeling of edges acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \ 1 \leq i \leq n-3\}$ and $(4P^2 + 16P + 15)^2$, $P \geqslant 1$ for $\{f^*(v_i'v_{i+1}') : \ m-n \leq i \leq m-2\}$. The remaining edges are labeled as follows,

$$\begin{aligned} f^*(vv_n) &= (f(v_n))^2,\ f^*(vv_1) = 4,\ f^*(vv_1') = (f(v_1'))^2 \\ f^*(v_{n-1}v_n) &= (f(v_{n-1}))^2\ (f(v_n))^2,\ f^*(v_1'v_2') = 9\ (f(v_{n-1}) + 2))^2 \\ if\ n = 5,\ f^*(v_2'v_3') &= (f(v_{n-1}) + 2))^2\ (f(v_3'))^2 \\ f^*(v_{i+2}'v_{i+3}') &= (4P^2 + 16P + 15)^2\ for\ 1 \leq i \leq m-3 \end{aligned}$$

Case 6: For $n = 4P^2+8P+3$ and $n = 4(P^2+2P+1)$ number of vertices where $P \geq 1$
$v_{n-1}^{'} = n+m-2$ and $v_{n-2}^{'} = n+m-1$ and if $f(v_i^{'})$ is labeled with even numbers before the label 3 for $n = 4(P^2+2P+1)$ then change the label with preceding vertex label. The remaining labels of both the edges and vertices are same as in the above cases. The labels of both vertices and edges are distinct in all cases, hence the result. □.

Example 3.1: The square product labeling of two cycles of any lengths n and m with a common cut vertex v is as shown in below figure.

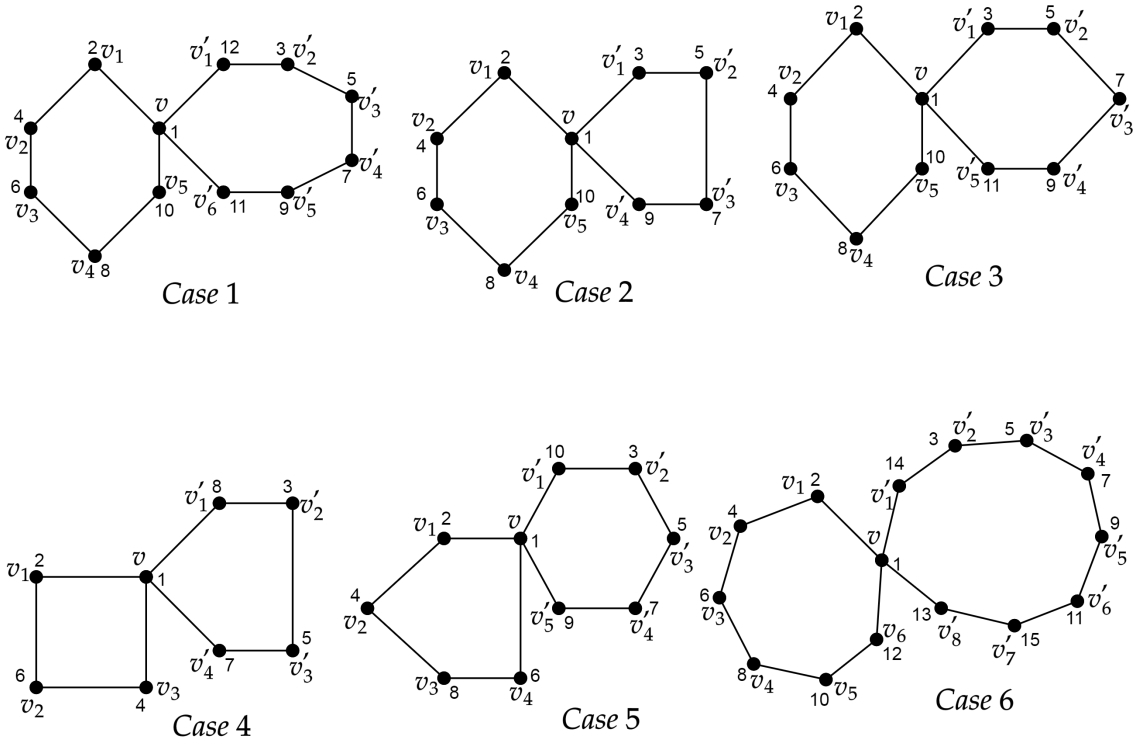

Figure 1: Square product labeling of two cycles of any lengths n and m with a common cutvertex v

Theorem 3.2. *For three cycles C_n, C_m, and C_l of any lengths n, m, and l with a common cut vertex v admits SPL.*

Proof. Consider three cycles C_n, C_m, and C_l having a common cut vertex v with fixed label as 1 and $V(C_n) = \{v_1, v_2, v_3, ..., v_n\}$, $V(C_m) = \{v_1^{'}, v_2^{'}, v_3^{'}, ..., v_m^{'}\}$, and $V(C_l) = \{v_1^{''}, v_2^{''}, v_3^{''}, ..., v_l^{''}\}$, hence the number of vertices are $(n+m+l)-2$ and edges are $3n$. We consider nine cases to prove the result and in each case

the labels of vertices of cactus graph starts from even numbers among the total number of vertices of that graph later the graph G is given odd numbers. let $f : V(G) \to \{1, 2, 3, \ldots, (n + m + l) - 2\}$, thus both the vertices and edges for all nine cases are labeled as follows,

Case 1: For $n = 2P(P + 1),\ P \geq 1$ and $n < m$

$$\begin{aligned}
&if\ n = 2P(P+1),\ P = 1 \\
f(v_1) &= 2,\ f(v_2) = 6,\ f(v_3) = 4,\ f(v_i^{'}) = f(v_2) + 2 \\
&i\ for\ 1 \leq i \leq m + n - l \\
f(v_{m+n-l}^{'} + i) &= 2i + 1\ for\ 1 \leq i \leq l - m,\ if\ l > m \\
&if\ n = 2P(P+1),\ P > 1 \\
f(v_i) &= 2i,\ 1 \leq i \leq (n-2),\ f(v_i^{'}) = f(v_{n-2}) + 2i \\
&for\ 1 \leq i \leq m + n - l \\
f(v_i^{''}) &= f(v_{n-1}^{'}) + 2i \\
&for\ 1 \leq i \leq n + l - m + 1 \\
f(v_{n-2}) &= 2n,\ f(v_{n-1}) = 2n - 2
\end{aligned}$$

The edge labels acquired here are distinct, as it forms a quadratic sequence of the form $(4P(P + 1))^2$, $P \geqslant 1$ for $\{f^*(v_i v_{i+1}) :\ 1 \leq i \leq n - 3\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_n) &= (f(v_n))^2,\ f^*(vv_1) = 4,\ f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\
f^*(v_1v_2) &= (f(v_1))^2\ (f(v_2))^2,\ f^*(v_2v_3) = (f(v_2))^2\ (f(v_3))^2 \\
f^*(vv_{n-1}^{'}) &= (f(v_{n-1}^{'}))^2,\ f^*(v_i^{'}v_{i+1}^{'}) = (f(v_{n-2}) + 2i))^2(f(v_i^{'}) + 2i))^2, \\
for\ 1 \leq i \leq m - 3& \\
f^*(v_{n-2}^{'}v_{n-1}^{'}) &= (f(v_{n-2}^{'}))^2(f(v_{n-1}^{'}))^2 \\
f^*(v_i^{''}v_{i+1}^{''}) &= (f(v_{n-2}^{'}) + 2i))^2(f(v_i^{''}) + 2i))^2\ for\ 1 \leq i \leq l + n - m
\end{aligned}$$

Case 2: For $n = 2P^2 + P + 1,\ P \geq 1$

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq \frac{n-1}{2},\ f(v_{n-2}) = (2n-2),\ f(v_{n-1}) = 2n - 4 \\
f(v_i^{'}) &= f(v_{n-2}) + 2i\ for\ 1 \leq i \leq m + n - l \\
f(v_i^{''}) &= f(v_{n-1}^{'}) + 2i\ for\ 1 \leq i \leq (n + l - m) + 1
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the form $(4P(P + 1))^2$, $P \geqslant 1$ for $\{f^*(v_i v_{i+1}) :\ 1 \leq i \leq n - 3\}$, $(4P^2 + 24P + 35))^2$, $P \geqslant 1$ for $\{f^*(v_i^{''} v_{i+1}^{''}) :\ 1 \leq i \leq l - 2\}$ and $(4P^2 - 1)^2$, $P \geqslant 1$ for

$\{f^*(v'_{(n+l-m)-1+i}v'_{(n+l-m)+1+(i-1)}) : 1 \leq i \leq m-l\}$ if $l < m$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv'_1) = (f(v'_1))^2 \\
f^*(v'_{n-2}v'_{n-1}) &= (f(v'_{n-2}))^2\ (f(v'_{n-1}))^2 \\
f^*(v'_{n-3}v'_{n-2}) &= (f(v'_{n-3}))^2\ (f(v'_{n-2}))^2 \\
f^*(vv'_{n-1}) &= (f(v'_{n-1}))^2,\ f^*(vv''_{n-1}) = (f(v''_{n-1}))^2 \\
f^*(v'_iv'_{i+1}) &= (f(v_{n-2}) + 2i))^2(f(v'_i) + 2i))^2 \\
for\ 1 \leq i \leq (l+n-m)-2 & \\
f^*(v'_{(n+l-m)-1}v'_{(n+l-m)+1)}) &= (f(v'_{(n+l-m)-1)}))^2(f(v'_{(n+l-m)+1}))^2
\end{aligned}$$

Case 3: For $n = m = l$

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq n-1,\ f(v'_i) = f(v_{n-1}) + 2i \\
for\ 1 \leq i \leq m-4 & \\
f(v'_{(m-4)+i}) &= 2i+1\ for\ 1 \leq i \leq m-4,\ f(v''_i) = f(v'_{n-1}) + 2i \\
for\ 1 \leq i \leq l-1 &
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the form $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v_{(m-3)+(i-1)}v_{(m-2)+(i-1)}) : 1 \leq i \leq m-5\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv'_1) = (f(v'_1))^2 \\
f^*(vv'_{n-1}) &= (f(v'_{n-1}))^2,\ f^*(vv''_{n-1}) = (f(v''_{n-1}))^2 \\
f^*(v'_iv'_{i+1}) &= (f(v_{n-1}) + 2i))^2(f(v'_i) + 2i))^2\ for\ 1 \leq i \leq m-5 \\
f^*(v_{m-4}v_{m-3}) &= (f(v_{m-4}))^2\ (f(v_{m-3}))^2 \\
f^*(v''_iv''_{i+1}) &= (f(v'_{m-1}) + 2i))^2(f(v'_i) + 2i))^2\ for\ 1 \leq i \leq l-2
\end{aligned}$$

Case 4: For $V(G) = 4P^2 + 8P + 3,\ P \geq 1, n < m\ and\ n \leqslant l$

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq n-1,\ f(v''_{n-2}) = (n+m+l-2) \\
f(v''_{n-1}) &= (n+m+l-4) \\
f(v'_i) &= f(v_{n-1}) + 2i\ for\ 1 \leq i \leq m-l-1 \\
f(v'_{m-(l+1)+i}) &= 2i+1\ for\ 1 \leq i \leq m-2 \\
f(v''_i) &= f(v'_{n-1}) + 2i\ for\ 1 \leq i \leq l-3
\end{aligned}$$

The edge labels acquired here are distinct, as it forms a quadratic sequence of the form $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : 1 \leq i \leq n-2\}$. The remaining

edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\
f^*(v_{n-2}^{''}v_{n-1}^{''}) &= (f(v_{n-2}^{''}))^2,\ f^*(v_1^{'}v_2^{'}) = (f(v_1^{'}))^2\ (f(v_2^{'}))^2 \\
f^*(vv_{n-1}^{'}) &= (f(v_{n-1}^{'}))^2,\ f^*(vv_{n-1}^{''}) = (f(v_{n-1}^{''}))^2,\ f^*(vv_1^{''}) = (f(v_1^{''}))^2 \\
f^*(v_i^{'}v_{i+1}^{'}) &= (f(v_{n-2}) + 2i))^2(f(v_i^{'}) + 2i))^2\ for\ 2 \leq i \leq m-2 \\
f^*(v_i^{''}v_{(i+1)}^{''}) &= (f(v_{(n-1)}^{'}) + 2i))^2(f(v_i^{'}) + 2i)^2\ for\ 1 \leq i \leq l-3
\end{aligned}$$

Case 5: For $V(G) = 4(P^2 + 2P + 1)$, $P \geq 1$

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq (n-1),\ f(v_i^{'}) = f(v_{n-1}) + 2i\ for\ 1 \leq i \leq m-l \\
f(v_{(m-4)+i}^{'}) &= 2i+1\ for\ 1 \leq i \leq m-4,\ f(v_{(m-l)+1}^{'}) = f(v_{(m-l)+4}^{'}) \\
f(v_{(m-l)+2}^{'}) &= f(v_{(m-l)+2}^{'}),\ f(v_i^{''}) = f(v_{n-1}^{'}) + 2i\ for\ 1 \leq i \leq l-3 \\
f(v_{n-2}^{''}) &= f(v_{(m-l)}^{'}) + 3,\ f(v_{n-1}^{''}) = f(v_{(n-1)}^{'}) - 2
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the form $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) :\ 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v_{(m-3)+(i-1)}^{'}v_{(m-2)+(i-1)}^{'}) :\ 1 \leq i \leq m-5\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv_1^{'}) = (f(v_1^{'}))^2 \\
f^*(v_{(m-l)+1}^{'}v_{(m-l)+2}^{'}) &= (f(v_{(m-l)}^{'}) + 4))^2\ (f(v_{(m-l)}^{'}) + 2))^2, \\
f^*(v_{m-4}^{'}v_{m-3}^{'}) &= (f(v_{m-4}^{'}))^2\ (f(v_{m-3}^{'}))^2 \\
f^*(vv_{n-1}^{'}) &= (f(v_{n-1}^{'}))^2,\ f^*(vv_{n-1}^{''}) = (f(v_{n-1}^{''}))^2 \\
f^*(v_i^{'}v_{i+1}^{'}) &= (f(v_{n-1}) + 2i))^2(f(v_i^{'}) + 2i))^2\ for\ 1 \leq i \leq m-l \\
f^*(v_i^{''}v_{(i+1)}^{''}) &= (f(v_{(n-1)}^{'}) + 2i)^2\ (f(v_i^{'}) + 2i))^2\ for\ 1 \leq i \leq l-2
\end{aligned}$$

Case 6: For $V(V) = 12P^2 + 4P + 1\ and\ 12P^2 + 4P + 1,\ P \geq 1$, where $n < m$

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq n-1 \\
f(v_i^{'}) &= f(v_{n-1}) + 2i\ for\ 1 \leq i \leq m-n \\
f(v_{(m-n)}^{'} + 1) &= (n+m+l-2),\ f(v_{(m-n)+2}^{'}) = (n+m+l-4) \\
f(v_{(m-n)+2}^{'} + i) &= 2i+1\ for\ 1 \leq i \leq (m-n)+2 \\
f(v_{n-2}^{''}) &= f(v_{(m-l)}^{'}) + 3,\ f(v_i^{''}) = f(v_{(n-1)}^{'}) + 2i\ for\ 1 \leq i \leq l-2
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) :\ 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$,

$P \geqslant 1$ for $\{f^*(v'_{(m-n)+2+i}v'_{(m-n)+4+i}) : \ 1 \leq i \leq (m-n)+1\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv'_1) = (f(v_{n-1})+2)^2 \\
f^*(vv''_1) &= (f(v'_{n-1})+2)^2 \\
f^*(vv'_{n-1}) &= (f(v'_{n-1}))^2,\ f^*(vv''_{n-1}) = (f(v''_{n-1}))^2 \\
f^*(v'_iv'_{i+1}) &= (f(v_{n-1})+2i))^2(f(v'_i)+2i))^2 \ for\ 1 \leq i \leq m-n \\
f^*(v''_iv''_{(i+1)}) &= (f(v'_{(n-1)})+2i)^2\ (f(v''_i)+2i))^2 \ for\ 1 \leq i \leq l-2 \\
f(v'_{(m-n)+1}v'_{(m-n)+2}) &= (n+m+l-2)^2\ (n+m+l-4)^2
\end{aligned}$$

Case 7: For $n+m = 2P^2+6P+5\ P \geq 1$ and $l = 2P^2+6P+4,\ P \geq 1$

$$\begin{aligned}
f(v_i) &= 2i \ for\ 1 \leq i \leq (n-2),\ f(v'_i) = f(v_{n-1})+2i,\ 1 \leq i \leq (m-3) \\
f(v'_{(n-2)}) &= 2(n+m)-2,\ f(v'_{(n-1)+2}) = 2(n+m)-4, \\
f(v''_i) &= 2i+1 \ for\ 1 \leq i \leq l-1
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the form $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \ 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v''_iv''_{i+1}) : \ 1 \leq i \leq l-2\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv'_1) = (f(v'_1)^2, \\
f^*(vv''_1) &= (f(v''_1)^2 \\
f^*(vv'_{n-1}) &= (f(v'_{n-1}))^2,\ f^*(vv''_{n-1}) = (f(v''_{n-1}))^2 \\
f^*(v'_iv'_{i+1}) &= (f(v_{n-1})+2i))^2(f(v'_i)+2i))^2 \ for\ 1 \leq i \leq m-4 \\
f^*(v'_{m-3}v'_{(m-2)}) &= (f(v'_{(m-3)}))^2\ (f(v'_{m-2}))^2 \\
f(v'_{(m-2)}v'_{(m-1)}) &= (2(n+m)-2)^2\ (2(n+m)-4)^2
\end{aligned}$$

Case 8: For $n+m = (2P^2+6P+4),\ P \geq 1$ and $l = (2P^2+6P+5),\ P \geq 1$

$$\begin{aligned}
f(v_i) &= 2i \ for\ 1 \leq i \leq (n-2), \\
f(v'_i) &= f(v_{n-1})+2i \ for\ 1 \leq i \leq m-2 \\
f(v''_2) &= 2(l-1),\ f(v''_1) = 3,\ f(v''_1) = 9 \\
f(v''_1) &= 5,\ f(v''_1) = 7,\ f(v''_{5+i}) = 2i+9 \ for\ 1 \leq i \leq l-6
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequences of the form $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \ 1 \leq i \leq n-2\}$ and $(2P+9)^2$,

$P \geqslant 1$ for $\{f^*(v''_{5+i}v''_{6+i}) : \ 1 \leq i \leq l-7\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv'_1) = (f(v_{(n-1)+2})^2 \\
f^*(vv''_1) &= (f(v''_1)^2 \\
f^*(vv'_{n-1}) &= (f(v'_{n-1}))^2,\ f^*(vv''_{n-1}) = (f(v''_{n-1}))^2 \\
f^*(v'_iv'_{i+1}) &= (f(v_{n-1})+2i))^2(f(v'_i)+2i))^2 \ for\ 1 \leq i \leq m-2 \\
f^*(v''_5v''_6) &= (f(v''_5))^2\ (f(v''_6))^2,\ f^*(v''_1v''_2) = (f(v''_1))^2,\ (f(v''_2))^2 \\
f^*(v''_3v''_4) &= (f(v''_3))^2\ (f(v''_4))^2,\ f^*(v''_4v''_5) = (f(v''_4))^2\ (f(v''_5))^2
\end{aligned}$$

Case 9: For all n, m, and l except the above cases

$$\begin{aligned}
f(v_i) &= 2i\ for\ 1 \leq i \leq (n-1), \\
f(v'_i) &= f(v_{n-1}) + 2i\ for\ 1 \leq i \leq n-m,\ if\ n > m\ and\ 1 \leq i \leq m-n \\
if\ n < m & \\
f(v'_{(n-m)+i}) &= 2i+1\ for\ 1 \leq i \leq m-2,\ f(v''_i) = f(v'_{n-1}) + 2i\ for\ 1 \leq i \leq l-1 \\
f(v''_i) &= 2i+1\ for\ 1 \leq i \leq l-1.
\end{aligned}$$

The edge labels acquired here are distinct, as it forms quadratic sequence of the forms $(4P(P+1))^2$, $P \geqslant 1$ for $\{f^*(v_iv_{i+1}) : \ 1 \leq i \leq n-2\}$ and $(4P^2-1)^2$, $P \geqslant 1$ for $\{f^*(v'_{(n-m)+(i-1))}v'_{(n-m+1)+(i-1)}) : \ 1 \leq i \leq m-3\}$. The remaining edges are labeled as follows,

$$\begin{aligned}
f^*(vv_{n-1}) &= (f(v_{n-1}))^2,\ f^*(vv_1) = 4,\ f^*(vv'_1) = (f(v'_1)^2,\ f^*(vv''_1) = (f(v''_1)^2 \\
f^*(vv'_{n-1}) &= (f(v'_{n-1}))^2,\ f^*(vv''_{n-1}) = (f(v''_{n-1}))^2 \\
f^*(v'_iv'_{i+1}) &= (f(v_{n-1})+2i))^2(f(v'_i)+2i))^2\ for\ 1 \leq i \leq n-m\ if\ n > m \\
f^*(v''_i\ v''_{(i+1)}) &= (f(v'_{(n-1)}+2i))^2\ (f(v''_i+2i))^2\ for\ 1 \leq i \leq l-2
\end{aligned}$$

In all the cases, the labeling pattern of edges and vertices are distinct so, for three cycles C_n, C_m, and C_l of any lengths n, m, and l with a common cut vertex v admits SPL. □

Example 3.2: The square product labeling of three cycles of any lengths n, m, and l with a common cut vertex v is shown in the below figure.

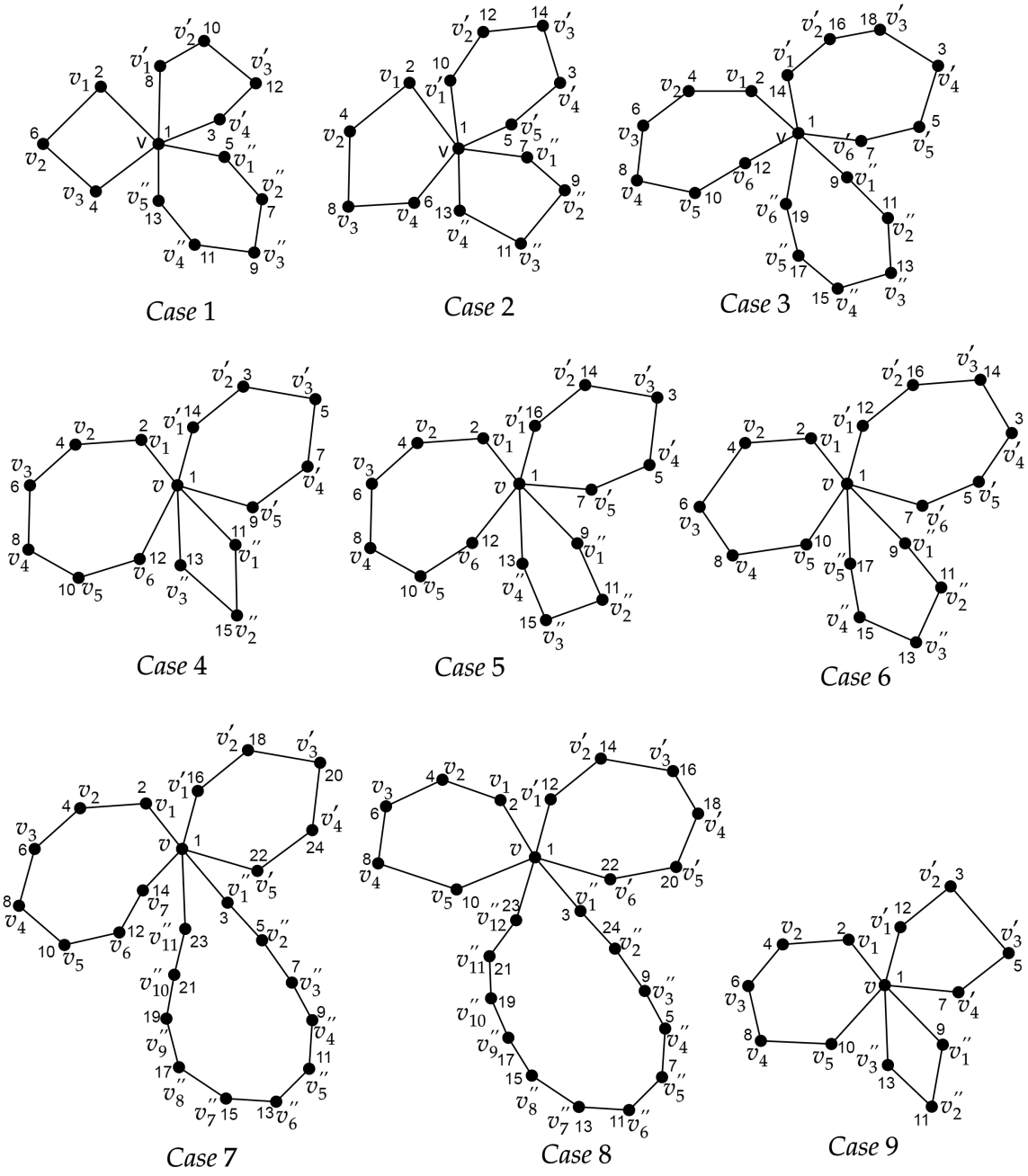

Figure 2: Three cycles of any lengths n, m, and l with a common cutvertex v

Theorem 3.3. *A graph G having r number of cycle of length n, with a common cutvertex v, except for the graph G with r number of cycle of length 3 having a common cutvertex v admits SPL.*

Proof. Consider the graph G having r number of cycles of length n with a common cutvertex v with $nr-(r-1)$ vertices and nr edges, here we consider two cases for prove the result with $f : V(G) \to \{1, 2, 3, \ldots, (nr-(r-1))\}$.
Case 1: Suppose G contains even copies of cycles of length n
In this case, the vertices of first half copies $\frac{r}{2}$ of cycle of length n are labeled by even numbers then next half copies by odd numbers. In first $\frac{r}{2}$ copies of cycle, the vertices are labeled as,

$$\begin{aligned}
f(v_1^1) &= 2,\ f(v_2^i) = f(v_1^i) + 2\ for\ 1 \le i \le \frac{r}{2} \\
f(v_{n-1}^i) &= f(v_1^i) + 4\ for\ 1 \le i \le \frac{r}{2} \\
f(v_3^i) &= f(v_{n-1}^i) + 2\ for\ 1 \le i \le \frac{r}{2},\ f(v_{3+i}^i) = f(v_3^i) + 2i\ for\ 1 \le i \le n-5 \\
f(v^(i+1)_1) &= f(v_{n-2}^i) + 2\ for\ 1 \le i \le \frac{r-2}{2}
\end{aligned}$$

In second $\frac{r}{2}$ copies of cycle, the vertices are labeled by,

$$\begin{aligned}
f\big(v_1^{\frac{r+2}{2}}\big) &= 3,\ f\big(v_2^{\frac{r+2}{2}}\big) = f\big(v_1^{\frac{r+2}{2}}\big) + 2\ for\ 1 \le i \le \frac{r}{2} \\
f\big(v_{n-1}^{\frac{r+2}{2}}\big) &= f\big(v_1^{\frac{r+2}{2}}\big) + 2\ for\ 1 \le i \le \frac{r}{2} \\
f\big(v_1^{\frac{r}{2}+(i-1)}\big) &= f\big(v_{n-2}^{\frac{r}{2}+i}\big) + 2\ for\ 1 \le i \le \frac{r-2}{2} \\
f\big(v_2^{\frac{r}{2}+i}\big) &= f\big(v_{n-1}^{\frac{r}{2}+i}\big) + 2\ for\ 1 \le i \le \frac{r}{2} \\
f\big(v_{2+i}^{\frac{r}{2}+i}\big) &= f\big(v_2^{\frac{r}{2}+i}\big) + 2\ for\ 1 \le i \le n-4
\end{aligned}$$

The labels of vertices of $f(v_1^i)$ and $f(v_{n-1}^i)$ with 1 yields distinct edge labels where $1 \le i \le r$, the remaining edge labels are as below

$$\begin{aligned}
f^*(v_1^i v_2^i) &= (f(v_1^i))^2\ (f(v_2^i))^2\ for\ 1 \le i \le \frac{r}{2}, \\
f^*(v_2^i v_3^i) &= (f(v_1^i)+2)^2\ (f(v_{n-1}^i)+2)^2\ for\ 1 \le i \le \frac{r}{2} \\
f^*(v_{n-2}^i v_{n-1}^i) &= (f(v_{n-2}^i))^2\ (f(v_1^i)+4)^2\ for\ 1 \le i \le n-4 \\
f^*(v_3^i v_{3+i}^i) &= (f(v_{n-1}^i+2))^2\ (f(v_3^i+2i))^2\ for\ 1 \le i \le n-5
\end{aligned}$$

The labels of edges of remaining $\frac{r}{2}$ copies of G are,

$$\begin{aligned}
f^*\big(v_1^{\frac{r}{2}+i} v_2^{\frac{r}{2}+i}\big) &= \big(f(v_1^{\frac{r}{2}+i})\big)^2\ \big(f(v_{n-1}^{\frac{r}{2}+i}+2)\big)^2\ for\ 1 \le i \le \frac{r-2}{2} \\
f^*\big(v_{n-2}^{\frac{r}{2}+i} v_{n-1}^{\frac{r}{2}+i}\big) &= \big(f(v_{n-2}^{\frac{r}{2}+i})\big)^2\ \big(f(v_1^{\frac{r}{2}+i}+2)\big)^2\ for\ 1 \le i \le \frac{r-2}{2} \\
f^*\big(v_2^{\frac{r}{2}+i} v_{2+i}^{\frac{r}{2}+i}\big) &= \big(f(v_{n-1}^{\frac{r}{2}+i}+2)\big)^2\ \big(f(v_2^{\frac{r}{2}+i}+2i)\big)^2\ for\ 1 \le i \le n-4
\end{aligned}$$

Case 2: Suppose G contains odd copies of cycles of length n
In this case, the vertices of r copies of cycle of length n are first labeled by even numbers then by odd numbers as follows,

$$\begin{aligned}
f(v_1^1) &= 2,\ f(v_2^i) = f(v_1^i) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v_1^{i+1}) &= f(v_{n-2}^i) + 2\ for\ 1 \le i \le \frac{r-3}{2} \\
f(v_{n-1}^i) &= f(v_1^i) + 4\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v_2^i) &= f(v_1^i) + 2i\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v^(i)_3) &= f(v_{n-1}^i) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v^(i)_{3+i}) &= f(v_3^i) + 2i\ for\ 1 \le i \le n-5 \\
f\big(v_1^{\frac{r+1}{2}}\big) &= f\big(v_{n-2}^{\frac{r-1}{2}}\big) + 2,\ f\big(v_{n-1}^{\frac{r+1}{2}}\big) = f\big(v_1^{\frac{r+1}{2}}\big) + 4 \\
f\big(v_2^{\frac{r-1}{2}+i}\big) &= f\big(v_1^{\frac{r-1}{2}+i}\big) + 2\ for\ 1 \le i \le n-6 \\
f\big(v_{n-4}^{\frac{r-1}{2}+i}\big) &= 2i+1\ for\ 1 \le i \le n-4 \\
f\big(v_1^{\frac{r+1}{2}+i}\big) &= f\big(v_{n-2}^{\frac{r+1}{2}+i}\big) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f\big(v_{n-1}^{\frac{r+1}{2}+i}\big) &= f\big(v_1^{\frac{r+1}{2}+i}\big) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f\big(v_{i+1}^{\frac{r+1}{2}+j}\big) &= f\big(v_1^{\frac{r+1}{2}+j}\big) + 4 + 2(i-1)\ for\ 1 \le i \le n-3\ and\ 1 \le j \le \frac{r-1}{2}
\end{aligned}$$

The labels of vertices of $f(v_1^i)$ and $f(v_{n-1}^i)$ with cutvertex label 1 yields distinct edge labels where $1 \le i \le r$, the remaining edge labels are as below.

$$\begin{aligned}
f^*(v_1^i v_2^i) &= (f(v_1^i))^2\ (f(v_1^i + 2))^2\ for\ 1 \le i \le \frac{r-1}{2}, \\
f^*(v_2^i v_3^i) &= (f(v_1^i + 2))^2\ (f(v_{n-1}^i + 2))^2\ for\ 1 \le i \le \frac{r-1}{2}, \\
f^*(v_{n-2}^i v_{n-1}^i) &= (f(v_{n-2}^i))^2\ (f(v_1^i + 4))^2\ for\ 1 \le i \le \frac{r-1}{2} \\
f^*(v_3^i v_{3+i}^i) &= (f(v_{n-1}^i + 2))^2\ (f(v_3^i + 2i))^2\ for\ 1 \le i \le n-4 \\
f^*\big(v_{n-2}^{\frac{r-1}{2}+i} v_{n-1}^{\frac{r-1}{2}+i}\big) &= \big(f(v_1^{\frac{r-1}{2}+2})\big)^2\ \big(f(v_1^{\frac{r-1}{2}+i} + 4)\big)^2\ for\ 1 \le i \le \frac{r-3}{2} \\
f^*\big(v_1^{\frac{r+1}{2}+i} v_2^{\frac{r+1}{2}+i}\big) &= \big(f(v_{n-2}^{\frac{r-1}{2}+i} + 2)\big)^2\ \big(f(v_{n-1}^{\frac{r-1}{2}+i} + 2)\big)^2\ for\ 1 \le i \le \frac{r-1}{2} \\
f^*\big(v_i^{\frac{r+1}{2}+i} v_{2+i}^{\frac{r+1}{2}+i}\big) &= \big(f(v_{n-1}^{\frac{r+1}{2}+i} + 2)\big)^2\ \big(f(v_2^{\frac{r+1}{2}+i} + 2i)\big)^2\ for\ 1 \le i \le n-3
\end{aligned}$$

Subcase 1: For cycle of length 4 with even cpoies

$$\begin{aligned}
f(v_1^1) &= 2,\ f(v_{n-1}^i) = f(v_1^i) + 2\ for\ 1 \le i \le \frac{r}{2} \\
f(v_1^{i+1}) &= f(v_2^i) + 2\ for\ 1 \le i \le \frac{r-2}{2} \\
f(v_2^i) &= f(v_{n-1}^i) + 2\ for\ 1 \le i \le \frac{r}{2},\ f\left(v_1^{\frac{r+2}{2}}\right) = 3 \\
f\left(v_{n-1}^{\frac{r+2}{2}}\right) &= f\left(v_1^{\frac{r}{2}+i}\right) + 2\ for\ 1 \le i \le \frac{r}{2},\ f\left(v_1^{\frac{r+2}{2}+i}\right) = f\left(v_2^{\frac{r}{2}}\right) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f\left(v_2^{\frac{r}{2}}\right) &= f\left(v_{n-1}^{\frac{r}{2}+i}\right) + 2\ for\ 1 \le i \le \frac{r}{2}
\end{aligned}$$

The labels of vertices of $f(v_1^i)$ and $f(v_{n-1}^i)$ with 1 yields distinct edge labels where $1 \le i \le r$, the remaining edge labels are as below

$$\begin{aligned}
f^*(v_1^i v_2^i) &= (f(v_1^i))^2\ (f(v_{n-1}^i + 2))^2\ for\ 1 \le i \le \frac{r}{2} \\
f^*(v_2^i v_{n-1}^i) &= (f(v_{n-1}^i + 2))^2\ (f(v_1^i + 2))^2\ for\ 1 \le i \le \frac{r}{2} \\
f^*\left(v_1^{\frac{r}{2}+i} v_2^i\right) &= \left(f(v_1^{\frac{r}{2}+i})\right)^2\ \left(f(v_{n-1}^{\frac{r}{2}+i} + 2)\right)^2\ for\ 1 \le i \le \frac{r}{2} \\
f^*\left(v_2^{\frac{r+2}{2}+i} v_{n-1}^{\frac{r}{2}+i}\right) &= \left(f(v_{n-1}^{\frac{r}{2}+i} + 2)\right)^2\ \left(f(v_1^{\frac{r}{2}+i} + 2)\right)^2\ for\ 1 \le i \le \frac{r}{2}
\end{aligned}$$

Subcase 2: For cycle of length 5 with odd cpoies

$$\begin{aligned}
f(v_1^1) &= 2,\ f(v_2^i) = f(v_1^i) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v_1^{i+1}) &= f(v_{n-2}^i) + 2\ for\ 1 \le i \le \frac{r-2}{2} \\
f(v_{n-1}^i) &= f(v_1^i) + 4\ for\ 1 \le i \le \frac{r-1}{2}, \\
f(v_2^i) &= f(v_1^i) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f(v_3^i) &= f(v_{n-1}^i) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f\left(v_1^{\frac{r+1}{2}}\right) &= f\left(v_3^{\frac{r-1}{2}}\right) + 2,\ f\left(v_3^{\frac{r+1}{2}}\right) = 3,\ f\left(v_{n-1}^{\frac{r+1}{2}}\right) = 7 \\
f\left(v_1^{\frac{r+1}{2}+i}\right) &= f\left(v_{n-1}^{\frac{r-1}{2}+i}\right) + 2\ for\ 1 \le i \le \frac{r-1}{2} \\
f\left(v_2^{\frac{r+3}{2}}\right) &= 9,\ f\left(v_3^{\frac{r+1}{2}+i}\right) = f\left(v_{n-1}^{\frac{r+3}{2}+i}\right) + 2i\ for\ 1 \le i \le \frac{r-1}{2} \\
f\left(v_2^{\frac{r+1}{2}+j}\right) &= f\left(v_1^{\frac{r+3}{2}+j}\right) + 2i\ for\ 1 \le i \le n-2\ and\ 1 \le j \le \frac{r-3}{2}
\end{aligned}$$

The labels of vertices of $f(v_1^i)$ and $f(v_{n-1}^i)$ with 1 yields distinct edge labels where $1 \leq i \leq r$, the remaining edge labels are as below

$$\begin{aligned}
f^*(v_1^i v_2^i) &= (f(v_1^i))^2\ (f(v_{n-1}^i+2))^2\ for\ 1 \leq i \leq \frac{r-1}{2} \\
f^*(v_2^i v_3^i) &= (f(v_1^i+2))^2\ (f(v_{n-1}^i+2))^2\ for\ 1 \leq i \leq \frac{r-1}{2} \\
f^*(v_{n-2}^i v_{n-1}^i) &= (f(v_{n-2}^i))^2\ (f(v_1^i+4))^2\ for\ 1 \leq i \leq \frac{r-1}{2} \\
f^*(v_3^i v_{3+i}^i) &= (f(v_{n-1}^i+2))^2\ (f(v_3^i+2i))^2\ for\ 1 \leq i \leq n-4 \\
f^*\big(v_1^{\frac{r+1}{2}} v_2^{\frac{r+1}{2}}\big) &= \big(f(v_{n-2}^{\frac{r-1}{2}}+2)\big)^2\ \big(f(v_1^{\frac{r+1}{2}}+2)\big)^2 \\
f^*\big(v_2^{\frac{r+1}{2}} v_3^{\frac{r+1}{2}}\big) &= 9\ \big(f(v_1^{\frac{r+1}{2}}+2)\big)^2 \\
f^*\big(v_2^{\frac{r+1}{3}} v_{n-1}^{\frac{r+1}{2}}\big) &= 9\ \big(f(v_{n-1}^{\frac{r+1}{2}})\big)^2 \\
f^*\big(v_1^{\frac{r+1}{2}} v_2^{\frac{r+1}{2}}\big) &= 25\ \big(f(v_{n-1}^{\frac{r+1}{2}}+2)\big)^2 \\
f^*\big(v_2^{\frac{r+3}{2}} v_3^{\frac{r+3}{2}}\big) &= \big(f(v_{n-1}^{\frac{r+1}{2}}+2)\big)^2\ \big(f(v_{n-1}^{\frac{r+3}{2}}+2)\big)^2 \\
f^*\big(v_2^{\frac{r+3}{3}} v_{n-1}^{\frac{r+3}{2}}\big) &= \big(f(v_{n-1}^{\frac{r+3}{2}}+2)\big)^2\ \big(f(v_{n-2}^{\frac{r+3}{2}}+2)\big)^2 \\
f^*\big(v_2^{\frac{r+3}{3}+i} v_2^{\frac{r+3}{2}+i}\big) &= \big(f(v_{n-1}^{\frac{r+1}{2}+i}+2)\big)^2\ \big(f(v_{n-2}^{\frac{r+3}{2}+i}+2i)\big)^2\ for\ 1 \leq i \leq \frac{r-3}{2}
\end{aligned}$$

For r copies of cycle of length 3 with common cut vertex v, $\mid V(G) \mid \geq (v_1^1)^2\ (v_2^1)^2$ are not square product graphs.
Consider a graph G containing r cycles of length 3 with fixed cut vertex labeled as 1, labeling of vertices carries distinct non negative integers and $V(G)$ is bijective. The cut vertex v is adjacent with all other vertices since the graph is C_3 with r copies. Hence, while labeling the edges the product of labels of vertices with the fixed cut vertex 1 is not injective for $\mid V(G) \mid \geq (v_1)^2\ (v_2)^2$. In the below figure, $f^*(v_1v_2) = 36\ and\ f^*(vv_1^2) = 36$ which is not injective.
□

Example 3.3: The square product labeling of even copies of C_6, odd copies of C_6, even copies of C_4, odd copies of C_5 and r copies of C_3 are shown in the below figures.

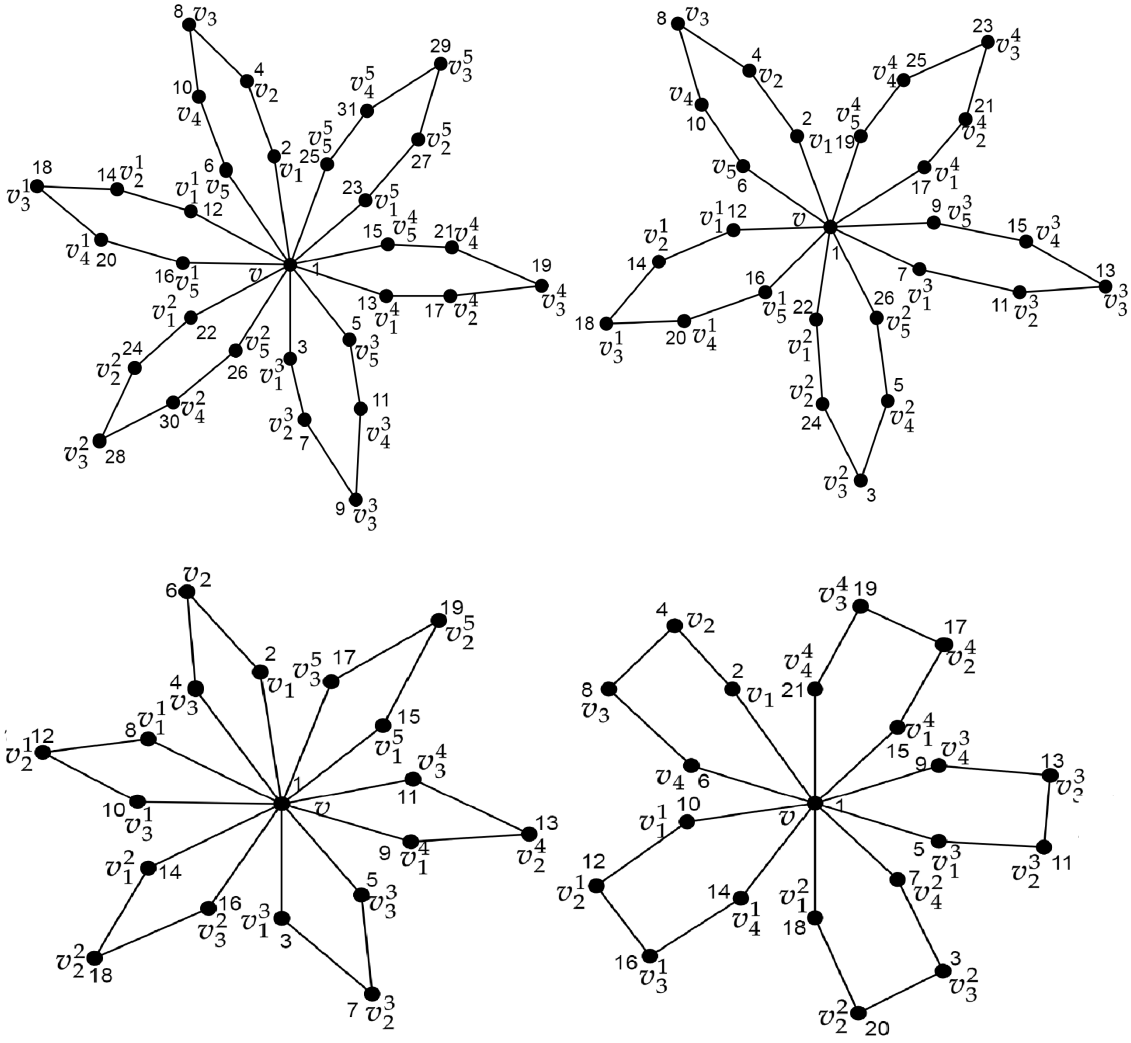

Figure 3: Square product labeling of even copies of C_6 and C_4 and odd copies of C_6 and C_5

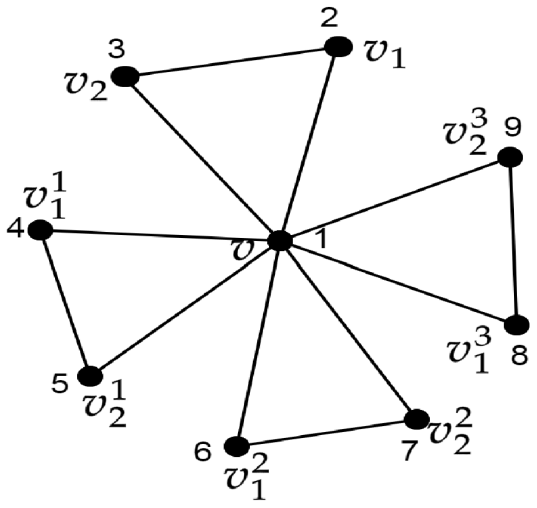

Figure 4: Square product labeling of r copies of C_3

4 Conclusion

In this article, results on square product labeling for cactus graphs with cycle blocks are established. Here the limitation is results can be established only for r copies of cycle of any length with a common cutvertex except for r copies of cycles of length 3 which are not square product graphs for $\mid V(G) \mid \geq (v_1)^2\ (v_2)^2$. In this article, the results are established only for cactus graphs with cycle blocks. The results on square product labeling can also be extended to trees.

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Amalgamated rings with m-nil clean properties

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Abstract

In this paper, we study the transfer of the notion of m-nil clean (i.e., a ring in which every element is a sum of a nilpotent and an m-potent elements) to the amalgamarted rings. We also find many sufficient and necessary conditions for the m-nil clean property to the amalgamated rings.
Keywords: Clean ring; nil-clean; m-clean; m-nil clean; m-potent.
2020 AMS subject classifications: 13A99, 13B99, 13A15, 16N20, 16N40. [1]

1 Introduction

In this paper, all our rings are assumed to be commutative with identity. Nicholson [1977] initially proposed the idea of clean rings in his research on Lifting idempotents and exchange rings. A ring is called clean if every element of the ring can be written as sum of a unit and an idempotent element. Clean rings are initially developed in Nicholson [1977] as a natural class of rings which have the exchange property. Various generalization of clean rings have been explored. Ye [2003] introduced and studied semiclean rings. Diesl [2013] initiated the notion of nil-clean ring where he replaced the unit element by a nilpotent element. S. Purkait and Kar [2020] introduced the notion of m-clean and strongly m-clean

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rings. Letting m be a positive integer greater than or equal to 2, a ring R is said to be m-clean if each element of R can be written as a sum of a unit and an m-potent element (i.e., an element r of a ring R is said to be m-potent if $r^m = r$). Following Diesl [2013], an element of a ring is called nil-clean if it is a sum of an idempotent and a nilpotent, and a ring is called nil-clean if every element is nil-clean. R. Barati and Abyzov [2022] gave a generalization to nil-clean called m-nil clean. An element r of a ring R is said to be m-nil clean if $r = n + f$, where f is m-potent in R and n is nilpotent in R. A ring R is called m-nil clean if each of its elements is m-nil clean. For a fixed natural number $m > 1$, the class of m-nil clean rings is a big subclass of the periodic rings which contains the class of nil-clean rings. A ring R is called periodic if for each $r \in R$ there exist distinct positive integers m and n such that $r^m = r^n$. A ring R is called potent if each element of R is potent (i.e., $r \in R$ is potent if $r^n = r$ for some $n \in \mathbb{R}$). Clearly, potent rings are periodic. For a fixed natural number $m > 1$, a ring R is called m-potent if $r^n = r$ for every $r \in R$. A ring R is UU-ring if every unit u in R is a unipotent, i.e., u can be expressed as $u = 1 + n$, where n is a nilpotent element of ring R.

Let R and S be rings, J be an ideal of S and $\phi : R \to S$ be a ring homomorphism. In this setting, we can consider the following subring of $R \times S$:

$$R \bowtie^{\phi} J := \{(r, \phi(r) + j) \mid r \in R, j \in J\}$$

called *the amalgamation of R with S along J with respect to ϕ* (introduced and studied by M. D'Anna and Fontana [2009], M. D'Anna and Fontana [2010]). This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in (D'Anna and Fontana [2007a], D'Anna and Fontana [2007b])).

In this paper, we investigate the structure of the rings of the form $R \bowtie^{\phi} J$, with a particular attention to the m-nil clean ring property. More precisely, we give a necessary and sufficient condition for the amalgamated rings to be m-nil clean. For the convenience of the reader, we denote $U(R)$, $M(R)$, $Idem(R)$, $Nil(R)$ and $J(R)$, the set of all unit elements of R, the set of all m-potent elements of R, the set of all idempotent elements of R, the set of all nilpotent elements of R and the Jacobson radical of R, respectively.

2 m-nil clean Ring

The aim of this section is to investigate general structure of m-nil clean rings which will be used in the sequel.

Definition 2.1. *R. Barati and Abyzov [2022] Let R be a ring. An element $r \in R$ is called m-nil clean if there exist a nilpotent $n \in R$ and an m-potent element $f \in R$ such that $r = n + f$. A ring R is called m-nil clean if every element of R is m-nil clean.*

Example 2.1. *The class of m-nil clean rings contains many familiar examples.*

1. *Every strongly m-nil clean ring is an m-potent ring.*

2. *Any queotient of an m-nil clean ring is m-nil clean.*

3. *Any finite product of m-nil clean ring is m-nil clean. But $R = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \cdots$ is an infinite product of m-nil clean rings, which is not m-nil clean. The element $(0, 2, 2, \cdots \in R)$ cannot be written as the sum of an m-potent and a nil potent element.*

4. *If R is m-nil clean, then so is the ring $R[[x]]/(x^n), n \in \mathbb{N}$*

Lemma 2.1. *If $r \in R$ be an m-potent element, then r^{m-1} is an idempotent element in R.*

Proof. Let $r \in R$ is an m-potent element. Then $(r^{m-1})^2 = r^{2(m-1)} = r^{2m-2} = r.r^{m-2} = r^{m-1}$. □

Proposition 2.1. *Let R be a commutative ring. Then*

(1) If $r \in R$ is a m-nil clean element, then r^n is a nil-clean element.

(2) If $r \in R$ is a m-nil clean element, then $r^n - r^{2n}$ is a nilpotent for some n.

Proof. (1) Assume that $r \in R$ is an m-nil clean element. Then, we have $r = f + n$ with $n \in Nil(R)$ and $f \in M(R)$. This implies $(r - n)^{m-1} = f^{m-1}$ is an idempotent by Lemma 2.1 . By applying Binomial theorem we get our desired result.
(2) Suppose $r \in R$ is a m-nil clean element. Then by (1), r^n is a nil-clean element. Then by [Y. Hirano and A.Yaqub, 1988, Theorem 3], $r^n - r^{2n}$ is nilpotent. □

Lemma 2.2. *Let R be a ring and $m > 1$ be an integer. Then R is strongly m-nil clean if and only if $r^m - r$ is nilpotent for all $r \in R$.*

Proof. It follows from [R. Barati and Abyzov, 2022, Theorem 1.10]. □

Proposition 2.2. *A ring R is m-nil clean if and only if $R/Nil(R)$ is an m-potent ring.*

Proof. Let $r + Nil(R) \in R/Nil(R)$. Since R is m-nil clean, r can be written as $r = n + f$, where $n \in Nil(R)$ and $f \in M(R)$. So, $r + Nil(R) = n + f + Nil(R) = f + Nil(R)$ and it is easy to see that $(f + Nil(R))^m = f + Nil(R)$. Therefore, $R/Nil(R)$ is an m-potent ring.
Conversely, assume that $R/Nil(R)$ is an m-potent ring. Let $r \in R$, then $r + Nil(R)$ is m-potent, i.e., $(r + Nil(R))^m = r + Nil(R)$. So $r^m - r \in Nil(R)$ and then, by Lemma 2.2, R is m-nil clean. □

The following proposition provides a characterizations of m-nil clean rings.

Proposition 2.3. *R is an m-nil clean ring if and only if for each $r \in R$, we have $r = f + u$, where $f \in M(R)$ and $u \in UU(R)$*

Proof. ($\Rightarrow$) Assume that R is an m-nil clean ring. Let $r \in R$. Then we have $r - 1 = n + f$, where $n \in Nil(R)$ and $f \in M(R)$. This implies $r = u + f$ such that $u = 1 + n \in UU(R)$ and $f \in M(R)$.
($\Leftarrow$) Let us assume that each element $r \in R$, we have $r = f + u$, where $u \in UU(R)$ and $f \in M(R)$. Let $r \in R$, then $r + 1 = u + f$ with $u \in UU(R)$ and $f \in M(R)$. This implies that $r + 1 = 1 + n + f$ such that $n \in Nil(R)$. Therefore, $r = n + f$. Hence, R is m-nil clean. □

Proposition 2.4. *Every m-nil clean ring is an m-clean ring.*

Proof. Assume that R is an m-nil clean ring. Let $r \in R$, then $r - 1 = n + f$, where $n \in Nil(R)$ and $f \in M(R)$. This implies $r = 1 + n + f$. Therefore, $r = u + f$, with $1 + n = u \in U(R)$. Hence, R is m-clean. □

The following example illustrate the converse of the Proposition 2.4 is not true.

Example 2.2. *Every infinite field is an m-clean ring, but not a m-nil clean.*

Proposition 2.5. *If R is m-clean and UU-ring, then R is an m-nil clean ring.*

Proof. Assume that R is both m-clean and UU-ring. Let $r \in R$, then $r - 1 = u + f$, where $u \in U(R)$ and $f \in M(R)$. Since R is UU-ring, $u = 1 + n$ for some $n \in Nil(R)$. Therefore, $r = n + f$. Hence, R is m-nil clean. □

Definition 2.2. *S. Purkait and Kar [2020] Let I be an ideal of a ring R. We say that m-potents lifted modulo I if for any $r \in R$ with $r - r^m \in I$ implies that there exists an m-potent f in R such that $f - r \in I$*

Proposition 2.6. *Let R be a UU-ring and I be an ideal of R such that $I \subset J(R)$. If R/I is m-nil clean and if m-potents in R can be lifted modulo I, then R is m-nil clean.*

Proof. Let us use $\overline{r}$ to denote $r + I$ in R/I. Let $r \in R$, then we can write $\overline{r} = \overline{n} + \overline{f}$, where $\overline{n} \in Nil(R/I)$ and $\overline{f} \in M(R/I)$. By assumption, we may assume $f^m = f$. By the fact that $\overline{1 + r - f} = \overline{1 + n}$ is a unit in R/I and $I \subseteq J(R)$. Now, it is easy to see that $1 + r - f$ is a unit in R. It follows immediately that $r = n + f$. □

Proposition 2.7. *Let R be a ring. Then the polynomial ring $R[x]$ is not an m-nil clean ring.*

Proof. Assume that $x = n + f$, where $f^m = f$ and $f \in R[x]$ and n is nilpotent in $R[x]$. By the commutativity of R, f must be in R. Thus $-f + x$ is nilpotent in $R[x]$. It follows immediately that 1 is a nilpotent element in R, which is impossible. □

Lemma 2.3. *Let R be an m-nil clean ring. Then $J(R)$ is nil.*

Proof. Let $r \in J(R)$. Then $r = f + n$, where $f \in M(R)$ and $n \in Nil(R)$. Suppose $n^k = 0$ for some k. Then $(r - f)^{mk+m-k} = n^{mk+m-k} = 0$; hence, $f^{mk+m-k} \in J(R)$. clearly $f^{mk+m-k} = (f^m)^k.f^{m-k} = f^k.f^{m-k} = f^m = f$. It follows that $f \in J(R)$, and so $f(f^{m-1} - 1) = 0$. This implies that $f = 0$; hence, $r = n \in Nil(R)$. Therefore, $J(R)$ is nil, as asserted. □

Theorem 2.1. *Let R be a ring such that $Idem(R) = \{0, 1\}$. If R is an m-nil clean ring, then each element of R is m-clean or sum of two units.*

Proof. Let $r \in R$. Since R is an $m$nil clean ring, $r - 1$ is an m-nil clean element, i.e., $r - 1 = n + f$, where $n \in Nil(R)$ and $f \in M(R)$. Then f^{m-1} is idempotent belongs to $Idem(R) = \{0, 1\}$, i.e., $f^{m-1} = 0$ or $f^{m-1} = 1$. Now if $f^{m-1} = 0$, then $r - 1 = n$. Since n is nilpotent then $r = n + 1$ is unit. If $f^{m-1} = 1$, then f is unit and $r = f + n + 1$, so r is the sum of two units. □

3 Amalgamated rings with m-nil clean properties

In this section, we investigate the m-nil clean properties of the amalgamated ring $R \bowtie^{\phi} J$. We start with an example to illustrate the definition.

Example 3.1. *Let $R = \mathbb{Z}_2$ and $S = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ be rings and $J = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ an ideal of S and $\phi : R \to S$ defined by $\phi(r) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$, where $r \in \mathbb{Z}_2$. Then R, S and $\phi(R) + J$ are m-clean rings. The amalgamated ring*

$R \bowtie^{\phi} J = \left\{ \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right), \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \right\}$ *is also an* m*-nil clean ring.*

In the following results, we prove that R and $\phi(R) + J$ are m-nil clean whenever the amalgamated ring $R \bowtie^{\phi} J$ is m-nil clean. We recall that a ring R is said to be uniquely m-nil clean if every element of the ring can be uniquely written as sum of nilpotent and m-potent element.

Lemma 3.1. *Let* R *be a ring and* $m > 1$ *be an integer. Then* R *is* m*-nil clean if and only if* $r^m - r$ *is nilpotent for all* $r \in R$*.*

Proof. It follows from [R. Barati and Abyzov, 2022, Theorem 1.10] □

Theorem 3.1. *let* $\phi : R \to S$ *be a ring homomorphism and let* J *be an ideal of* S*. Then the following conditions are equivalent :*

1. $R \bowtie^{\phi} J$ *is* m*-nil clean.*
2. R *and* $\phi(R) + J$ *are* m*-nil clean.*

Proof. $(1) \Rightarrow (2)$: If $R \bowtie^{\phi} J$ is m-nil clean, then R and $\phi(R) + J$ are homomorphic images of $R \bowtie^{\phi} J$ (see [M. D'Anna and Fontana, 2009, Proposition 5.1],). Hence R and and $\phi(R) + J$ are m-nil clean.
$(2) \Rightarrow (1)$: Let $(r, \phi(r)+j) \in R \bowtie^{\phi} J$. Then R and and $\phi(R)+J$ are m-nil clean implies $r^m - r$ and $(r, \phi(r)+j)^m - (r, \phi(r)+j)$ are nilpotent elements in R and and $\phi(R)+J$, respectively. Therefore, it is easy to see that $(r, \phi(r)+j)^m - (r, \phi(r)+j)$ is nilpotent in $R \bowtie^{\phi} J$. Hence $R \bowtie^{\phi} J$ is m-nil clean. □

Example 3.2. *Let* $R = \mathbb{Z}_6, S = \mathbb{Z}_3 \times \mathbb{Z}_3, J = 0 \times \mathbb{Z}_3$ *and* $\phi : R \to S$ *defined by* $\phi(0) = \phi(3) = (0,0), \phi(1) = \phi(4) = (1,1)$ *and* $\phi(2) = \phi(5) = (2,2)$*. It is easy to see that* R *and* $\phi(R)+j = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$ *are* m*-nil clean rings. Then by Theorem 3.1, the amalgamated ring* $R \bowtie^{\phi} J = \{(0,(0,0)),(0,(0,1)),(0,(0,2)),(1,(1,0)),(1,(1,1)),(1(1,2)),(2,(2,0)),(2,(2,1)),(2,(2,2))(3,(0,0)),(3,(0,1)),(3,(0,2)),(4,(1,0)),(4,(1,1)),(4,(1,2))(5,(2,0)),(5,(2,1)),(5,(2,2))\}$ *is* m*-nil clean.*

Corolary 3.1. *Let* $\phi : R \to S$ *be a ring homomorphism and* J *be an ideal of* S*. Then the following conditions are equivalent:*

1. $R \bowtie^{\phi} J$ *is* m*-nil clean.*
2. *Any proper homomorphic image of* $R \bowtie^{\phi} J$ *is* m*-nil clean.*

Proof. $(1) \Rightarrow (2)$ Assume that $R \bowtie^{\phi} J$ is m-nil clean. It is well known that any homomorphic image of a m-nil clean ring is m-nil clean (due to the fact that any homomorphic image of m-potent is m-potent and homomorphic image of nilpotent is nilpotent).
$(2) \Rightarrow (1)$ Assume that any homomorphic image of $R \bowtie^{\phi} J$ is m-nil clean. Since R and $\phi(R) + J$ are proper homomorphic image of $R \bowtie^{\phi} J$, so R and $\phi(R) + J$ are m-nil clean by Theorem 3.1, as desired. □

Corolary 3.2. *Let $\phi : R \to S$ be a ring homomorphism and R is m-nil clean and J be an ideal of S with $J \subset Nil(S)$, then $R \bowtie^{\phi} J$ is m-nil clean.*

Proof. By Theorem 3.1, it is enough to prove that $\phi(R) + J$ is m-nil clean. Let $(r, \phi(r) + j) \in R \bowtie^{\phi} J$. Since R is m-nil clean, $r = f + n$, where f is an m-potent and n is a nilpotent. Therefore, $\phi(r) + j = \phi(f) + \phi(n) + j$. Since $J \subset Nil(S), j \in Nil(S)$ and $\phi(n) \in Nil(R)$ implies that $\phi(n) + j \in Nil(\phi(R) + J)$. Therefore, $\phi(r) + j$ is a sum of m-potent $\phi(f)$ and nilpotent $\phi(n) + j$ in $\phi(R) + J$, as desired. □

Corolary 3.3. *let $\phi : R \to S$ be a ring homomorphism and J be an ideal of S such that $J \cap M(B) = 0$ and $M(R \bowtie^{\phi} J) = \{(f, \phi(f)) | f \in M(R)\}$. Then the following conditions are equivalent:*

1. *$R \bowtie^{\phi} J$ is m-nil clean.*
2. *R is m-nil clean and $J \subset Nil(S)$*

Proof. By Theorem 3.1, R is m-nil clean and now it remains to prove that $J \subset Nil(S)$.
Let $j \in J$, without loss of generality, we may assume that $j \neq 0$. Hence $(0, j)$ is an m-nil clean element of $R \bowtie^{f} J$ and so by [M. Chhiti and Tamekkante, 2015, Lemma 2.10], $(0, j) = (f, \phi(f)) + (n, \phi(n) + k)$, where $f \in M(R), n \in Nil(R)$ and $k \in Nil(S) \cap J$. Since $f + n = 0$, $f = -n \in M(R) \cap Nil(R) = \{0\}$ and so $j = \phi(f) + \phi(n) + k = k$. Therefore $J \subset Nil(S)$.
Conversely, assume that R is m-nil clean and $J \subset Nil(S)$. Then, by Corollary 3.2, $R \bowtie^{\phi} J$ is m-nil clean. □

Theorem 3.2. *let $\phi : R \to S$ be a ring homomorphism and J be an ideal of S and $J \subseteq Nil(S)$. Then we have the following statements:*

1. *Assume that ϕ is one-one. If $\phi(R) + J$ is m-nil clean, then $R \bowtie^{\phi} J$ is m-nil clean.*
2. *Assume that J is nilpotent. Then $R \bowtie^{\phi} J$ is m-nil clean if and only if R is m-nil clean.*

Proof.
(1) As $J \subseteq Nil(S)$ and ϕ is one-one, $\phi^{-1}(J) \subseteq Nil(R)$. On the other hand $\frac{R\bowtie^{\phi} J}{\phi^{-1}(J)\{0\}} \cong \phi(R) + J$. The remaining proof is similar to (1).
(2) The forward direction is shown in Theorem 3.1.
Conversely, let $(r, \phi(r) + j) \in R \bowtie^{\phi} J$. By the fact that $\frac{R\bowtie^{\phi} J}{\{0\}\times J} \cong R$, we get $\overline{(r, \phi(r) + j)} \in \frac{R\bowtie^{\phi} J}{\{0\}\times J}$ is an m-nil clean element, as R is m-nil clean. □

Theorem 3.3. *Let $\phi : R \to S$ be a ring homomorphism and let J be an ideal of S. Set $\overline{R} = R/Nil(R), \overline{S} = S/Nil(S), \pi : S \to \overline{S}$ the canonical projection and $\overline{J} = \pi(J)$ and consider the ring homomorphism $\overline{\phi} : \overline{R} \to \overline{S}$ such that $\overline{\phi}(\overline{r}) = \overline{\phi(x)}$. Then, $R \bowtie^{\phi} J$ is m-nil clean if and only if $\overline{R} \bowtie^{\overline{\phi}} \overline{J}$ is an m-potent ring.*

Proof. It is easy to see that $\overline{\phi}$ is well defined and it is a ring homomorphism. The map $\xi : R \bowtie^{\phi} J/Nil(R \bowtie^{\phi} J) \to \overline{R} \bowtie^{\overline{\phi}} \overline{J}$ defined by $\xi(\overline{r, f(r) + j}) = (\overline{r}, \overline{\phi}(\overline{r}) + \overline{j})$ is an isomorphism.
Assume that $R \bowtie^{\phi} J$. Then by Proposition 2.2, $R \bowtie^{\phi} J/Nil(R \bowtie^{\phi} J)$ is m-potent and so $\overline{R} \bowtie^{\overline{\phi}} \overline{J}$ is m-potent.
Conversely, assume that $\overline{R} \bowtie^{\overline{\phi}} \overline{J}$ is m-potent. Hence $R \bowtie^{\phi} J/Nil(R \bowtie^{\phi} J)$ is m-potent and so $R \bowtie^{\phi} J$ is m-nil clean by Proposition 2.2. □

Example 3.3. *Let T be a ring, J be an ideal of T, and let D be a subring of T such that $J \cap D = (0)$. If $J \subset Nil(T)$, then the ring $D + J$ is m-nil clean if and only if D is m-nil clean.*

Proof. By [M. D'Anna and Fontana, 2009, Proposition 5.1 (3)], $D + J$ is isomorphic to the ring $D \bowtie^{i} J$, where $i : D \hookrightarrow J$ is the natural embedding. Thus, by Theorem 3.2, $D + J$ is m-nil clean if and only if D is m-nil clean. □

4 Conclusions

In this paper, we provide the necessary and sufficient condition for amalgamated ring to be a m-nil clean ring. This study will further help in studying the m-nil clean properties in other classical constructions in the ring theory. In future, there is a scope to study the generalization of amalgamated ring namely bi-amalgamated ring with m-nil clean properties.

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Fork-decomposition of strong product of graphs

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Abstract

Decomposition of arbitrary graphs into subgraphs of small size is assuming importance in the literature. There are several studies on the isomorphic decomposition of graphs into paths, cycles, trees, stars, sunlet etc. Fork is a tree obtained by subdividing any edge of a star of size three exactly once. In this paper, we investigate the necessary and sufficient for the fork-decomposition of Strong product of graphs.
Keywords: Decomposition, Fork, Product graph, Strong Product.
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1 Introduction

We consider only simple, finite and undirected graphs. Let K_n denote the complete graph on n vertices and $K_{m,n}$ denote the complete bipartite graph with parts of sizes m and n. Let P_k denote the path of length $k-1$ and S_k denote the star of size $k-1$. Let $L = \{H_1, H_2, ..., H_r\}$ be a family of subgraphs of G. An L-decomposition of G is an edge-disjoint decomposition of G into positive integers α_i copies of H_i where $i \in \{1, 2, ..., r\}$. Furthermore, if each $H_i (i \in \{1, 2, ..., r\})$ is isomorphic to a graph H, then we say that G has an H-decomposition. The obvious necessary condition for the existence of a $\{H_1, H_2, ..., H_r\}$ - decomposition of G is

$$\sum_{i=1}^{r} \alpha_i e(H_i) = e(G) \tag{1}$$

We call this equation as necessary sum condition.

Terms not defined here are used in the sense of Bondy and Murty Bondy and Murty [2008].

Decomposition of arbitrary graphs into subgraphs of small size are assuming importance in the literature. There are several studies on the isomorphic decomposition of graphs into paths qing Zhai and hong Lu [2006], Kumar [2003], cycles Alspach and Gavlas [2001], trees Barat and Gerbner [2014], stars Zhao and Wu [2015], sunlet Akwu and Ajayi [2013] etc. The general problem of H-decompositions was proved to be NP-complete for any H of size greater than 2 by Dor and Tarsi Dor and Tarsi [1997].

Fork is a tree obtained by subdividing any edge of a star of size three exactly once. A tree with degree sequence $(1, 1, 1, 2, 3)$ is unique and is nothing but the fork defined above. This graph was defined by Simone and Sassano in the name of *chair graph* in 1993, when they studied the stability number of bull and chair-free graphs Simone and Sassano [1993]. In 2014, Barat and Gerbner Barat and Gerbner [2014] studied decomposition of 191-edge connected graphs which can be decomposed into forks as a possible attempt to solve the following conjecture:

Conjecture 1. Barat and Thomassen [2006] For each tree T, there exists a natural number k_T such that the following holds: if G is a k_T -edge-connected simple graph such that $|E(T)|$ divides $|E(G)|$, then G has a T-decomposition.

The edge-connectivity constants in the solved cases of Conjecture 1 are seemingly far from best possible. There is very little known about lower bounds. This motivated us to concentrate on strong product graphs which have lower edge-connectivity.

If a graph G admits a H-decomposition, then $|E(H)|$ divides $|E(G)|$. Since the size of a fork is 4, for a fork-decomposition the obvious necessary condition

is

$$|E(G)| \equiv 0 \ (mod\ 4) \tag{2}$$

In Devi and Joseph [2013] they studied the P_4 Decomposition of Product graphs. Decomposition of complete bipartite graphs, complete graphs and corona graphs into Fork was studied in Joseph and Issacraj [2022]. Fork-decomposition of cartesian product and direct product of graphs were studied in Issacraj and Joseph [2023] & Issacraj and Joseph [2022] respectively. In this paper, we investigate the decomposition of strong product of graphs into forks.

Definition 1.1. *Hammack et al. [2011] The strong product of G and H is the graph denoted as $G \boxtimes H$, and defined by $V(G \boxtimes H) = \{(g,h)/g \in V(G)$ and $h \in V(H)\}$, $E(G \boxtimes H) = E(G \square H) \cup E(G \times H)$. The number of edges in $G \boxtimes H$ is $|V(G)||E(H)| + |V(H)||E(G)| + 2|E(G)||E(H)|$.*

Notation 1.1. *We observe that $K_2 \boxtimes K_2 = K_4$. The product of K_2 with itself produces the shape $\boxtimes$. This explains the reason behind the symbol for this product.*

The following results are used in the subsequent section.

Theorem 1.1. *Joseph and Issacraj [2022] The complete bipartite graph $K_{m,n}$ is fork-decomposable if and only if $mn \equiv 0 \ (mod\ 4)$ except $K_{2,4i+2} (i = 1, 2, ...)$.*

Theorem 1.2. *Joseph and Issacraj [2022] $C_n \circ \overline{K_m}$ is fork-decomposable if and only if $m = 1$ and $n = 2k$ or $m = 3$.*

Theorem 1.3. *Issacraj and Joseph [2023] $P_n \square P_m$ is fork-decomposable if and only if $m \equiv n \ (mod\ 4)$ where $n \leq m$.*

Theorem 1.4. *Issacraj and Joseph [2023] The graph $K_m \square P_n$ is fork-decomposable if it satisfies any one of the following conditions.*

1. $m \equiv 0 \ (mod\ 8)$
2. $n \equiv 0 \ (mod\ 2)$ and $m \equiv 0 \ (mod\ 4)$
3. $n \equiv 2 \ (mod\ 4)$ and $m \equiv 2 \ (mod\ 4)$
4. $n \equiv 3 \ (mod\ 4)$ and $m \equiv 5 \ (mod\ 8)$
5. $n \equiv 1 \ (mod\ 4)$ and $m \equiv 1 \ (mod\ 8)$

Theorem 1.5. *Issacraj and Joseph [2023] The graph $K_m \square C_n$ is fork-decomposable if and only if it satisfies any one of the following conditions.*

1. n is even and $m \equiv 0 \ (mod\ 4)$ or $m \equiv -1 \ (mod\ 4)$

2. n *is odd and* $m \equiv 0\ (mod\ 8)$ *or* $m \equiv -1\ (mod\ 8)$

3. $n \equiv 0\ (mod\ 4)$

Theorem 1.6. *Issacraj and Joseph [2022] For* $m, n \geq 3$, $P_m \times P_n$ *is fork-decomposable if and only if* $(m-1)(n-1)$ *is a multiple of 4.*

Theorem 1.7. *Issacraj and Joseph [2022] For* $m \geq 3$, $K_m \times P_n$ *is fork-decomposable if and only if*

1. n *is even and* $m = 4k$ *or* $4k+1$.

2. n *is odd.*

Theorem 1.8. *Issacraj and Joseph [2022] For* $m \geq 3$, $K_m \times C_n$ *is fork-decomposable if and only if*

1. n *is odd and* $m = 4k$ *or* $4k+1$.

2. n *is even.*

In this paper, we investigate the necessary and sufficient condition for the decomposition of strong product of graphs into forks.

2 Strong product of two paths

In this section, we investigate the necessary and sufficient condition for the decomposition of strong product of paths into forks. First let us see an example for the fork-decomposition of strong product of paths.

Example 2.1. $P_2 \boxtimes P_4$ *is fork-decomposable.*

Proof. Let $V(P_2) = \{x_1, x_2\}$ and $V(P_4) = \{y_1, y_2, y_3, y_4\}$. Then $V(P_2 \boxtimes P_4) = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4)\}$. Rename the vertices $(x_1, y_i) = u_i$, and $(x_2, y_i) = v_i$, for all $1 \leq i \leq 4$. Fork-decomposition of $P_2 \boxtimes P_4$ is given in Figure 1.

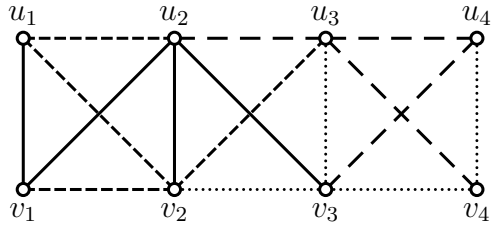

Figure 1: $P_2 \boxtimes P_4$ □

The following two lemmas are needed for proving the theorem.

Lemma 2.1. *For $m \geq 2$, $P_2 \boxtimes P_m$ is fork-decomposable if and only if $m \equiv 0\ (mod\ 4)$.*

Proof. Total number of edges in $P_2 \boxtimes P_m$ is $5m-4$. If $P_2 \boxtimes P_m$ is fork - decomposable, then $5m-4 \equiv 0\ (mod\ 4)$. This implies that $m \equiv 0\ (mod\ 4)$.

Conversely, assume that $m \equiv\ 0(mod\ 4)$.

Let $V(P_2) = \{x_1, x_2\}$ and $V(P_m) = \{y_1, y_2, \ldots, y_m\}$. Then $V(P_2 \boxtimes P_m) = \{(x_1, y_1),\ (x_1, y_2), \ldots,\ (x_1, y_m),\ (x_2, y_1),\ (x_2, y_2),\ \ldots,\ (x_2, y_m)\}$. Rename the vertices $(x_1, y_i) = u_i$, and $(x_2, y_i) = v_i$, for all $1 \leq i \leq m$.

If $m = 4$, $P_2 \boxtimes P_4$ is fork-decomposable by Example 2.1.

Fork-decomposition of the graph obtained after removing $P_2 \boxtimes P_4$ from $P_2 \boxtimes P_m$ is given by $\{u_{4i+1}v_{4i},\ u_{4i+1}v_{4i+1},\ u_{4i+1}v_{4i+2},\ v_{4i+2}u_{4i+2}\}$, $\{v_{4i+1}u_{4i},\ v_{4i+1}v_{4i},\ v_{4i+1}v_{4i+2},\ u_{4i}u_{4i+1}\}$, $\{u_{4i+2}u_{4i+1},\ u_{4i+2}v_{4i+1},\ u_{4i+2}v_{4i+3},\ v_{4i+3}v_{4i+2}\}$, $\{u_{4i+3}u_{4i+2},\ u_{4i+3}v_{4i+3},\ u_{4i+3}v_{4i+4},\ v_{4i+4}v_{4i+3}\}$, $\{u_{4i+4}v_{4i+4},\ u_{4i+4}v_{4i+3},\ u_{4i+4}u_{4i+3},\ u_{4i+3}v_{4i+2}\}$, for $1 \leq i \leq \frac{m-4}{4}$. Hence the graph $P_2 \boxtimes P_m$ is fork-decomposable.

□

Lemma 2.2. *For $m \geq 2$, $P_4 \boxtimes P_m$ is fork-decomposable if and only if $m \equiv 2\ (mod\ 4)$.*

Proof. Let $G = P_4 \boxtimes P_m$. Total number of edges in $P_4 \boxtimes P_m$ is $17m-10$. If G is fork-decomposable, then $17m - 10 \equiv 0\ (mod\ 4)$ which implies that $8(2m-1) + m - 2 \equiv 0\ (mod\ 4)$. Then, $m - 2 \equiv 0\ (mod\ 4)$. This implies that $m \equiv 2\ (mod\ 4)$.

Conversely, assume that $m \equiv 2\ (mod\ 4)$.

Let $V(P_4) = \{x_1, x_2, x_3, x_4\}$ and $V(P_m) = \{y_1, y_2, \ldots, y_m\}$. Then $V(P_4 \boxtimes P_m) = \{(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_m), (x_2, y_1), (x_2, y_2), \ldots, (x_2, y_m), (x_3, y_1), (x_3, y_2), \ldots, (x_3, y_m), (x_4, y_1), (x_4, y_2), \ldots, (x_4, y_m)\}$.
Rename the vertices $(x_1, y_i) = a_i$, $(x_2, y_i) = b_i$, $(x_3, y_i) = c_i$ and $(x_4, y_i) = d_i$, for all $1 \leq i \leq m$.

For $m = 2$. The graph is isomorphic to $P_2 \boxtimes P_4$ which is fork-decomposable by Lemma 2.1. Let H_1 be the graph obtained by removing $P_4 \boxtimes P_2$ from $P_4 \boxtimes P_m$. Then a fork-decomposition of H_1 is given by $\{a_{i+1}a_i,\ a_{i+1}b_{i+1},\ a_{i+1}a_{i+2},\ b_{i+1}c_i\}$, $\{b_ia_{i+1},\ b_ib_{i+1},\ b_ic_{i+1},\ b_{i+1}a_i\}$, $\{d_{i+1}d_i,\ d_{i+1}c_{i+1},\ d_{i+1}c_{i+2},\ b_{i+1}c_{i+1}\}$, $\{c_{i+1}c_i,\ c_{i+1}d_i,\ c_{i+1}c_{i+2},\ c_{i+2}b_{i+1}\}$, $\{b_{i+2}a_{i+1},\ b_{i+2}b_{i+1},\ b_{i+2}c_{i+1},\ b_{i+1}a_{i+2}\}$, $\{d_{i+2}c_{i+1},\ d_{i+2}d_{i+1},\ d_{i+2}c_{i+3},\ d_{i+1}c_i\}$, $\{a_{i+2}a_{i+3},\ a_{i+2}b_{i+2},\ a_{i+2}b_{i+3},\ b_{i+2}c_{i+2}\}$, $\{c_{i+2}b_{i+3},\ c_{i+2}c_{i+3},\ c_{i+2}d_{i+2},\ b_{i+3}b_{i+2}\}$, $\{d_{i+3}c_{i+2},\ d_{i+3}c_{i+3},\ d_{i+3}d_{i+4},\ c_{i+3}b_{i+3}\}$, $\{a_{i+3}b_{i+2},\ a_{i+3}b_{i+3},\ a_{i+3}b_{i+4},\ b_{i+2}c_{i+3}\}$, $\{a_{i+4}a_{i+3},\ a_{i+4}b_{i+3},\ a_{i+4}b_{i+4},\ b_{i+3}c_{i+4}\}$, $\{b_{i+4}b_{i+3},\ b_{i+4}c_{i+3},\ b_{i+4}c_{i+4},\ c_{i+3}d_{i+4}\}$, $\{c_{i+4}c_{i+3},\ c_{i+4}d_{i+3},\ c_{i+4}d_{i+4},\ d_{i+3}d_{i+2}\}$,

for $i \equiv 2 \ (mod\ 4)$. $G = H_1 \cup (P_4 \boxtimes P_2)$. Since H_1 and $P_4 \boxtimes P_2$ are fork-decomposable, G is fork-decomposable. □

In the following theorem, we investigate the necessary and sufficient condition for the fork-decomposition of strong product of paths.

Theorem 2.1. *For $m, n \geq 2$, $P_m \boxtimes P_n$ is fork-decomposable if and only if it satisfies any one of the following conditions:*

1. $m \equiv 0 \ (mod\ 4)$ *and* $n \equiv 2 \ (mod\ 4)$
2. $m \equiv 1 \ (mod\ 4)$ *and* $n \equiv 1 \ (mod\ 4)$
3. $m \equiv 3 \ (mod\ 4)$ *and* $n \equiv 3 \ (mod\ 4)$.

Proof. Total number of edges in $P_m \boxtimes P_n$ is $n(m-1)+m(n-1)+2(m-1)(n-1) = 4mn - 3(m+n) + 2$.

If $P_m \boxtimes P_n$ is fork-decomposable, then $4mn-3(m+n)+2$ must be a multiple of 4. That means $3(m+n)-2$ must be a multiple of 4. Then $3(m+n)-2 = 4p$, where p is any arbitrary positive integer. Then $3(m+n) = 4p+2 = 2(2p+1)$ which implies that $m+n$ must be even. Hence m and n must be both even or both odd. Let the values of m and n be $(4k, 4k+1, 4k+2, 4k+3)$ and $(4l, 4l+1, 4l+2, 4l+3)$ respectively. If m is $4k$ then n can take the values $4l$ or $4l+2$. If m is $4k+1$ then n can take the values $4l+1$ or $4l+3$. If m is $4k+2$ then n can take the values $4l$ or $4l+2$. If m is $4k+3$ then n can take the values $4l+1$ or $4l+3$.

Suppose $m = 4k$ and $n = 4l$, then $3(m+n)-2 = 3(4k+4l)-2 = 12(k+l)-2$ which is not a multiple of 4. Hence this condition does not hold.

Suppose $m = 4k$ and $n = 4l+2$, then $3(m+n)-2 = 3(4k+4l+2)-2 = 12(k+l)+6-2$ which is a multiple of 4. Hence $m \equiv 0 \ (mod\ 4)$ and $n \equiv 2 \ (mod\ 4)$ which is a condition 1.

Suppose $m = 4k+1$ and $n = 4l+1$, then $3(m+n)-2 = 3(4k+1+4l+1)-2 = 12(k+l)+6-2$ which is a multiple of 4. Hence $m \equiv 1 \ (mod\ 4)$ and $n \equiv 1 \ (mod\ 4)$ which is a condition 2.

Suppose $m = 4k+1$ and $n = 4l+3$, then $3(m+n)-2 = 3(4k+1+4l+3)-2 = 12(k+l)+12-2$ which is not a multiple of 4. Hence this condition does not hold.

Suppose $m = 4k+2$ and $n = 4l+2$, then $3(m+n)-2 = 3(4k+2+4l+2)-2 = 12(k+l)+12-2$ which is not a multiple of 4. Hence this condition does not hold.

Suppose $m = 4k+3$ and $n = 4l+3$, then $3(m+n)-2 = 3(4k+3+4l+3)-2 = 12(k+l)+18-2$ which is a multiple of 4. Hence $m \equiv 3 \ (mod\ 4)$ and $n \equiv 3 \ (mod\ 4)$ which is a condition 3.

Now we shall prove the converse part.

Let $V(P_m) = \{x_1, x_2, \ldots, x_m\}$ and $V(P_n) = \{y_1, y_2, \ldots, y_n\}$.

Then $V(P_m \boxtimes P_n) = \{(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_m), (x_2, y_1), (x_2, y_2), \ldots, (x_2, y_m), \ldots, (x_n, y_1), (x_n, y_2), \ldots, (x_n, y_m)\}$. Rename the vertices $(x_i, y_j) = u_{ij}$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Suppose $m \equiv 0\ (mod\ 4)$ and $n \equiv 2\ (mod\ 4)$. The induced subgraph $\langle x_{i1}, x_{i2}, \ldots, x_{in}\rangle$ $i = 1, 2$ is isomorphic to $P_2 \boxtimes P_n$ which is fork-decomposable, by Lemma 2.1. The induced subgraph $\langle x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{(i+1)1}, x_{(i+1)2}, x_{(i+1)3}, x_{(i+1)4}, \ldots, x_{(i+4)1}, x_{(i+4)2}, x_{(i+4)3}, x_{(i+4)4}\rangle$, $-\langle x_{i1}, x_{i2}, x_{i3}, x_{i4}\rangle$, $i \equiv 2\ (mod\ 4)$ is isomorphic to H_1 (given in Lemma 2.2) which is fork-decomposable.

After removing $P_2 \boxtimes P_n$ and H_1 from $P_m \boxtimes P_n$, the graph H_2 given in the figure 2 is obtained.

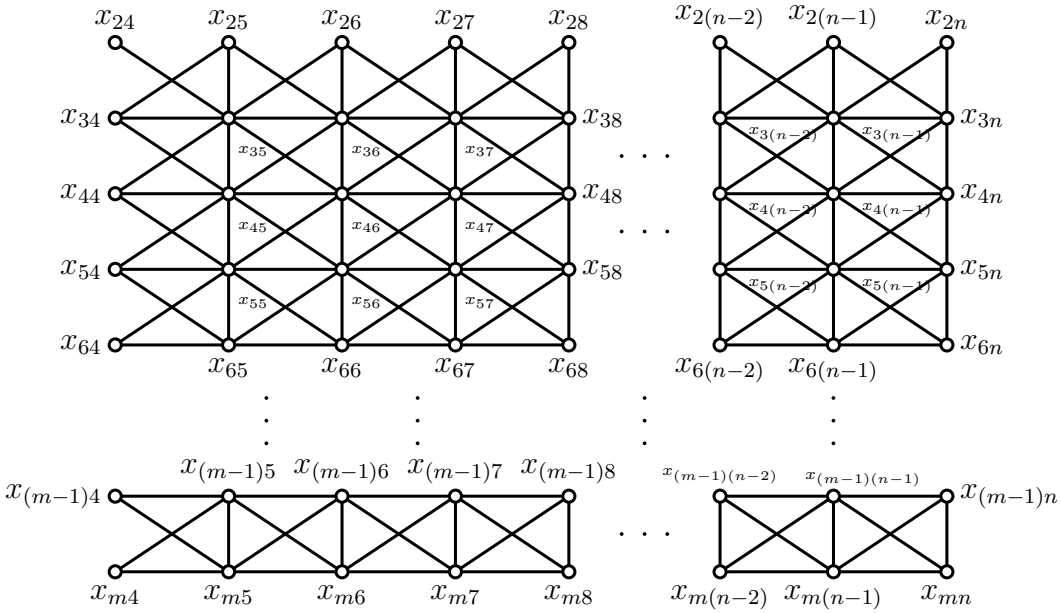

Figure 2: The graph H_2.

The graph H_2 can be decomposed into $P_{m-1} \times P_{n-3}$ and collection of forks $\{x_{i(j+1)}x_{ij},\ x_{i(j+1)}x_{(i-1)(j+1)},\ x_{i(j+1)}x_{i(j+2)},\ x_{i(j+2)}x_{(i-1)(j+2)}\}$ where $i = 3, 4, \ldots, m$, $j \equiv 0\ (mod\ 2)$, $4 \leq j \leq m-2$. The graph $P_{m-1} \times P_{n-3}$ is fork-decomposable by Theorem 1.6, since $m - 1 \equiv 0\ (mod\ 4) \Rightarrow m \equiv 1\ (mod\ 4)$ and $n - 3 - 2 \equiv 0\ (mod\ 4) \Rightarrow n \equiv 5\ (mod\ 4) \Rightarrow n \equiv 1\ (mod\ 4)$. The graph $P_m \boxtimes P_n = (P_2 \boxtimes P_n) \cup H_1 \cup H_2$ and hence $P_m \boxtimes P_n$ is fork-decomposable.

The graph $P_m \boxtimes P_n = (P_m \square P_n) \cup (P_m \times P_n)$. By Theorem 1.3 and 1.6, $P_m \boxtimes P_n$ is fork-decomposable for conditions (2) and (3). $\square$

3 Strong product of path and cycle

In the following theorem, we investigate the necessary and sufficient condition for the decomposition of strong product of path and cycle into forks.

Theorem 3.1. *For $m \geq 3$, $C_m \boxtimes P_n$ is fork-decomposable if and only if $m \equiv 0\ (mod\ 4)$.*

Proof. Total number of edges in $C_m \boxtimes P_n$ is $4mn - 3m$.

If $C_m \boxtimes P_n$ is fork-decomposable, by Equation (2) $4mn - 3m \equiv 0\ (mod\ 4)$. This implies that $m(4n - 3) \equiv 0\ (mod\ 4)$. Then, $m \equiv 0\ (mod\ 4)$ or $4n - 3 \equiv 0\ (mod\ 4)$. The second condition does not hold and hence $m \equiv 0\ (mod\ 4)$.

Conversely, assume that $m \equiv 0\ (mod\ 4)$.

First let us prove that $C_m \boxtimes P_2$ is fork-decomposable.

Let $V(C_n) = \{w_1, w_2, \ldots, w_n\}$ and $V(P_2) = \{x_1, x_2\}$. Then $V(C_m \boxtimes P_2) = \{(w_1, x_1), (w_2, x_1), \ldots, (w_n, x_1), (w_1, x_1), (w_2, x_2), \ldots, (w_n, x_2)\}$. Rename the vertices $(w_j, x_1) = u_j$, $(w_k, x_2) = v_k, 1 \leq j, k \leq m$. Then the fork-decomposition of $C_m \boxtimes P_2$ is given by $\{u_iv_i, u_iv_{i+1}, u_iv_{i-1}, v_{i+1}u_{i+1}\}$, $\{v_iv_{i+1}, v_iu_{i-1}, v_iv_{i-1}, u_{i-1}u_i\}$, $\{u_{i+1}u_i, u_{i+1}v_i, u_{i+1}v_{i+2}, v_{i+2}v_{i+3}\}$, $\{u_{i+2}u_{i+1}, u_{i+2}v_{i+2}, u_{i+2}v_{i+3}, v_{i+2}v_{i+1}\}$, $\{u_{i+3}u_{i+2}, u_{i+3}v_{i+3}, u_{i+3}v_{i+2}, u_{i+2}v_{i+1}\}$ for $i \equiv 1\ (mod\ 4)$. The subscripts are taken modulo m.

The graph $C_m \boxtimes P_n$ can be decomposed into one copy of $C_m \boxtimes P_2$ and $n - 3$ copies of $C_n \circ \overline{K_3}$. $C_n \circ \overline{K_3}$ is fork-decomposable by Theorem 1.2. Hence $C_m \boxtimes P_n$ is fork-decomposable. □

In the following theorem, we investigate the necessary and sufficient condition for the decomposition of strong product of cycles into forks.

Theorem 3.2. *For $m, n \geq 3$, $C_m \boxtimes C_n$ is fork-decomposable for all values of m and n.*

Proof. Total number of edges in $C_m \boxtimes C_n$ is $4mn$. If $C_m \boxtimes C_n$ is fork - decomposable, then $|E(C_m \boxtimes C_n)|$ satisfies Equation (2) for all values of m and n.

The graph $C_m \boxtimes C_n$ can be decomposed into n copies of $C_n \circ \overline{K_3}$. $C_n \circ \overline{K_3}$ is fork-decomposable by Theorem 1.2. Hence $C_m \boxtimes C_n$ is fork-decomposable. □

4 Strong product of complete graph and path

In the following theorem, we state the necessary and sufficient condition for the fork-decomposition of strong product of complete graph and path.

Theorem 4.1. *The graph $K_m \boxtimes P_n$ if fork-decomposable if and only if satisfies any one of the following conditions:*

1. $m \equiv 0\ (mod\ 8)$
2. $n \equiv 1\ (mod\ 4)$ *and* $m \equiv 1\ (mod\ 8)$
3. $n \equiv 2\ (mod\ 4)$ *and* $m \equiv 0\ (mod\ 4)$
4. $n \equiv 3\ (mod\ 4)$ *and* $m \equiv 5\ (mod\ 8)$
5. $n \equiv 0\ (mod\ 4)$ *and* $m \equiv 0\ (mod\ 2)$

Proof. Total number of edges in $K_m \boxtimes P_n$ is $m(n-1) + n\left(\frac{m(m-1)}{2}\right) + 2(n-1)\left(\frac{m(m-1)}{2}\right) = \frac{3m^2n-2m^2-mn}{2} = \frac{m}{2}\left(m(3n-2)-n\right)$.

If the graph $K_m \boxtimes P_n$ is fork - decomposable, then by Equation (2), $\frac{m}{2}\left(m(3n-2)-n\right) \equiv 0\ (mod\ 4)$. That is,

$$m(m(3n-2)-n) \equiv 0\ (mod\ 8) \tag{3}$$

Clearly, it satisfies $m \equiv 0\ (mod\ 8)$ which is a condition (1).

Assume that $n \equiv 0\ (mod\ 4)$. That is $n = 4k$, where k is any arbitrary positive integer. Then (3) $\Rightarrow m(m(12k-2)-4k) \equiv 0\ (mod\ 8) \Rightarrow m(m(6k-1)-2k) \equiv 0\ (mod\ 4) \Rightarrow m^2(6k-1) - 2km \equiv 0\ (mod\ 4)$. Clearly, $2km$ is even and $6k-1$ is odd. Hence m^2 must be even. That is $m \equiv 0\ (mod\ 2)$, which is a condition (5).

Assume that $n \equiv 1\ (mod\ 4)$. That is $n = 4k+1$, where k is any arbitrary positive integer. Then (3) $\Rightarrow m(m(12k+3-2)-4k-1) \equiv 0\ (mod\ 8) \Rightarrow m^2(12k+1)-4mk-m \equiv 0\ (mod\ 8) \Rightarrow 12m^2k+m^2-4mk-m \equiv 0\ (mod\ 8) \Rightarrow 4mk(3m-1)+m(m-1) \equiv 0\ (mod\ 8)$. Clearly, $4mk(3m-1) \equiv 0\ (mod\ 8)$, since $m(3m-1)$ is even. So $m(m-1) \equiv 0\ (mod\ 8)$. That implies $m \equiv 0\ (mod\ 8)$, which is a condition (1) and $m \equiv 1\ (mod\ 8)$ which is a condition (2).

Assume that $n \equiv 2\ (mod\ 4)$. That is $n = 4k+2$, where k is any arbitrary positive integer. Then (3) $\Rightarrow m(m(12k+6-2)-4k-2) \equiv 0\ (mod\ 8) \Rightarrow m(m(12k+4)-4k-2) \equiv 0\ (mod\ 8) \Rightarrow m(4(m(3k+1)-k)-2) \equiv 0\ (mod\ 8) \Rightarrow m(12mk+4m-4k-2) \equiv 0\ (mod\ 8) \Rightarrow m(6mk+2m-2k-1) \equiv 0\ (mod\ 4)$. Clearly, $m \equiv 0\ (mod\ 4)$, since $6mk+2m-2k-1$ is odd, which is a condition (3).

Assume that $n \equiv 3\ (mod\ 4)$. That is $n = 4k+3$, where k is any arbitrary positive integer. Then (3) $\Rightarrow m(m(12k+9-2)-4k-3) \equiv 0\ (mod\ 8) \Rightarrow m(m(12k+7)-4k-3) \equiv 0\ (mod\ 8) \Rightarrow 12mk+7m-4k-3 \equiv 0\ (mod\ 8) \Rightarrow 8mk+4mk-4k+8m-8-m+5 \equiv 0\ (mod\ 8) \Rightarrow 4k(m-1)-m+5 \equiv 0\ (mod\ 8)$. Clearly, m is odd and it satisfies when $m \equiv 5\ (mod\ 8)$ which is a condition (4).
Conversely,

1. The condition (1) is proved by using conditions 1.4 (1) and 1.7 (1) and (2).
2. The condition (2) is proved by using conditions 1.4 (5) and 1.7 (2).

3. The condition (3) is proved by using conditions 1.4 (2) and 1.7 (1).

4. The condition (4) is proved by using conditions 1.4 (4) and 1.7 (2).

5. Consider $n \equiv 0 \ (mod\ 4)$ and $m \equiv 0 \ (mod\ 2)$. The case (5) can be stated into two subcases as $n \equiv 0 \ (mod\ 4)$, $m \equiv 0 \ (mod\ 4)$ and $n \equiv 0 \ (mod\ 4)$, $m \equiv 2 \ (mod\ 4)$. For $n \equiv 0 \ (mod\ 4)$, $m \equiv 0 \ (mod\ 4)$, $K_m \boxtimes P_n$ is fork-decomposable by 1.4 (2) and 1.7 (1).

Now, let us prove the result for $n \equiv 0 \ (mod\ 4)$, $m \equiv 2 \ (mod\ 4)$. Consider $K_m \boxtimes P_4$. For $m = 2$, the graph $K_2 \boxtimes P_4$ is fork-decomposable by Lemma 2.1. $K_m \boxtimes P_4$ can be decomposed into one copy of $K_2 \boxtimes P_4$ and 10 copies of $K_{2,m-2}$. $K_{2,m-2}$ is fork-decomposable by Theorem 1.1. $K_m \boxtimes P_n$ can be decomposed into $\frac{n}{4}$ copies of $K_m \boxtimes P_4$ and $\frac{n}{4} - 1$ copes of $K_{m,m}$ which is fork-decomposable by Theorem 1.1. □

5 Strong product of complete graph and cycle

In the following theorem, we state the necessary and sufficient condition for the fork-decomposition of strong product of complete graph and cycle.

Theorem 5.1. *The graph $K_m \boxtimes C_n$ if fork-decomposable if and only if satisfies any one of the following conditions.*

1. $n \equiv 0 \ (mod\ 4)$
2. $m \equiv 0, 3 \ (mod\ 8)$
3. $m \equiv 4, 7 \ (mod\ 8)$ *and* $n \equiv 0 \ (mod\ 2)$

Proof. Total number of edges in $K_m \boxtimes P_n$ is $\frac{mn(3m-1)}{2}$.

If the graph $K_m \boxtimes C_n$ is fork-decomposable, then $\frac{mn(3m-1)}{2}$ satisfies Equation (2).This implies that $mn(3m-1) = 8k$, where k is any arbitrary positive integer. Hence our possibilities are $m = 8k$ or $n = 8k$ or $3m - 1 = 8k$ or n and m are even.

$m = 8k$, is the first part of condition (2).

Clearly, $m(3m-1)$ is even and hence $n = 4k$ which is a condition (1).

Consider $3m - 1 = 8k$ which implies $m = \frac{8k+1}{3}$. Since m is an integer, $m \equiv 3 \ (mod\ 8)$, which is a second part of condition (2).

Let m and n are even and consider $n \equiv 0 \ (mod\ 2)$. Then $m(m-1)$ is a multiple of 4 which implies that either $m = 4k$ or $3m - 1 = 4k$. If $m = 4k$, $m \equiv 0 \ (mod\ 4)$, which is a first part of condition (3).

Also if, $3m - 1 = 4k$ which implies $m = \frac{4k+1}{3}$. Since m is an integer, m takes the values $3, 7, 11, 15, 19, \ldots$. Hence $m \equiv 3 \ (mod\ 4)$. This congruence can be stated in two ways as $m \equiv 3 \ (mod\ 8)$ and $m \equiv 7 \ (mod\ 8)$. The condition

$m \equiv 3\ (mod\ 8)$ is satisfied for all n which is a condition 2. Hence when $n \equiv 0\ (mod\ 1)$. we have $m \equiv 7\ (mod\ 8)$, which is the second part of condition (3). Conversely,

1. Consider $n \equiv 0\ (mod\ 4)$. This condition is satisfied by using conditions 1.5 (3) and 1.8 (2).

2. Consider $m \equiv 0, 3\ (mod\ 8)$. Consider $m \equiv 0\ (mod\ 8)$. This condition is satisfied by using conditions 1.5 (3) and 1.8 (2). Consider $m \equiv 3\ (mod\ 8)$. The graph $K_m \boxtimes C_n$ can be decomposed into $K_{m-3} \boxtimes C_n$, $K_3 \boxtimes C_n$ and n copies of $K_{3,3(m-3)}$. $K_3 \boxtimes C_n$ is fork-decomposable by Theorem 3.2 and $K_{3,3(m-3)}$ is fork-decomposable by Theorem 1.1.

3. Consider $m \equiv 4, 7\ (mod\ 8)$ and $n \equiv 0\ (mod\ 2)$. Consider $m \equiv 4\ (mod\ 8)$. This condition is satisfied by using conditions 1.5 (1) and 1.8 (2). Consider $m \equiv 7\ (mod\ 8)$. This condition is satisfied by using conditions 1.5 (1) and 1.8 (2). □

6 Discussion and Conclusion

In this paper, we have reviewed the literature on decomposition of graphs with special reference to the subgraph fork. Fork-decomposition of 191-edge connected graphs has already been studied in the literature. But this constant is far from best possible. Very little is known about lower bounds. In this paper, we have investigated and characterized some class of lower edge connectivity graphs for fork-decomposition. In Section 2, we have characterized the fork-decomposition of strong product of two paths. In Section 3, we have characterized the fork-decomposition of strong product of path and cycle and strong product of two cycles. In Section 4, we have characterized the fork-decomposition of strong product of complete graph and path. In Section 5, we have characterized the fork-decomposition of strong product of complete graph and cycle. A similar characterization for fork-decomposition of $G \boxtimes H$ where $G, H \in \{K_n, K_{m,n}, W_n\}$ seems to be an interesting open problem for further research.

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Results on Binary Soft Topological Spaces

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Abstract

The Baire space is used in the proof of results in many areas of analysis and geometry, including some of the fundamental theorems of functional analysis. The concept of Baire spaces has been studied extensively in general topology in [5, 8, 9, 10, 11, 15]. Thangaraj and Anjalmose [35] studied Baire spaces in the context of fuzzy theory. In this paper, we discussed the notions of the binary soft nowhere dense, binary soft dense, binary soft G_δ-set, binary soft first and second category sets, binary soft Baire spaces. Many of their properties are revealed and different characterizations of each are given. In conclusion, we determined some conditions under which the subspace property of a Baire space is preserved.
Keywords: Binary soft set, binary soft nowhere dense set, binary soft dense set, binary soft G_δ-set, binary soft Baire space.
2020 AMS subject classifications: 54A05, 54E52. [1]

1 Introduction

The idea of soft sets was introduced by Molodtsov [14] in 1999. Soft set theory allows researchers to choose the type of parameters they need, which greatly simplifies decision-making and makes the method more productive in the absence of partial data. Later, Shabir et al. [19] started researching on soft topological spaces. Many researchers continued their work on soft topology, including

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Aygunoghu [4], Ahmad [2], Maji [13] and Hussain [12]. In 2016, Acikgöz et al. [1] presented the idea of binary soft set theory on two universal sets and investigated various features. Later, Patil et al. [16] introduced the separation axioms in binary soft topological spaces such as $n-T_0$, $n-T_1$ and $n-T_2$ spaces.

Baire space is a space through which applications of completeness are made in analysis. Also, it is used in well-known theorems as the closed graph theorem, the open mapping theorem and the uniform boundedness theorem. The Baire category theorem (BCT), which specifies sufficient conditions for a topological space to be a Baire space, is a significant result in general topology and functional analysis. Further, BCT is used to prove Hartogs's theorem, a fundamental result in the theory of several complex variables.

The Baire space was initially introduced in Bourbaki's [7] Topologie Generale Chapter lX and was named in honour of Rene Louis Baire. Baire spaces have been extensively studied in classical topology [5, 8, 9, 10, 11, 15]. The first and second category sets were first established by Rene Louis Baire in 1899 [6]. Thangaraj and Anjalmose introduced and studied the concept of Baire spaces in fuzzy topology [20]. Riaz and Fatima [18] studied the concept of soft Baire spaces in soft metric spaces. Ameen and Khalaf investigated the soft Baire invariance using soft semicontinuous, soft somewhat continuous and soft somewhat open functions in [3].

In this article, we introduced a class of binary soft topological space called binary soft Baire space and studied some topological properties that are preserved by proper subspaces such as binary soft nowhere dense, binary soft dense, binary soft first and second category sets and binary soft Baire spaces. We obtained some conditions under which subspace property is preserved. Later, we shall show that binary soft compact, $n-T_2$ space falls in the class of binary soft Baire space.

We have referred the paper [1] for the definition of binary soft (briefly BS) set, binary absolute soft set, binary null soft set, union of two BS sets, intersection of two BS sets and [16, 17] for BS topology, binary soft topological space (briefly BSTS), BS subspace and BS relative topology.

2 Binary Soft Nowhere Dense Set

Definition 2.1. *A BS subset (F,E) of a BSTS (X_1,X_2,τ_b,E) is called a BS dense set (resp. BS co-dense set) if $\overline{\overline{(F,E)}}=\tilde{\tilde{E}}$ (resp. $(F,E)^{\circledcirc}=\tilde{\tilde{\phi}}$).*

Definition 2.2. *A BS subset (F,E) of a BSTS (X_1,X_2,τ_b,E) is called a BS nowhere dense set if $\left(\overline{\overline{(F,E)}}\right)^{\circledcirc}=\tilde{\tilde{\phi}}$.*

Definition 2.3. *A BS subset (F,E) of a BSTS (X_1,X_2,τ_b,E) is called a BS G_δ-set if it is countable BS intersection of BS open sets.*

Example 2.1. *Let $X_1 = R$ and $X_2 = Q$ be two initial universal sets and $E = \{e_1, e_2\}$ be a set of parameters and $\tau_b = \{\tilde{\tilde{E}}, \tilde{\tilde{\phi}}, (F_1, E), (F_2, E), (F_3, E)\}$, where*
$(F_1, E) = \{(e_1, (N, N)), (e_2, (Q, Q))\}$,
$(F_2, E) = \{(e_1, (Q^{'}, \phi)), (e_2, (\phi, \phi))\}$,
$(F_3, E) = \{(e_1, (N \cup Q^{'}, N)), (e_2, (Q, Q))\}$
then BS closed sets are $\tilde{\tilde{E}}, \tilde{\tilde{\phi}}$
$(F_1, E)^{'}$=$\{(e_1, (R \setminus N, Q \setminus N)), (e_2, (Q^{'}, \phi))\}$,
$(F_2, E)^{'}$=$\{(e_1, (Q, Q)), (e_2, (R, Q))\}$,
$(F_3, E)^{'}$=$\{(e_1, (Q \setminus N, Q \setminus N)), (e_2, (Q^{'}, \phi))\}$.
Consider the BS sets
$(F_3, E) = \{(e_1, (Q^{'} \cup N, N)), (e_2, (Q, Q))\}$ *then,* $\overline{\overline{(F_3, E)}} = \tilde{\tilde{E}}$. *Therefore,* (F_3, E) *is a BS dense set in* (R, Q, τ_b, E).
$(G_1, E) = \{(e_1, (Q^{-}, Q^{-})), (e_2, (Q^{'}, \phi))\}$ *then,* $(G_1, E)^{\circledcirc} = \tilde{\tilde{\phi}}$ *and* $\left(\overline{\overline{(G_1, E)}}\right)^{\circledcirc} = \tilde{\tilde{\phi}}$. *Therefore,* (G_2, E) *BS co-dense and BS nowhere dense set* (R, Q, τ_b, E).
$(G_2, E) = \{(e_1, (Q^{+} \setminus N, Q^{+} \setminus N)), (e_2, (Q^{'}, \phi))\}$ *then,* $(G_2, E)^{\circledcirc} = \tilde{\tilde{\phi}}$. *Therefore* (G_2, E) *is also a BS nowhere dense set* (R, Q, τ_b, E).

Theorem 2.1. *Let (F, E) be a BS subset of a BSTS (X_1, X_2, τ_b, E) then the following assertions are interchangeable:*

1. *(F, E) is BS nowhere dense in (X_1, X_2, τ_b, E).*
2. *$\tilde{\tilde{E}} - \overline{\overline{(F, E)}}$ is BS dense in (X_1, X_2, τ_b, E).*
3. *For each non-empty BS open set (G, E) in (X_1, X_2, τ_b, E), there exist a non-empty BS open set (H, E) in (X_1, X_2, τ_b, E) such that $(H, E) \tilde{\tilde{\subseteq}} (G, E)$ and $(H, E) \tilde{\tilde{\cap}} (F, E) = \phi$.*

Proof. $(1) \Rightarrow (2)$, Let (F, E) is BS nowhere dense set in (X_1, X_2, τ_b, E) then $\left(\overline{\overline{(F, E)}}\right)^{\circledcirc} = \tilde{\tilde{\phi}}$. Let (W, E) be a BS open subset of (X_1, X_2, τ_b, E). Since, $\left(\overline{\overline{(F, E)}}\right)^{\circledcirc} = \tilde{\tilde{\phi}}$ therefore (W, E) intersects $\tilde{\tilde{E}} - (\overline{\overline{(F, E)}})$ this holds for all BS open subsets of (X_1, X_2, τ_b, E). Therefore, $\tilde{\tilde{E}} - (\overline{\overline{(F, E)}})$ is BS dense set in (X_1, X_2, τ_b, E).
$(2) \Rightarrow (3)$, Let $\tilde{\tilde{E}} - (\overline{\overline{(F, E)}})$ be a BS dense set in (X_1, X_2, τ_b, E) and (G, E) be non-empty BS open subset of (X_1, X_2, τ_b, E) then, $(G, E) \tilde{\tilde{\cap}} (\tilde{\tilde{E}} - \overline{\overline{(F, E)}}) \neq \phi$. Let $(H, E) = (G, E) \tilde{\tilde{\cap}} (\tilde{\tilde{E}} - \overline{\overline{(F, E)}})$ then clearly $(H, E) \tilde{\tilde{\subseteq}} (G, E)$ and $(H, E) \tilde{\tilde{\cap}} (F, E) = \phi$.
$(3) \Rightarrow (1)$, Suppose (F, E) is not BS nowhere dense set in (X_1, X_2, τ_b, E). That

is $\left(\overline{\overline{(F,E)}}\right)^{\circledcirc} \neq \tilde{\tilde{\phi}}$ then, any non-empty BS open subset of $\left(\overline{\overline{(F,E)}}\right)^{\circledcirc}$ would intersect (F,E), which is a contradiction. Therefore, (F,E) is BS nowhere dense in (X_1, X_2, τ_b, E).

The following theorem gives the relation between BS dense and BS nowhere dense sets.

Theorem 2.2. *If (F,E) is a BS open and dense subset of (X_1, X_2, τ_b, E) then, $(F,E)^c$ is BS nowhere dense set.*

Proof. Let (F,E) is a BS open and dense subset of (X_1, X_2, τ_b, E) then, $\overline{\overline{(F,E)}} = \tilde{\tilde{E}}$, this implies $\tilde{\tilde{E}} - \overline{\overline{(F,E)}} = \tilde{\tilde{\phi}}$, $(\tilde{\tilde{E}} - (F,E))^{\circledcirc} = \tilde{\tilde{\phi}}$, $((F,E)^c)^{\circledcirc} = \tilde{\tilde{\phi}}$. Since, $(F,E)^c$ is BS closed subset. Therefore, $(F,E)^c$ is BS nowhere dense set.

Remark 2.1. *The converse of the preceding theorem is generally untrue. It can be demonstrated using the example below.*

Example 2.2. *Let* $X_1 = \{a_1, a_2, a_3\}$, $X_2 = \{b_1, b_2, b_3\}$, $E = \{e_1, e_2\}$ *and*
$\tau_b = \{\tilde{\tilde{E}}, \tilde{\tilde{\phi}}, (F_1,E), (F_2,E), (F_3,E), (F_4,E), (F_5,E), (F_6,E), (F_7,E), (F_8,E)\}$
where $(F_1,E) = \{(e_1, (\{a_1\}, \{b_1\})), (e_2, (\{a_2\}, \{b_2\}))\}$,
$(F_2,E) = \{(e_1, (\{a_2\}, \{b_2\})), (e_2, (\{a_3\}, \{b_1\}))\}$,
$(F_3,E) = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_2, a_3\}, \{b_1, b_2\}))\}$,
$(F_4,E) = \{(e_1, (\{a_1\}, \{b_2\})), (e_2, (\{a_2\}, \{b_1\}))\}$,
$(F_5,E) = \{(e_1, (\{a_1\}, \{b_1, b_2\})), (e_2, (\{a_2\}, \{b_1, b_2\}))\}$,
$(F_6,E) = \{(e_1, (\{a_1\}, \phi)), (e_2, (\{a_2\}, \phi))\}$,
$(F_7,E) = \{(e_1, (\{a_1, a_2\}, \{b_2\})), (e_2, (\{a_2, a_3\}, \{b_1\}))\}$,
$(F_8,E) = \{(e_1, (\phi, \{b_2\})), (e_2, (\phi, \{b_1\}))\}$
then binary soft closed sets are $\tilde{\tilde{E}}, \tilde{\tilde{\phi}}$
$(F_1,E)'$=$\{(e_1, (\{a_2, a_3\}, \{b_2, b_3\})), (e_2, (\{a_1, a_3\}, \{b_1, b_3\}))\}$,
$(F_2,E)'$=$\{(e_1, (\{a_1, a_3\}, \{b_1, b_3\})), (e_2, (\{a_1, a_2\}, \{b_2, b_3\}))\}$,
$(F_3,E)'$=$\{(e_1, (\{a_3\}, \{b_3\})), (e_2, (\{a_1\}, \{b_3\}))\}$,
$(F_4,E)'$=$\{(e_1, (\{a_2, a_3\}, \{b_1, b_3\})), (e_2, (\{a_1, a_3\}, \{b_2, b_3\}))\}$,
$(F_5,E)'$=$\{(e_1, (\{a_2, a_3\}, \{b_3\})), (e_2, (\{a_1, a_3\}, \{b_3\}))\}$,
$(F_6,E)'$=$\{(e_1, (\{a_2, a_3\}, \{b_1, b_2, b_3\})), (e_2, (\{a_1, a_3\}, \{b_1, b_2, b_3\}))\}$,
$(F_7,E)'$=$\{(e_1, (\{a_3\}, \{b_1, b_3\})), (e_2, (\{a_1\}, \{b_2, b_3\}))\}$,
$(F_8,E)'$=$\{(e_1, (\{a_1, a_2, a_3\}, \{b_1, b_3\})), (e_2, (\{a_1, a_2, a_3\}, \{b_2, b_3\}))\}$.
Let $(H,E) = \{(e_1, (\{a_3\}, \phi)), (e_2, (\{a_3\}, \{b_3\}))\}$ *be a binary soft subset of* (X_1, X_2, τ_b, E). *Clearly,* $\overline{\overline{(H,E)}} = \{(e_1, (\{a_2, a_3\}, \{b_3\})), (e_2, (\{a_1, a_3\}, \{b_3\}))\}$ *then* $\left(\overline{\overline{(H,E)}}\right)^{\circledcirc} = \tilde{\tilde{\phi}}$. *Therefore,* (H,E) *is BS nowhere dense set in* (X_1, X_2, τ_b, E) *but,* $(H,E)^c = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$ *is not BS open set.*

Theorem 2.3. *If (F,E) is a BS closed and nowhere dense subset of (X_1,X_2,τ_b,E) then, $(F,E)^c$ is BS dense subset of (X_1,X_2,τ_b,E).*

Proof: Let (F,E) is a BS closed and nowhere dense subset of (X_1,X_2,τ_b,E) then, $\left(\overline{\overline{(F,E)}}\right)^{\circledcirc}=\tilde{\tilde{\phi}}$, this implies $(F,E)^{\circledcirc}=\tilde{\tilde{\phi}}$, $\tilde{\tilde{E}}-(F,E)^{\circledcirc}=\tilde{\tilde{E}}$, $\overline{\overline{(\tilde{\tilde{E}}-(F,E))}}=\tilde{\tilde{E}}$, $\overline{\overline{((F,E)^c)}}=\tilde{\tilde{E}}$. Therefore, $(F,E)^c$ is BS dense set.

Remark 2.2. *The previous theorem's converse is generally not true. The example below can be used to illustrate it.*

Example 2.3. *In the example 2.2 consider,*
$(G,E)=\{(e_1,(\{a_1,a_2,a_3\},\{b_1,b_2,b_3\})),(e_2,(\{a_1,a_2\},\{b_1,b_2,b_3\}))\}$ *be a BS dense subset of* (X_1,X_2,τ_b,E) *but,* $(G,E)^c=\{(e_1,(\phi,\phi)),(e_2,(\{a_3\},\phi))\}$ *is not BS closed set.*

Theorem 2.4. *Let (Y,τ_{b_y},E) be a BS subspace of (X_1,X_2,τ_b,E) and (F,E) be a BS subset of $\tilde{\tilde{Y}}$. If (F,E) is BS nowhere dense set in $\tilde{\tilde{Y}}$, then (F,E) is BS nowhere dense set in (X_1,X_2,τ_b,E).*

Proof: Suppose (F,E) is a BS nowhere dense subset of $\tilde{\tilde{Y}}$, Let (G,E) be a BS open subset of (X_1,X_2,τ_b,E) then, there exists a non-empty BS open set (H,E) in $\tilde{\tilde{Y}}$ such that $(H,E)\tilde{\tilde{\subseteq}}(G,E)\tilde{\tilde{\cap}}(Y,E)$ and $(H,E)\tilde{\tilde{\cap}}(F,E)=\tilde{\tilde{\phi}}$. Now there exists a BS open set (W,E) in (X_1,X_2,τ_b,E) such that $(H,E)=(W,E)\tilde{\tilde{\cap}}(Y,E)$ thus $(W,E)\tilde{\tilde{\subseteq}}(G,E)$ and $(W,E)\tilde{\tilde{\cap}}(F,E)=\tilde{\tilde{\phi}}$. Therefore, (F,E) is BS nowhere dense set in (X_1,X_2,τ_b,E).

Remark 2.3. *The previous theorem's converse is generally not true. The example below can be used to illustrate it.*

Example 2.4. *In the example 2.2, consider the BS subspace*
$\tilde{\tilde{Y}}=\{(e_1,(\{a_3\},\{b_3\})),(e_2,(\{a_3\},\{b_3\}))\}$ *with BS subspace topology*
$\tau_{b_y}=\{\tilde{\tilde{Y}},\tilde{\tilde{\phi}},(G_1,E)\}$, *where* $(G_1,E)=\{(e_1,(\phi,\phi)),(e_2,(\{a_3\},\phi))\}$ *and*
$(G_1,E)'=\{(e_1,(\{a_1,a_2,a_3\},\{b_1,b_2,b_3\})),(e_2,(\{a_1,a_2\},\{b_1,b_2,b_3\}))\}$.
Let $(H,E)=\{(e_1,(\{a_3\},\phi)),(e_2,(\{a_3\},\{b_3\}))\}$ *be a binary soft subset of* (X_1,X_2,τ_b,E). *Clearly,* (H,E) *is a BS nowhere dense set in* (X_1,X_2,τ_b,E). *In BS subspace* (Y,τ_{b_y},E), $\overline{\overline{(H,E)}}^{y}=\tilde{\tilde{Y}}$ *then* $\left(\overline{\overline{(H,E)}}^{y}\right)^{\circledcirc_y}=(\tilde{\tilde{Y}})^{\circledcirc_y}\neq\tilde{\tilde{\phi}}$. *Therefore,* (H,E) *is not BS nowhere dense set in* $(\tilde{\tilde{Y}},\tau_{b_y},E)$.

Theorem 2.5. *Let $(\tilde{\tilde{Y}},\tau_{b_y},E)$ is BS open or BS dense subspace of (X_1,X_2,τ_b,E) and (F,E) is BS nowhere dense in (X_1,X_2,τ_b,E) then, (F,E) is BS nowhere dense in $(\tilde{\tilde{Y}},\tau_{b_y},E)$.*

Proof: Let $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ be a BS open subspace of (X_1, X_2, τ_b, E) and (F, E) is BS nowhere dense in (X_1, X_2, τ_b, E). Let (G, E) is BS open subset of $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ then (G, E) is BS open subset of (X_1, X_2, τ_b, E). Therefore, there exists a non-empty BS open subset (H, E) of (G, E) such that $(H, E)\tilde{\tilde{\cap}}(F, E) = \tilde{\tilde{\phi}}$, (H, E) is also BS open subset of $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$. Therefore, (F, E) is also BS nowhere dense in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$.
The proof is similar in the case of $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ is BS dense subspace of (X_1, X_2, τ_b, E).

Corolary 2.1. *The above theorem need not be true for BS closed subspaces. That is, if $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ is a BS closed subspace of a BS topological space (X_1, X_2, τ_b, E) and (F, E) is BS nowhere dense set in (X_1, X_2, τ_b, E) then, (F, E) need not to be BS nowhere dense set in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$.*

Example 2.5. *Let $X_1 = \{a_1, a_2, a_3\}$, $X_2 = \{b_1, b_2, b_3\}$, $E = \{e_1, e_2\}$ and*
$\tau_b = \{\tilde{\tilde{E}}, \tilde{\tilde{\phi}}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E), (F_7, E), (F_8, E), (F_9, E), (F_{10}, E), (F_{11}, E)\}$
where $(F_1, E) = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$,
$(F_2, E) = \{(e_1, (\{a_3\}, \{b_3\})), (e_2, (\{a_3\}, \{b_3\}))\}$,
$(F_3, E) = \{(e_1, (\{a_2\}, \{b_1\})), (e_2, (\{a_1\}, \{b_2\}))\}$,
$(F_4, E) = \{(e_1, (\{a_2, a_3\}, \{b_1, b_3\})), (e_2, (\{a_1, a_3\}, \{b_2, b_3\}))\}$,
$(F_5, E) = \{(e_1, (\{a_3\}, \{b_1\})), (e_2, (\{a_3\}, \{b_2\}))\}$,
$(F_6, E) = \{(e_1, (\{a_1, a_2, a_3\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2, a_3\}, \{b_1, b_2\}))\}$,
$(F_7, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \phi))\}$,
$(F_8, E) = \{(e_1, (\phi, \{b_2\})), (e_2, (\phi, \{b_1\}))\}$,
$(F_9, E) = \{(e_1, (\{a_3\}, \{b_1, b_3\})), (e_2, (\{a_3\}, \{b_2, b_3\}))\}$,
$(F_{10}, E) = \{(e_1, (\{a_2, a_3\}, \{b_1\})), (e_2, (\{a_1, a_3\}, \{b_2\}))\}$,
$(F_{11}, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \{b_2\}))\}$
then binary soft closed sets are $\tilde{\tilde{E}}, \tilde{\tilde{\phi}}$
$(F_1, E)' = \{(e_1, (\{a_3\}, \{b_3\})), (e_2, (\{a_3\}, \{b_3\}))\}$,
$(F_2, E)' = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\},$
$(F_3, E)' = \{(e_1, (\{a_1, a_3\}, \{b_2, b_3\})), (e_2, (\{a_2, a_3\}, \{b_1, b_3\}))\}$,
$(F_4, E)' = \{(e_1, (\{a_1\}, \{b_2\})), (e_2, (\{a_2\}, \{b_1\}))\}$,
$(F_5, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_3\}))\}$,
$(F_6, E)' = \{(e_1, (\phi, \{b_3\})), (e_2, (\phi, \{b_3\}))\}$,
$(F_7, E)' = \{(e_1, (\{a_1, a_2, a_3\}, \{b_2, b_3\})), (e_2, (\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}))\}$,
$(F_8, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2\})), (e_2, (\{a_1, a_2\}, \{b_1\}))\}$,
$(F_9, E)' = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2, b_3\}))\}$,
$(F_{10}, E)' = \{(e_1, (\{a_1\}, \{b_2, b_3\})), (e_2, (\{a_2\}, \{b_1, b_3\}))\}$,
$(F_{11}, E)' = \{(e_1, (\{a_1, a_2, a_3\}, \{b_2, b_3\})), (e_2, (\{a_1, a_2, a_3\}, \{b_1, b_3\}))\}$.

Let $\tilde{\tilde{Y}} = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2, b_3\}))\}$ *be a BS closed subset of* (X_1, X_2, τ_b, E) *then the BS subspace topology is*
$\tau_y = \{\tilde{\tilde{Y}}, \tilde{\tilde{\phi}}, (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E), (G_6, E), (G_7, E)\}$
where $(G_1, E) = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$,
$(G_2, E) = \{(e_1, (\phi, \{b_3\})), (e_2, (\phi, \{b_3\}))\}$,
$(G_3, E) = \{(e_1, (\{a_2\}, \{b_1\})), (e_2, (\{a_1\}, \{b_2\}))\}$,
$(G_4, E) = \{(e_1, (\{a_2\}, \{b_1, b_3\})), (e_2, (\{a_1\}, \{b_2, b_3\}))\}$,
$(G_5, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \{b_2\}))\}$,
$(G_6, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \phi))\}$,
$(G_7, E) = \{(e_1, (\phi, \{b_1, b_3\})), (e_2, (\phi, \{b_2, b_3\}))\}$
then BS closed sets are $\tilde{\tilde{Y}}, \tilde{\tilde{\phi}}$ $(G_1, E)' = \{(e_1, (\phi, \{b_3\})), (e_2, (\phi, \{b_3\}))\}$,
$(G_2, E)' = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$,
$(G_3, E)' = \{(e_1, (\{a_1\}, \{b_2, b_3\})), (e_2, (\{a_2\}, \{b_1, b_3\}))\}$,
$(G_4, E)' = \{(e_1, (\{a_1\}, \{b_2\})), (e_2, (\{a_2\}, \{b_1\}))\}$,
$(G_5, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_3\}))\}$,
$(G_6, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2, b_3\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2, b_3\}))\}$,
$(G_7, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2\})), (e_2, (\{a_1, a_2\}, \{b_1\}))\}$.
Consider the BS set $\{(e_1, (\phi, \{b_3\})), (e_2, (\phi, \{b_3\}))\}$ *which is BS nowhere dense set in* (X_1, X_2, τ_b, E)*, but it is not BS nowhere dense set in the BS closed subspace* $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$.

3 Binary Soft Baire Spaces

Definition 3.1. *A BS subset* (F, E) *of a BSTS* (X_1, X_2, τ_b, E) *is called BS first category set if it is the BS union of countable family of BS nowhere dense sets.*

Definition 3.2. *A BS subset* (F, E) *of a BSTS* (X_1, X_2, τ_b, E) *is called BS second category set if it is not BS first category set.*

Definition 3.3. *A BS Baire space is a BSTS such that every non-empty BS open subset is BS second category.*

Example 3.1. *Consider the example 2.1, the BS set*
$(H, E) = \{(e_1, (Q \setminus N, Q \setminus N)), (e_2, (Q', \phi))\}$ *is a BS first category set, because* $(H, E) = (G_1, E)\tilde{\cup}(G_2, E)$. *Also,* (R, Q, τ_b, E) *is a BS Baire space.*

Theorem 3.1. *The following assertions are interchangeable for a BSTS* (X_1, X_2, τ_b, E)*:*

1. (X_1, X_2, τ_b, E) *is a BS baire space.*

2. *The BS intersection of any sequence of BS dense open sets is BS dense in* (X_1, X_2, τ_b, E).

3. *The BS complement of any BS first category set in* (X_1, X_2, τ_b, E) *is BS dense in* (X_1, X_2, τ_b, E).

4. *Every countable BS union of BS closed sets with empty BS interior in* (X_1, X_2, τ_b, E) *has empty BS interior in* (X_1, X_2, τ_b, E).

Proof:$(1) \Rightarrow (2)$, Suppose (X_1, X_2, τ_b, E) is a BS Baire space. Let $\{(G_i, E)\}$ be a sequence of BS dense open sets in (X_1, X_2, τ_b, E). Let us assume that their BS intersection $\tilde{\tilde{\cap}}(G_i, E)$ is not BS dense in (X_1, X_2, τ_b, E), then there exists a BS open set (H, E) such that it doesnot intersect $\tilde{\tilde{\cap}}(G_i, E)$. That is $(H, E) = \tilde{\tilde{E}} - \tilde{\tilde{\cap}}(G_i, E)$, $(H, E) = \tilde{\tilde{\cup}}(\tilde{\tilde{E}} - (G_i, E))$ where each $\tilde{\tilde{E}} - (G_i, E))$ is BS nowhere dense set in (X_1, X_2, τ_b, E). Therefore, (H, E) is of BS first category set. Which is contradiction to (X_1, X_2, τ_b, E) is a BS baire space. Therefore, $\tilde{\tilde{\cap}}(G_i, E)$ is a BS dense set in (X_1, X_2, τ_b, E).
$(2) \Rightarrow (3)$, Let $\{(F_i, E)\}$ be a sequence of BS closed nowhere dense subsets of (X_1, X_2, τ_b, E), then their BS union $\tilde{\tilde{\cup}}(F_i, E)$ is a BS first category set in (X_1, X_2, τ_b, E), we have to prove $\tilde{\tilde{E}} - \tilde{\tilde{\cup}}(F_i, E)$ is BS dense in (X_1, X_2, τ_b, E). Let us assume that it is not dense in (X_1, X_2, τ_b, E). Consider a BS open set (G, E) in (X_1, X_2, τ_b, E) such that it does not intersect with $\tilde{\tilde{E}} - \tilde{\tilde{\cup}}(F_i, E) = \tilde{\tilde{\cap}}(\tilde{\tilde{E}} - (F_i, E))$, where each $(\tilde{\tilde{E}} - (F_i, E))$ is BS dense openset in (X_1, X_2, τ_b, E) then their binary soft intersection is not BS dense in (X_1, X_2, τ_b, E), which is contradiction. Therefore, $\tilde{\tilde{E}} - \tilde{\tilde{\cup}}(F_i, E)$ is BS dense in (X_1, X_2, τ_b, E).
$(3) \Rightarrow (4)$, Let $\{(F_i, E)\}$ be a sequence of BS closed sets with empty interiors. Suppose $\tilde{\tilde{\cup}}(F_i, E)$ does not has empty BS interior, then $\tilde{\tilde{E}} - \tilde{\tilde{\cup}}(F_i, E)$ would not be BS dense in (X_1, X_2, τ_b, E), which is a contradiction. Therefore, $\tilde{\tilde{\cup}}(F_i, E)$ has empty BS interior.
$(4) \Rightarrow (1)$, Let $\{(F_i, E)\}$ be a sequence of BS nowhere dense subsets in (X_1, X_2, τ_b, E). Suppose $\tilde{\tilde{\cup}}(F_i, E)$ is a BS open set, then each $cl(F_i, E)$ has no BS interior point, that is $\left(\overline{\overline{(F_i, E)}}\right)^{\circledcirc} = \tilde{\tilde{E}}$, then $\tilde{\tilde{\cup}}\overline{\overline{(F_i, E)}} = \overline{\overline{(\tilde{\tilde{\cup}}(F_i, E))}}$, since $\tilde{\tilde{\cup}}(F_i, E)$ is BS open, this implies it contains all its interior points, therefore $\tilde{\tilde{\cup}}\overline{\overline{(F_i, E)}}$ has BS interior points, which is contradiction to (4), therefore $\tilde{\tilde{\cup}}(F_i, E)$ is not BS openset. Therefore, BS opensets cannot be written as BS union of BS nowhere dense sets. Therefore, (X_1, X_2, τ_b, E) is a BS Baire space.

Theorem 3.2. *the following assertions are interchangeable for a BSTS* (X_1, X_2, τ_b, E).

1. (X_1, X_2, τ_b, E) *is a BS baire space.*
2. $(G, E)^{\circledcirc} = \tilde{\tilde{\phi}}$ *for every BS first category set* (G, E) *in* (X_1, X_2, τ_b, E).
3. $\overline{\overline{\overline{(H, E)}}} = \tilde{\tilde{E}}$ *for every BS residual set* (H, E) *in* (X_1, X_2, τ_b, E).

Proof: $(1) \Rightarrow (2)$, *Let* (X_1, X_2, τ_b, E) *be a BS baire space, let* (G, E) *be a BS first category set then* $(G, E) = \tilde{\tilde{\cup}}(G_i, E)$ *where* (G_i, E)*'s are BS nowhere dense sets then* $(G, E)^{\circledcirc} = \left(\tilde{\tilde{\cup}}(G_i, E)\right)^{\circledcirc} = \tilde{\tilde{\phi}}$. *Therefore,* $(G, E)^{\circledcirc} = \tilde{\tilde{\phi}}$

$(2) \Rightarrow (3)$, *Let* (H, E) *be a BS residual set then* $(H, E)'$ *is BS first category set i.e.* $\tilde{\tilde{E}} - (H, E)$ *is BS first category set, then* $(\tilde{\tilde{E}} - (H, E))^{\circledcirc} = \tilde{\tilde{\phi}}$, *then* $\tilde{\tilde{E}} - \overline{\overline{\overline{(H, E)}}} = \tilde{\tilde{\phi}}$. *Therefore,* $\overline{\overline{\overline{(H, E)}}} = \tilde{\tilde{E}}$.

$(3) \Rightarrow (1)$, *Let* (F, E) *be a BS first category set then* $\tilde{\tilde{E}} - (F, E)$ *is BS residual set then* $\overline{\overline{\overline{(\tilde{\tilde{E}} - (F, E))}}} = \tilde{\tilde{E}}$, *then* $\tilde{\tilde{E}} - (F, E)^{\circledcirc} = \tilde{\tilde{E}}$, $(F, E)^{\circledcirc} = \tilde{\tilde{\phi}}$ *therefore* $\left(\tilde{\tilde{\cup}}(F_i, E)\right)^{\circledcirc} = \tilde{\tilde{\phi}}$ *where* (F_i, E)*'s are BS nowhere dense sets. Therefore,* (X_1, X_2, τ_b, E) *is a BS Baire space.*

Theorem 3.3. *Every BS open or BS dense subspace of a BS Baire space is a BS Baire space.*

Proof: Let (X_1, X_2, τ_b, E) be a BS baire space, let $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ be a BS open subspace of (X_1, X_2, τ_b, E), consider (F_i, E) be a sequence of BS nowhere dense sets in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$, by theorem 2.4 (F_i, E) is also a sequence of BS nowhere dense sets in (X_1, X_2, τ_b, E). Since (X_1, X_2, τ_b, E) is a BS baire space then $\tilde{\tilde{\cup}}(F_i, E)$ is also BS nowhere dense set in (X_1, X_2, τ_b, E). Let us assume that $\tilde{\tilde{\cup}}(F_i, E)$ is not BS nowhere dense set in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$, then $\left(\overline{\overline{\overline{(\tilde{\tilde{\cup}}(F_i, E))}}}^{y}\right)^{\circledcirc_y} \neq \tilde{\tilde{\phi}}$ then there exists a BS open set (G, E) in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ such that $(G, E)\tilde{\tilde{\cap}}(\tilde{\tilde{\cup}}(F_i, E)) \neq \tilde{\tilde{\phi}}$, since $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ is a BS open subspace of (X_1, X_2, τ_b, E). Thus (G, E) is a BS open set in (X_1, X_2, τ_b, E) then $(G, E)\tilde{\tilde{\cap}}(\tilde{\tilde{\cup}}(F_i, E)) \neq \tilde{\tilde{\phi}}$, which is contradiction that (X_1, X_2, τ_b, E) is a binary soft baire space, therefore $\left(\overline{\overline{\overline{(\tilde{\tilde{\cup}}(F_i, E))}}}^{y}\right)^{\circledcirc_y} = \tilde{\tilde{\phi}}$, thus $\tilde{\tilde{\cup}}(F_i, E)$ is BS nowhere dense set in $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$. Therefore $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ is also a BS Baire space.
The proof is similar for a BS dense subspace.

Example 3.2. *Consider the example 2.5,*
let $\tilde{\tilde{Y}} = \{(e_1, (\{a_1, a_2\}, \{b_1, b_2\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$ *be a binary soft open*

subset of a BS Baire space (X_1, X_2, τ_b, E) *then the BS subspace topology is*
$\tau_{b_y} = \{\tilde{\tilde{Y}}, \tilde{\tilde{\phi}}, (G_1, E), (G_2, E), (G_3, E)\}$
where $(G_1, E) = \{(e_1, (\{a_2\}, \{b_1\})), (e_2, (\{a_1\}, \{b_2\}))\}$,
$(G_2, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \{b_2\}))\}$,
$(G_3, E) = \{(e_1, (\phi, \{b_1\})), (e_2, (\phi, \phi))\}$
then BS closed sets are $\tilde{\tilde{Y}}, \tilde{\tilde{\phi}}$, $(G_1, E)' = \{(e_1, (\{a_1\}, \{b_2\})), (e_2, (\{a_2\}, \{b_1\}))\}$,
$(G_2, E)' = \{(e_1, (\{a_1, a_2\}, \{b_2\})), (e_2, (\{a_1, a_2\}, \{b_1\}))\}$,
$(G_3, E)' = \{(e_1, (\{a_1, a_2\}, \{b_1\})), (e_2, (\{a_1, a_2\}, \{b_1, b_2\}))\}$.
Since, (X_1, X_2, τ_b, E) *is a BS Baire space and* $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ *is a BS open subspace of* (X_1, X_2, τ_b, E)*, then* $(\tilde{\tilde{Y}}, \tau_{b_y}, E)$ *is also BS Baire space.*

Corolary 3.1. *A BS compact and BS* $n - T_2$ *space is BS regular.*

Theorem 3.4. *If* (X_1, X_2, τ_b, E) *is a BS compact and BS* $n - T_2$ *space then* (X_1, X_2, τ_b, E) *is a BS baire space.*

Proof: Let (X_1, X_2, τ_b, E) is a BS compact and BS $n-T_2$ space, let $\{(F_i, E)\}$ be a countable collection of closed sets of (X_1, X_2, τ_b, E) having empty BS interiors, to prove $\tilde{\tilde{\cup}}(F_i, E)$ has empty BS interior, that is $\tilde{\tilde{\cup}}(F_i, E)$ does not contain any BS open set. Consider a BS openset (G_1, E) of (X_1, X_2, τ_b, E), then we must find a pair of points $(x_0, y_0) \in (G_1, E)$ but does not lie in any of the sets (F_i, E). Let us first consider the set (F_1, E), by hypothesis (F_1, E) does not contain (G_1, E), that is there exist a point $(x_1, y_1) \in (G_1, E)$ but $(x_1, y_1) \notin^s (F_1, E)$, since (X_1, X_2, τ_b, E) is a BS compact and BS $n - T_2$ space then by corollary 3.1 it is a BS n-regular space, thus for any $(x_1, y_1) \in (G_1, E)$ there exist a BS open set (H_1, E) such that $(x_1, y_1) \in (H_1, E)$ $\tilde{\tilde{\subseteq}}\overline{\overline{(H_1, E)}}$ $\tilde{\tilde{\subseteq}}(G_1, E)$ also for any $(x_1, y_1) \in X_1 \times X_2$ and $(x_1, y_1) \notin^s (F_1, E)$ then, there is a BS open set (H_1, E) such that $(x_1, y_1) \in (H_1, E)$ and $\overline{\overline{(H_1, E)}}\tilde{\tilde{\cap}}(F_1, E) = \tilde{\tilde{\phi}}$ [16], $\overline{\overline{(H_1, E)}}$ $\tilde{\tilde{\subseteq}}(G_1, E)$. There fore given any nonempty binary soft open set G_n, E, we choose a point $(x_n, y_n) \in (G_n, E)$ and it does not lie in the BS closed set (F_i, E) for all i, then we choose a BS open set (H_n, E) such that $\overline{\overline{(H_n, E)}}\tilde{\tilde{\cap}}(F_n, E) = \tilde{\tilde{\phi}}$ and $\overline{\overline{(H_n, E)}}$ $\tilde{\tilde{\subseteq}}(G_n, E)$. Since (X_1, X_2, τ_b, E) is a BS compact space then consider $\{\overline{\overline{(H_n, E)}}\}$ be a family of BS closed sets, by finite intersection property of BS closed sets [16], we have $\tilde{\tilde{\cap}}\overline{\overline{(H_i, E)}} \neq \tilde{\tilde{\phi}}$, let $(x_0, y_0) \in \tilde{\tilde{\cap}}\overline{\overline{(H_i, E)}}$, that is $(x_0, y_0) \in \overline{\overline{(H_i, E)}}$ for all i, this implies $(x_0, y_0) \in (G_i, E)$ also $(x_0, y_0) \notin^s (F_i, E)$ for all i, thus $(x_0, y_0) \notin^s \tilde{\tilde{\cup}}(F_i, E)$, that is $\tilde{\tilde{\cup}}(F_i, E)$ is also has empty BS interior, There fore (X_1, X_2, τ_b, E) is a BS Baire space.

Remark 3.1. *The converse of the preceding theorem is generally untrue. It can be demonstrated using the example below.*

Example 3.3. *Let* $X_1 = \{a_1, a_2\}$, $X_2 = \{b_1, b_2\}$, $E = \{i, ii\}$ *and*
$\tau_b = \{\tilde{\tilde{E}}, \tilde{\tilde{\phi}}, (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E), (G_6, E), (G_7, E)\}$
where $(G_1, E) = \{(i, (\{a_1, a_2\}, \{b_1\})), (ii, (\{a_1, a_2\}, \{b_1\}))\}$,
$(G_2, E) = \{(i, (\{a_2\}, \{b_1, b_2\})), (ii, (\{a_2\}, \{b_1, b_2\}))\}$,
$(G_3, E) = \{(i, (\{a_1, a_2\}, \{b_2\})), (ii, (\{a_1, a_2\}, \{b_2\}))\}$,
$(G_4, E) = \{(i, (\{a_2\}, \{b_1\})), (ii, (\{a_2\}, \{b_1\}))\}$,
$(G_5, E) = \{(i, (\{a_2\}, \{b_2\})), (ii, (\{a_2\}, \{b_2\}))\}$,
$(G_6, E) = \{(i, (\{a_1, a_2\}, \phi)), (ii, (\{a_1, a_2\}, \phi))\}$,
$(G_7, E) = \{(i, (\{a_2\}, \phi)), (ii, (\{a_2\}, \phi))\}$.
Then, (X_1, X_2, τ_b, E) *is a BS Baire space and BS compact space as it is a finite space. But, it is not* $n-T_2$ *space because,* $(a_1, b_1) \neq (a_2, b_2) \in X_1 \times X_2$ *and there does not exist BS disjoint open sets* (G, E) *and* (F, E) *such that* $(a_1, b_1) \in (G, E)$ *and* $(a_2, b_2) \in (F, E)$.

4 Discussion and Conclusion

This paper contributes to the area of Baire spaces in the BS topological spaces. We defined BS nowhere dense, BS dense, BS first, second category sets and obtained their properties. Also, we have introduced BS Baire space and obtained their characterizations. We determined some conditions under which the subspace property of a Baire space is preserved and established a relation between BS compact and BS Baire space. To support the obtained result and relations we have provided examples.

In the future, we intend to investigate the further properties of BS topological spaces with Baire property.

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Deep Features and Clustering Based Keyframes Selection with Security

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Abstract

The digital world is developing more quickly than ever. Multimedia processing and distribution become vulnerable issues due to the enormous quantity and significance of vital information. Therefore, extensive technologies and algorithms are required for the safe transmission of messages, images, and video files. This paper proposes a secure framework by acute integration of video summarization and image encryption. Three parts comprise the proposed cryptosystem framework. Firstly, the useful informative frames are first extracted using an efficient and lightweight technique that make use of the color histogram-clustering (RGB-HSV) approach's processing capabilities. Each frame of a video is represented by deep features, which are based on an enhanced pre-trained Inception-v3 network. After that summary is obtain using the K-means optimal clustering algorithm means the representative key frames extracted using the clusters highest possible entropy nodes. Experimental validation on two well-known standard datasets demonstrates the proposed methods superiority to numerous state-of-the-art approaches. Finally, the proposed framework performs an efficient image encryption and decryption algorithm by employing a general linear group function GL_n(F). The analysis and testing outcomes prove the superiority of proposed adaptive RSA.

Keywords: Clustering; Color Histogram; General Linear Group; Image

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Encryption and Decryption; Inception-V3; Video Summarization.[1]

1 Introduction

Multimedia content is expanding at an exponential rate in the digital era and has taken on significant importance in the formulation of modern life's methods. Processing mechanisms in this situation have high operating costs and limited resources. Practically every firm needs videos, encompassing social networks, web conferencing, traffic adaptive control, environmental monitoring, and security monitoring. Video data must be processed in a significant way, primarily to recognize the instructive objects and scenes. On the other hand, the storage and transmission of data alongside security has grown to be a major concern. The system creates enormous amounts of data, all of which must be processed securely and in real time. Thus, a few academics presented a variety of strategies, including video summaries and cryptography methods, to overcome computational complexity and security Muhammad et al. [2020], Hamza et al. [2019].

Video summarization (VS) is the practice of extracting significant and brief video clips that accurately convey the entire overarching story line of the original video Sahu and Chowdhury [2020]. The generated summary must have two essential characteristics: it must always include high-priority information and be devoid of any duplication scenes. As a result, the demand for automated and effective image retrieval solutions is rising. These technologies are designed to automatically identify redundant images Rani and Kumar [2020]. A traditional image retrieval method compares the feature vectors of different images based on various distance metrics in order to assess how similar the images are to one another. The most comparable images will be retrieved once a query image feature vector is compared to those in the database. The first and most prominent visual feature in image retrieval and indexing is the color feature da Silva Torres and Falcao [2006]. The ability to demonstrate visual content in images, the simplicity and high reliability of extracting color information from the images, the relative strength in separating images from one another, the relative robustness to background complexity, and independence from image size and orientation are the most significant benefits of color feature Alamdar and Keyvanpour [2011]. One of the most extensively used color feature extractors is the color histogram because it is easily implemented, computationally efficient, and invariant to rotation and slight changes in viewing position Li and Jiang [2016].

The important scene that was highlighted inside the original video is represented by the key frames that were extracted, but subject to issues with confidentiality, integrity,

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authenticity, and other factors, such information is not safe to share online. So, it is strongly advised to secure frames Muhammad et al. [2020], Hamza et al. [2019], Sixing et al. [2012], Mukhedkar et al. [2015], Ramanujam and Karuppiah [2011], Zia et al. [2022]. As a result, the focus is applying the security algorithm to solve these kinds of issues in our work. The innovative concept uses a linear group to make the key space more difficult. It consists of exponential modulus as a function of group for the encryption network and hides all features of the original image. The decryption procedure, which is the inverse operation of the encryption procedure, is comparable to conventional encryption-decryption techniques.

The rest of the paper is organized as follows: Theoretical information related to this research is discussed in section 2. Section 3 proposes a solution to the current challenge in secure video summarization, while Section 4 outlines the experiments with a comparison of proposed and existing methods. Section V concludes with conclusions and future scope.

2 Related Work

The multimedia device intends to automatically perform an activity or provide real-time information from the data centers. In other words, the key goal is to collect accurate, intelligent data that performs via device mastery approaches under extraordinary circumstances. Numerous different kinds of research have been done and are still being pursued for this goal. A technique to directly measure pixel differences between two frames was introduced by Wu and Xu [2013]. On the basis of creation differentiable histograms and histogram-based loss functions, Avi-Aharon et al. [2020] introduced the DeepHist, a unique deep learning approach for image-to-image translation.

To address the shortcomings of existing methodologies that neglect to consider for variations in shot complexity, Liang et al. [2021] proposed a news report summarization scheme based on SURF features and an enhanced clustering algorithm. The detection of the shot's abrupt and gradual boundaries was accomplished using SURF features. The color histogram of the video frames within the shot was then clustered using an enhanced clustering algorithm. The proposed approach recorded an average accuracy of 93.33 percent and a recall rate of 97.22 percent in shot boundary detection when tested on news video datasets. In addition, Gygli [2018] proposed an FCNN to discover shot detection end-to-end, from pixels to final shot boundaries and enabled to use a large temporal context without any need to repetitively sequence frames. Souček and Lokoč [2020] presented TransNet-V2, a deep network for identification of prevalent shot transitions, that further represented a significant preliminary step of video analysis procedures.

Both the computer vision and multimedia industries are now interested in researching egocentric video summarization. Gygli et al. [2014] segmented the clip into seg-

ments based on motion cues and gave each section a score. Finally, the best segments from the summary were chosen within time limits. As a solution to the aforementioned issues, Kumar and Shrimankar [2017] developed a local-alignment-based FASTA strategy to summarize the events in multi-view videos. The FASTA algorithm was then used to capture interview dependencies between different video views via local alignment before object tracking that used to extract the frames with low activity.

The presented work highlight some well-known research on clustering-based video summarization (Sahu and Chowdhury [2020], Rani and Kumar [2020]) because that is the basis of our solution. The most widely used clustering-based approaches are as follows: Sahu and Chowdhury [2020] introduced CSMIK K-means, which was K-means based on a center-surround model (CSM) and an Integer Knapsack type formulation (IK). CSMIK K-Means validated various cluster groupings and calculates the optimal number of clusters as well as the associated summary. In the continuation, authors Rani and Kumar [2020] presented a keyframe extraction technique focused on 4 visual features, and a clustering analysis based on the Kohonen Self-Organizing Map to obtained the most prominent frames from the list of frames generated after fusion. Papadopoulos et al. [2013] proposed another fully automated VS method in which the final number of clusters for summarization calculated by dynamic calculation process.

Due to the quick development of digital video technology, a lot of data is produced through video calls, or internet video conferencing, for which participants set up an active session for contact with one another. Web video conferencing, which is used in online seminars or webinars, means that the sender sends their information on the web. As a result, before outsourcing, it is very necessary to encrypt the confidential information using an effective cryptographic technique. In a protected digital world, the ciphertext pair and the integrity of the ciphertext pair data have a significant impact on network efficiency. To obtain a strong security statement, it must be able to withstand all known cryptanalysis. Against Simon and Simeck, authors Lu et al. [2022] developed neural distinguishers (NDs) and related key neural distinguishers (RKNDs). Simon32/64's ND and RKND achieved 11- and 11-round accuracy of 59.55 percent and 97.90 percent, respectively. In 13 rounds of Simon64/128, the ND achieved an accuracy of 60.32 percent, while the RKND achieved an accuracy of 95.49 percent. In order to continue ensuring the secure and efficient transmission of video data. According to Cheng et al. [2020] the important semantic elements (IPM, MVD, residual coefficient, and delta QP) were encrypted in each slice. This method made use of the segmentation capabilities of the H.264 encoding and procedure of selective video encryption was based on video coding technology and a four-dimensional hyperchaotic scheme. The authors also examined the perceived quality of encrypted video using several reference videos sequences with motion, texture, and objects.

Further, to ensure image security, Gaherwar et al. [2022] presented a method for safeguarding images by selective alteration (SISA). The technique used selective en-

cryption or blurring to encrypt an image, which reduced processing time with no compromise on security. By applying deep learning techniques, authors Ding et al. [2020] suggested a method for encrypting and decrypting medical images (specifically DLED-Net). To encrypt and decrypt the medical image, the Cycle-GAN network was chosen as the learning network. The technique for modelling was directed by a target domain in order to implement the encryption procedure. Additionally, a ROI-mining network was developed to extract the ROI from medical images that had been encrypted, allowing DLEDNet to segment the relevant organs or tissues in ciphertext environments without having to first decrypt the images.

As opposed to the normal video acquired from a third-person view, the egocentric video often captured by first-person equipment presents a number of issues. The same items could disappear and reappear in succeeding frames in an egocentric clip since the camera is always moving. Thus, creating automated, instructive summaries from egocentric videos has grown to be a very difficult challenge. In this study, the best clustering method for egocentric video summaries are introduced. The goal of the solution workflow is to obtain better accuracy while using minimal processing resources. However, certain solutions do not have sufficient robustness when it comes to securing key-frames. Another issue is the use of mathematical transformation for noise and cropping-resistant image encryption. Consequently, additional in-depth evaluations to choose the best cryptosystem are always necessary.

3 Proposed Methodology

The four key stages of our proposed framework are as follows: a) Preprocessing; b) Feature extraction using the modified Inception model and obtaining a set of a number of clusters (k values) for key-frame selection; c) Key-frame security. An overview of our proposed solution is described in the form of a block diagram in figure 1.

3.1 Pre-processing

The video is comprised of multiple frames. Therefore, choosing a precise number of frames from a sequence of video at fixed intervals is necessary for preprocessing. Hence, the video sequence V_{id} is the set of number of frames F_N at time 't', represented by equation (1), in which F(t+i) refers to a frame at time (t+i).

$$V_{id} = \sum_{i=1}^{N} F(t+i) \tag{1}$$

The size of the V_{id} is extremely large as it contains useful and useless frames, which affects the computational time. The elimination of useless frames from V_{id} is mandatory.

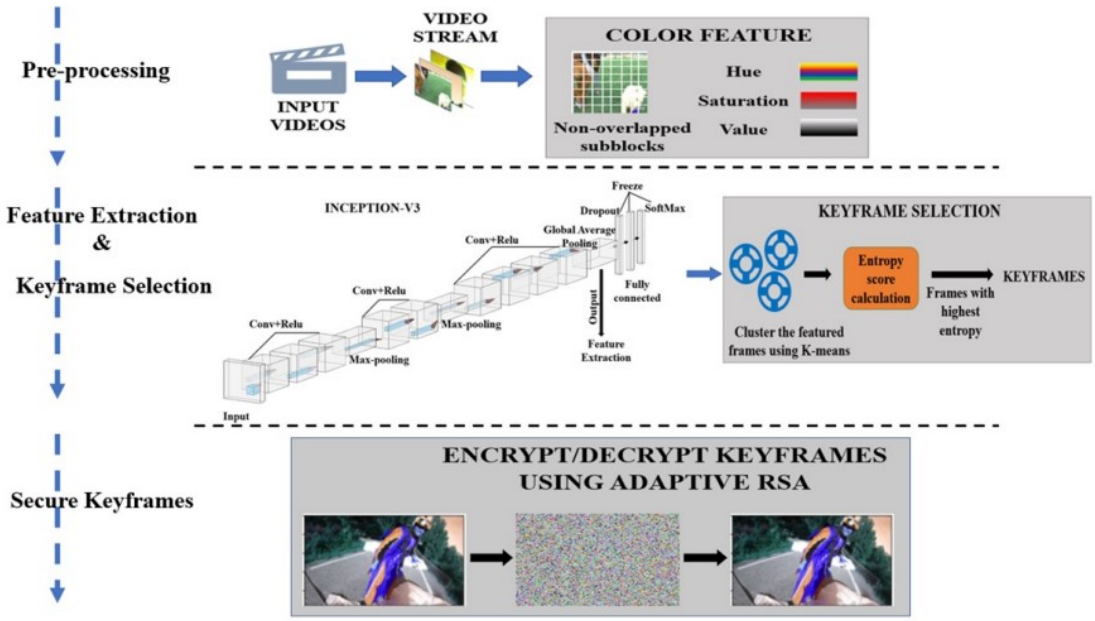

Figure 1: Block Diagram of Proposed Methodology

Hence, global color histogram which analyze statistical color frequency in an image by solving the problems of change in translation, rotation and angle of view Srivastava et al. [2015] is used. For histogram computation, the frame is divided into three color channels (R, G, B), and each color channel is further split into non-overlapping sub-blocks. Let F(x),F(x+1),F(x+3),. . . . . .,F_N are the frames of dimension P*Q. The frames are further divided into number of sub-blocks S_B with dimension M*N, where $M < P$ and $N < Q$. The histogram difference of each and every channel is measured by histogram intersection as given in equation (2-4).

$$H_R(F(x),F(x+i)) = \frac{1}{S_B}\sum_{k=1}^{S_B}\left(\left(1-\sum_{i=1}^{24} min\left(\left(H_{x,k}(l)\right)_R,\left(H_{x+1,k}(l)\right)_R\right)\right)\right) \quad (2)$$

$$H_G(F(x),F(x+i)) = \frac{1}{S_B}\sum_{k=1}^{S_B}\left(\left(1-\sum_{i=1}^{24} min\left(\left(H_{x,k}(l)\right)_G,\left(H_{x+1,k}(l)\right)_G\right)\right)\right) \quad (3)$$

$$H_B(F(x),F(x+i)) = \frac{1}{S_B}\sum_{k=1}^{S_B}\left(\left(1-\sum_{i=1}^{24} min\left(\left(H_{x,k}(l)\right)_B,\left(H_{x+1,k}(l)\right)_B\right)\right)\right) \quad (4)$$

where, $(H_{x,k}(l))$, $(H_{x+1,k}(l))$ are the color histogram of k_{th} section of three RGB channels of frame F(x), F(x+1) respectively.

Finally, the average of all channel differences, D_H is computed as shown in equations (5) and it is compared to the input threshold T_H in function (6) for the elimination of useless frames.

$$D_{H-RGB} = \frac{\sum_{(} H_R) + (H_G) + (H_B)}{3} \tag{5}$$

$$\{D_{H-RGB} < T_H\} \tag{6}$$

If above condition satisfy then frames declared as useful informative frames.

3.2 Feature and Keyframe Extraction

To obtain visual descriptors, the presented method uses Inception-V3 (Szegedy et al. [2016]). In comparison to dppLSTM (Zhang et al. [2016]), M-AVS (Ji et al. [2019]), GoogleNet (Sahu and Chowdhury [2020]), and SUM/GAN (Mahasseni et al. [2017]), the model has the best feature to decrease the number of resources and enhance speed. The fine-tuning of all the convolutional layers till the global average pooling as the end of the network occur in this work. The final three layers are removed, and the features generated by the bottleneck layer are used. After that, the well-known k-means algorithm (Ahmed et al. [2020]) is applied to cluster featured frames. K-means separates the area of focus from the background and utilizes a color palette to represent the visual of frame as the human eye would perceive it. The number of clusters to be set 15 percent and 20 percent for SUMME, TVSUM datasets respectively. After that, the frame with the highest channel entropy in the cluster is declared as key-frame. The entropy being discussed here is Shannon Entropy, which quantifies the informational content of an image and specifies how much ambiguity or randomness there is in an image. The information that an image contains can be calculated using the equation (7) to determine its quality, in which 'k' is the color space of an image and 'p' is the ratio of the number of occurrences of intensity level to total pixels.

$$H(I) = \sum_{i \in I} p_i(i) * \log \frac{1}{p_i(i)} \tag{7}$$

where p_i = p(k=i) ; i=1,2,3,...,'m'

Since then, the entropy computed from channels and HSV for hue, saturation, and intensity value has been used for the colour histogram because it effectively divides RGB into luminosity and chromaticity Nazir et al. [2018] and simulates how humans perceive colour using formulas (8-10).

$$H = \cos^{-1} \frac{\frac{1}{2}(R-G)+(R-B)}{\sqrt{(R-G)^2-(G-B)(R-B)}} \tag{8}$$

$$S = 1 - \left(\frac{3\,[min(R,G,B)]}{R+G+B} \right) \tag{9}$$

$$V = \left(\frac{R+G+B}{3} \right) \tag{10}$$

where 'H' is the hue dominant wavelength of color in image collection, 'S' is the saturation magnitude of white light with in image and 'V' is the intensity value or lightness that describe how dark or light a color. Each RGB pixel in a colorful image is 3 bytes in size, with each color having an intensity range of 0 to 255 and a total of 256*256*256 colors that can be displayed. As a result, information entropy for frame F of size M*N with RGB channel account of A_{RGB} given in equation (11).

$$F_{RGB} = -\log_2 256^{M*N*A_{RGB}} \tag{11}$$

3.3 Keyframes Security

The unique characteristics of video, such as its enormous size and volume, make it impossible to ensure video security, and the majority of currently used security methods struggle to handle complicated data in real-time. Therefore, an adaptive RSA encryption and decryption algorithm is proposed in this research as given in algorithm 1. The ARSA uses three prime numbers because multiplying three significant prime numbers quickly yields the desired result, but doing the opposite requires a lot of computing power. The exponential modulus (g) taken as the number of elements in general linear group GL_n(F). Under matrix multiplication, the GL_n(F) is a set of integrate n*n matrices with elements in finite field F. The number of elements in $GL_n(F_p)$ is $\Pi_{k=0}^{n-1} \left(p^n - p^k\right)$. Suppose, there are n*n matrices with linearly independent rows. The first row can be anything other than the zero row, so there are p^n-1 possibilities. The second row must be linearly independent from the first, so there are p^n-p possibilities. The i^{th} row also be linearly independent from the first $i-1$ rows, so total possibilities are $\left(p^n - p^{i-1}\right)$. Therefore, $(p^n-1).(p^n-p).....\left(p^n - p^{i-1}\right)$ equals to $\Pi_{k=0}^{n-1} \left(p^n - p^k\right)$.

Algorithm 1 : Adaptive RSA

1. Choose three prime numbers p,q and r randomly and independent to each other.
2. Calculate : n = p*q*r and g = g_p + g_q +g_r

(i) $g_p = (p^2 - 1) . (p^2 - p)$

(ii) $g_q = (q^2 - 1) . (q^2 - q)$

(iii) $g_r = (r^2 - 1) . (r^2 - r)$

3. Choose an integer e, $1 < e <$ g , such that gcd(e,g) = 1
4. Compute the private exponent d, $1 < d <$ g , such that ed $\equiv$ 1 mod g
5. Encryption : 'M' is the original RGB image vector
   C = M^e(mod n)
6. Decryption : 'C' is the encrypted RGB image vector
   M = C^d(mod n)

4 Experimental Results

4.1 Datasets Description

The SUMME and TVSUM datasets were tested in order to validate the method's effectiveness. The TVSUM provides 50 videos from various genres, including vlogs, news, and egocentrics, although the SUMME has 25 videos in the sports, holidays, and events categories. Table 1 summaries the characteristics of both datasets.

Table 1: Characteristics Details of Datasets

Dataset	**Number of clips**	**Resolution**	**Shots**	**Size**
SUMME	25	720*1280	390	1hour 10 minutes
TVSUM50	50	360*540	1720	3hours 50 minutes

In the SUMME and TVSUM databases, the maximum number of user summaries (Sahu and Chowdhury [2020]) per video is 15 and 20 respectively. The figure 2 shows some extracted key-frames of the clips of scuba from SUMME and bike-tricks, grooming animals from TVSUM datasets. There are no redundant frames in the key-frames that were extracted, which accurately reflect the majority of the information in the video.

4.2 Qualitative Analysis

User summary for summarization from Sahu and Chowdhury [2020] and Zhang et al. [2020] are considered as the ground truth for both datasets in experiments. This work used the scripts and other helpful information Sahu and Chowdhury [2020] to get the findings and summaries. Another potentially useful piece of work was completed

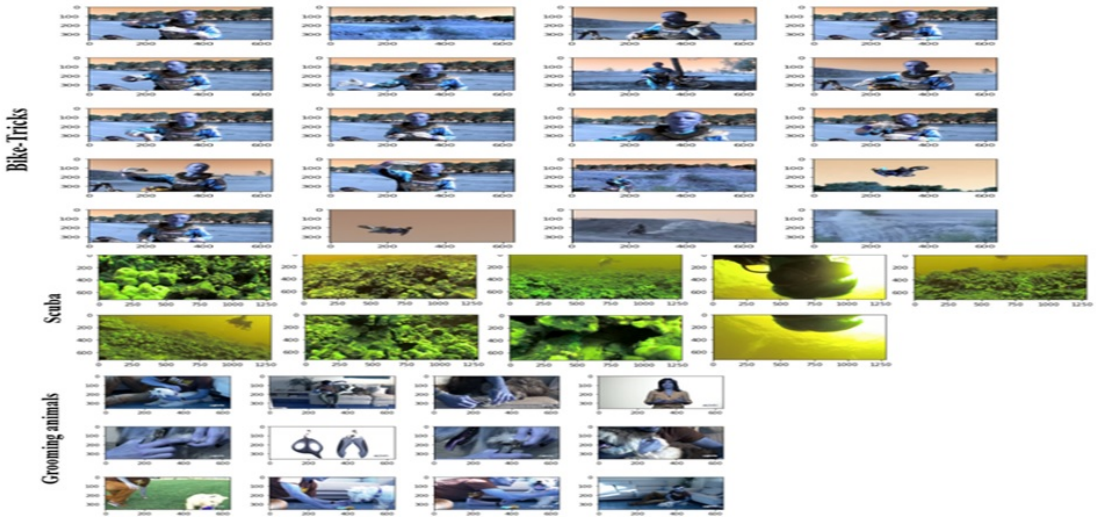

Figure 2: Few frames of summary generated by proposed approach

by Zhang et al. Zhang et al. [2020], who used the SUMME database to generate their outcomes and summary. The scripts and some other beneficial information are obtained, such as reviewing the generated frames for video highlights since the strategy is reliable for generating frames. The results of qualitative evaluation are displayed in figure 3, and these clearly demonstrate that the proposed technique is capable of summarizing key-frames that contain representative objects, such as the instant a person is jumping or flying. The CSMIK k-means and online motion techniques, on the other hand, are more likely to choose frames with limited information. The same is accurate for the bike-polo key-frames of the proposed method given in figure 4 as compared to Gygli [2018], which includes every action such as setting up in the goal, giving the ball to a player, receiving a shot, tracking a player hopping into position, and quickly crossing a player.

4.3 Quantitative Analysis

For performance evaluation, recall, precision, and f1-score are used as metrics. The term ”precision” relates to a method’s accuracy, and it can be determined by counting the key-frames that were improperly extracted. The recall value refers to the possibility of each key-frame existing in the ground truth. In equation (12), $KF_{matched}$ indicates how many key-frames were matched to the ground truth, and $KF_{extracted}$ indicates how many key-frames were extracted overall using our method. The number of key-frames in the ground truth summary is denoted by $KF_{groundtruth}$ in equation (13) and f1-score is

Figure 3: Representative frames of Base-jumping category of SUMME dataset: 1st row (Proposed framework), 2nd row Sahu and Chowdhury [2020] and 3rd row Zhang et al. [2020]

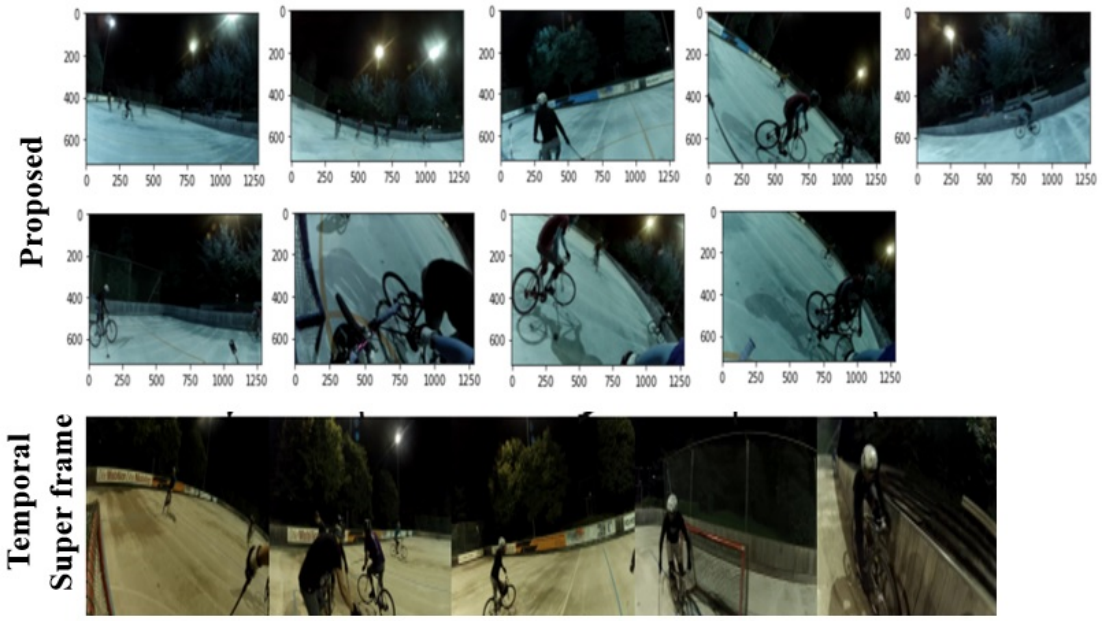

Figure 4: Representative frames of summarized events Bike-polo: 1st row (Proposed framework), 2nd row Gygli [2018]

computed via equation (14).

$$Precision = \frac{KF_{matched}}{KF_{extracted}} \tag{12}$$

$$Recall = \frac{KF_{matched}}{KF_{groundtruth}} \tag{13}$$

$$F1 - score = \frac{2 * (Precision * Recall)}{(Precision + Recall)} \tag{14}$$

The analysis of recall and precision under various types of videos in a comparison Table 2 with the algorithms by Liang et al. [2021] and TransNetV2 Souček and Lokoč [2020], and Table 3 below displays the f-score comparison results with Sahu and Chowdhury [2020], Huang and Wang [2019], Gygli [2018], Zhao et al. [2018], Khan et al. [2019]. The comparison results clearly confirm the effectiveness of the algorithm.

4.4 Security Analysis

To withstand extensive attacks, key-frames encryption and decryption are performed using general linear group GL_n(F) algorithms. Here, GL_n(F) of degree 'n' over prime numbers is the set of n*n convertible matrices with entries from prime numbers. The

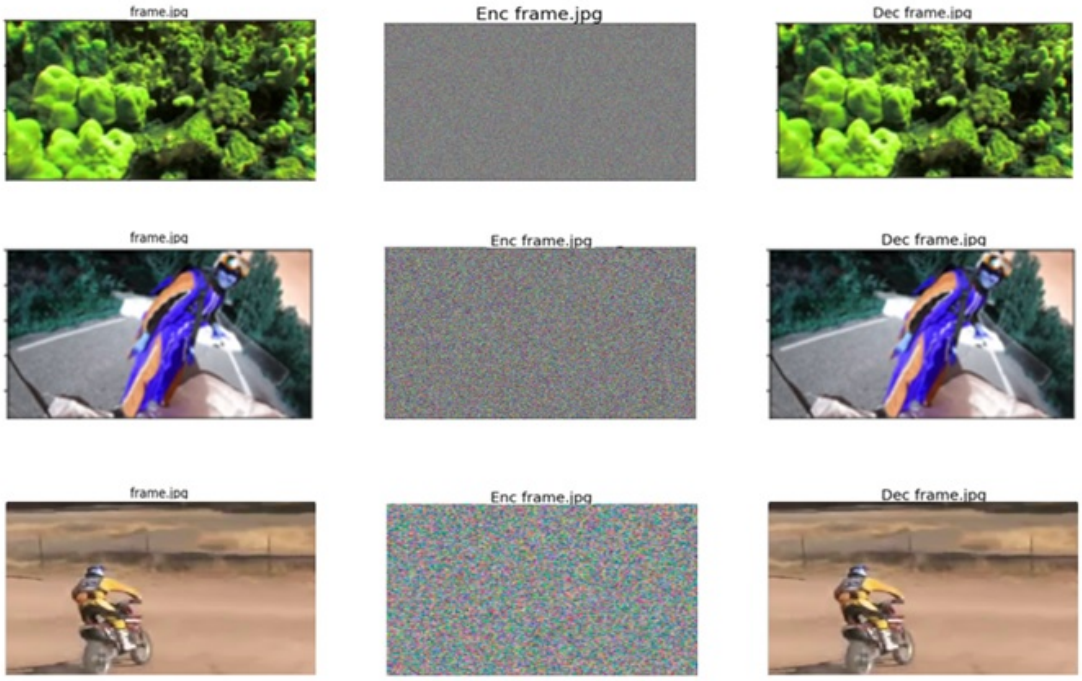

Figure 5: Encryption and Decryption of few representative key-frames using adaptive RSA

Table 2: Quantitative Evaluation of Precision and Recall

Methods	**SUMME**		**TVSUM**	
	Precision	**Recall**	**Precision**	**Recall**
SURF Liang et al. [2021]	-	-	87.91	95.93
TransNetV2 Souček and Lokoč [2020]	-	-	92.02	92.56
Proposed	**87.50**	**93.33**	**90.47**	**95.04**

Table 3: Quantitative Evaluation of F1-score

Methods	**SUMME**	**TVSUM**
CNN+CSMIK K-means Sahu and Chowdhury [2020]	0.452	0.629
CapsNet Huang and Wang [2019]	0.89	0.87
FCNN Gygli [2018]	0.87	0.85
HSA-RNN Zhao et al. [2018]	0.89	-
CNN+LSTM Khan et al. [2019]	-	0.84
Proposed	**0.89**	**0.92**

RGB channel of the image determines the group degree, with n set to 3. Two keys were created using the RSA exponential modulus function of adaptive asymmetric encryption. These keys can invert each other's encrypted data but cannot decrypt their own data. The technique offers a high degree of protection for the key-frames by hiding all features of the original key-frames as shown in figure 5. The technique is extremely vulnerable to the secret key as value obtained by function of GL_n(F). As a result, the proposed image cryptosystem does not provide hackers with any helpful information, thus efficiently validating security.

5 Conclusion and Future work

A significant portion of redundant video data is produced as a result of recent improvements in digital networks. Its management, analysis, and transmission are complex. Hence, there is a need of image prioritization. In this study, the informative frames are first extracted using an effective video summarizing technique. The experimental results support our algorithm's superior performance in terms of precision, recall, and f1-score metrics when compared to existing state-of-the-art approaches. The security and privacy of the retrieved key frames during communication are most importance because these are necessary for subsequent analysis. As a result, proposed adaptive RSA to encrypt key-frames before transmission. Here, the exponential modulus that assists in the creation of two keys is calculated using the general linear group.

The algorithm provides a high degree of safety as it hides all the featured infor-

mation. The key drawback of the proposed methodology is that it keeps the visual representation of the decrypted data while encrypting numerous frames at once rather than one-to-one. Therefore, in order to increase the security of overall system, future development will focus on dynamic keys rather than traditional encryption keys.

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Products on Maximal Compact Frames

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Abstract

In topological spaces, many topological properties such as separation properties, paracompactness etc. are preserved under the act of taking product of topological spaces. The well known Tychonoff theorem in topological spaces which states that product of compact spaces is compact. Many of these results can be extended to the "generalized topological spaces", known as locales(frames). According to Tychonoff product theorem for locales, locale product(coproduct of frames) of compact locales(frames) is compact. In this paper, we examine whether the coproduct of maximal compact frames is maximal compact. We examine the case for a finite coproduct and for an arbitrary coproduct of maximal compact frames. Every subframe of a compact frame is compact but a sublocale need not be. We provide a characterization for a sublocale of a maximal compact frame to be a maximal compact sublocale.

Keywords: Frame, locale, spatial frame, maximal compact frame, subframe, sublocale.

2020 AMS subject classifications: 06D22, 54E [1]

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1 Introduction

Garrett Birkhoff, in 1936, pointed out the notion of the comparison of two different topologies on the same basic set. He had done this by ordering these topologies as a lattice under set inclusion. A topological space (X, T) with property R is said to be maximal R if T is a maximal element in the set $R(X)$ of all topologies on the set X having property R with the partial ordering of set inclusions. The set of all topologies sharing a given property may not have a greatest element, but it may have maximal elements.

In topological spaces, a closed subspace of a compact space is compact and a compact subspace of a Hausdorff space is closed. Thus in a compact Hausdorff space, closed subspaces coincide with compact subspaces. "A topological space is maximal compact if and only if its compact subsets are precisely the closed sets", as proved in [12]. Norman Levine named those spaces in which closed subsets coincide with compact subsets as *C-C Spaces*. A detailed analysis of its properties are discussed in [10].

A locale is a categorical extension of a topological space whereas a frame is an object in its opposite category. In 1972, through the paper [3], J.R.Isbell pointed out the need for a seperate terminology for the dual category of frames. The objects of this dual category are named as "locales" by him and they are actually the "generalized spaces". A pointfree analogue of the maximal compact space and its associated properties are discussed in [6]. A maximal compact frame is a compact frame which is not properly contained in any other compact frame. A characterization of the maximal compact frame is that a subframe A of any non-compact frame L is maximal compact if and only if the closed sublocales of A are exactly the compact sublocales of A. It is also proved that a compact Hausdorff frame is maximal compact.

We know that many of the topological properties such as separation properties, paracompactness etc. are preserved under the act of taking product of topological spaces. The well known Tychonoff theorem in topological spaces which states that product of compact spaces is compact is equivalent to the axiom of choice by [9]. Coproducts(products) in frames(locales) are the categorical counterpart of product of topological spaces. Frame theory has the advantage that many results in topology requiring *Axiom of Choice* or some of its variants can be proved without its use. The Tychonoff's theorem for locales is constructively proved by [7] without using the axiom of choice.

In this paper, we examine whether the coproduct of maximal compact frames is maximal compact. We examine the case for a finite coproduct and for an arbitrary coproduct of maximal compact frames. Every subframe of a compact frame is compact but a sublocale need not be. We provide a characterization for a sublocale of a maximal compact frame to be a maximal compact sublocale.

2 Preliminaries

The term frame was coined by C.H. Dowker. A remarkable study on frames was done by them in [1]. A *frame* is a complete lattice L in which the infinite distributive law $a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$ holds for all $a \in L, S \subseteq L$. A map between frames that preserves arbitrary joins and finite meets is called a *frame homomorphism*. Associated with a frame homomorphism $h : M \to L$ is its right adjoint $h_* : L \to M$ given by $h_*(b) = \bigvee \{x \in M : h(x) \leq b\}$. We denote the top element and the bottom element of a frame by 1 and 0 respectively. The category of frames and frame homomorphisms is denoted by **Frm**. The dual category $\mathbf{Frm}^{op}$ is referred to as the category of locales denoted by **Loc**. The morphisms in **Loc**, called *localic maps*, are given by the right adjoints of frame homomorphisms between two objects. A frame is said to be *spatial*, if it is isomorphic to the topology ΩX of a topological space $(X, \Omega X)$.

A subset of a frame which is closed under arbitrary joins and finite meets in that frame is called a *subframe*. A *sublocale* M of a locale L can be represented in terms of an onto frame homomorphism $h : L \to M$ in the sense that the image of M under the right adjoint $h_* : M \to L$ will represent that sublocale. For a locale L, denote $\uparrow a = \{x \in L : x \geq a\}$ and $\downarrow b = \{x \in L : x \leq b\}$. Then the sublocale given by the frame homomorphism $j : L \to \uparrow a$ defined by $x \to a \vee x$ for any $a \in L$ is called a *closed sublocale* of L. A *cover* in a frame L is a subset S of L with $\bigvee S = 1_L$. A frame L is said to be *compact* if each cover A of L has a finite subcover.

The construction of coproducts in frames which was first presented in [1]. The construction is provided below through the following definitions.

Definition 2.1. *Let $R \subseteq A \times A$ be an arbitrary binary relation on a frame A. An element $s \in A$ is* R-saturated *if $aRb \Rightarrow (a \wedge c \leq s \Leftrightarrow b \wedge c \leq s)\ for all\ a, b, c \in A$*

Let A/R denotes the set of all saturated elements of A. Define $\nu : A \to A/R$ by $\nu(a) = \nu_R(a) = \bigwedge \{s \in A : a \leq s\}$ where s is saturated. Then ν is a surjective frame homomorphism. For a semilattice A, define $\mathcal{D}(A) = \{U \subseteq A : \phi \neq U = \downarrow U\}$. Then $(\mathcal{D}(A), \subseteq)$ is a frame. Define $\lambda_A : A \to \mathcal{D}A$ by $\lambda_A(a) = \downarrow a$ which is a semilattice homomorphism between them. Let $A_i, i \in I$ be frames. Set $\prod_{i\in I}^{'} A_i = \{(a_i)_{i\in I} \in \prod_{i\in I} A_i : a_i = 1$ for all but finitely many i $\} \bigcup \{(0)_{i\in I}\}$ Define $\gamma_j : A_j \to \prod_{i\in I}^{'} A_i$ by setting

$$(\gamma_j(a))_i = \begin{cases} a & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$$

Consider the frame $\mathcal{D}(A)$ where $A = \prod_{i\in I}^{'} A_i$.
$R = \left\{(\lambda_A\gamma_j(\bigvee_{m\in M} a_m), \bigvee_{m\in M} \lambda_A\gamma_j(a_m)) : j \in I, a_m \in A_j\right\}$ where M is any set is a relation.

Definition 2.2. *The frame $\bigoplus_{i\in I} A_i = \mathcal{D}(\prod'_{i\in I} A_i)/R$, containing all R-saturated elements of the frame $\mathcal{D}(\prod'_{i\in I} A_i)$ is called the* frame coproduct.

Definition 2.3. *The mapping $\nu : \mathcal{D}(\prod'_{i\in I} A_i) \to \bigoplus_{i\in I} A_i$ be as defined above. The maps $p_j = \nu \circ \lambda \circ \gamma_j : A_j \to \bigoplus_{i\in I} A_i$ are frame homomorphisms, called* coproduct injections. *Also $\bigwedge_{i\in I} p_i(a_i) = \oplus(a_i)_{i\in I}$. Thus the set of all elements of the form $\oplus(a_i)_{i\in I}$ is a join basis of $\oplus(A_i)_{i\in I}$. The right adjoint of p_j denoted by ${p_j}^*$ is called the* projection *of the locale $\bigoplus_{i\in I} A_i$ to the locale A_j.*

A frame A is called a *Hausdorff frame* if for any $U \in A \oplus A$, the codiagonal $\nabla : A \oplus A \to A$ defined by

$$\nabla(U) = \bigvee \{a \wedge b : (a, b) \in U\}$$

is a closed sublocale.

The frame L is said to be a *regular* frame if $a = \bigvee \{x \in L/x \prec a\}$ for all $a \in L$.
For a detailed reading regarding frames we refer to [11].

3 Maximal Compact Frames and Frame coproducts

An intense research has been done on maximal and minimal topologies so far. The properties of maximal topologies for compactness, countable compactness, and sequential compactness etc. are investigated in [2] and the relations between these spaces are investigated. The paper also presented discussions on some interesting product theorems. Analogously maximal compactness in the pointfree counterpart is studied by [6]. A maximal compact frame is defined here as given below

Definition 3.1. *[6] A frame A is said to be* maximal compact *if,*

1. *A is compact,*
2. *if A is a proper subframe of the frame L, then L is not compact.*

A characterization for maximal compact frames is obtained in [6].

Theorem 3.1. *[6] A frame is maximal compact if and only if its closed sublocales are exactly the compact sublocales.*

For a topological space X, the frame of open sets is denoted by ΩX. A relation between a C-C space and the associated frame of open sets is established in the following result.

Theorem 3.2. *[6] The topological space $(X, \Omega X)$ is a C-C space if and only if ΩX is a maximal compact frame.*

A compact Hausdorff topological space is regular and the same is true locales too.

Theorem 3.3. *[11] A compact Hausdorff locale is regular.*

The property of being a C-C spaces is productive if the component spaces are Hausdorff. The product Hausdorff C-C spaces are C-C and vice-versa by the following result.

Theorem 3.4. *[10] Let (X, τ) be a topological space and let $(X \times X, \tau^*)$ be the cartesian product of (X, τ) with itself. Then $(X \times X, \tau^*)$ is C-C if and only if (X, τ) is C-C and Hausdorff.*

The following three results regarding productive properties of compact frame as well as regular frames are proved in [8]. They are as expected as one learned in topological spaces.

Theorem 3.5. *A compact regular frame(locale) is spatial.*

Theorem 3.6. *The coproduct(product) of compact frames(locales) is compact.*

Theorem 3.7. *The coproduct(product) of regular frames(locales) is regular.*

The frame coproduct of component spaces is isomorphic to frame of open sets of the product topology of these spaces.

Theorem 3.8. *[11] Let $X_i, i \in I$ be family of spaces. Then $\oplus_i(\Omega X_i)$ is isomorphic to $\Omega(\oplus X_i)$ if and only if it is a spatial frame.*

3.1 Coproducts on Maximal Compact Frames

Most of the topological properties are preserved under the act of taking product. The productive properties of *C-C Spaces* are discussed and a characterization is obtained in [10]. The paper also provides some results associated with the productive properties of *C-C Spaces*. The previous section contains characterization for a maximal compact frame and some results associated with it. In this section, we are conducting a study on productive properties of maximal compact frames.

The inverse image of a compact sublocale of a compact frame under the localic projection map in the coproduct a given nonempty family of nonempty frames is again compact by the following result.

Theorem 3.9. *Let $\{A_i : i \in I\}$ be a nonempty family of nonempty compact frames and let A be the frame coproduct. If C_j is a compact sublocale of A_j, then ${p_j}^{*-1}(C_j)$ is compact in A.*

Proof. Take $A_j = C_j$ in $\bigoplus_{i\in I} A_i$, then we have ${p_j}^*(\bigoplus_{i\in I} A_i) = C_j$. Thus ${p_j}^{*-1}(C_j) = \bigoplus_{i\in I} A_i$ where $A_j = C_j$. Since C_j is a compact sublocale and all the other A_is are compact, by Tychonoff theorem for locales, $\bigoplus_{i\in I} A_i$ is compact. Hence ${p_j}_*^{-1}(C_j)$ is compact in A. □

Theorem 3.10. *A locale L is compact if and only if the product projection $p^* : L \oplus M \to M$(the coproduct injection $p : M \to L \oplus M$ in frame language) is closed for every locale M*

The above result is known as the Kuratowski-Mrowka Theorem for locales[11] and is used for proving the next result..

Theorem 3.11. *Let $\{A_i : i \in I\}$ be a non empty family of non empty compact frames and let A be the frame coproduct. If A is a maximal compact frame, then each A_i is a maximal compact frame.*

Proof. If C_j is closed in A_j, then it is compact as a closed sublocale of a compact frame is compact. Now suppose that C_j is a compact sublocale of A_j. Then by *Theorem* 3.9, $C = {p_j}^{*-1}(C_j)$ is compact in A and hence it is closed in A as A is a CCE Frame. Then ${p_j}^*(C) = C_j$ and since A is compact by Tychonoff theorem for locales C_j is closed as the projections of A to A_i being closed maps by*Theorem* 3.10. Hence each A_i is a maximal compact frame. □

The converse of the above result need not be true. We prove this through the next theorem which tells that the coproduct of a compact frame A with itself is a maximal compact frame if and only if A is a maximal compact frame that is Hausdorff. Hence, if the condition Hausdorffness is dropped, then the coproduct may not be a maximal compact frame.

Theorem 3.12. *Let A be any compact frame and let $A \oplus A$ be the coproduct of A with itself. Then $A \oplus A$ is a maximal compact frame if and only if A is maximal compact and Hausdorff.*

Proof. Suppose that $A \oplus A$ is a maximal compact Frame. Then A is a maximal compact frame by *Theorem* 3.11. Let $\Delta(U) = \bigvee \{a \wedge b : (a, b) \in U\}$ where $U \in A \oplus A$. Then $\Delta : A \oplus A \to A$ called the codiagonal is a surjective frame homomorphism and hence its right adjoint Δ^* is a sublocale map by which A is a sublocale of $A \oplus A$. Since A is a maximal compact frame, it is compact. Hence A is a compact sublocale of $A \oplus A$. Since $A \oplus A$ is a maximal compact frame, A is

closed in $A \oplus A$, by *Theorem* 3.1. Thus A is Hausdorff as the diagonal Δ embeds A as a closed sublocale of $A \oplus A$, by definition of a Hausdorff frame.

Assume that A is a maximal compact frame and Hausdorff. A compact Hausdorff frame is regular by *Theorem* 3.3. Since a compact regular frame is spatial by *Theorem* 3.5, A is spatial. Then the topological space corresponding to A is a C-C space by *Corollary* 3.2. Also the space corresponding to a regular frame is regular and hence Hausdorff. Thus A is a C-C space that is Hausdorff and hence by *Theorem* 3.4, the product topological space is C-C. Then by *Corollary* 3.2, the product topology is a maximal compact Frame. Since A is compact and regular, by *Theorem* 3.6 and *Theorem* 3.7, $A \oplus A$ is compact and regular. Then, by *Theorem* 3.5, $A \oplus A$ is spatial. Now, by *Theorem* 3.8, $A \oplus A$ is isomorphic to the product topology. Hence $A \oplus A$ is a maximal compact frame. □

We know that every subframe of a compact frame is compact. But every sublocale of a compact locale need not be compact. It happens when the sublocale becomes a closed sublocale. We know that a sublocale is different from a subframe. A sublocale is a quotient frame and hence it cannot be regarded as a subframe of a frame. Hence a maximal compact frame can have a sublocale which in its own respect may become a maximal compact frame. In the next theorem, we prove that the above situation occurs when the sublocale is closed.

Theorem 3.13. *Let A be a maximal compact frame. A sublocale K of A is maximal compact if and only if K is closed in A.*

Proof. Suppose that the sublocale K of A is maximal compact. Then it is a compact sublocale of A and hence closed, as A is a maximal compact frame.

Conversely, suppose that K is a closed sublocale of A. Then K is a compact sublocale of A as it is maximal compact. Thus any closed sublocale of K is compact. Now assume that K_1 is a compact sublocale of K. Then it is a compact sublocale of A. Since A is a maximal compact Frame K_1 is closed in A. Therefore $K_1 = \uparrow_A a$ where $a \in A$. Since K_1 is a sublocale of K, we have $\uparrow_A a = \uparrow_K a$. Hence K_1 is closed in K. Thus K is a maximal compact locale. □

4 Conclusions

As discussed above, the study of productive properties in topological spaces were extensively studied in [2]. But the extension of these results to the "generalized spaces" were not done to that extent. At first we characterized the maximal compact frames in [6] and consequently took the first step to examine the productive property of maximal compact frames. We conclude that if an arbitrary coproduct of compact frames is maximal compact then each component frame is

maximal compact and the converse need not hold. We also examined the same result in finite case and observed that Hausdorffness of the component frames is necessary for the two way existence of the result.

A frame is said to be *reversible*[5], if every order preserving self bijection is a frame isomorphism. A characterization for reversible frames is given in [5]. It is also proved that a frame that is maximal or minimal with respect to some frame isomorphic property is reversible. Hence the characterization for maximal compact frames can be used as a method to identify reversible frames . Thereby one can get some information about the automorphism groups possible on finite frames that are maximal compact[4]. The possible automorphism groups on infinite frames is still an open problem.

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Feebly r-clean ring and feebly $*$-r-clean ring

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Abstract

In this article, we introduce the concept of feebly r-clean ring and feebly $*$-r-clean ring. A ring R is defined to be feebly r-clean, if every element a can be written as $a = r + e - f$, where u is a regular and e, f are orthogonal idempotents and A $*$ ring R is defined to be feebly $*$-r clean if every element a can be written as $a = r + p - q$, where r is regular element and p, q are orthogonal projections. Further we generalise this concept of feebly r-clean ring to feebly $g(x) - r$-clean ring, where $g(x) \in C(R)[x]$ and $C(R)$ is the centre of ring R.
Keywords: r-clean rings, feebly r-clean ring, $*$-clean ring, feebly $*$-clean ring, $*$-r- feebly clean ring.
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1 Introduction and preliminaries

Throughout this paper, all rings are assumed to be associative with identity. As defined by Nicholson [1977], an element a in a ring R is clean, if a can be written as sum of an idempotent and a unit. A ring R is clean ring, if every element is clean. Arora and Kundu [2017] defined, an element a of a ring R is called feebly clean if $a = u + e - f$ where u is a unit of R and e, f are orthogonal idempotents of R. A ring R is feebly clean ring if every element of R is feebly clean. Recall that, an element r of a ring R is a regular (Von Neumann), if there exists $y \in R$ such that $r = ryr$. E. Osba and Alkam [2004] defined, a ring R is Von Neumann local ring if for every $a \in R$, at least one of a or $1 - a$ has a Von Neumann inverse. Ashrafi and Nasibi [2013a] defined, an element a of a ring R is r-clean if $a = r + e$ where r is a regular element of R and e is an idempotent of R. A ring R is r-clean

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ring if every element of a ring R is r-clean. Motivated by these ideas we define an element a of a ring R as feebly r-clean if a can be written as $a = r + e - f$ where r is regular element of R and e, f are orthogonal idempotents of R. A ring R is feebly r-clean ring if every element of a ring R is feebly r-clean. For $\emptyset \neq S = \{0, 1\} \subseteq Id(R)$, R is S-feebly r-clean ring if each $a \in R$ can be written as $a = r + e - f$, where r is regular and e, f are orthogonal idemptents from S. In this Paper, we introduce the family of feebly r-clean rings and feebly $*$-r-clean rings. we prove that direct product of feebly r-clean ring is r-clean ring and discussed the properties of feebly r-clean rings with examples. Further we prove that, let $M =_B M_A$ be a bi-module and $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ is a formal triangular matrix is feebly r-clean, then A and B are feebly r-clean ring. We show that incase of an abelian ring this concept of feebly r-clean rings coincides with that of feebly clean rings and hence with that of exchange rings. Also we generalise these results to $g(x)$-r-clean rings. Further we discuss some interesting properties of feebly $g(x)$-r-clean ring, where $g(x)$ is in the center of a ring R.

Recall that, a ring $R[[x, \alpha]]$ is the ring of skew formal power series over R, which means, all formal power series in x with coeffients from R and α is an ring endomorphism. Multiplication is defined by $xr = \alpha(r)r$, for all $r \in R$. In particular $R[[x]] = R[[x; I_R]]$ is the ring of formal power series over R. we prove that, let R be an abelian ring with identity, if R is feebly r-clean, then $R[[x]]$ is feebly r-clean. Consider a ring $R[I, J]$ constructed in Tamer Kosan and Zhou [2016], for any subring J of a ring I, we prove the result, which gives the relation between $R[I, J]$ with I and J about feebly r-cleanness and prove that, if R is a feebly r-clean ring with no non-trivial idempotents, then the center of R is also a feebly r-clean ring.

Next, we introduce the concept of feebly $*$-clean ring and feebly $*$-r-clean ring. Recall that, A ring R is $*$-ring, if there exists an operation $*$:$R \to R$ such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$, for all $x, y \in R$. An element p of a $*$-ring is projection if $p^2 = p = p^*$. Obviously 0 and 1 are projections of $*$-ring. We define an element a in a $*$-ring R is feebly $*$-clean element, if $a = u + p - q$ where $u \in U(R)$ and p, q are orthogonal projections of R. A ring $*$-ring R is feebly $*$-clean if every element of $*$-ring is feebly $*$-clean. Finally, we define an element a in a $*$-ring R is feebly $*$- r-clean element, if $a = r + p - q$ where $r \in Reg(R)$ and p, q are orthogonal projections of R. A ring $*$-ring R is feebly $*$-r-clean if every element of $*$-ring is feebly $*$-r-clean. Further properties of feebly $*$-r-clean rings are studied, some of them analogous to those for feebly $*$-clean rings.

For a ring R, the set of units, the set of idempotents, the set of regular elements, the set of projections and the centre of a ring are denoted by $U(R)$, $Id(R)$, $Reg(R)$, $P(R)$ and $C(R)$, respectively.

2 Feebly r-clean ring

definition 2.1. *An element x in a ring R is called feebly r-clean if there exist a regular element $r \in Reg(R)$ and orthogonal idempotents e, $f \in Id(R)$ such that $x = r + e - f$. A ring R is called feebly r-clean ring if every element of R is feebly r-clean. An element x of a ring R is called $S = \{0, 1\}$ feebly r- clean if $x = r + e - f$, where r is a regular and e, f are orthogonal idempotents from $S = \{0, 1\}$.*

Proposition 2.1. *Every homomorphic image of feebly r-clean ring is feebly r-clean.*

Theorem 2.1. *Let $\{R_i\}$ be a family of rings. Then the direct product $R = \prod R_i$ is feebly r-clean ring if and only if each R_i is feebly r-clean ring ring*

Proof. ($\Rightarrow$) This is immediate from the homomorphic image of a idempotent (resp., regular) is a idempotent (resp., regular).

($\Leftarrow$) Suppose each R_i is feebly r-clean. Let $a = (a_i) \in \prod R_i$. For each i, there exist $r_i \in Reg(R_i)$ and orthogonal idempotents $e_i, f_i \in Id(R_i)$ such that $a_i = r_i + e_i - f_i$. Then $a = r + e - f$, where $r = (r_i) \in Reg(\prod R_i)$ and $e = (e_i), f = (f_i)$ are orthogonal idmepotents of $\prod R_i$. Hence $\prod R_i$ is feebly r-clean ring. □

Lemma 2.1. *Let R be a ring with no zero divisor. Then R is feebly clean if and only if R is feebly r-clean.*

Proof. ($\Rightarrow$) Suppose R is a feebly clean ring. For $a \in R$, then there exist $u \in U(R)$ and orthogonal idempotents $e, f \in Id(R)$ such that $a = u + e - f$. Since $u \in Reg(R)$, hence R is feebly r-clean ring.

($\Leftarrow$) Let R be a feebly r-clean ring. For $a \in R$, there exist $r \in Reg(R)$ and orthogonal idempotents $e, f \in Id(R)$ such that $a = r + e - f$. Let $r(\neq 0) \in Reg(R)$, then there exists $y \in R$ such that $r = ryr$, which implies $r(1 - yr) = 0$, thus r is a unit. Therefore, R is a feebly clean ring. □

Lemma 2.2. *Let R be a ring and $e^2 = e \in R$. $a = u - f \in eRe$, where $u \in U(eRe)$, $f \in Id(eRe)$. Then there exist element $v \in U(R)$ and $\overline{f} \in Id(R)$ such that $a = v + e - \overline{f}$.*

Proof. Let $u \in U(eRe)$, then there exists $w \in U(eRe)$ such that $uw = e = wu$. Assume $v = u+(1-e)$. Since $v(1-e) = 0 = (1-e)w$, then $v(w+(1-e)) = e = w(1+(1-e))v$. Thus $a-v-e = u+e-f-u-(1-e)-e = -f-(1-e)$. Also $(f+(1-e))^2 = (f+(1-e)$. Hence $a = v+e-\overline{f}$, where $\overline{f} = (f+(1-e))$. □

Proposition 2.2. *If a is feebly clean of a ring R, then $-a$ is also feebly clean in R.*

Proposition 2.3. *Let R be an abelian ring and $a \in R$ is feebly clean element then ae is feebly clean for any idempotent $e \in R$.*

Proof. Let $a \in R$ be a feebly clean element. Then $a = u + e_1 - e_2$, where $u \in U(R)$ and e_1, e_2 are orthogonal of $Id(R)$. Since R is abelian, then $ue \in U(Re)$ and $e_1e, e_2e \in Id(Re)$ with $e_1ee_2e = e_2ee_1e = 0$. Hence $ae = ue + e_1e - e_2e$. Therefore, ae is feebly clean. □

Lemma 2.3. *Let R be an abelian ring such that a and $-a$ both are clean. Then the following hold*

1. *a and $-a$ are feebly clean element of R.*
2. *$a - e$ is feebly clean for any idempotent e of R.*

Proof. (1) Since a is clean element of R, then there exist unit $u \in U(R)$ and idempotent $e \in Id(R)$ such that $a = u + e$. Take $f = 0 \in Id(R)$, then $a = u + e - f$. The similar argument for $-a$, thus $-a = u^{'} + e^{'} - f^{'}$, as we desired.

(2) Let $a = u_1 + e_1$ and $-a = u_2 + e_2$, where $u_1, u_2 \in U(R)$ and $e_1, e_2 \in Id(R)$. Then

$$\begin{aligned} a - e &= (ae + a(1-e)) - e \\ &= ae + a - ae - e \\ &= (a-1)e + a(1-e) \\ &= -(1-a)e + a(1-e) \\ &= -(1 - e_1 - u_1)e + (-u_2 - e_2)(1-e) \\ &= -(1-e_1)e + u_1e - u_2(1-e) - e_2(1-e) \\ &= -\{(1-e_1)e + e_2(1-e)\} + \{u_1e - u_2(1-e)\}. \end{aligned}$$

Therefore, $a - e$ is feebly clean. □

Theorem 2.2. *Let R be an abelian ring. Then R is S-feebly clean if and only if R is S-feebly r-clean.*

Proof. ($\Rightarrow$) This is immediate from feebly clean rings are feebly r-clean.

($\Leftarrow$) Let R be a S-feebly r-clean ring and $x \in R$. So $x = r + e$ or $x = r - f$, where $r = ryr$ for some $y \in R$ and orthogonal idempotents $e, f \in Id(R)$. If $x = r + e$, Assume $yr = e'$, then $(e')^2 = (yr)^2 = yryr = yr = e'$, thus $e' \in Id(R)$. Also

$$\begin{aligned} (ye' + (1-e'))(re' + (1-e')) &= ye're' + ye'(1-e') + (1-e')re' + (1-e')^2 \\ &= ye're' + ye' - ye' + re' - re' + 1 - e' \\ &= yr \\ &= e' + 1 - e' \\ &= 1. \end{aligned}$$

Take $u = (re' + (1 - e'))$, then $u \in U(R)$ and $ue' = r$. Also, $ue' + f' \in U(R)$, where $(1 - e') = f'$. Hence $-r = -(ue' + f') + f'$. Since a and $-a$ are clean, then $a + e$ and $a - e$ are clean. Thus $r + e$ is clean. If $x = r - f$, By Lemma 2.3, $r - f$ is S-feebly clean. Therefore R is S-feebly clean. □

Example 2.1. *For p prime, $R = \mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b\}$ is feebly r-clean ring.*

Example 2.2. *For distinct primes p, q, $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)} = \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b, q \nmid b\}$ is feebly r-clean.*

Example 2.3. *Some feebly r-clean rings are not r-clean. Consider $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} = \{\frac{m}{n} \in \mathbb{Q} \mid p \nmid n, q \nmid n\}$. Here $Id(\mathbb{Z}_{(5)}) = \{0, 1\}$. Take $a \in R$, then $a = \frac{x}{y}$ for some $x, y \in \mathbb{Z}$, where $y \neq 0$, 3 doesnot divides y, 5 doesnot divides y. If $\frac{x}{y}$ is a regular, then $a = \frac{x}{y} = \frac{x}{y} + 0 - 0$, thus a is feebly r clean. If $a = \frac{x}{y}$ is not regular, then $a = \frac{x}{y} - 1$ or $a = \frac{x}{y} + 1$ is regular. Hence $a = \frac{x}{y} - 1 + 1 - 0$ or $a = \frac{x}{y} - 1 + 0 - 1$. Therefore, a is feebly r-clean. But which is not r-clean, because, take $\frac{3}{8} \in R$, then $\frac{3}{8} = \frac{3}{8} + 0$ or $\frac{3}{8} = \frac{-5}{8} + 1$ in both cases $\frac{3}{8}$, $\frac{-5}{8}$ are not units in R.*

Example 2.4. *Let $R = (\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)})[i]$ is feebly r-clean only if $p, q \equiv 3(mod 4)$, where p, q are odd primes.*

Theorem 2.3. *Let R be a ring with no non-trivial idempotents, then R is feebly r-clean if and only if R is Von-Neumann local.*

Proof. ($\Rightarrow$) Suppose R is feebly r-clean ring and 0 and 1 are the only idempotents in R. Then for any $x \in R$, either x or $x - 1$ is regular. Hence R is Von-Neumann local.

($\Leftarrow$) Since R is Von-Neumann local, for $a \in R$, then either a or $1-a$ is regular. If a is regular, then $a = a + 0 - 0$, a is feebly r-clean. If $1 - a$ is regular, then $a - 1$ is also regular, write $a = a - 1 + 1 - 0$. Therefore, a is feebly r-clean. □

Corolary 2.1. *Let R be a feebly r-clean ring with no non-trivial idempotents. Then R is exchange.*

Proof. Since every regular element x of a ring is an exchange element, which means there exists idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Also, von-neumann local rings are exchange rings by Theorem 2.3, as we desired. □

Theorem 2.4. *If R is abelian ring then R is S-feebly r-clean ring if and only if R is an exchange ring*

Proof. ($\Rightarrow$) It is immediate from Corollary 2.1.

($\Leftarrow$) Suppose R is exchange ring. Let $a \in R$, then there exists idempotent $e \in R$ such that $1 - e \in (1 - a)R$. If $e = ax$ we assume $ex = x$ so that $xax = x$.

Since R is abelian ring, then all idempotents are central, then $xa = x(ax)a = xa(ax) = (xa)ax = a(xa)x = ax$. Also $1 - e = y(1 - a)$, where $(1 - e)y = y$ and $y(1 - a) = (1 - a)y$. Then

$$\begin{aligned}
[a - (1 - e)](x - y) &= ax - (1 - e)x - ay + (1 - e)y \\
&= ax - 0 - ay + y \\
&= ax + (1 - a)y \\
&= ax + 1 - e \\
&= ax + 1 - ax \\
&= 1.
\end{aligned}$$

Also, $(x - y)[a - (1 - e)] = 1$. Therefore, R is S-feebly r-clean ring. □

Proposition 2.4. *Let R be an abelian ring with identity. If R is feebly r-clean, then $R[[x]]$ is feebly r-clean.*

Proof. ($\Leftarrow$) Since $R = \frac{R[[x]]}{<x>}$, it is immediate from homomorphic image of a idempotent (resp., regular) is a idempotent (resp., regular).

($\Rightarrow$) Suppose R is feebly r-clean. Let $h = a_0 + a_1x + a_2x^2 + ... \in R[[x]]$. By assumption, let $a_0 = r + e - f \in R$, then $r' = r + a_1x + a_2x^2 + ...$ is regular in $R[[x]]$. Hence $h = u' + e - f$ is feebly r-clean in $R[[x]]$

□

Corolary 2.2. *Let R be an abelain ring and α be an endomorphism of R. Then the following are equivalent*

1. *R is feebly r-clean ring.*
2. *The formal power series ring $R[[x]]$ over R is feebly r-clean.*
3. *The skew power series ring $R[[x; \alpha]]$ over R is feebly r-clean.*

Proposition 2.5. *Let J be a subring of a ring I then $R[I, J]$ is feebly r-clean if and only if I is r-clean and J is feebly r-clean ring.*

Proof. ($\Rightarrow$) Since $R[I, J]$ is feebly r-clean, then $R[I, J] = I \otimes I$ by Theorem 2.1, I is r-clean, Clearly J is feebly r-clean ring by Proposition 2.1.

($\Leftarrow$) Assume I is r-clean and J is feebly r-clean. Let $x = (i_1, i_2, ..., i_n, j_1, J_2, ...) \in R[I, J]$. Since I is feebly r-clean ring, then $j = r + e - f$, where $r \in Reg(R)$ and orthogonal idempotents $e, f \in Id(R)$. If $j = r + e - f$, write $j_i = r_i + e_i - f_i$. Then $x = \overline{r} + \overline{e} - \overline{f}$ where $\overline{r} = (r_1, r_2, ...r_n, e, e, ...)$ is regular element of $R[I, J]$ and $\overline{e} = (e_1, e_2, ...e_n, e, e, ...)$, $\overline{f} = (f_1, f_2, ...f_n, e, e, ...)$ are orthogonal idempotents of $R[I, J]$. □

Note that every abelian semi regular ring is feebly clean and hence feebly r-clean. However the converse is not true as shown by the following example.

Example 2.5. *Let $\mathbb{Q}$ be a field of rational number and $\mathbb{Q}^{'}$ the ring of all rational number with odd denominators. Then by [Ashrafi and Nasibi [2013b], Example 2.7] and Theorem 2.2, $R[\mathbb{Q}, \mathbb{Q}^{'}]$ commutative exchange ring and hence $R[\mathbb{Q}, \mathbb{Q}^{'}]$ is feebly r-clean ring but not semi regular.*

If R is feebly r-clean ring then R/I is feebly r-clean being homomorphic images of feebly r-clean ring. The theorem is partial converse of this statement.

Theorem 2.5. *If I is a regular ideal of ring R with idempotent can be lifted modulo I. Then R is S-feebly r-clean if and only if R/I is S-feebly r-clean.*

Proof. The proof of theorem is follows by [Ashrafi and Nasibi [2013b], Theorem 2.8]. It is enough to prove that for any $a \in R$ if $a + I$ has $a = r - e$ feebly r-clean decomposition in R/I then a is feebly r-clean in R. Let $a+I \in R/I$ has $a = r-e$ feebly r-clean decomposition in R/I, then there exists idempotent $e + I \in R/I$ such that $(a + I) + (e + I) \in Reg(R/I)$, which shows $(a + e) + I \in Reg(R/I)$. Therefore, $((a + e) + I)(x + I)((a + e) + I) = (a + e) + I$ for some $x \in R$, thus $(a + e)x(a + e) - (a + e) \in I$ but I is regular ideal. Hence $(a + e)$ is regular by [Brown and McCoy [1949], Lemma 1]. Since idempotent can be lifted modulo I, Take e is an idempotent of R, as required □

Proposition 2.6. *Let $M =_B M_A$ be a bimodule. If $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ a formal triangular matrix ring is feebly r-clean then A and B are feebly r-clean ring.*

Proof. Let $a \in A$, $b \in B$ and $m \in M$. Take $t = \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \in T$, Then $t = \begin{pmatrix} a & 0 \\ m & b \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ r_2 & r_3 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ e_2 & e_3 \end{pmatrix} - \begin{pmatrix} f_1 & 0 \\ f_2 & f_3 \end{pmatrix}$, where $\begin{pmatrix} r_1 & 0 \\ r_2 & r_3 \end{pmatrix} \in Reg(T)$ and $\begin{pmatrix} e_1 & 0 \\ e_2 & e_3 \end{pmatrix}, \begin{pmatrix} f_1 & 0 \\ f_2 & f_3 \end{pmatrix}$ are orthogonal idempotents of T. Clearly, e_1, f_1 are orthogonal idempotents in A and e_3, f_3 are orthogonal idempotents in B respectively. Also r_1, r_2 regular element in A and B respectively. Then $a = r_1 + e_1 - f_1$ and $b = r_3 + e_3 - f_3$. Hence A and B are both feebly r-clean rings. □

Theorem 2.6. *Let R be a feebly r-clean ring with no non-trivial idempotents. Then the center of R is also a feebly r-clean ring.*

Proof. Let $a \in C(R)$, then there exist regular element $r \in R$ such that either $x = r$ or $x = r + 1$ or $x = r - 1$. If $x = r$, then $r \in Reg(R)$, we write $x = r + 0 - 0$ is feebly r-clean in $C(R)$. If $x = r + 1$, then $x - 1 = r$ is regular

and $x-1 \in C(R)$, hence $x = x-1+1-0$ is feebly r-clean in $C(R)$. If $x = r-1$, then $x+1 = r$, Also $x+1 \in C(R)$, we write $x = x+1+0-1$ is feebly r-clean in $C(R)$. Therefore, R is feebly r-clean. □

3 Feebly $g(x)$-r-clean ring

definition 3.1. *Let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $x \in R$ is called feebly $g(x)$-r-clean if $x = r+e-f$, where $r \in Reg(R)$ and $g(e) = g(f) = 0$. We say that R is feebly $g(x)$-r-clean if every element is feebly $g(x)$-r-clean.*

If $g(x) = x^2-x$, then feebly $g(x)$-r-clean ring is similar to feebly r-clean ring. An element s of R is called root of the polynomial $g(x) \in C(R)[x]$ if $g(s) = 0$. For $g(x) = x^2 - x$, the ring $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is feebly $g(x)$-r-clean ring.

Let R and S be a rings and $\theta : C(R) \to C(S)$ be a ring homomorphism with $\theta(1) = 1$, then θ induces a map $\theta^{'}$ from $C(R)[x]$ to $C(S)[x]$ such that for $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$, $\theta^{'}(g(x)) := \sum_{i=0}^{n} \theta(a_i)x^i \in C(S)[x]$.

Proposition 3.1. *Let $\theta : R \to S$ be a ring epimorphism. If R is feebly $g(x)$-r-clean ring then S is feebly $\theta^{'}(g(x))$-r-clean ring.*

Proof. Let $g(x) = a_o + a_1 + ... + a_n x^n \in C(R)[x]$, then $\theta^{'}(g(x)) = \theta(a_0) + \theta(a_1)x + ... + \theta(a_n)x^n \in C(S)[x]$. Since θ is a ring epimorphism so for any $s \in S$, there exists $x \in R$ such that $\theta(x) = s$. Let $x = r + s_0 - t_0$, where $r \in Reg(R)$ and $g(s_0) = g(t_0) = 0$, as R is feebly $g(x)$-r-clean ring. Now $s = \theta(x) = \theta(r + s_0 - t_0) = \theta(r) + \theta(s_0) - \theta(t_0)$. Clearly $\theta(r) \in Reg(S)$ and

$$\begin{aligned}\theta^{'}(g(\theta(s_0))) &= \theta(a_0) + \theta(a_1)\theta(s_0) + ... + \theta(a_n) + (\theta(s_0))^n \\ &= \theta(a_0 + a_1 s_0 + ... + a_n s_0^n) \\ &= \theta(0) \\ &= 0\end{aligned}$$

Hence S is feebly $\theta^{'}(g(x))$-r-clean ring. □

Theorem 3.1. *Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i \in I}$ be a family of rings. Then $\prod_{i \in 1} R_i$ is feebly $g(x)$-r-clean ring if and only if R_i's are feebly $g(x)$-r-clean ring*

Proof. ($\Rightarrow$) It is immediate from Theorem 2.1.

($\Leftarrow$) Suppose R_i is feebly $g(x)$-r-clean. Let $a = (a_i) \in \prod R_i$. For each i, $a_i = r_i + e_i - f_i$, where $r_i \in Reg(R_i)$ and e_i, f_i are orthogonal idempotent of $Id(R_i)$. Hence $a = r+e-f$, where $r = r_i \in (\prod R_i)$ and $e = e_i, f = f_i \in (\prod R_i)$ with $ef = fe = 0$. Therefore, $\prod R_i$ is feebly $g(x)$-r-clean ring. □

Theorem 3.2. *Let R be a ring and s, $t \in R$, Then R is feebly $(sx^{2n} - tx)$-r-clean ring if and only if R is feebly $(sx^{2n} + tx)$-r-clean ring for $n \in \mathbb{N}$*

Proof. Suppose R is feebly $(sx^{2n} - tx)$-r-clean ring. For each $a \in R$, then $a = r + e - f$ where $r \in Reg(R)$ and e, f are orthogonal idempotent of R with $(se^{2n} - te) = (sx^{2n} - tx) = 0$. Since $a \in R$, $-a = r + e - f$, then $a = (-r) - e + f$ with $s(-e)^{2n} - t(-e) = (se^{2n} + ex) = 0$. Hence R is feebly $(sx^{2n} + tx)$-r-clean. The similar argument work for the converse part. □

Proposition 3.2. *Let R be an abelian ring and $g(x) \in C(R)[x]$. Then if a is $g(x)$-clean element in R, then ae^{n-1} is also $g(x)$-clean element in R for any root e in $g(x)$.*

Proof. Suppose a is $g(x)$-r-clean in R. For $a \in R$, then there exist $r \in Reg(R)$ and orthogonal idempotents $e_1, e_2 \in Id(R)$ such that $a = r + e_1 - e_2$ with $g(e_1) = g(e_2) = 0$. Take any e be any root of $g(x)$ in R, then $ae^{n-1} = re^{n-1} + e_1e^{n-1} - e_2e^{n-1}$, hence $re^{n-1} \in Reg(e^{n-1}Re^{n-1})$. Also $(e_1e^{n-1})^n = e_1^ne^{n-1} = e_1e^{n-1}$ and $(e_2e^{n-1})^n = e_2^ne^{n-1} = e_2e^{n-1}$ with $e_1e^{n-1} = e^{n-1}e_2 = 0$. Therefore, ae^{n-1} is $g(x)$-r-clean in R.

□

4 Feebly $*$-clean rings and feebly $*$-r-clean ring

definition 4.1. *An element x of a $*$-ring R is called feebly $*$-clean if $x = u + p - q$ where $u \in U(R)$ and p, q are orthogonal projection of R. A $*$-ring R is called feebly $*$-clean ring if every element of R is feebly $*$-clean*

Proposition 4.1. *Homomorphic image of feebly $*$-clean ring is feebly $*$-clean.*

Lemma 4.1. *Let R ba a boolean $*$-ring. Then R is S-feebly $*$-clean if and only if $* = 1_R$ is the identity map of R.*

Proof. ($\Rightarrow$) Note that boolean rings are feebly clean. Assume that R is S-feebly-$*$-clean, for $a \in R$, then there exist $u \in U(R)$ and orthogonal projection $p, q \in P(R)$ such that $a = u + p - q$. By Assumption, $a = u + p$ or $a = u - q$, hence $a = 1 + p$ or $a = 1 - q$. Also $1 + p, 1 - q \in P(R)$. Thus $a^* = a$, which shows $* = 1_R$.

($\Leftarrow$) Suppose $* = 1_R$ is the identity map of R. Then every idempotent of R is a projection, as desired. □

Example 4.1. *Some feebly clean rings are not feebly $*$-clean. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with involution $*$ defined by $(x, y)^* = (y, x)$.*

Since $Id(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \{(0,0),(0,1),(1,0),(1,1)\}$ *and* $U(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \{(1,1)\}$, *hence every element of* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *can be written as sum of units and difference of orthogonal idempotents. Therefore,* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *is feebly clean but not feebly* $*$*-clean, because* $(0,1)$ *is cannot be written as sum units and difference of orthogonal projections.*

Corolary 4.1. *Let* R *be a* $*$*-ring. If* $2 \in U(R)$*, the following are equivalent*

1. R is S-feebly clean and every unit of R is self-adjoint.
2. R is S-feebly $*$-clean and $* = 1_R$.

Proof. (1) $\Rightarrow$ (2) Suppose R is S-feebly clean and every unit of R is self-adjoint. Let $a \in R$, then there exist unit $u \in U(R)$ and orthogonal projection $p, q \in P(R)$ sucht that $a = u + p$ or $a = u - q$. Since $1 - 2p, 1 - 2q \in U(R)$, then $(1-2p)^* = (1-2p)$ which implies $p^* = p$ and $(1-2q)^* = (1-2q)$, also $q^* = q$. Note that R be a $*$-ring, If $2 \in U(R)$, then for any $u^2 = 1$, $u^* = u \in R$ if and only if all idempotents are projection. Hence $a^* = (u+p)^* = u + p = a$ and $a^* = (u-q)^* = u - q = a$, as required.

(2) $\Rightarrow$ (1) It is immediate from every S-feebly $*$-clean ring is S-feebly clean. □

For a $*$-ring R, an element $x \in R$ is called self-adjoint square root of 1 if $x^2 = 1$ and $x^* = x$. Also $*$ induces an involution in the power series ring $R[[x]]$ defined by $(\sum_i^\infty a_i x^i)^* = \sum_i^\infty a_i^* x^i$.

Theorem 4.1. *Let* R *be a* $*$*-ring, the following are equivalent*

1. R *is* S*-feebly* $*$*-clean and* $2 \in U(R)$.
2. *Every element of* R *is a sum of unit and a self adjoint square root of* 1 *or an element of the form* $2p + 1$*, where* $p^2 = p = p^*$.

Proof. (1) $\Rightarrow$ (2) Suppose R is S-feebly clean and $2 \in U(R)$. Let $a \in R$, then there exist $u \in U(R)$ and orthogonal projection $p, q \in P(R)$ such that $\frac{a-1}{2} = u+p$ or $\frac{a-1}{2} = u - q$. If $\frac{a-1}{2} = u + p$, then $a = 2u + (1 + 2p)$, where $p^2 = p = p^*$. If $\frac{a-1}{2} = u - q$, then $a = 2u + (1 - 2q)$, where $2u \in U(R)$ and $1 - 2q$ is a self-adjoint square root of 1.

(2) $\Rightarrow$ (1) Let $a \in R$, then $a = u + x$ or $a = u + (2p + 1)$ where $u \in U(R)$, $p \in P(R)$ and x is self-adjoint square root of 1. Thus $1 = u + x$ or $1 = u + (2p+1)$. If $1 = u + x$ then $(1 - u)^2 = x^2 = 1$, which implies $u^2 = 2u$, hence $2 \in U(R)$ by Wang and Cui [2016]. If $1 = u + (2p + 1)$, then $u = -2p$, also $2 \in U(R)$. Next we prove that R is feebly $*$-clean ring. For $a \in R$, $2a + 1 = u + x$ or

$2a + 1 = u + (2p + 1)$ where $u \in U(R)$, x is self-adjoint square root of 1 and $p \in P(R)$. If $2a+1 = u+x$, then $a = \frac{u}{2} - \frac{1-x}{2}$. Also $(\frac{1-x}{2})^2 = \frac{1-x}{2} = (\frac{1-x}{2})^*$ and $2 \in U(R)$, a is feebly $*$ feebly clean. If $2a + 1 = u + (2p + 1)$, then $a = \frac{u}{2} + p$, a is feebly $*$-clean. □

Lemma 4.2. *Let R be a feebly $*$-clean ring. If I is a $*$-invariant ideal of R Then R/I is feebly $*$-clean. In particular, $R/J(R)$ is feebly $*$-clean ring.*

Proof. Since the homomorphic image of a projection (resp., unit) is a projection (resp., unit), hence the result. Next we prove that $J(R)$ is invariant. Take $x^* \in (J(R))^*$, then $x^* \in J(R)$. Also $x \in J(R)$. For any $a \in R$, then $1 - a^*x \in U(R)$. Hence $1 - a^*x = (1 - a^*x)^*$ is unit of R, as we desired. □

Proposition 4.2. *Let R be a $*$-ring. Then $R[[x]]$ is S-feebly $*$-clean if and only if R is S-feebly $*$-clean.*

Proof. ($\Rightarrow$) Suppose $R[[x]]$ is feebly clean, then as a homomorphic copy of $R[[x]]$, R is feebly $*$-clean.

($\Leftarrow$) Let R be a feebly$*$-clean. Take $f(x) = \sum a_i x^i \in R[[x]]$. For $a_0 \in R$, there exist $r_0 \in Reg(R)$ and orthogonal projection $p_0, q_0 \in R$ such that $a_0 = r_0 + p_0 - q_0$. Then $f(x) = \sum a_i x^i = p_0 - q_0 + r_0 + a_0 + a_1x + a_2x^2 + ..$ where $r_0 + a_1x + a_2x^2 + ... \in Reg(R[[x]])$ and $p_0, q_0 \in P(R) \subseteq P(R[[x]])$. Therefore, $R[[x]]$ is feebly $*$-clean.

□

Theorem 4.2. *Let $\{R_\alpha\}$ be a colection of $*$-rings. Then the direct product $R = \prod_\alpha R_\alpha$ is feebly $*$-clean if and only if each R_α is feebly $*$-clean ring.*

Proof. Similar to the proof of Theorem 2.1. □

definition 4.2. *An element x in a ring $*$-ring R is called feebly $*$-r-clean if $x = r + p - q$, where $r \in Reg(R)$ and orthogonal projection p, $q \in P(R)$. A $*$-ring R is called feebly $*$-r-clean ring if every element of R is feebly $*$-r-clean.*

Proposition 4.3. *Let R be an abelian $*$-ring. If a is feebly $*$-r-clean element in R. Then ae is feebly $*$-r-clean, for any $e \in P(R)$*

Proof. Let $a = u + p - q$, where $r \in Reg(R)$ and orthogonal projection p, $q \in P(R)$. For $e \in P(R)$, $ae = ue + pe - qe$. Clearly $ue \in U(eRe)$ and pe, qe are orthogonal projection of $P(eRe)$. Hence ae is feebly $*$-clean. □

Proposition 4.4. *Let R ba an abelian $*$-ring and a be a S-feebly $*$-clean element in R and $e \in P(R)$. If $-a$ is S-feebly $*$-clean Then $a + e$ is also S-feebly $*$-clean.*

Proof. Since a and $a+1$ are S-feebly $*$-clean, then $1-a$ is also S-feebly $*$-clean. For a is S-feebly $*$-clean, then $a = u + p$ or $a = u - q$ where $u \in U(R)$ and orthogonal projection $p, q \in P(R)$. If $a = u + p$, then $1 + a = v + q$ where $v \in U(R)$. Now $a+e = (1+a)e+a(1-e) = (ve+u(1-e))+(ge+f(1-e))$. Thus $ve + u(1 - e) \in U(R)$ and $ge + f(1 - e) \in P(R)$, as we required. The similar way will work for $a = u - q$. Hence the result. □

Lemma 4.3. *Let R be an abelian $*$-ring where every idempotent of the form rx or xr is projection for any regular element r, then r is S-feebly $*$-clean.*

Proof. Let $r \in Reg(R)$ then $r = rxr$, for some $x \in R$. So $p = xr \in P(R)$. Take $e = xr$, then

$$\begin{aligned} e &= xr \\ &= xr + r - r \\ &= xr + r - rxr \\ &= xr + rxr - rxrxr \\ &= xr + rp - rpp \\ &= xr + (1-p)rxr \\ &= (x + (1-p)rx)r \\ &= ar, \end{aligned}$$

where $a = y + (1 - p)rx$. Here $e^2 = e \in P(R)$ and $(1 - e) = (1 - p)(1 - r) = b(1 - r) \in R(1 - r)$, here $y = (1 - p)$. Assume $ea = a$, then $ara = a$. By R is an abelian $*$-ring, $ra = r(ar)a = ra(ra) = (ra)ra = a(ra)r = ar$ we write $1 - e = y(1 - r)$, where $(1 - e)y = y$ and $y(1 - r) = (1 - r)y$. Then $(1 - r)$ is the inverse of $x - (1 - e)$. as we required. □

Theorem 4.3. *Let R be an abelian $*$-clean, where any idempotent of the form $e = rx$ or xr is projection, for any $r \in Reg(R)$. Then R is S-feebly $*$-r-clean if and only if R is feebly $*$-clean.*

Proof. ($\Leftarrow$) This is immediate from S-feebly $*$-clean rings are S-feebly-$*$-r-clean

($\Rightarrow$) Suppose R is S-feebly $*$-r-clean. Let $a \in R$, then there exist $r \in Reg(R)$ and orthogonal projection $p, q \in P(R)$ such that $a = r + p$ or $a = r - q$. For $r \in Reg(R)$, $r = rbr$ for some $b \in R$. Take $e = rb$, then $(re + (1 - e))(be + (1 - e)) = 1 = (be+(1-e)(re+(1-e)) = 1$ and $e \in P(R)$. Thus $u = re+(1-e)$ is a unit and $r = eu$. Set $p' = 1 - e$, then $-(eu + p')$ is a unit and $p' \in P(R)$. Hence $-r = -(ue + p') + p'$ is S-feebly-$*$-clean by Lemma 4.3, r is S-feebly-$*$-clean. By Proposition 4.4, $x = r + p'$ is S-feebly $*$-clean. The similar argument will work for $x = r - p$. Hence the result. □

5 Conclusions

In this article, feebly $*$-r-clean rings are further discussed with an emphasis on their relations with feebly-r-clean rings and we investigate the properties of them with examples. The future scope of this study is to investigate the ideals of a feebly-r-clean rings and feebly $*$-r-clean rings.

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Optimization of fuzzy two-level production inventory system with persuasive exertion-reliant demand

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Abstract

Dynamic manufacturing-inventory system's intelligent production design develops a crucial problem in the business reactivity of indecisions. This investigation examines a two-level production system under fuzzy parameters and decision variables by implementing a pentagonal fuzzy quantity. As designed, the fuzzy system is defuzzified through the graded mean technique, and then the Kuhn-Tucker technique is utilised to obtain the optimum production size and shortage level. The effectual algorithms are established to project an intellectual industrial approach such as optimal production quantity, shortage level, trade group's exertions, and then minimum integrated expected total cost. The evaluation of the proposed fuzzy system is primed to fit numerical examples compared to the crisp system. This suggested fuzzy system is likewise compared with a particular case of the prior model. Numerical illustrations and sensitiveness studies are delivered in the direction of exhibiting the applicability based on the offered procedure and the attained outcomes.

Keywords: Optimal production; Shortage level; Trade group's exertions demand; Pentagonal fuzzy quantity; Kuhn-Tucker method.

2010 AMS subject classification: 94D05, 35Q93.

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‡ Received on March 9, 2023. Accepted on August 10, 2023. Published on September 3, 2023. doi: 10.23755/rm.v48i0.1152. ISSN: 1592-7415. eISSN: 2282-8214. © R. Vithyadevi and K. Annadurai. This paper is published under the CC-BY license agreement.

1. Introduction

An old-style isolated tactic for commercial industries does not work in the current violent and forceful international commercial situation. Advanced tactics enable revenue enhancements, price reductions, and tractability when addressing commercial indecisions. Through supply chain management, that chance can be realised through cooperative associations that exploit anticipated facility stages, decrease inventory prices, and create customer advantages. Additionally, current inventory models presume that all products are good. A specific ratio of products manufactured is in poor condition, regardless of the industry. In order to address the problem of imperfect quality, studies are currently being conducted to improve the inventory model.

Zadeh [13] presented the idea of fuzzy collections as an attempt at developing a set of concepts and techniques for dealing in a systematic way with imprecision objects that are not sharply defined. Dubois and Prade [3] devoted a structured exposition to system-oriented, realistic topics dealing with the fuzzy situation. Zimmermann [7] presented the fuzzy set concept, and its presentation part initially stretches the fundamentals of uncertain mathematics, while the next is concerned with applications of fuzzy set theory. Chen [19] presented a relationship function and the arithmetic operations of fuzzy numbers. Chang et. al. [18] considered the fuzzy state in the traditional inventory by backorder model due to some tangible circumstances. It is simple to fuzzify q, s or together. The common fuzzy number's graded mean defuzzification method was initially hosted by Chen and Hseih [23]. Chen et al. [24] extended the comprehensive L-R type fuzzy quantity through graded mean integration depiction and proved certain ranking possessions. Goyal and Cardenas-Barron [20] dealt with an inventory typical that is framed with a fuzzy differential equation, then the parallel inventory expenditures are also determined by fuzzy Riemann-integration. Yao and Chiang [11] investigated how to obtain the minimum entire price and the optimum order size through defuzzifying the whole price over the centroid process and the same mechanism executed via the signed distance process. Chiang et al. [9] proposed an inventory system to estimate the whole price in the ambiguous logic. Two circumstances of the results are fuzzy loading volume and order volume by way of triangular fuzzy quantities used numerically through the Nedler-Mead procedure.

Kazemi et al. [14] developed an inventory system for shortages in an uncertain environment by hiring binary categories of fuzzy quantities, which are triangular and trapezoidal. The prime solution for the established typical is calculated via the Kuhn-Tucker technique. An inventory typical designed aimed at objects using damaged levels, lack of backorder over fuzzy situations, was investigated by Mahata and Goswami [5]. Guchhait et al. [15] developed an integrating uncertain production rate in flawed production practise. A fuzzy inventory, typical of declining things along with time-oscillating requests, lacks under fully backlogged conditions, as framed by Harish Nagar and Priyanka Surana [8]. The Graded mean technique exists to utilise defuzzifying entire price tasks and the outcomes determined through this method.

Priyan et al. [22] assumed demand rate is depends on sensitive to persuasive actions or trade group initiatives. The inventory system considers manufacture levels as uncertain, but conceivable to define by fuzzy triangular quantity. Kamble [10] discussed the idea of pentagonal uncertain numbers. Established pentagonal fuzzy quantities are considered by the resources of internal math operations by consuming alpha-cut processes. Chakraborty et. al. [1] explored nonlinear membership roles for observing both symmetric and asymmetric pentagonal fuzzy systems. Vithyadevi and Annadurai [17] derived a combined inventory system through gathering cost decline reliant on lead interval in an uncertain situation by hiring a trapezoidal fuzzy number. The optimum result used for the established model is calculated by the Lagrangian method.

Taha [6], Hillier and Lieberman [4] gave uncertain problem-solving methods in operation research. Mathematical programming and stochastic models are introduced in operations research. Kaufmann and Gupta [2] delivered an introduction to fuzzy numbers and the procedures usage and gave the basic definitions and operations with many examples. Cardenas-Barron et al.'s [12] derived inventory manufacture system designed to dual-level transfer sequence involving one manufacturer and one vendor with the demand is profoundly persuasive in the actions of sales group's initiatives. Priyan and Uthayakumar [21] reported a probabilistic imperfect retailer-buyer combined inventory model using the deliberation of worthy investigation faults on the purchaser's and structure price as the principal asset. Joint inventory typical is extended to conclude the optimum solutions of decision variables after the seller towards the retailer in a single production run. Sundara Rajan and Uthayakumar [16] derived the typical economic order size of inventory, which was measured subject to persuasive exertions. They found a unique optimal replenishment schedule.

The residual parts of the paper were assembled as plans. In Section 2, preliminaries are presented. Section 3 hosts notations and assumptions about the typical extension of inventory manufacture. In Section 4, the crisp production-inventory system remains framed. Section 5 examines the fuzzy inventory production system. Section 6 demonstrates the crisp and fuzzy production-inventory system through a numerical sample. Also in this section, a sensitiveness test is delivered. In Section 7, comparative study is offered. Finally, it affords a conclusion and future research directions.

2. Preliminaries

The fuzzy-integrated expected total cost in a two-level production inventory system constructed on the graded mean integration technique and preliminary results are given below:

2.1 Defuzzification for Pentagonal Fuzzy Number (Harish Nagar and Priyanka Surana [8])

In Figure 1, the graded mean integration technique for $\tilde{\alpha}$ is defined by $\tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ as a pentagonal fuzzy number. Then the defuzzification

$$P(\tilde{\alpha}) = \frac{1}{2}\frac{\int_0^1 \frac{h}{2}[\alpha_1 + \alpha_2 + (\alpha_3 - \alpha_1)h + \alpha_4 + \alpha_5 - (\alpha_5 - \alpha_3)h]dh}{\int_0^1 hdh} \tag{1}$$

$$P(\tilde{\alpha}) = \frac{1}{12}(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + \alpha_5). \tag{2}$$

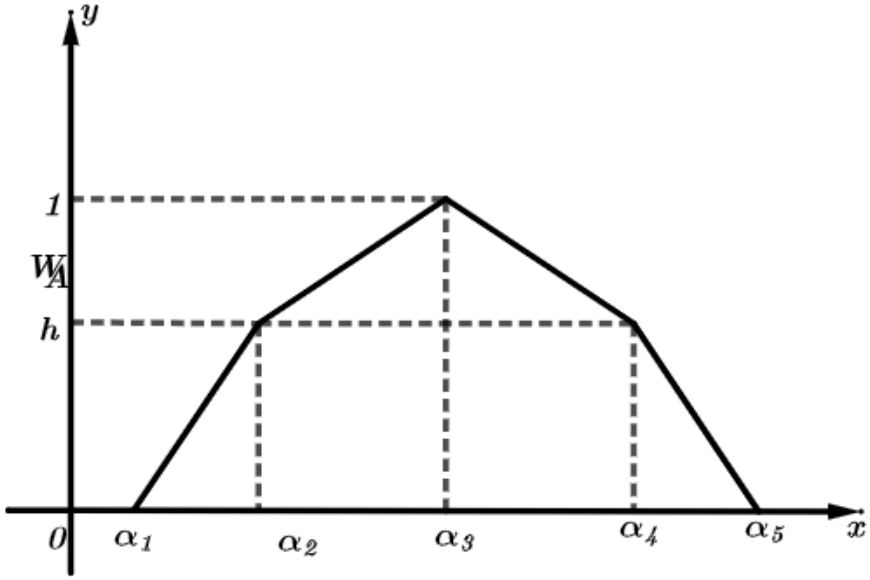


Figure 1: Pentagonal fuzzy number

2.2. Arithmetic operations on pentagonal fuzzy numbers (Kamble [10])

We define some fuzzy arithmetic operations for the pentagonal fuzzy numbers as follows.

Let $\tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ and $\tilde{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ be two pentagonal fuzzy numbers, then

1. Sum of $\tilde{\alpha}$ and $\tilde{\beta}$ indicated in $\tilde{\alpha} \oplus \tilde{\beta} = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3, \alpha_4 + \beta_4, \alpha_5 + \beta_5)$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ are any real numbers.
2. Product of $\tilde{\alpha}$ and $\tilde{\beta}$ indicated in $\tilde{\alpha} \otimes \tilde{\beta} = (\alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3, \alpha_4\beta_4, \alpha_5\beta_5)$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ remain any positive real numbers.
3. If $-\tilde{\beta} = (-\beta_5, -\beta_4, -\beta_3, -\beta_2, -\beta_1)$, then the subtraction of $\tilde{\beta}$ from $\tilde{\alpha}$ is $\tilde{\alpha}\,!\ \tilde{\beta} = (\alpha_1 - \beta_5, \alpha_2 - \beta_4, \alpha_3 - \beta_3, \alpha_4 - \beta_2, \alpha_5 - \beta_1)$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ are any real numbers.

4. $\frac{1}{\tilde{\beta}} = \tilde{\beta}^{-1} = \left(\frac{1}{\beta_5}, \frac{1}{\beta_4}, \frac{1}{\beta_3}, \frac{1}{\beta_2}, \frac{1}{\beta_1}\right)$, where $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ are entirely non-zero positive real numbers, then division of $\tilde{\alpha}$ and $\tilde{\beta}$ is $\tilde{\alpha} \% \tilde{\beta} = \left(\frac{\alpha_1}{\beta_5}, \frac{\alpha_2}{\beta_4}, \frac{\alpha_3}{\beta_3}, \frac{\alpha_4}{\beta_2}, \frac{\alpha_5}{\beta_1}\right)$.

5. Any real number g, $g \otimes \tilde{\alpha} = \begin{cases} (g\alpha_1, g\alpha_2, g\alpha_3, g\alpha_4, g\alpha_5), g \geq 0 \\ (g\alpha_5, g\alpha_4, g\alpha_3, g\alpha_2, g\alpha_1), g < 0. \end{cases}$

2.3. The Kuhn–Tucker Conditions

Theorem 1. Adopt an objective function $\Phi(\Upsilon)$ and the constraints $\mathrm{E}_1(\Upsilon), \mathrm{E}_2(\Upsilon), ..., \mathrm{E}_t(\Upsilon)$, are differentiable satisfying definite regularity situations. Then $\Upsilon^* = (\gamma_1^*, \gamma_2^*, ..., \gamma_s^*)$ is the optimum determination for the nonlinear encoding problem only if there exists t numbers $\lambda_1, \lambda_2, ..., \lambda_t$, such that entirely the following settings are satisfied: $\frac{\partial \Phi}{\partial \gamma_j} - \sum_{i=1}^{t} \lambda_i \frac{\partial \mathrm{E}_i}{\partial \gamma_j} \geq 0$ at $\gamma = \gamma^*$ for $j = 1, 2, ..., s$.

$1. \gamma_j^* \frac{\partial \Phi}{\partial \gamma_j} - \sum_{i=1}^{r} \lambda_i \frac{\partial \mathrm{E}_i}{\partial \gamma_j} = 0$	at $\gamma = \gamma^*$ for $j = 1, 2, ..., s$.
$2. \mathrm{E}_i(\gamma)^* - \beta_i \leq 0$	for $i = 1, 2, ..., t$.
$3. \lambda_i(\mathrm{E}_i(\gamma)^* - \beta_i) = 0$	for $j = 1, 2, ..., s$.
$4. \gamma_j^* \geq 0$	for $i = 1, 2, ..., t$.
$\lambda_i \geq 0$	

3. Notations and assumptions

We adopt the following notations and assumptions to develop the mathematical model:

3.1 Notations

y	Production quantity,
φ	Trade group's exertions (1-point, 2- points, etc.),
K	Shortage level,
$D(\varphi)$	Demand proportion of the goods,
D_f	The principal portion of the demand proportion and that liberated of the trade group's exertions (φ),
D_s	A scale factor of the secondary part of the demand, which differs with the

	trade group's exertions (φ),
n	Price per unit effort of the trade group's exertions,
R	Production proportion,
M	Manufacture's setup cost,
O	Vendor's ordering cost,
H_f	Manufactured item's holding cost per unit/unit period,
C_1	Retrieve cost containing removal cost/unit,
C_3	Shortage cost / unit /unit period,
H_r	Raw material's holding cost /unit / unit period,
r	Flexibility factor,
s	Checking ratio / unit,
C_2	Checking cost / unit,
θ	Distribution parameter $(0 \le \theta \le 1)$ cost of trade group's exertions,
ζ	Probability mass function $f(\zeta)$ within faulty products proportionate probability,
δ	Probability mass function $f(\zeta)$ within faultless products $(\delta = 1-\zeta)$ proportionate probability,
μ_0	Time at adjusting the shortages,
μ_1	Time span for satisfies the demand,
$\Psi(y, K, \varphi)$	Crisp integrated expected total cost per unit quantity,
$\tilde{\Psi}(\tilde{y}, \tilde{K}, \varphi)$	Fuzzy integrated expected total cost per unit quantity.

3.2 Assumptions

1. A single supply chain between manufacturer and vendor is deliberated in addition to denoting a combined system, which produces optimal policies integrated to attain a minimum cost for the supply chain.
2. The manufacturer's shortages are unacceptable, but the vendor's shortages are acceptable and these are entirely backlogged.

4. Mathematical formulation

In the manufacturing inventory system, the rate of demand for the final consumers is specified through the succeeding manifestation $D(\varphi) = D_f + D_s(1-\frac{1}{1+\varphi})$ is associated with Cardenas-Barron and Sana [12].

4.1. Individual total cost for manufacturer

The faultless-condition basic resources inventory cost is $\frac{H_r}{2}\delta y T_1 = \frac{H_r}{2R}\delta^2 y^2$. At the same time the fault-condition basic resources inventory cost is $H_r \zeta y \frac{y}{s} = \frac{y^2 H_r \zeta}{s}$. Furthermore, the completed holding goods inventory cost $\frac{H_f}{2R}\delta^2 y^2$ is obtainable. The fault-condition basic resources removal cost for the manufacturer is $\zeta y C_1$. The checking cost is yC_2. The manufacturer's setup cost is M. The trade group's exertions distribution cost is $n\theta\varphi^r$. The manufacturer's inventory outline is displayed in Figure 2. Hence, the unit quantity's expected total cost, which is the summation of the above-mentioned costs for the manufacturer, is considered as:

$$\Psi_M = \frac{M}{E[\delta]y} + H_r\left(\frac{yE[\delta]}{2R} + \frac{yE[\zeta]}{sE[\delta]}\right) + \frac{H_f yE[\delta]}{2R} + \frac{E[\zeta]C_1}{E[\delta]} + \frac{C_2}{E[\delta]} + \frac{n\theta\varphi^r}{E[\delta]y}. \quad (3)$$

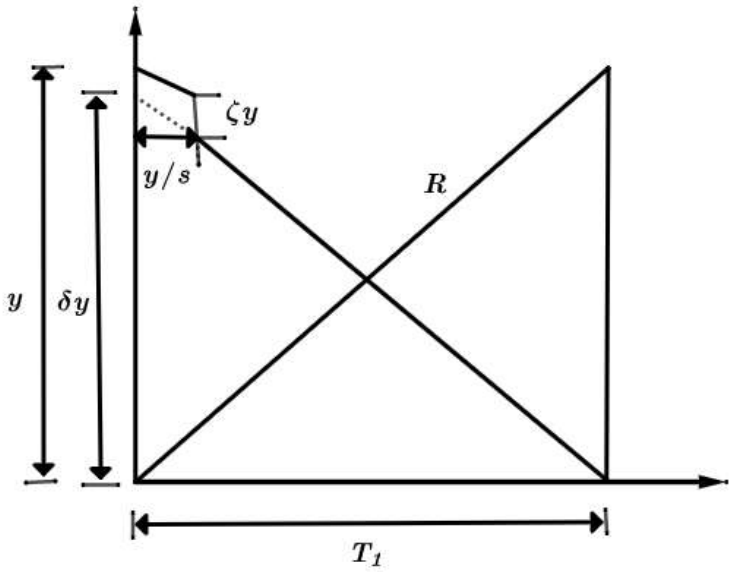


Figure 2: The manufacturer's inventory position versus time.

4.2. Individual total cost for vendor

The holding cost of the vendor is $\frac{1}{2}H_f\{\delta y - K\}\mu_1 = \frac{1}{2D}H_f\{\delta y - K\}^2$, the shortage cost is $\frac{1}{2}KC_3\mu_0 = \frac{1}{2D}C_3K^2$, the vendor's ordering cost is O, and then trade group's

exertions distribution cost is $(1-\theta)n\varphi^r$. In Figure 3, the outline of vendor inventory is exposed. Henceforth, we define the total expected cost per unit quantity, which includes ordering, holding, sharing initiatives with the vendor and shortage costs is specified by

$$\Psi_V = \frac{O}{E[\delta]y} + H_f \frac{\{E[\delta]y - K\}^2}{2DE[\delta]y} + \frac{C_3K^2}{2DE[\delta]y} + \frac{n(1-\theta)\varphi^r}{E[\delta]y}. \tag{4}$$

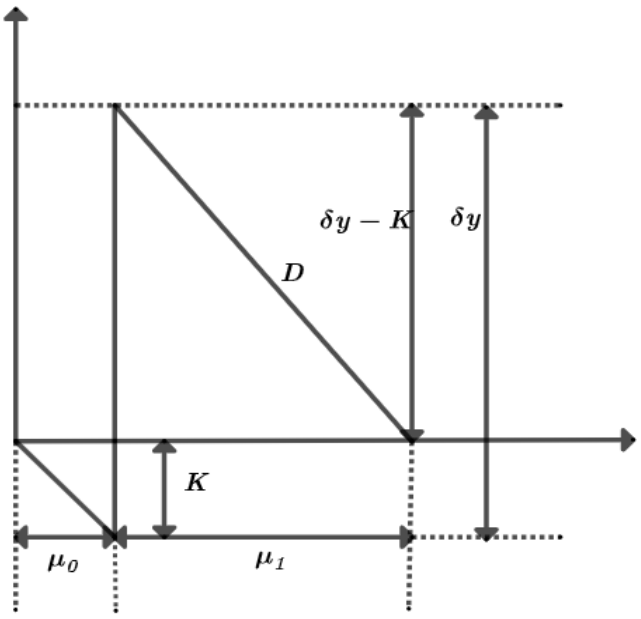


Figure 3: The vendor's inventory position versus time.

4.3. Integrated total cost of the manufacture and vendor

Affording into the assumptions, then the equations (3) to (4) defined above, (Priyan et.al. [22]) the system's integrated expected total cost represented in $\Psi(y,K,\varphi)$ comprise total expected cost of manufacturer Ψ_M and expected total cost of vendor Ψ_V. The crisp system's integrated expected total cost perhaps mathematically expressed as

$$\Psi(y,K,\varphi) = \Psi_M + \Psi_V = \frac{M}{y\delta} + H_r\left(\frac{\delta y}{2R} + \frac{\zeta y}{s\delta}\right) + \frac{H_f\delta y}{2R} + \frac{\zeta C_1}{\delta} + \frac{C_2}{\delta} + \frac{n\theta\varphi^r}{y\delta} + \frac{O}{y\delta} + H_f\frac{\{\delta y - K\}^2}{2D\delta y} + \frac{C_3K^2}{2Dy\delta} + \frac{(1-\theta)n\varphi^r}{y\delta},$$

$$\Psi(y,K,\varphi) = \frac{M+O}{\delta y} + \frac{(H_r+H_f)\delta y}{2R} + \frac{H_r\zeta y}{s\delta} + \frac{(\zeta C_1 + C_2)}{\delta} + \frac{H_f}{2D}\left[\delta y + \frac{K^2}{\delta y} - 2K\right] + \frac{C_3K^2}{2D\delta y} + \frac{n\varphi^r}{\delta y}. \tag{5}$$

Here, the integrated expected total cost $\Psi(y,K,\varphi)$ in which the terms δ and ζ in equation (5) are reformed by their expected values of $E[\delta]$ and $E[\zeta]$ correspondingly.

For the crisp case given in equation (5), the system's integrated total cost is $\Psi(y,K,\varphi)$ if entire basic resources of order lot sizes are faultless $\zeta = 0$.

$$\Psi(y,K,\varphi) = \frac{(M+O)}{y} + \frac{(H_r + H_f)y}{2R} + \frac{H_r y}{s} + C_2 + \frac{H_f}{2D}(y + \frac{K^2}{y} - 2K) + \frac{C_3 K^2}{2D} + \frac{n\varphi^r}{y}. \quad (6)$$

4.4. Inventory model in crisp case

In the first case, we handle the production inventory system with persuasive exertion-reliant demand in a crisp manner.

4.4.1. Solution procedure

A synchronization problem involves minimizing equation (5) and determining optimal values for production lot-size y, backlog K and sales team initiatives φ.

Property: 1 As a static y and φ, $\Psi(y,K,\varphi)$ is convex in K.

Proof. Partial derivatives of $\Psi(y,K,\varphi)$ with respect to K, produce

$$\frac{\partial \Psi(y,K,\varphi)}{\partial K} = -\frac{H_f}{D} + \frac{K(H_f + C_3)}{D\delta y}, \quad (7)$$

and

$$\frac{\partial^2 \Psi(y,K,\varphi)}{\partial K^2} = \frac{(H_f + C_3)}{D\delta y} > 0. \quad (8)$$

And so, for static y and φ, $\Psi(y,K,\varphi)$ is convex in K.

For stable values of K and y, $\Psi(y,K,\varphi)$ is a convex function in φ. In this way, the search for optimal sales initiatives is reduced to an inspect for a locally optimal solution. Besides, by Property 1 $\Psi(y,K,\varphi)$ is subsequently convex in K for specified values φ with y. The distinct optimum based on K (indicated as K^*) feasible to acquire in equation (7) to zero as

$$K^* = (\frac{H_f}{H_f + C_3})\delta y^*. \quad (9)$$

Property: 2 As a static φ, $\Psi(y,K,\varphi)$ is convex in y.

Proof. Partial derivatives of $\Psi(y,K,\varphi)$ with respect to y, produce

$$\frac{\partial \Psi(y,K,\varphi)}{\partial y} = -\frac{(M+O)}{\delta y^2} + \frac{(H_f + H_r)\delta}{2R} + \frac{H_r \zeta}{s\delta} + \frac{H_f}{2D}(\delta - \frac{K^2}{\delta y^2}) - \frac{C_3 K^2}{2D\delta y^2} - \frac{n\varphi^r}{\delta y^2}. \quad (10)$$

Substitute equation (9) in equation (10) then we get,

$$\frac{\partial \Psi(y,K,\varphi)}{\partial y} = -\frac{(M+O)}{\delta y^2} + \frac{(H_f + H_r)\delta}{2R} + \frac{H_r \zeta}{s\delta} + \frac{H_f \delta}{2D} - \frac{{H_f}^2 \delta}{2D(H_f + C_3)} - \frac{n\varphi^r}{\delta y^2} \tag{11}$$

and

$$\frac{\partial^2 \Psi(y,K,\varphi)}{\partial y^2} = \frac{2(M+O+n\varphi^r)}{\delta y^3} > 0. \tag{12}$$

Therefore, for static φ, $\Psi(y,K,\varphi)$ is convex in y.

In addition, by Property 2 $\Psi(y,K,\varphi)$ is convex in y as specified value φ. The distinct optimal production lot size value of y (indicated to y^*) can be derived through equation (11) to zero as

$$y^* = \left\{ \frac{2(M+O+n\varphi^r)}{\delta^2 \left(\frac{(H_r + H_f)}{R} + \frac{2H_r \zeta}{s\delta^2} + \frac{H_f}{D}\left(1 - \left(\frac{H_f}{H_f + C_3}\right)\right)\right)} \right\}^{1/2}. \tag{13}$$

The subsequent algorithm is utilized into determine the optimum results for y, K and φ as the projected crisp case.

Algorithm: 1

Step 1. Fix the trade group's exertions $\varphi = 1$.

Step 2. Compute $y_{(\varphi)}$ by using equation (13).

Step 3. Calculate $K_{(\varphi)}$ beginning at equation (9) through $y_{(\varphi)}$.

Step 4. Find the consequent $\Psi(y_{(\varphi)}, K_{(\varphi)}, \varphi)$ by equation (5).

Step 5. Fix $\varphi = \varphi + 1$, recurrence Steps 2–4 to acquire $\Psi(y_{(\varphi)}, K_{(\varphi)}, \varphi)$.

Step 6. If $\Psi(y_{(\varphi)}, K_{(\varphi)}, \varphi) \geq \Psi(y_{(\varphi-1)}, K_{(\varphi-1)}, \varphi - 1)$, then continue to Step 7 else proceed to Step 5.

Step 7. Fix $(y_{(\varphi-1)}, K_{(\varphi-1)}, \varphi - 1) = (y^*, K^*, \varphi^*)$ then $\Psi(y^*, K^*, \varphi^*)$ is minimum integrated expected total cost, (y^*, K^*, φ^*) is optimum result of crisp inventory model.

5. Fuzzy model

In the second case, we handle the production inventory system with persuasive exertion-reliant demand in a fuzzy manner.

5.1 Solution procedure

The fuzzy integrated inventory model is obtainable through the equation (5); i.e., by fuzzifying the decision variables (y, K, φ) and the input parameters $(C_1, C_2, H_r, H_f, s, O, M, n, R)$.

The fuzzy form of integrated expected total cost in equation (5) is given as

$$\begin{aligned}\tilde{\Psi}(\tilde{y},\tilde{K},\varphi) = {} & ((\tilde{M}\oplus\tilde{O})\%(\delta\otimes\tilde{y}))\oplus(((\tilde{H}_r\oplus\tilde{H}_f)\otimes(\delta\otimes\tilde{y}))\%(2\otimes\tilde{R}))\oplus((\tilde{H}_r\otimes\zeta\otimes\tilde{y})\% \\ & (\tilde{s}\otimes\delta))\oplus(((\zeta\otimes\tilde{C}_1)\oplus\tilde{C}_2)\%\delta)\oplus(((\tilde{H}_f)\%(2\otimes D))\otimes[(\tilde{y}\otimes\delta)\oplus(\tilde{K}^2\%\delta\otimes\tilde{y}) \\ & !\ (\tilde{K}\otimes 2)])\oplus((C_3\otimes\tilde{K}^2)\%(2\otimes D\otimes\delta\otimes\tilde{y}))\oplus((\tilde{n}\otimes\varphi^r)\%(\delta\otimes\tilde{y})),\end{aligned} \tag{14}$$

where $\%, \oplus, !$, and $\otimes$ are the fuzzy arithmetical operatives under function principle. Here, we adopt that each input parameter $\tilde{R} = (R_1, R_2, R_3, R_4, R_5)$. $\tilde{C}_1 = (C_{11}, C_{12}, C_{13}, C_{14}, C_{15})$, $\tilde{O} = (O_1, O_2, O_3, O_4, O_5)$, $\tilde{H}_r = (H_{r1}, H_{r2}, H_{r3}, H_{r4}, H_{r5})$, $\tilde{H}_f = (H_{f1}, H_{f2}, H_{f3}, H_{f4}, H_{f5})$, $\tilde{s} = (s_1, s_2, s_3, s_4, s_5)$, $\tilde{C}_2 = (C_{21}, C_{22}, C_{23}, C_{24}, C_{25})$, $\tilde{M} = (M_1, M_2, M_3, M_4, M_5)$, $\tilde{n} = (n_1, n_2, n_3, n_4, n_5)$, are positive pentagonal fuzzy number including the five modules. Furthermore we adopt the decision variables which are fuzzified affording to the pentagonal rule as: $\tilde{K} = (K_1, K_2, K_3, K_4, K_5)$, $\tilde{y} = (y_1, y_2, y_3, y_4, y_5)$.

We initiate through, fuzzy integrated expected total cost $\tilde{\Psi}(\tilde{y},\tilde{K},\varphi)$ which is given by equation (14). Then

$$\begin{aligned}\tilde{\Psi}(\tilde{y},\tilde{K},\varphi) = \Bigg(& \frac{M_1+O_1}{\delta y_5}+\frac{(H_{r1}+H_{f1})\delta y_1}{2R_5}+\frac{H_{r1}\zeta y_1}{s_5\delta}+\frac{(\zeta C_{11}+C_{21})}{\delta}+\frac{H_{f1}}{2D}[\delta y_1+\frac{K_1^{\,2}}{\delta y_5}-2K_5]+\frac{C_3K_1^{\,2}}{2D\delta y_5}+\frac{n_1\varphi^r}{\delta y_5}, \\ & \frac{M_2+O_2}{\delta y_4}+\frac{(H_{r2}+H_{f2})\delta y_2}{2R_4}+\frac{H_{r2}\zeta y_2}{s_4\delta}+\frac{(\zeta C_{12}+C_{22})}{\delta}+\frac{H_{f2}}{2D}[\delta y_2+\frac{K_2^{\,2}}{\delta y_4}-2K_4]+\frac{C_3K_2^{\,2}}{2D\delta y_4}+\frac{n_2\varphi^r}{\delta y_4}, \\ & \frac{M_3+O_3}{\delta y_3}+\frac{(H_{r3}+H_{f3})\delta y_3}{2R_3}+\frac{H_{r3}\zeta y_3}{s_3\delta}+\frac{(\zeta C_{13}+C_{23})}{\delta}+\frac{H_{f3}}{2D}[\delta y_3+\frac{K_3^{\,2}}{\delta y_3}-2K_3]+\frac{C_3K_3^{\,2}}{2D\delta y_3}+\frac{n_3\varphi^r}{\delta y_3}, \\ & \frac{M_4+O_4}{\delta y_2}+\frac{(H_{r4}+H_{f4})\delta y_4}{2R_2}+\frac{H_{r4}\zeta y_4}{s_2\delta}+\frac{(\zeta C_{14}+C_{24})}{\delta}+\frac{H_{f4}}{2D}[\delta y_4+\frac{K_4^{\,2}}{\delta y_2}-2K_2]+\frac{C_3K_4^{\,2}}{2D\delta y_2}+\frac{n_4\varphi^r}{\delta y_2}, \\ & \frac{M_5+O_5}{\delta y_1}+\frac{(H_{r5}+H_{f5})\delta y_5}{2R_1}+\frac{H_{r5}\zeta y_5}{s_1\delta}+\frac{(\zeta C_{15}+C_{25})}{\delta}+\frac{H_{f5}}{2D}[\delta y_5+\frac{K_5^{\,2}}{\delta y_1}-2K_1]+\frac{C_3K_5^{\,2}}{2D\delta y_1}+\frac{n_5\varphi^r}{\delta y_1}\Bigg).\end{aligned} \tag{15}$$

We defuzzify $\tilde{\Psi}(\tilde{y},\tilde{K},\varphi)$ by equation (2) then get through the graded mean integration technique of $\tilde{\Psi}(\tilde{y},\tilde{K},\varphi)$

$$
\begin{aligned}
P(\tilde{\Psi}(\tilde{y},\tilde{K},\varphi)) = & \left(\frac{1}{12}\left(\frac{M_1+O_1}{\delta y_5}+\frac{(H_{r1}+H_{f1})\delta y_1}{2R_5}+\frac{H_{r1}\zeta y_1}{s_5\delta}+\frac{(\zeta C_{11}+C_{21})}{\delta}+\frac{H_{f1}}{2D}[\delta y_1+\frac{K_1^{\,2}}{\delta y_5}-2K_5]+\frac{C_3K_1^{\,2}}{2D\delta y_5}+\frac{n_1\varphi^r}{\delta y_5}\right)\right.\\
& +\frac{3}{12}\left(\frac{M_2+O_2}{\delta y_4}+\frac{(H_{r2}+H_{f2})\delta y_2}{2R_4}+\frac{H_{r2}\zeta y_2}{s_4\delta}+\frac{(\zeta C_{12}+C_{22})}{\delta}+\frac{H_{f2}}{2D}[\delta y_2+\frac{K_2^{\,2}}{\delta y_4}-2K_4]+\frac{C_3K_2^{\,2}}{2D\delta y_4}+\frac{n_2\varphi^r}{\delta y_4}\right)\\
& +\frac{4}{12}\left(\frac{M_3+O_3}{\delta y_3}+\frac{(H_{r3}+H_{f3})\delta y_3}{2R_3}+\frac{H_{r3}\zeta y_3}{s_3\delta}+\frac{(\zeta C_{13}+C_{23})}{\delta}+\frac{H_{f3}}{2D}[\delta y_3+\frac{K_3^{\,2}}{\delta y_3}-2K_3]+\frac{C_3K_3^{\,2}}{2D\delta y_3}+\frac{n_3\varphi^r}{\delta y_3}\right) \qquad (16)\\
& +\frac{3}{12}\left(\frac{M_4+O_4}{\delta y_2}+\frac{(H_{r4}+H_{f4})\delta y_4}{2R_2}+\frac{H_{r4}\zeta y_4}{s_2\delta}+\frac{(\zeta C_{14}+C_{24})}{\delta}+\frac{H_{f4}}{2D}[\delta y_4+\frac{K_4^{\,2}}{\delta y_2}-2K_2]+\frac{C_3K_4^{\,2}}{2D\delta y_2}+\frac{n_4\varphi^r}{\delta y_2}\right)\\
& \left.+\frac{1}{12}\left(\frac{M_5+O_5}{\delta y_1}+\frac{(H_{r5}+H_{f5})\delta y_5}{2R_1}+\frac{H_{r5}\zeta y_5}{s_1\delta}+\frac{(\zeta C_{15}+C_{25})}{\delta}+\frac{H_{f5}}{2D}[\delta y_5+\frac{K_5^{\,2}}{\delta y_1}-2K_1]+\frac{C_3K_5^{\,2}}{2D\delta y_1}+\frac{n_5\varphi^r}{\delta y_1}\right)\right),
\end{aligned}
$$

where $0<K_1\le K_2\le K_3\le K_4\le K_5$ and $0<y_1\le y_2\le y_3\le y_4\le y_5$. We esteem $P(\tilde{\Psi}(\tilde{y},\tilde{K},\varphi))$ in this way determine for the integrated expected total cost per unit as the fuzzy sense. We can exchange inequality conditions $0\le K_1\le K_2\le K_3\le K_4\le K_5$, $0\le y_1\le y_2\le y_3\le y_4\le y_5$ without changing the meaning of equation (16). Thus, the optimal solution of given in equation (16), subject to the succeeding dissimilarity constraints: $y_4-y_5\le 0,\ y_3-y_4\le 0,\ y_2-y_3\le 0,\ y_1-y_2\le 0,\ -y_1<0$ and $K_4-K_5\le 0,\ K_3-K_4\le 0,\ K_2-K_3\le 0,\ K_1-K_2\le 0,\ -K_1<0.$

An optimum production, shortage level of $P(\tilde{\Psi}(\tilde{y},\tilde{K},\varphi)$ is found by applying the Kuhn–Tucker conditions subject to ten inequalities as imposed conditions. Kuhn–Tucker conditions on Theorem 1 as follows $\lambda\ge 0$,

$$\nabla P(\tilde{\Psi}(\tilde{y},\tilde{K},\varphi)-\lambda\nabla \mathrm{E}(\tilde{y},\tilde{K},\varphi)=0,\lambda \mathrm{E}(\tilde{y},\tilde{K},\varphi)=0,\mathrm{E}(\tilde{y},\tilde{K},\varphi)\le 0.$$

These conditions simplify to the following $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10}\ge 0,$

$$
\begin{aligned}
&\nabla P(\tilde{\Psi}(\tilde{y},\tilde{K},\varphi)-\lambda_1(K_1-K_2)-\lambda_2(K_2-K_3)-\lambda_3(K_3-K_4)-\lambda_4(K_4-K_5)-\lambda_5(-K_1)\\
&-\lambda_6(y_1-y_2)-\lambda_7(y_2-y_3)-\lambda_8(y_3-y_4)-\lambda_9(y_4-y_5)-\lambda_{10}(-y_1)=0, \qquad (17)\\
&\frac{1}{12}\left[\frac{H_{f1}K_1}{D\delta y_5}+\frac{C_3K_1}{D\delta y_5}-\frac{H_{f5}}{D}\right]-\lambda_1+\lambda_5=0, \qquad (18)\\
&\frac{3}{12}\left[\frac{H_{f2}K_2}{D\delta y_4}+\frac{C_3K_2}{D\delta y_4}-\frac{H_{f4}}{D}\right]-\lambda_2+\lambda_1=0, \qquad (19)\\
&\frac{4}{12}\left[\frac{H_{f3}K_3}{D\delta y_3}+\frac{C_3K_3}{D\delta y_3}-\frac{H_{f3}}{D}\right]-\lambda_3+\lambda_2=0, \qquad (20)\\
&\frac{3}{12}\left[\frac{H_{f4}K_4}{D\delta y_2}+\frac{C_3K_4}{D\delta y_2}-\frac{H_{f2}}{D}\right]-\lambda_4+\lambda_3=0, \qquad (21)\\
&\frac{1}{12}\left[\frac{H_{f5}K_5}{D\delta y_1}+\frac{C_3K_5}{D\delta y_1}-\frac{H_{f1}}{D}\right]+\lambda_4=0, \qquad (22)
\end{aligned}
$$

$$\frac{1}{12}\left[-\frac{(M_5+O_5)}{\delta y_1^{\,2}}+\frac{(H_{f1}+H_{r1})\delta}{2R_5}+\frac{H_{r1}\zeta}{s_5\delta}+\frac{H_{f1}\delta}{2D}-\frac{H_{f5}K_5^{\,2}}{2D\delta y_1^{\,2}}-\frac{C_3K_5^{\,2}}{2D\delta y_1^{\,2}}-\frac{n_5\varphi^r}{\delta y_1^{\,2}}\right]-\lambda_6+\lambda_{10}=0 \quad (23)$$

$$\frac{3}{12}\left[-\frac{(M_4+O_4)}{\delta y_2^{\,2}}+\frac{(H_{f2}+H_{r2})\delta}{2R_4}+\frac{H_{r2}\zeta}{s_4\delta}+\frac{H_{f2}\delta}{2D}-\frac{H_{f4}K_4^{\,2}}{2D\delta y_2^{\,2}}-\frac{C_3K_4^{\,2}}{2D\delta y_2^{\,2}}-\frac{n_4\varphi^r}{\delta y_2^{\,2}}\right]-\lambda_7+\lambda_6=0 \quad (24)$$

$$\frac{4}{12}\left[-\frac{(M_3+O_3)}{\delta y_3^{\,2}}+\frac{(H_{f3}+H_{r3})\delta}{2R_3}+\frac{H_{r3}\zeta}{s_3\delta}+\frac{H_{f3}\delta}{2D}-\frac{H_{f3}K_3^{\,2}}{2D\delta y_3^{\,2}}-\frac{C_3K_3^{\,2}}{2D\delta y_3^{\,2}}-\frac{n_3\varphi^r}{\delta y_3^{\,2}}\right]-\lambda_8+\lambda_7=0 \quad (25)$$

$$\frac{3}{12}\left[-\frac{(M_2+O_2)}{\delta y_4^{\,2}}+\frac{(H_{f4}+H_{r4})\delta}{2R_2}+\frac{H_{r4}\zeta}{s_2\delta}+\frac{H_{f4}\delta}{2D}-\frac{H_{f2}K_2^{\,2}}{2D\delta y_4^{\,2}}-\frac{C_3K_2^{\,2}}{2D\delta y_4^{\,2}}-\frac{n_2\varphi^r}{\delta y_4^{\,2}}\right]-\lambda_9+\lambda_8=0 \quad (26)$$

$$\frac{1}{12}\left[-\frac{(M_1+O_1)}{\delta y_5^{\,2}}+\frac{(H_{f2}+H_{r2})\delta}{2R_1}+\frac{H_{r5}\zeta}{s_1\delta}+\frac{H_{f5}\delta}{2D}-\frac{H_{f1}K_1^{\,2}}{2D\delta y_5^{\,2}}-\frac{C_3K_1^{\,2}}{2D\delta y_5^{\,2}}-\frac{n_1\varphi^r}{\delta y_5^{\,2}}\right]+\lambda_9=0 \quad (27)$$

$$K_i-K_{i+1}\le 0, \qquad i=1,2,3,4, \quad (28)$$

$$K_1>0, \quad (29)$$

$$y_i-y_{i+1}\le 0, \qquad i=1,2,3,4, \quad (30)$$

$$y_1>0, \quad (31)$$

$$\lambda_i(K_i-K_{i+1})=0, \qquad i=1,2,3,4, \quad (32)$$

$$\lambda_5(K_1)=0, \quad (33)$$

$$\lambda_{5+i}(y_i-y_{i+1})=0, \qquad i=1,2,3,4, \quad (34)$$

$$\lambda_{10}(y_1)=0, \quad (35)$$

$$K_i\ge 0,\ y_i\ge 0,\ \ i=1,2,3,4,5 \text{ and } \lambda_j\ge 0,\ \ j=1,2,.....,10. \quad (36)$$

Because $K_1>0,$ and $\lambda_5K_1=0,$ then $\lambda_5=0.$ If $\lambda_1=\lambda_2=\lambda_3=\lambda_4=0,$ then $K_5<K_4<K_3<K_2<K_1,$ it does not satisfy the constraints $K_5\ge K_4\ge K_3\ge K_2\ge K_1>0.$ Therefore, $K_5=K_4, K_4=K_3, K_3=K_2, K_2=K_1,$ that is $K_5=K_4=K_3=K_2=K_1=\tilde{K}^*.$ Since $y_1>0,$ and $\lambda_{10}y_1=0,$ then $\lambda_{10}=0.$ If $\lambda_6=\lambda_7=\lambda_8=\lambda_9=0,$ then $y_5<y_4<y_3<y_2<y_1,$ that cannot satisfy the constraints $y_5\ge y_4\ge y_3\ge y_2\ge y_1>0.$ Hence, $y_5=y_4, y_4=y_3, y_3=y_2, y_2=y_1,$ it is $y_5=y_4=y_3=y_2=y_1=\tilde{y}^*.$ Henceforth, from equations (18)–(27), we determine the fuzzy optimum production quantity $\tilde{y}^*$ and shortage level quantity $\tilde{K}^*$ as

$$\tilde{y}^*=\left\{\frac{P+3[S]+4[T]+3[U]+V}{\delta^2(E+3F+4G+3H+I)}\right\}^{1/2}, \quad (37)$$

$$\tilde{K}^*=\left(\frac{H_{f5}\delta y_5+3H_{f4}\delta y_4+4H_{f3}\delta y_3+3H_{f2}\delta y_2+H_{f1}\delta y_1}{(H_{f1}+C_3)+3(H_{f2}+C_3)+4(H_{f3}+C_3)+3(H_{f4}+C_3)+(H_{f5}+C_3)}\right). \quad (38)$$

Where, $P = 2(M_5 + O_5 + n_5\varphi^r)$, $S = 2(M_4 + O_4 + n_4\varphi^r)$, $T = 2(M_3 + O_3 + n_3\varphi^r)$, $U = 2(M_2 + O_2 + n_2\varphi^r)$, $V = 2(M_1 + O_1 + n_1\varphi^r)$, $E = \frac{(H_{r1} + H_{f1})}{R_5} + \frac{2H_{r1}\zeta}{s_5\delta^2} + \frac{H_{f1}}{D}(1 - (\frac{H_{f1}}{H_{f5} + C_3}))$, $F = \frac{(H_{r2} + H_{f2})}{R_4} + \frac{2H_{r2}\zeta}{s_4\delta^2} + \frac{H_{f2}}{D}(1 - (\frac{H_{f2}}{H_{f4} + C_3}))$, $G = \frac{(H_{r3} + H_{f3})}{R_3} + \frac{2H_{r3}\zeta}{s_3\delta^2} + \frac{H_{f3}}{D}(1 - (\frac{H_{f3}}{H_{f3} + C_3}))$, $H = \frac{(H_{r4} + H_{f4})}{R_2} + \frac{2H_{r4}\zeta}{s_2\delta^2} + \frac{H_{f4}}{D}(1 - (\frac{H_{f4}}{H_{f2} + C_3}))$, $I = \frac{(H_{r5} + H_{f5})}{R_1} + \frac{2H_{r5}\zeta}{s_1\delta^2} + \frac{H_{f5}}{D}(1 - (\frac{H_{f5}}{H_{f1} + C_3}))$.

The optimum integrated expected total cost $P(\tilde{\Psi}(\tilde{y}, \tilde{K}, \varphi))$ is found through direct substitution of equations (37) and (38) into equation (16).

It is noted that when the parameters $(C_1, C_2, H_r, H_f, s, O, M, n, R)$ are real numbers, that is, $C_{15} = C_{14} = C_{13} = C_{12} = C_{11} = C_1$, $C_{25} = C_{24} = C_{23} = C_{22} = C_{21} = C_2$, $H_{r5} = H_{r4} = H_{r3} = H_{r2} = H_{r1} = H_r$, $H_{f5} = H_{f4} = H_{f3} = H_{f2} = H_{f1} = H_f$, $s_5 = s_4 = s_3 = s_2 = s_1 = s$, $O_5 = O_4 = O_3 = O_2 = O_1 = O$, $M_1 = M_2 = M_3 = M_4 = M_5 = M$, $n_5 = n_4 = n_3 = n_2 = n_1 = n$, $R_5 = R_4 = R_3 = R_2 = R_1 = R$. Assume that the decision variables (y, K) are real numbers, $y_5 = y_4 = y_3 = y_2 = y_1 = y$, $K_5 = K_4 = K_3 = K_2 = K_1 = K$ then equations (37) and (38) condensed as equations (9) and (13) respectively as

$$K^* = (\frac{H_f}{H_f + C_3})\delta y^*, \tag{39}$$

$$y^* = \left\{ \frac{2(M + O + n\varphi^r)}{\delta^2 \left(\frac{(H_r + H_f)}{R} + \frac{2H_r\zeta}{s\delta^2} + \frac{H_f}{D}(1 - (\frac{H_f}{H_f + C_3})) \right)} \right\}^{1/2}. \tag{40}$$

The subsequent algorithm is utilized into determine the optimum results for $\tilde{y}, \tilde{K}$ and φ as the projected fuzzy inventory case.

Algorithm: 2

Step 1. Fix the trade group's exertions $\varphi = 1$.

Step 2. Compute $\tilde{y}_{(\varphi)}$ by equation (37).

Step 3. Calculate $\tilde{K}_{(\varphi)}$ from equation (38) by $\tilde{y}_{(\varphi)}$.

Step 4. Find the consequent $P(\tilde{\Psi}(\tilde{y}_{(\varphi)}, \tilde{K}_{(\varphi)}, \varphi))$ by equation (16).

Step 5. Fix $\varphi = \varphi + 1$, recurrence Steps 2–4 to acquire $P(\tilde{\Psi}(\tilde{y}_{(\varphi)}, \tilde{K}_{(\varphi)}, \varphi))$.

Step 6. If $P(\tilde{\Psi}(\tilde{y}_{(\varphi)}, \tilde{K}_{(\varphi)}, \varphi)) \geq P(\tilde{\Psi}(\tilde{y}_{(\varphi-1)}, \tilde{K}_{(\varphi-1)}, \varphi - 1))$, then continues Step 7 else proceed Step 5.

Step 7. Fix $(\tilde{y}_{(\varphi-1)}, \tilde{K}_{(\varphi-1)}, \varphi-1) = (\tilde{y}^*, \tilde{K}^*, \varphi^*)$ then $P(\tilde{\Psi}(\tilde{y}^*, \tilde{K}^*, \varphi^*))$ is minimum integrated expected total cost, $(\tilde{y}^*, \tilde{K}^*, \varphi^*)$ is optimum result of projected fuzzy inventory model.

6. Numerical and sensitivity analysis

6.1. Numerical analysis

In the present division, a numerical sample is taken towards demonstrate that exceeding result technique. Let us deliberate the structure through the crisp data used in Priyan et.al. [22] the result towards this sample is acquired through MatLab software. The input of crisp and fuzzy parameters are proceeds: $H_r = \$5$/unit/year, $C_1 = \$15$/unit, $H_f = \$3$/unit/year, $D_s = 10$units/year, $D_f = 100$units/year, $C_3 = \$25$/unit/year, $C_2 = \$0.5$/unit, $s = 175200$, $M = \$400$/setup, $O = \$100$/order, $n = \$1500$/unit, $r = 1$, $\theta = 50\%$, $R = 150$units/year, $\tilde{H}_r = \$(1, 3, 5, 7, 9)$/unit/year, $\tilde{H}_f = \$(1, 2, 3, 4, 5)$/unit/year, $\tilde{C}_1 = \$(5, 10, 15, 20, 25)$/unit, $\tilde{C}_2 = \$(0.1, 0.3, 0.5, 0.7, 0.9)$/unit, $\tilde{s} =$ (154200, 164700, 175200, 185700, 196200), $\tilde{O} = \$(27, 63.5, 100, 136.5, 173)$/unit/year, $\tilde{M} =$ $(230, 315, 400, 485, 570)$/setup, $\tilde{n} = \$(584, 1042, 1500, 1958, 2416)$/unit, $\tilde{R} =$ (65, 107.5, 150, 192.5, 235)units/year, and ζ is random variable as uniformly distributed, $U_D(\alpha, \beta) =$ (0,0.04).

Here $E(\zeta) = (\alpha + \beta)/2 = 0.02$ and $E[(1-\zeta)^2] = E[\delta^2] = \dfrac{(\alpha^2 + \beta^2 + \alpha\beta)}{3} + 1 - \alpha - \beta =$ 0.96053. Currently using the Algorithm 2 technique produces the succeeding optimal solutions for the fuzzy model: $\tilde{y}^* = 211.0$ units, $\tilde{K}^* = 22.4$ units, $\varphi^* =$ 1 point, $P(\tilde{\Psi}(\tilde{y}^*, \tilde{K}^*, \varphi^*)) = \$18.0$/unit, and $P(\tilde{\Psi}(\tilde{y}^*, \tilde{K}^*, \varphi^*)) \times \tilde{y}^* = \$3795.0$. Similarly, we use Algorithm 1 procedure to get the following optimal solutions for crisp model: $y^* =$ 229.8 units, $K^* = 24.1$ units, $\varphi^* =$ 1 point, $\Psi(y^*, K^*, \varphi^*) = \$18.6$/unit, and $\Psi(y^*, K^*, \varphi^*) \times y^* = \$4269.0$.

6.2. Sensitivity analysis

We study the distinct responses of $H_r, H_f, C_1, C_2, s, O, M, n, R, C_3, D_f$, and D_s on the optimal y, K and φ to clarify the crisp and fuzzy cases and algorithms. The sensitiveness study is implemented via varying the parameters $H_r, H_f, C_1, C_2, s,$ O, M, n, R, C_3, D_f, $\tilde{H}_r, \tilde{H}_f, \tilde{C}_1, \tilde{C}_2, \tilde{s}, \tilde{O}, \tilde{M}, \tilde{n}, \tilde{R}, C_3, D_f$, and D_s by +90%, +60%, +30%, –30%, –60%, and –90%. The outcomes for crisp and fuzzy cases are offered in Tables 1 to 3 and the corresponding total cost curves are plotted in Figure 4.

Parameters	changing %	Crisp case					Fuzzy case					Fuzzy case Savings %		
		y^*	K^*	φ^*	Ψ	$\Psi \times y^*$	$\tilde{y}^*$	$\tilde{K}^*$	φ^*	$P(\tilde{\Psi})$	$P(\tilde{\Psi}) \times \tilde{y}^*$	$\tilde{y}^*$	$\tilde{K}^*$	$P(\tilde{\Psi}) \times \tilde{y}^*$
D_f	+90	249.2	26.2	1	17.2	4284.8	225.4	24.3	1	16.7	3764.2	9.6	7.0	12.1
	+60	244.7	25.7	1	17.5	4281.1	222.1	23.9	1	17.0	3771.8	9.2	7.1	11.9
	+30	238.6	25.0	1	17.9	4276.1	217.6	23.2	1	17.4	3781.6	8.8	7.3	11.6
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	216.3	22.7	1	19.7	4257.9	200.5	21.0	1	19.0	3814.2	7.3	7.5	10.4
	−60	192.1	20.2	1	22.1	4238.2	181.2	18.6	1	21.2	3843.5	5.7	7.6	9.3
	−90	134.0	14.1	1	31.3	4190.7	130.7	13.1	1	29.7	3888.3	2.4	7.1	7.2
D_s	+90	231.4	24.3	1	18.5	4270.2	212.2	22.5	1	17.9	3792.7	8.3	7.4	11.2
	+60	230.9	24.2	1	18.5	4269.8	211.8	22.5	1	17.9	3793.5	8.3	7.4	11.2
	+30	230.3	24.2	1	18.5	4269.4	211.4	22.4	1	17.9	3794.3	8.2	7.4	11.1
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	229.3	24.1	1	18.6	4268.5	210.6	22.3	1	18.0	3795.8	8.2	7.4	11.1
	−60	228.7	24.0	1	18.7	4268.1	210.2	22.2	1	18.1	3796.7	8.1	7.4	11.0
	−90	228.2	24.0	1	18.7	4267.6	209.8	22.2	1	18.1	3797.5	8.1	7.4	11.0
O & $\tilde{O}$	+90	234.9	24.7	1	19.0	4456.8	215.7	22.8	1	18.3	3957.9	8.2	7.4	11.2
	+60	233.2	24.5	1	18.8	4394.2	214.1	22.7	1	18.2	3903.7	8.2	7.4	11.2
	+30	231.5	24.3	1	18.7	4331.6	212.6	22.5	1	18.1	3849.4	8.2	7.4	11.1
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	228.1	23.9	1	18.4	4206.3	209.4	22.2	1	17.9	3740.7	8.2	7.4	11.1
	−60	226.3	23.8	1	18.3	4143.7	207.8	22.0	1	17.7	3686.4	8.2	7.3	11.0
	−90	224.6	23.6	1	18.2	4081.0	206.2	21.9	1	17.6	3632.0	8.2	7.3	11.0
R & $\tilde{R}$	+90	278.8	29.3	1	15.5	4308.9	261.1	27.1	1	14.9	3890.4	6.3	7.6	9.7
	+60	266.0	27.9	1	16.2	4298.5	247.8	25.8	1	15.6	3864.4	6.9	7.5	10.1
	+30	250.2	26.3	1	17.1	4285.6	231.5	24.3	1	16.6	3833.2	7.5	7.5	10.6
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	202.4	21.2	1	21.0	4246.5	184.1	19.7	1	20.4	3747.0	9.0	7.2	11.8
	−60	161.9	17.0	1	26.0	4213.5	145.7	15.8	1	25.3	3683.3	10.0	6.9	12.6
	−90	86.3	9.1	1	48.1	4151.8	76.6	8.5	1	46.8	3585.5	11.3	6.3	13.6
M & $\tilde{M}$	+90	249.6	26.2	1	20.1	5019.8	229.2	24.3	1	19.5	4478.0	8.2	7.2	10.8
	+60	243.2	25.5	1	19.6	4769.7	223.3	23.7	1	19.0	4250.5	8.2	7.3	10.9
	+30	236.6	24.8	1	19.1	4519.4	217.2	23.0	1	18.5	4022.9	8.2	7.3	11.0
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	222.8	23.4	1	18.0	4018.4	204.6	21.7	1	17.4	3567.0	8.2	7.4	11.2
	−60	215.6	22.6	1	17.5	3767.6	197.9	20.9	1	16.9	3338.7	8.2	7.5	11.4
	−90	208.1	21.9	1	16.9	3516.6	191.1	20.2	1	16.3	3110.1	8.2	7.6	11.6

Table 1: Changing of parameters $D_f, D_s, O, \tilde{O}, R, \tilde{R}, M$ and $\tilde{M}$ on optimum result.

m e n g -	Crisp case	Fuzzy case	Fuzzy case Savings

												%		
		y^*	K^*	φ^*	Ψ	$\Psi \times y^*$	$\tilde{y}^*$	$\tilde{K}^*$	φ^*	$P(\Psi)$	$P(\Psi)\times\tilde{y}^*$	$\tilde{y}^*$	$\tilde{K}^*$	$P(\Psi)\times\tilde{y}^*$
n & $\tilde{n}$	+90	297.4	31.2	1	23.8	7079.1	273.1	28.9	1	23.0	6278.8	8.2	7.5	11.3
	+60	276.7	29.1	1	22.2	6143.9	254.1	26.9	1	21.5	5452.2	8.2	7.4	11.3
	+30	254.4	26.7	1	20.5	5207.3	233.5	24.7	1	19.8	4624.5	8.2	7.4	11.2
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	202.3	21.2	1	16.5	3328.2	185.8	19.7	1	16.0	2963.4	8.2	7.3	11.0
	−60	170.4	17.9	1	14.0	2383.9	156.5	16.6	1	13.6	2128.3	8.2	7.2	10.7
	−90	131.0	13.8	1	10.9	1433.4	120.3	12.8	1	10.7	1287.4	8.2	7.0	10.2
C_1 & $\tilde{C}_1$	+90	229.8	24.1	1	18.9	4332.3	211.0	22.4	1	18.3	3853.2	8.2	7.4	11.1
	+60	229.8	24.1	1	18.8	4311.2	211.0	22.4	1	18.2	3833.8	8.2	7.4	11.1
	+30	229.8	24.1	1	18.7	4290.1	211.0	22.4	1	18.1	3814.4	8.2	7.4	11.1
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	229.8	24.1	1	18.5	4247.9	211.0	22.4	1	17.9	3775.7	8.2	7.4	11.1
	−60	229.8	24.1	1	18.4	4226.8	211.0	22.4	1	17.8	3756.3	8.2	7.4	11.1
	−90	229.8	24.1	1	18.3	4205.6	211.0	22.4	1	17.7	3736.9	8.2	7.4	11.1
C_2 & $\tilde{C}_2$	+90	229.8	24.1	1	19.0	4374.5	211.0	22.4	1	18.4	3891.9	8.2	7.4	11.0
	+60	229.8	24.1	1	18.9	4339.3	211.0	22.4	1	18.3	3859.6	8.2	7.4	11.1
	+30	229.8	24.1	1	18.7	4304.1	211.0	22.4	1	18.1	3827.3	8.2	7.4	11.1
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	229.8	24.1	1	18.4	4233.8	211.0	22.4	1	17.8	3762.7	8.2	7.4	11.1
	−60	229.8	24.1	1	18.3	4198.6	211.0	22.4	1	17.7	3730.5	8.2	7.4	11.1
	−90	229.8	24.1	1	18.1	4163.4	211.0	22.4	1	17.5	3698.2	8.2	7.4	11.2
s & $\tilde{s}$	+90	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	+60	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	+30	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−60	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−90	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
C_3	+90	227.9	13.3	1	18.7	4267.4	209.2	12.3	1	18.1	3792.1	8.2	7.3	11.1
	+60	228.3	15.6	1	18.7	4267.7	209.6	14.5	1	18.1	3792.7	8.2	7.3	11.1
	+30	228.9	19.0	1	18.6	4268.2	210.1	17.6	1	18.1	3793.6	8.2	7.3	11.1
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	231.5	33.2	1	18.4	4270.3	212.5	30.7	1	17.9	3797.9	8.2	7.4	11.1
	−60	235.1	53.2	1	18.2	4273.3	216.1	49.2	1	17.6	3805.5	8.1	7.5	10.9
	−90	250.6	133.9	1	17.1	4285.9	233.8	123.7	1	16.5	3861.4	6.7	7.7	9.9

Table 2: Changing of parameters $n, \tilde{n}, C_1, \tilde{C}_1, C_2, \tilde{C}_2, s, \tilde{s}$ and C_3 on optimum result.

Parameters	changing %	Crisp case					Fuzzy case					Fuzzy case Savings %		
		y^*	K^*	φ^*	Ψ	$\Psi \times y^*$	$\tilde{y}^*$	$\tilde{K}^*$	φ^*	$P(\tilde{\Psi})$	$P(\tilde{\Psi}) \times \tilde{y}^*$	$\tilde{y}^*$	$\tilde{K}^*$	$P(\tilde{\Psi}) \times \tilde{y}^*$
H_r & $\tilde{H}_r$	+90	195.6	20.5	1	21.7	4241.0	177.4	19.1	1	21.0	3728.1	9.3	6.9	12.1
	+60	205.3	21.6	1	20.7	4248.9	186.7	20.0	1	20.1	3746.3	9.0	7.1	11.8
	+30	216.5	22.7	1	19.7	4258.1	197.8	21.1	1	19.1	3768.2	8.7	7.2	11.5
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	245.9	25.8	1	17.4	4282.1	227.3	23.9	1	16.8	3829.0	7.6	7.5	10.6
	−60	266.0	27.9	1	16.2	4298.5	248.2	25.8	1	15.6	3873.4	6.7	7.7	9.9
	−90	292.0	30.7	1	14.8	4319.7	276.0	28.3	1	14.3	3934.5	5.5	7.8	8.9
H_f & $\tilde{H}_f$	+90	189.8	34.5	1	22.3	4236.3	176.5	31.9	1	21.6	3808.0	7.0	7.8	10.1
	+60	200.4	31.6	1	21.2	4245.0	185.7	29.2	1	20.5	3804.4	7.3	7.7	10.4
	+30	213.4	28.2	1	19.9	4255.5	196.9	26.1	1	19.3	3800.3	7.7	7.5	10.7
	0	229.8	24.1	1	18.6	4269.0	211.0	22.4	1	18.0	3795.0	8.2	7.4	11.1
	−30	251.6	19.1	1	17.0	4286.7	229.4	17.7	1	16.5	3787.5	8.8	7.1	11.6
	−60	282.3	12.7	1	15.3	4311.8	254.8	11.8	1	14.8	3774.6	9.8	6.6	12.5
	−90	330.4	3.8	1	13.2	4351.0	292.9	3.6	1	12.8	3747.1	11.3	5.5	13.9

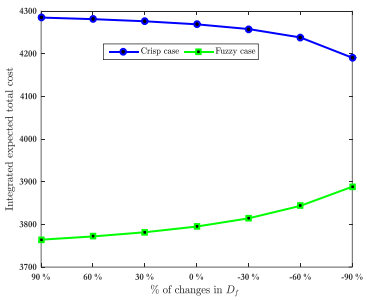
Table 3: Changing of parameters $H_r, \tilde{H}_r, H_f$ and $\tilde{H}_f$ on optimum result.

6.3. Managerial effects

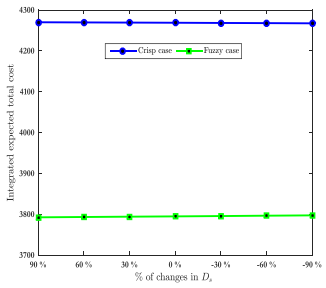
The specified certain managerial effects of the suggested case established in that arithmetical and sensitiveness study.

- According to Tables 1 and 2, together of optimal production quantities y, and $\tilde{y}$, shortage level K, and $\tilde{K}$, are raise as the first part of the demand rate D_f rises. The reason for that when demand increases, the production quantity and the shortage level also increase, resulting in a success situation for all parties involved. Compared to the crisp case, we find $P(\tilde{\Psi}) \times \tilde{y}^*$ falls with an increase in demand rate D_f. This arises owing toward the effect of shortage, holding rate, etc., on the inventory phases.
- Increasing the second portion of demand ratio D_s decreases mutually the optimal production quantity, shortage level of crisp and fuzzy case.
- As ordering costs O and $\tilde{O}$ decrease, corresponding crisp and fuzzy shortage levels K, and $\tilde{K}$, as well as optimal production quantity y, and $\tilde{y}$, integrated expected total costs $\Psi \times y^*$ and $P(\tilde{\Psi}) \times \tilde{y}^*$ decrease without affecting sales team initiatives φ^*. Similarly, it gives the same effect for the production rates R and $\tilde{R}$, the setup costs M and $\tilde{M}$, the sales teams' initiatives cost n and $\tilde{n}$.

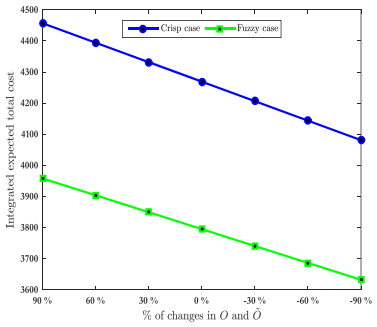
- There is a stable relationship between the crisp and fuzzy optimal production lot sizes and shortage levels when salvage costs C_1 and $\tilde{C}_1$, screening costs C_2 and $\tilde{C}_2$ decrease. Furthermore, the situation is stimulating toward annotation of unit quantity's integrated expected total cost declines though the integrated expected total cost and rises uninterrupted the trade group's exertions φ^*.
- The effects of Tables 2, 3 are clarified that the decision variables are constant while the screening rates s and $\tilde{s}$ decrease.
- Lot-sizes y and $\tilde{y}$, shortage levels K and $\tilde{K}$ increase as shortage costs C_3 and raw material's holding costs H_r and $\tilde{H}_r$ decrease. Further, a remarkable trend is observed in which the expected total costs of unit's declines while the integrated expected total costs increase without having an impact on the trade group's exertions φ^*.
- As a result of a decrease manufactured holding costs H_f and $\tilde{H}_f$ the optimal production quantity y and $\tilde{y}$ increase, shortage level K and $\tilde{K}$ decline. Moreover, it is notable these unit quantity's expected total cost decreases whereas the integrated expected total cost rises in crisp case and decreases in fuzzy case without affecting the trade group's exertions φ^*.
- Based on the fuzzy model, we observe shortage level, optimum production quantity, minimum integrated expected total cost savings, ranging from 5.5% to 7.8%, 2.4% to 11.3%, and 7.2% to 13.2%, respectively.
- Additionally, it is shown that the optimal production quantity, shortage level, and minimum integrated expected total cost of the fuzzy model are significantly different from their crisp model. The graphical demonstration of the sensitiveness study is exposed in Figure 4.
- Also, the minimum integrated expected total cost of the fuzzy model is highly significant from the results of the crisp model.



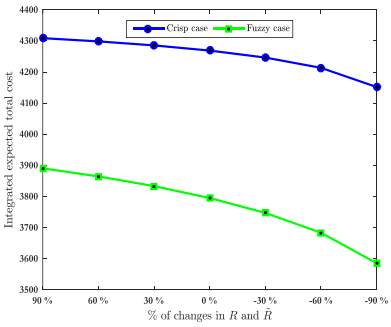
(a) Sensitivity analysis for D_f.



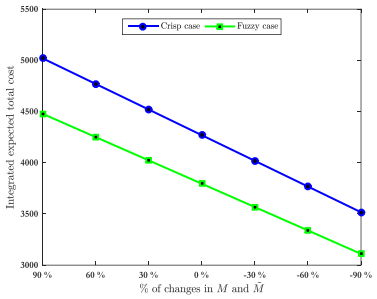
(b) Sensitivity analysis for D_s.



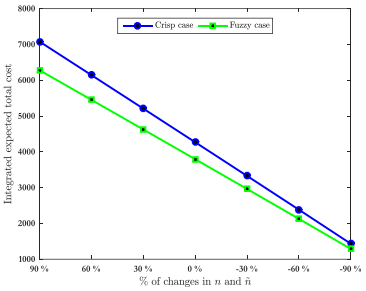
(c) Sensitivity analysis for O and $\tilde{O}$.



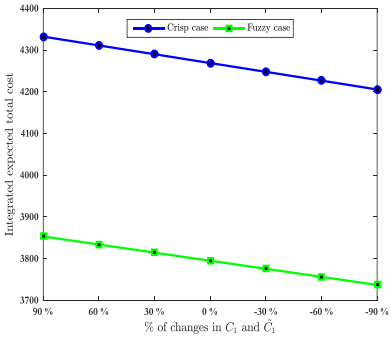
(d) Sensitivity analysis for R and $\tilde{R}$.



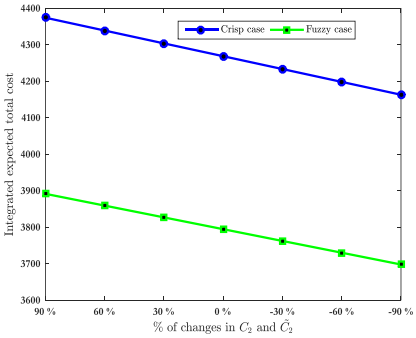
(e) Sensitivity analysis for M and $\tilde{M}$.



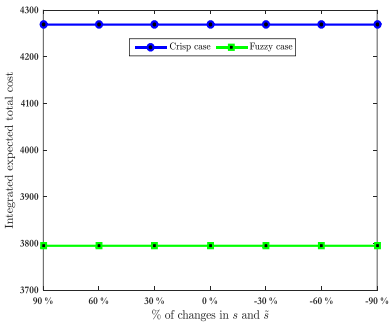
(f) Sensitivity analysis for n and $\tilde{n}$.



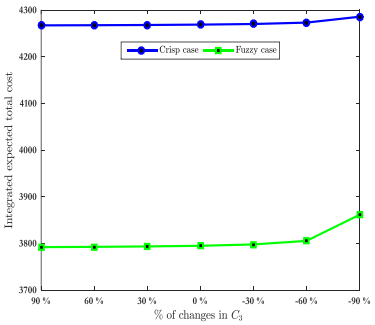
(g) Sensitivity analysis for C_1 and $\tilde{C}_1$.



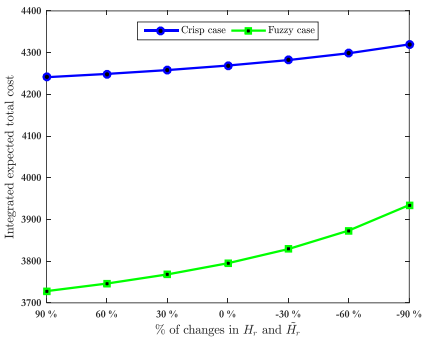
(h) Sensitivity analysis for C_2 and $\tilde{C}_2$.



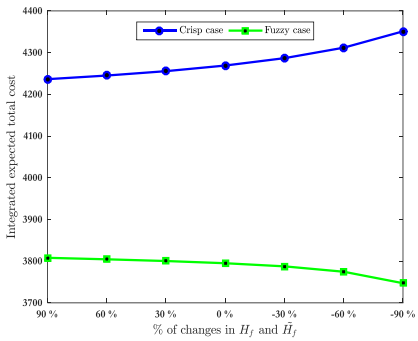
(i) Sensitivity analysis for s and $\tilde{s}$.



(j) Sensitivity analysis for C_3.



(k) Sensitivity analysis for H_r and $\tilde{H}_r$.



(l) Sensitivity analysis for H_f and $\tilde{H}_f$.

Figure 4(a)-(l): Parameters variation on the integrated expected total cost.

7. Comparative study

In Table 4, the projected model is compared with those of shortage level, optimal production quantity, and unit quantity's integrated expected total cost. As the measurement deviation of enactment methods among the cases declines, the integrated expected total cost also decreases. It is evident that the model with fuzzy optimal production lot sizes, shortage levels, and integrated expected total costs is more advanced in terms of consumption than the model with crisp parameters. All enactment measures of the model are significantly influenced by the hypothesis regarding fuzzy minimum integrated total costs. Additionally, the fuzzy inventory case is related to a particular case in the preceding model. The fuzzy inventory model with Kuhn-Tucker conditions and the graded mean integration method has a vital influence on the coordination enactment processes, and this fuzzy inventory model can forecast the enactment processes powerfully.

The proposed model followed procedures to find the minimum integrated expected total cost in fuzzy situation. We compare the proposed model with the previous model and the outcomes are tabulated. Hence a proficient outcome for the proposed fuzzy model is achieved.

Table 4 shows the result of the Priyan et. al. [22] inventory model in a fuzzy environment. They used production rate as a fuzzy number and got the shortage level, optimal production lot size, unit quantity's integrated expected total cost, and integrated expected total cost, which values 24.3 units, 231.1 units, \$18.5/unit, and \$4270.0, respectively. In our proposed fuzzified inventory model, we get the shortage level, optimal production lot size, unit quantity's integrated expected total cost, and integrated expected total cost, which values 22.4 units, 211.0 units, \$18.0/unit, and \$3795.0, respectively. Based on two inventory models, it is observed that the integrated expected total cost varies significantly.

Comparisons	Shortage level	Optimal production quantity	Unit quantity's integrated expected total cost	Integrated expected total cost
Crisp inventory model	24.1	229.8	18.6	4269.0
Priyan et.al. [22] model production rate considered as fuzzy number	24.3	231.1	18.5	4270.0
This proposed fuzzy inventory model	22.4	211.0	18.0	3795.0
Savings (%) with Priyan et.al.[22] model	8.0	8.7	2.8	11.1

Table 4: Summary of the comparisons.

In this way, our fuzzy integrated inventory model provides the business with the ability to deal with indeterminate inventory cost constraints. It is perceived that uncertain cost parameters give 8.0%, 8.7%, 2.8%, and 11.1% of the savings in shortage level, optimal production lot-size, integrated unit quantity's expected total cost, and the integrated expected total cost, respectively. Hence, the proposed fuzzified inventory model is the finest way to manipulate the inventory models.

8. Conclusions

Modern global developments in manufacturing skills have brought about an alteration in business. Rapid-fluctuating skills on the manufactured article have become essential for a likewise fast reaction from manufacturing businesses. Towards these ends, manufacturing industries need to choose suitable manufacturing policies, invention plans, industrial methods, effort section and instrument resources, machinery, inventory control, and tools. The assortment of conclusions is difficult, and resolution-making is also challenging at the present time. Decision architects in the manufacturing segment frequently face the difficult task of producing and managing inventory goods in an indeterminate atmosphere. This study discussed a two-level production-inventory system under fuzzy parameters and decision variables by implementing pentagonal fuzzy numbers. At this point, the manufacturer and the vendor effort under a combined station, in which an integrated result architect makes entirely conclusions towards, optimising the decision variables and minimising the integrated expected total cost for the whole supply chain.

An inventory model with scheduled fuzzy parameters and decision variables is offered in the proposed system. The pentagonal fuzzy inventory model was solved using Kuhn-Tucker conditions and framed the algorithm to examine the optimum result for decision variables and total cost for the proposed inventory model. According to the results, there was a variation in the cost between crisp and fuzzy models in response to

fluctuations in the decision variables. Hence, the research exposes that in the model, the fuzzy model provided a commercial profit-making model for the minimum integrated expected total cost. For future investigation on this problem, we would consider multi-item with multi-objective of the system to deliberate the setup cost or ordering cost, for instance, as a decide variable and deliberate the special properties of decreasing ordering cost or setup cost changeability.

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Fuzzy dominator colouring on fuzzy soft graphs

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Abstract

A fuzzy soft dominator colouring of a fuzzy soft graph G^S(T,V) is an appropriate fuzzy soft colouring such that every single vertex of G^S(T,V) dominate entire vertex of a colour group. In the present work, we initiate fuzzy dominator colouring on fuzzy soft path, fuzzy soft cycle, complete fuzzy soft graph, complete fuzzy soft bipartite graph and its fuzzy soft dominator chromatic number is presented as well as bounds for fuzzy soft dominator chromatic number on fuzzy soft graph is established.

Keywords: Fuzzy Soft Graph, Fuzzy dominator chromatic number, Fuzzy soft path, Fuzzy soft cycle, Strong arcs.
2020 AMS subject classifications: 05C72; 05C15. [1]

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[1]Received on March 23, 2023. Accepted on August 1, 2023. Published on October 14, 2023. DOI: 10.23755/rm.v48i0.1153. ISSN: 1592-7415. eISSN: 2282-8214. ©Jahir Hussain Rasheed and Afya farhana Mohamed Sheik Dawood. This paper is published under the CC-BY licence agreement.

1 Introduction

Fuzzy soft graphs are a useful mathematical tool for simulating the ambiguity of the actual world in view of parameters. Fuzzy soft graphs, which are often utilised in many different disciplines, including decision-making issues, combine fuzzy soft sets and the graph model. Scheduling problems are just one of many practical issues involving the allocation of scarce resources for which graph colouring is used as a model. Graph colouring also has an important place in discrete mathematics and combinatorial optimization.

To address ambiguous issues in the fields of engineering, social science, economics, medical research, and environment, **?** created the idea of soft set theory. Fuzzy soft sets, a blend of a fuzzy set and a soft set, were first introduced by Maji **?**. Fuzzy soft graphs were independently introduced by **?** as well. **?** presented fuzzy soft graphs and examined their operations as well as several other graph theoretical ideas.

Domination is a fast growing area of graph theory study, and the numerous ways it is used to networks, distributed computers, social networks, and online graphs helps to explain why there is more interest in this subject. The dominator colouring problem in graphs was first described by **?**. An appropriate colouring of a graph G with the extra characteristic that each vertex in the graph dominates an entire class is known as a dominator colouring of G. The smallest number of colour classes in a graph's dominator colouring is known as the dominator chromatic number. We want to reduce the number of colour categories. **?** explored the dominator chromatic number for the hypercube and more broadly for bipartite graphs. He also presented dominator colorings in bipartite graphs.**?** studied dominator colorings in some classes of graphs. The idea of fuzzy dominator colouring in fuzzy graphs was created by **?**. They investigated the fuzzy dominator chromatic number for a number of fuzzy graphs and explored its boundaries. Domination in fuzzy soft graphs was the subject of study done by **?**.
Fuzzy dominator coloring applied to fuzzy soft graph yields fuzzy soft dominator coloring which concentrates on minimizing the number of color classes of FSG. A fuzzy dominator FSG's colouring should be done in such a way that every node of a colour group is dominated by a vertex of G^S(T,V). The proposed fuzzy soft dominator coloring concentrates on strength of connectedness, strong arc and strong neighbour in classes of fuzzy soft graph in view of parameters from the existing method.

In this paper, we introduce fuzzy soft dominator coloring to fuzzy soft graphs and its fuzzy soft dominator chromatic numbers are determined with bounds.

2 Preliminaries

Definition 2.1. *Let V =* $\{x_1, x_2, x_3, \cdots, x_n\}$ *is a non-empty set, R is a parameter set and T* $\subseteq R$. *Also let*
(i) α*: T* $\to F(V)$ *(set of all fuzzy subsets in V)*
e $\dashv \alpha$*(e) =* $\alpha_e(say)$
α_e *: V* $\to [0, 1]$
(T,α*) : Fuzzy Soft Vertex*

(ii) β*: T* $\to$ *F(V*$\times$ *V) (set of all fuzzy subsets in V*$\times$ *V)*
e $\dashv \beta$*(e) =* β_e *(say)*
β_e *: V*$\times$ *V* $\to [0, 1]$
(T,β*) : Fuzzy Soft Edge*
Then ((T,α*),(T,*β*)) is called fuzzy soft graph (FSG) iff* $\beta_e(x, y) \leq \alpha_e(x) \wedge \alpha_e(y)$ *for all e* $\in T$ *and this fuzzy soft graphs are symbolized by* $G^S(T, V)$. *Additionally, a FSG is a parametrized family unit of fuzzy graphs.*

Definition 2.2. *A series of distinctive points* $x_1, x_2, \cdots x_n$ *in a FSG is mean to be a Path such that* $\forall$ *e* $\in T$ *and* β_e $(x_{i-1}, x_i) > 0$ $\forall$ *i = 1,2,3,*$\cdots n$.

Definition 2.3. *A FSG* $G^S(T, V)$ *is called Fuzzy Soft Cycle if it contain above one lowest arc,* $\forall$ $e \in T$.

Definition 2.4. *A fuzzy soft graph G*^{S}*(T,V) is complete if* $\beta_e(x, y) = \alpha_e(x) \wedge \alpha_e(y)$, *for every x,y* $\in \alpha^*$, *e* $\in T$.

Definition 2.5. *A FSG G*^{S}*(T,V) is supposed to be a fuzzy soft bipartite if the node set V can be divided into 2 non-empty sets* V_1^e *and* V_2^e *such that* V_1^e *and* V_2^e *are fuzzy independent sets. These sets are called fuzzy bipartition of V, thus each efficient arc of FSG has one end in* V_1^e *and other end in* V_2^e, $\forall$ $e \in T$.

Definition 2.6. *A fuzzy soft bipartite graph is complete if for an individual node* V_1^e, *each single node of* V_2^e *is an efficient neighbour,* $\forall$ $e \in T$.

3 Fuzzy Soft Dominator Colouring on Fuzzy Soft Graphs

Definition 3.1. *The strength of connectedness in a FSG G*^{S}*(T,V) joining two nodes x,y is define as the greatest strength of the entire paths among x and y and is indicated by* $\beta_e^{\infty}(x, y), \forall$ $e \in T$.

Definition 3.2. *An arc (x,y) in FSG G^S(T,V) is said to be a strong arc iff $\beta_e(x, y)$ = β_e^∞ (x,y) e $\in$ T and x, y is said to be strong adjacent vertices (or) efficiently adjoining nodes.*

Definition 3.3. *A k-fuzzy soft vertex colouring of a fuzzy soft graph G^S(T,V) is an allotment of k –colours, commonly mean as 1,2,$\cdots$ k to the vertices of G^S(T,V). A fuzzy soft vertex colouring is stated as appropriate colouring if two strong adjacent vertices receive distinct colours.*

Definition 3.4. *A fuzzy soft dominator coloring (FSDC) of a FSG G^S(T,V) is an appropriate fuzzy soft colouring such that every single vertex of G^S(T,V) dominate entire vertex of a colour group.*

Definition 3.5. *A fuzzy dominator chromatic number (FDCN) of a FSG is the least quantity of colour groups in a fuzzy soft dominator colouring of G^S(T,V) and it is indicated as χ_{fd}^e (G^S(T,V)), e $\in$ T.*

4 Main Results

Proposition 4.1. *If G^S(T,V) is a fuzzy soft graph such that $G^{S^*}(T, V) = (V, E)$ is a soft path of size n $\geq$ 3, in that case $\chi_{fd}^e(G^S(T, V))$= 1 + $\lceil \frac{n}{3} \rceil$ for n=3,4 and 5 and 2+$\lceil \frac{n}{3} \rceil$ for n $\geq$ 6, $\forall$ $e \in T$.*

Proof. Consider a fuzzy soft graph G^S(T,V) of whom fundamental graph is a soft path of size n.

$\Longrightarrow$ All interior points of G^S(T,V) dominate themselves and their connecting points. Hence, it is clear that the fuzzy soft dominator chromatic number of G^S(T,V) for n =3,4 and 5 is 1+$\lceil \frac{n}{3} \rceil$.

Now we have to prove for the case n $\geq$ 6 as 2+$\lceil \frac{n}{3} \rceil$.

case (i) n$\equiv$ 0 (mod 3)

Consider a fuzzy soft graph G^S(T,V) with points v_1,v_2,...,v_{3_k}. To obtain the fuzzy soft dominator chromatic number of G^S(T,V) we have to use no less than 3k uncommon colours. With no loss of generality, pick the mid point of every 3 nodes, then we have v_2,v_5,...,v_{3_k-1} having k = $\lceil \frac{n}{3} \rceil$ colour groups of G^S(T,V).

Now colour the resting nodes of G^S(T,V). We need two colour groups to colour the resting nodes v_3 and v_4 to obtain appropriate fuzzy soft colouring since v_3 and v_4 are strong arcs. So allot the colour k+1 to v_1,v_2,...,v_{3_k} and assign colour k+2 to v_4,v_7,...,v_{3_k-2}

In the end, we have k+2 =$\lceil \frac{n}{3} \rceil$+2 colour groups in a least fuzzy soft dominator colouring.

case (ii) n$\equiv$ 2 (mod 3)

We should colour an additional node 3k+2 in this case, because it is not strong adjoin to any node in the equal colour group. So put on this node in the colour group v_1,v_2,...,v_{3_k}. Hence we get a least fuzzy soft dominator colouring of G^S(T,V) as k+2 =$\lceil \frac{n}{3} \rceil$+2.

Proposition 4.2. *If G^S(T,V) is a fuzzy soft cycle of length n, then $\chi^e_{fd}(G^S(T,V))$= $\lceil \frac{n}{3} \rceil$ for n=4 and $\lceil \frac{n}{3} \rceil$+2 for n $\geq$ 5,$\forall$ $e \in T$.*

Proof. Let C^e_4 = $v_1v_2v_3v_4v_1$ be a fuzzy soft cycle of size n = 4, for every e $\in T$. If the cycle be even, we have two fuzzy independent sets which are v_1,v_3 and v_2,v_4. Then allot colour 1 to the nodes v_1,v_3 and colour 2 to the nodes v_2,v_4. Hence we obtain the least fuzzy soft dominator colouring of C^e_n with $\chi^e_{fd}(G^S(T,V))$= $\lceil \frac{n}{3} \rceil$ = $\lceil \frac{4}{3} \rceil$ = 1.3= 2.
Now we examine the case n$\geq$ 5 of C^e_n.

Consider a sub graph P^e_n of C^e_n, $e \in T$ with nodes v_1,v_2,...,v_n with the equal membership function as in C^e_n. By Proposition 4.1, we have $\chi^e_{fd}(P^e_n)$ = 2+$\lceil \frac{n}{3} \rceil$, $e \in T$ and also there exists a least fuzzy soft dominator colouring such that the points v_1 and v_n are dominated by themselves or by its adjacent nodes v_2 and v_{n-1}.

From the sub graph P^e_n we get the fuzzy soft cycle C^e_n by joining an edge v_1v_n. If the colours of v_1v_n are alike, say colour m ($1 \leq m \leq \chi^e_{fd}(P^e_n)$)) $e \in T$ which is one of the repeated colour in fuzzy dominator colouring of P^e_n.

Because $\chi^e_{fd}(G^S(T,V))$ = 2+$\lceil \frac{n}{3} \rceil$, $e \in T$ there is at least one other colour that repeats say n ($1 \leq n \neq m \leq \chi^e_{fd}(P^e_n)$) allot colour n to v_1. Now the colour v_2 is a non-consecutive colour since v_1 need to dominate a few colour group in P^e_n and so the colour v_1 and v_2 are distinct. The resting nodes have an appropriate fuzzy soft colouring since it was transmitted from P^e_n. The new fuzzy soft colouring is a fuzzy soft dominator colouring because it was a fuzzy soft dominator colouring of P^e_n and the transform in the additional arc were analyzed.

Suppose, if the colours of v_1 and v_n were uncommon colours in a fuzzy soft dominator colouring of P^e_n, later the colouring would not be optimal. And so, at the most one of the node v_1 and v_n will have an uncommon colour say v_n. Then certainly, v_{n-1} has a common colour in an optimal fuzzy soft dominator colouring of P^e_n. Then joining an edge v_1 v_n will maintain the fuzzy soft dominator colouring of P^e_n of C^e_n. Thus $\chi^e_{fd}(G^S(T,V))$ = 2+$\lceil \frac{n}{3} \rceil$ for n $\geq$ 5, $\forall$ $e \in T$.

Proposition 4.3. *If G^S(T,V) is a complete fuzzy soft graph then $\chi^e_{fd}(G^S(T,V))$ = n, $\forall$ $e \in T$.*

Proof. Given that G^S(T,V) is complete, then every vertex of G^S(T,V) dominates itself and its adjacent vertices, since all arcs are strong arcs in complete fuzzy soft graph. Hence every node of G^S(T,V) dominates a colour group. Therefore G^S(T,V) consist of n colour groups v_1,v_2,...,v_n, $\chi^e_{fd}(G^S(T,V))$ = n, $\forall$ $e \in T$.

Proposition 4.4. *If G^S(T,V) is a complete fuzzy soft bipartite graph then* $\chi^e_{fd}(G^S(T,V))$ = 2, $\forall e \in T$.

Proof. Since $\chi^e_{fd}(G^S(T,V))$ = 2 , which means the FSG contains 2 colour groups say V^e_1 and V^e_2. This implies every point of V^e_1 dominates each point ofV^e_2, then we have a bipartition as V^e_1 and V^e_2 respectively, $\forall\ e \in T$.
Therefore G^S(T,V) must be a complete fuzzy soft bipartite graph.

4.1 Bounds on Fuzzy Dominator Chromatic Number of Fuzzy Soft Graphs

Let G^S(T,V) be a connected fuzzy soft graph of size n$\geq$ 2. Then we need no less than two different colours in a fuzzy soft dominator colouring since there are two nodes which are strong adjacent to each other. Futhermore, if each node gains its unique colour, we also obtain a fuzzy soft dominator colouring.Thus

$$2\leq \chi^e_{fd}(G^S(T,V)) \leq \text{n}, \forall\ e \in T.$$

and these bounds are sharp for complete fuzzy soft bipartite graph and complete fuzzy soft graph.

Proposition 4.5. *Let G^S(T,V) be a connected fuzzy soft graph. Then* $\chi^e_{fd}(G^S(T,V))$ = 2, $\forall\ e \in T$ *iff G^S(T,V) is a complete fuzzy soft bipartite graph.*

Proof. Consider a fuzzy soft graph with $\chi^e_{fd}(G^S(T,V))$ = 2 (i.e.) the graph contains 2 colour groups V^e_1 and V^e_2 (say). If $|V^e_1|$ = 1 (or)$|V^e_2|$ = 1, $\forall\ e \in T$.
$\implies$ G^S(T,V) is complete fuzzy soft bipartite graph with bipartition $|X^e_1|$=1 and $|X^e_2|$= n-1, since G^S(T,V) is connected. Suppose we have $|V^e_1| \geq 2$ and $|V^e_2| \geq 2$,$\forall e \in T$. Let v $\in V^e_1$. Since V^e_1 is a fuzzy independent set and $|V^e_1| \geq 2$, it implies that the vertex v is not dominated by any node in V^e_1. This results that the vertex v is efficiently adjoin to every point in V^e_2. Equivalently, for any random point in V^e_2, that point is efficiently adjoin to each point in V^e_1. As a result, G^S(T,V) is a complete fuzzy soft bipartite graph. Converse part follows from the proposition 4.4.

Proposition 4.6. *Let G^S(T,V) be a connected fuzzy soft graph. Then* $\chi^e_{fd}(G^S(T,V))$ = n, $\forall\ e \in T$ *if and only if G^S(T,V) is a fuzzy soft graph whose fundamental graph is complete and the arcs are strong arcs.*

Proof. Consider a connected fuzzy soft graph G^S(T,V) with fuzzy soft dominator chromatic number n. SupposeG^S(T,V) is not a fuzzy soft graph with complete graph as an underlying crisp graph with strong arcs. Then there exists at the minimum of one couple of points u,v such that u and v are not efficiently

adjoin. Represent fuzzy soft colouring on G^S(T,V) in which u and v taking the same colour and the rest of the nodes meet with unique colour. Therefore,G^S(T,V) receives a fuzzy soft dominator colouring with $\chi^e_{fd}(G^S(T,V))$ = n-1 which is a contradictory statement for $\chi^e_{fd}(G^S(T,V))$ = n, $\forall\ e \in T$. Converse part follows from proposition 4.3.

Proposition 4.7. *A fuzzy soft graph G^S(T,V) have a point of efficiently adjoin degree n-1 then $\chi^e_{fd}(G^S(T,V)) = \chi^e_f(G^S(T,V)), \forall\ e \in T$.*

Proof. Let G^S(T,V) be a connected fuzzy soft graph with n points. Suppose that G^S(T,V) having a node of strong adjacent degree n-1. Let v_1, v_2 ... v_n be the points of G^S(T,V). With no loss of generality, let v_i for some i (i=1,2,... n) be the point of degree n-1 which is strong. Then v_i is efficiently adjoin to every point of G^S(T,V). If c_1, c_2, c_3 ... c_k be the colour groups of G^S(T,V) such that $\chi^e_f(G^S(T,V))$ = k, $\forall e \in T$. Clearly the node v_i is in one of the colour group say c_i. This is a fuzzy soft colouring which is appropriate and each node of G^S(T,V) dominates at least one colour group (c_i.) Hence we have a fuzzy soft dominator colouring of k colour groups. Therefore $\chi^e_{fd}(G^S(T,V))$ = k = $\chi^e_f(G^S(T,V))$, $\forall$ $e \in T$.

Corolary 4.1. *If the fuzzy soft domination number of G^S(T,V) is equal to 1, then $\chi^e_{fd}(G^S(T,V)) = \chi^e_f(G^S(T,V)), \forall\ e \in T$.*

Proposition 4.8. *Let G^S(T,V) be a connected fuzzy soft graph. Then, max ($\chi^e_f(G^S(T,V))$,$\gamma^e(G^S(T,V))$) $\leq \chi^e_{fd}(G^S(T,V)) \leq \chi^e_f(G^S(T,V))$ + $\gamma^e(G^S(T,V))$, $\forall\ e \in T$.*

Proof. Let G^S(T,V) be a connected fuzzy soft graph, since every fuzzy soft dominator colouring must be an appropriate fuzzy soft colouring, we have $\chi^e_f(G^S(T,V)) \leq \chi^e_{fd}(G^S(T,V))$. In addition, let c be a least fuzzy soft dominator colouring ofG^S(T,V). For every colour group ofG^S(T,V) let u_i be a point in the group i with $1 \leq\ i\ \leq \chi^e_{fd}(G^S(T,V))$. Then S = u_i:1$\leq$ i $\leq \chi^e_{fd}(G^S(T,V))$ is a dominating set which gives $\gamma^e(G^S(T,V)) \leq |S|$ = c =$\chi^e_{fd}(G^S(T,V))$. Therefore max ($\chi^e_f(G^S(T,V))$,$\gamma^e(G^S(T,V))$) $\leq \chi^e_{fd}(G^S(T,V))$, $\forall\ e \in T$.

For the upper bound, let c be an appropriate fuzzy soft colouring of G^S(T,V) with $\chi^e_f(G^S(T,V))$ colours. Now assign colours $\chi^e_f(G^S(T,V))$ + 1, $\chi^e_f(G^S(T,V))$ +2,... $\chi^e_f(G^S(T,V))$ + $\gamma^e(G^S(T,V))$ to the points of least dominating set of G^S(T,V) departing the remaining points coloured already. Clearly this is a fuzzy soft dominator colouring of G^S(T,V). Hence $\chi^e_{fd}(G^S(T,V)) \leq \chi^e_f(G^S(T,V))$ + $\gamma^e(G^S(T,V))$, $\forall\ e \in T$.

Corolary 4.2. *If G^S(T,V) be complete fuzzy soft bipartite graph, then the lower bound max ($\chi^e_f(G^S(T,V))$),γ^e(G^S(T,V)) $\leq \chi^e_{fd}(G^S(T,V))$,$\forall e \in T$ is sharp.*

Proof. Let G^S(T,V) be complete fuzzy soft bipartite graph, then max ($\chi^e_f(G^S(T,V))$,$\gamma^e(G^S(T,V))$) = max(2,2) = 2 $\leq \chi^e_{fd}(G^S(T,V))$= 2, $\forall\ e \in T$.

Corolary 4.3. *If G^S(T,V) be a fuzzy soft path of length n≥6, then the upper bound $\chi^e_{fd}(G^S(T,V)) \leq \chi^e_f(G^S(T,V))$ + $\gamma^e(G^S(T,V))$, $\forall\ e \in T$ is sharp.*

Proof. Let G^S(T,V) be a fuzzy soft path of length n≥ 6, then $\chi^e_{fd}(G^S(T,V))$=2+$\lceil \frac{n}{3} \rceil \leq \chi^e_f(G^S(T,V))$ + $\gamma^e(G^S(T,V))$=2 + $\lceil \frac{n}{3} \rceil$, $\forall\ e \in T$.

5 Conclusion

In this work, we defined fuzzy soft dominator colouring for classes of fuzzy soft graphs and its fuzzy soft dominator chromatic numbers through bounds were studied. In this study, we analysed the bounds for the connected fuzzy soft graph of size n $\geq$ 2 is 2 $\leq \chi^e_{fd}(G^S(T,V)) \leq$ n, $\forall\ e \in T$ which is sharp for complete fuzzy soft bipartite graph and complete fuzzy soft graph. And also, the lower bound for complete fuzzy soft bipartite and the upper bound for fuzzy soft path is sharp in terms of domination number. We further extend this study on some more types of fuzzy soft graphs.

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Non Deterministic Zero Divisor Graph

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Abstract

A non-deterministic zero divisor graph refers to an element in a ring or algebraic structure that can multiply with another element to give zero, but the specific outcome of the multiplication is not uniquely determined. In other words, there may be multiple elements that can multiply with the given element to produce zero. This concept is typically encountered in non-commutative rings or algebras where the order of multiplication matters. In such structures, the existence of non-deterministic zero divisors can complicate calculations and lead to different results depending on the order of operations. In this paper, we examine that all the zerodivisor graphs are nondeterministic graph but the converse need not. We manifest the nondeterministic zero divisor graph is possible only with nonprimes. Hereby, we study about weighted graphs, Weiner index and golden ratio rule. Also we provide an algorithm for zero divisor graph of n parameters and thereby, explore the given graph is either deterministic or nondeterministic graph using python.

Keywords: Zero divisor graphs, Non deterministic graph, Weiner index, Weighted graph, Golden ratio rule.
2020 AMS subject classifications:94B05, 05C50, 81P73. [1]

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1 Introduction

The concept of zero divisor graph was first introduced by Beck in 1988 [2]. Anderson and Livingston [1] were the first who simplify Beck's zero divisor graph. In this paper, we find the zero divisors Z_p and Z_q, where p and q are either prime or composite. In 2002, Redmond [11] extended the definition of zero divisor graph to the ring. In 2005, Costa and Faridi [3] introduced Zero-divisor graphs of noncommutative rings can have various structures and properties. They may be either connected or disconnected or have specific coloring properties. In 2018, Fan and Liu [4] suggested the zero-divisor graphs from a commutative ring theory perspective.

Deterministic and nondeterministic graphs are mathematical structures used in computer science to model systems that have a finite number of states and transitions between those states. These graphs are used in a wide range of applications, including natural language processing, machine learning, and computer vision. The concept of deterministic graphs was introduced by Stephen Kleene in the 1950s, as part of his work on regular expressions and finite automata. A deterministic graph is a graph in which each state has exactly one transition for each input symbol. This means that, given a current state and an input symbol, there is only one possible next state. In the 1960 s, researchers such as Michael O. Rabin and Dana S. Scott [10] developed the concept of nondeterministic finite automata (NFA), which are more powerful than deterministic finite automata (DFA) because they allow for multiple possible transitions from one state to another. This led to the development of algorithms for converting NFA into DFA, which are more efficient to work with.

A nondeterministic graph is a graph in which each state may have multiple transitions for the same input symbol. This means that, given a current state and an input symbol, there may be multiple possible next states. Nondeterministic graphs are more expressive than deterministic graphs, but they are also more difficult to work with computationally.

Stuart A. Kurtz [9] wrote a paper in 1973 titled "Deterministic and Nondeterministic Graphs," which presented a unified framework for deterministic and nondeterministic graphs. This paper provided a foundation for further research on the topic and helped to establish the importance of deterministic and nondeterministic graphs in computer science.

Today, deterministic and nondeterministic graphs are important concepts in computer science and are used in a wide range of applications. Researchers continue to explore new algorithms and applications for these graphs, making them an active area of research in computer science.

Throughout this paper, we consider the zerodivisor graph z_n.

In section 2, the basic definition and preliminary results of the contents are pro-

vided. In section 3, we examined that the nondeterministic zero divisor graph with suitable illustrations. In section 4, we provide an algorithm for zero divisor graph at n parameters and verify the given graph is either deterministic or non deterministic graph using python.

2 Preliminaries

We will step over some fundamental definitions in this section that are related to our main concept.

Definition 2.1. [1] Let z_n be graph with a vertex set v. The two distinct vertices v_1 and v_2 are adjacent if and only if $v_1 v_2 = 0$. It is said to be a *zero divisor graph.*

Definition 2.2. [1] *Incidence matrix* is a two-dimensional Boolean matrix, in which the rows represent the vertices and columns represent the edges. The entries indicate whether the vertex at a row is incident to the edge at a column. Incidence matrix is one of the ways to represent a graph.

Definition 2.3. [9] A *non-deterministic graph* is a mathematical representation of a system in which the outcome of certain actions or events is not uniquely determined. In a non-deterministic graph, multiple valid paths or results may exist.

Definition 2.4. [11] The *Wiener index* (W) is a graph invariant in mathematics that is defined as the sum of the distances between all pairs of vertices in a graph.

Definition 2.5. [6] A *weighted graph* is a type of graph where a numerical value, called a weight, is assigned to each edge in the graph.

Definition 2.6. [6] The *golden ratio rule* is a design principle that uses the mathematical ratio of approximately 1.618 to create aesthetically pleasing and balanced compositions, by dividing a line or shape into two unequal parts in a specific way.

3 Non deterministic zero divisor graph

An element in a ring or other mathematical structure that can multiply with another component to yield zero is also referred to as a non-deterministic zero divisor graph because the exact outcome of the multiplication cannot be determined in advance. The order of multiplication matters in non-commutative rings or algebras, wherever this topic is frequently discussed. The existence of non-deterministic zero divisor graphs in such structures can render calculations more difficult and generate different outcomes depending on the order in which operations are performed.

Definition 3.1. Let z_n be a zero divisor graph, represented by a directed graph with vertices V and edges E. z_n is said to be *non deterministic zero divisor graph* if, for every vertex set $S \subseteq V$, there exists a specific set of choices C such that the outcome of certain actions or events is not uniquely determined.

Lemma 3.1. *If n is nonprime, then z_n is non deterministic zero divisor graph.*

Proof. Let $n = p_1, p_2, \ldots\ldots, p_n$ are distinct non primes. Then the zerodivisor z_n can be partitioned into disjoint sets. Then, $|A_1|$ = (P_2 - 1)(P_3 - 1), $|A_2|$ = (P_1 - 1)(P_3 - 1), $|A_3|$ = (P_3 - 1)(P_2 - 1),$|A_4|$ = P_3 - 2,$|A_5|$ = (P_5 - 1)(P_4 - 2). Hence, $|V|$ $\neq \sum_{i=1}^{n} |A_i|$. The nonprime numbers of zero divisor in which every edges are not uniquely determined. □

Example 3.1 Let n=15 be non prime. The zero divisor graph of z_{15} is as follows:

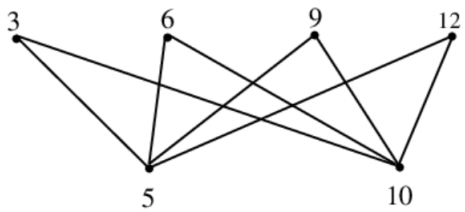

Fig 3.1 : z_{15}

The graph is nondeterministic. Therefore, z_{15} is nondeterministic zerodivisor graph.

Lemma 3.2. *If z_n is nondeterministic zero divisor graph, then n is nonprime.*

Proof. Let z_n be a non deteministic zero divisor graph. The incidence matrix M is defined as:

$$M = [m_{ij}] = \begin{cases} 1, & \text{if the } i^{th} \text{ vertex is incident to the } j^{th} \text{ edge,} \\ 0, & \text{otherwise.} \end{cases}$$

By using weiner index of incidence matrix, we concluded that n is non prime. □

Example:3.2 The incidence matrix for z_{15} is,

$$\text{M}= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Here, the Weiner index(W) is 14. Hence,it is nonprime.

Theorem 3.1. *If z_n is nondeterministic zero divisor graph iff n is nonprime.*

Proof. Clearly,from the above lemmas. □

Theorem 3.2. *If n is nonprime iff z_n is nondeterministic zerodivisor graph.*

Proof. Let n be nonprimes. The fibonacci sequence is the sequence formed by $z(n)^2 = 2z(n-1)^2 + 2z(n-2)^2 - z(n-3)^2$. By using, golden ratio rule, z_n=$\emptyset^n(1-\emptyset)^n/\sqrt{5}$ (where z_n denotes the the sequence) n denotes the number of the sequence. $\emptyset\rightarrow$ Golden ratio. The value of $\emptyset = 1.618$. Hence, z_n is nondeterministic zerodivisor graph. Conversely, Let z_n be a nondeterministic zerodivisor graph. Let us take a multiple of two vertices which are adjacent to be the weighted graph. Clearly, the sum of weighted graph is nonprime. □

Example:3.2
To find the 9^{th} Fibonacci number using the golden ratio formula, we can substitute n = 9 into the formula: F_n = ($\emptyset^n$ - (1-$\emptyset$)^n) / sqrt(5).
where $\emptyset$ (phi) is the golden ratio, which is approximately equal to 1.61803398875.
Thus, we have: F_9 = ($\emptyset^9$ - (1-$\emptyset$)^9) / sqrt(5) = (34.091 - 0.188) / sqrt(5) = 21.000
Therefore, the 9^{th} Fibonacci number is 21, which is the same as the value obtained by the recursive or iterative methods.

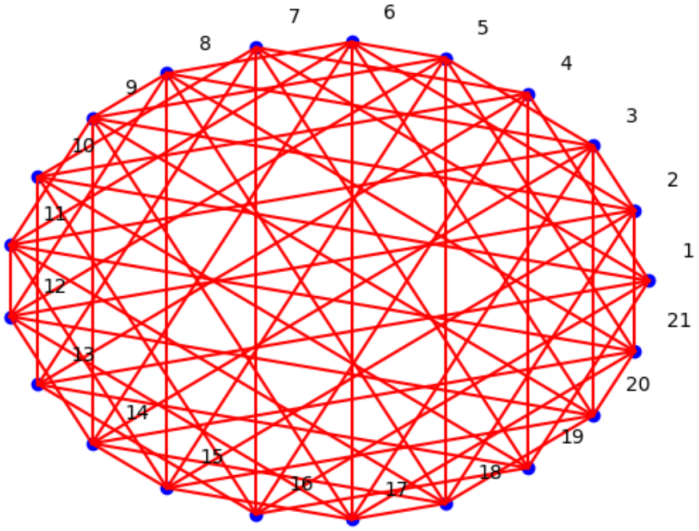

Fig 3.2 : z_{21}

Additionally, we have used the nondeterministic zerodivisor to demonstrate that n is not prime. The result of the figure is depicted above.
In fig 3.3, the multiple of two vertices and noted as weighted graph. The sum of

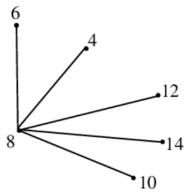

Fig 3.3 : z_{16}

weighted graph is nonprime.

4 Algorithm

4.1 The algorithm for zero divisor graph for a given n parameters

In this section, we provide an algorithm for zero divisor graph using the matplotlib library in Python(pseudocodes)
1. Initializing num-values, as any non prime number
2. Define the nodes and their positions
nodes = list(range(1, num-nodes+1))
theta = evenly spaced values between(0, 2 * Pi, num-nodes, excluding endpoint)
3.calculate positions using polar coordinates(theta)
4.Define the edges and their weights
edges = []
weights = []
for i = 0 to num-nodes - 1 do:
for j = i + 1 to num-nodes - 1 do:
if (i + j) add edge(edges, i + 1, j + 1)
add weight(weights, 0)
5.Draw the nodes
figure, axis = create figure and axis()
for each node, position in zip(nodes, positions) do:
plot node(axis, position)
6. Draw the edges representing zero divisors in red
for each edge, weight in zip(edges, weights) do:
if weight == 0 then:
node1 = positions[find index of node(nodes, edge[0])]
node2 = positions[find index of node(nodes, edge[1])]
plot edge(axis, node1, node2, 'red')
7. Show the graph
turn off axis(axis)
show figure(figure)

4.2 Manifest the graph is either deterministic or nondeterministic graph

In this section, we provide an algorithm to explore the given graph is either deterministic or nondeterministic graph using python(pseudocodes).
1. Initializing the states, alphabets and transitions.

2. Assigning the set of initial states and accepting states
3. Define a function to stimulate the automaton
function simulate(inputString):
current states $<$ set(initial states)
next states $<$ set()
for each character in input string:
for each state in current states:
if (state, character) is a key in transitions:
next states $<$ next states union transitions[(state, character)]
current states $<$ next states
next states $<$ set()
4. Test the automaton for each state in current States
current states $<$ set(initial states)
next states $<$ set()
print(simulate('ab')) :"true"
print(simulate('bab')) :"true"
print(simulate('aabb')) :"false"
print(simulate('bbab')) :"false"

5 Conclusions

In this paper, we introduced deterministic and nondeterministic zero divisor graph over z_n, where n is nonprime. Throughout this paper we considered the graph is zerodivisor graph z_n, where n is nonprime. We examined that the zerodivisor graph is either deterministic zerodivisor graph or nondeterministic zero divisor graph. Finally, we determined that the zero divisor graph z_n is nondeterministic onaccount of nonprime n but, the converse need not be true. We proved the above result with suitable illustrations. we provided an algorithm for constructing the zero divisor graph with n parameters and also we furnished an algorithm to manifest the given graph is either deterministic or nondeterministic graph. The implementation of the algorithm overdone in the Python programming language. The Applications of nondeterministic zero divisor graphs in Neural Network and coding theory are under our consideration.

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Fuzzy Soft Set Connected Mappings

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Abstract

In this paper, the concepts of fuzzy soft connectedness between fuzzy soft sets and fuzzy soft set connected mappings in fuzzy soft topological spaces has been introduced. It is shown that a fuzzy soft topological space is fuzzy soft connected if and only if it is fuzzy soft connected between every pair of its nonempty fuzzy soft sets. Every fuzzy soft continuous mapping is fuzzy soft set-connected a counter example is given to show the converse may not be true. Several properties of fuzzy soft set-connected mappings in fuzzy soft topological spaces have been studied.

Keywords: Fuzzy soft sets; Fuzzy soft connectedness; Fuzzy soft connectedness between fuzzy soft sets and Fuzzy soft set-connected mappings.
2020 AMS subject classifications:54A40, 54D05, 54C08. [1]

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1 Introduction

The notion of a fuzzy set was introduced by Zadeh [1965] as a generalization of classical set in the year 1965. Chang [1968] gave the definition of fuzzy topology and extended some topological concepts to fuzzy sets. In 1999, Molodtsov [1999] introduced the concept of soft sets to deal with uncertainties while modelling the problems with incomplete information. In 2011, Shabir and Naz [2011] initiated the study of soft topological spaces as a generalization of topological spaces. The hybrid structure of fuzzy sets and soft set called fuzzy soft set was created by Maji et al. [2001]. To continue the investigation on fuzzy soft sets, Ahmad and Kharal [2009] presented some more properties of fuzzy soft sets and introduced the notion of a mapping on fuzzy soft sets. Tanay and Kandemir [2011] introduced the fuzzy soft topology as an extension of fuzzy topology and soft topology. Fuzzy topological spaces are further investigated by Varol and Aygun [2012] and shown that a fuzzy soft topological space gives a parametrized family of fuzzy topological spaces. Roy and Samanta [2011] and Tridiv et al. [2012] are also investigated various topological concepts in fuzzy soft topological spaces. The study of connectedness in fuzzy soft topology was initiated by Karatas et al. [2015] and further studied by Kandil et al. [2017]. Connectedness between sets and set connected mappings is one of the important topic of research in Topology. In 2018, Thakur and Rajput [2018] extended and studied these concepts to soft topology. Till the date these concepts are not studied in fuzzy soft topology. Therefore to fill up this gap, the present paper introduces the concept of connectedness between fuzzy soft sets and studied some of its properties in fuzzy soft topological spaces. Further the concepts of fuzzy soft set-connected mappings are defined and established some theorems related to its characterizations and properties. The results established in this paper generalized many results which are already available in the literature.

2 Preliminaries

Throughout this paper, X refers to an initial universe, E is the set of all parameters for X, I = [0,1] and I^X is the set of all fuzzy sets on X. The reader should refer Molodtsov [1999] and Zadeh [1965] for the basic concepts on fuzzy sets and soft sets.

Definition 2.1. *(Maji et al. [2001]) Let $A \subset E$. A pair (f,A) is called a fuzzy soft set (in short* $\mathbf{FSS}$*) over X, where $f : A \to I^X$ defined by , (f,A)(e) = $\mu^e_{(f,A)}$,where $\mu^e_{(f,A)} = \tilde{0}$ if $e \notin A$ and $\mu^e_{(f,A)} \neq \tilde{0}$ if $e \in A$.*

The family of all $\mathbf{FSS}$s over (X,E) will be denoted by $\mathbf{FS}(\mathbf{X}, \mathbf{E})$. For the

notations, basic operations and properties of **FSS**s the reader should refer Ahmad and Kharal [2009], Maji et al. [2001].

Definition 2.2. *(Tanay and Kandemir [2011]) A subfamily τ of FS(X, E) is called a fuzzy soft topology on X if:*

(1) $\widetilde{\phi}$, $\widetilde{X}$ belong to τ.

(2) The union of any number of soft sets in τ belongs to τ.

(3) The intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a fuzzy soft topological space (in short **FSTS***). The members of τ are called fuzzy soft open sets in X and their complements called fuzzy soft closed sets in X.*

The basic fuzzy soft topological concepts can be seen in Kandil et al. [2017], Karatas et al. [2015], Roy and Samanta [2011], Tridiv et al. [2012] and Varol and Aygun [2012]).

Lemma 2.1. *(Roy and Samanta [2011]) Let (f,A) and (g,B) be two* **FSS***s. Then, $(f, A) \subseteq (g, B)^c \Leftrightarrow (f, A)\tilde{q}(g, B)$. Where $(f, A)\tilde{q}(g, B)$ means (f,A) is not quasi-coincident with (g,B).*

Lemma 2.2. *(Tridiv et al. [2012]) Let (Y, τ_Y, E) be a fuzzy soft subspace of a* **FSTS** *(X, τ, E) and (F, E) be a fuzzy soft open set in Y. If $\tilde{Y} \in \tau$ then (F, E) $\in \tau$.*

Lemma 2.3. *(Tridiv et al. [2012]) Let (X,τ,E) be a* **FSTS** *and (Y, τ_Y,E) be a fuzzy soft subspace of (X, τ, E), then a fuzzy soft closed set (F_Y, E) of Y is fuzzy soft closed in X if and only if $\tilde{Y}$ is fuzzy soft closed in X.*

3 Connectedness between fuzzy soft sets

Throughout this paper fuzzy soft clopen means fuzzy soft closed open.

Definition 3.1. *(Kandil et al. [2017], Karatas et al. [2015]) A* **FSTS** *(X,τ,E) is fuzzy soft connected if and only if there is no nonempty* **FSS** *of (X,τ,E) which is both fuzzy soft open and fuzzy soft closed in (X,τ,E).*

Definition 3.2. *A* **FSTS** *(X ,τ, E) is said to be fuzzy soft connected between* **FSS***s (f_1,E) and (f_2,E) if and only if there is no fuzzy soft clopen set (f,E) over X such that (f_1,E) $\subset$ (f,E) and (f,E) $\tilde{q}$ (f_2,E).*

Theorem 3.1. *A* **FSTS** *(X,τ,E) is fuzzy soft connected between* **FSS***s (f_1,E) and (f_2,E) if and only if there is no fuzzy soft clopen set (f,E) over X such that (f_1,E) $\subset$ (f,E) $\subset (f_2, E)^c$.*

Proof. *Follows from Definition 3.2. and Lemma 2.1.*□

Theorem 3.2. *If a* **FSTS** *(X,τ,E) is fuzzy soft connected between* **FSS***s (f_1,E) and (f_2,E) then (f_1,E) $\neq \phi \neq$ (f_2,E).*

Proof. *If any* **FSS** *(f_1,E) = ϕ, then (f_1,E) is a fuzzy soft clopen set over X such that (f_1,E) $\subset$ (f_2,E) and (f_1,E) $\tilde{q}$ (f_2,E) and hence (X,τ,E) can not be fuzzy soft connected between* **FSS***s (f_1,E) and (f_2,E), which is contradiction.*□

Theorem 3.3. *If a* **FSTS** *(X,τ,E) is fuzzy soft connected between* **FSS***s (f_1,E) and (f_2,E) and if (f_1,E) $\subset$ (f_3,E) and (f_2,E) $\subset$ (f_4,E) then (X,τ,E) is fuzzy soft connected between* **FSS***s (f_3,E) and (f_4,E).*

Proof. *Suppose* **FSTS** *(X,τ,E) is not fuzzy soft connected between* **FSS***s (f_3,E) and (f_4,E) then there is a fuzzy soft clopen set (f,E) over X such that (f_3,E) $\subset$ (f,E) and (f,E) $\tilde{q}$ (f_4,E) .Clearly (f_1,E) $\subset$ (f,E). Now we claim that (f,E) $\tilde{q}$ (f_2,E) .If (f,E) q (f_2,E) then there exists a point x $\in$ X such that $\mu^e_{(f,E)}$(x) + $\mu^e_{(f_2,E)}$ (x) $\succ$ 1. Therefore $\mu^e_{(f,E)}$(x) + $\mu^e_{(f_4,E)}$ (x) $\succ$ $\mu^e_{(f,E)}$(x) + $\mu^e_{(f_2,E)}$ (x) $\succ$ 1 and (f,E) q (f_4,E), a contradiction. Consequently, (X,τ,E) is not fuzzy soft connected between* **FSS***s (f_1,E) and (f_2,E).*□

Theorem 3.4. *A* **FSTS** *(X,τ,E) is fuzzy soft connected between* **FSS***s (f_1,E) and (f_2,E) if and only if (X,τ,E) is fuzzy soft connected between* **FSS***s Cl(f_1,E) and Cl(f_2,E).*

Proof. *Necessity : Follows from Theorem 3.3.*

Sufficiency : Suppose **FSTS** *(X,τ,E) is not fuzzy soft connected between* **FSS***s (f_1,E) and (f_2,E), then there exists fuzzy soft clopen set (f,E) over X such that (f_1,E) $\subset$ (f,E) and (f,E) $\tilde{q}$ (f_2,E). Since (f,E) is fuzzy soft closed, Cl(f_1,E) $\subset$ Cl(f,E) = (f,E). Clearly, by Lemma 2.1, (f,E) $\tilde{q}$ (f_2,E) $\Leftrightarrow$ (f,E) $\subset (f_2, E)^c$. Therefore (f,E) = Int(f,E) $\subset$ Int($(f_2, E)^c$) = $(Cl(f_2, E))^c$. Hence, (f,E) $\tilde{q}$ Cl(f_2,E) and (X,τ,E) is not fuzzy soft connected between* **FSS***s Cl(F_1,E) and Cl(F_2,E).*□

Theorem 3.5. *If (f_1,E) and (f_2,E) are two* **FSS***s in* **FSTS** *(X,τ,E) and (f_1,E) q (f_2,E), then (X,τ,E) is fuzzy soft connected between (f_1,E) and (f_2,E).*

Proof. *If (f,E) is any fuzzy soft clopen set over X such that (f_1,E) $\subset$ (f,E), then (f_1,E) q (f_2,E) $\Rightarrow$ (f,E) q (f_2,E). This proves the theorem.*□

Remark 3.1. *The converse of Theorem 3.5 need not be true .*

Example 3.1. *Let X = $\{a,b\}$ be universe set and E = $\{e_1$,e_2 $\}$ be the set of*

parameters. The **FSS***s Let (f,E), (f_1,E) and (f_2,E) over X are defined as follows:*

$$f_1(e_1) = \{(a, 0.3), (b, 0.4)\} \qquad f_1(e_2) = \{(a, 0.2), (b, 0.3)\}$$
$$f_2(e_1) = \{(a, 0.5), (b, 0.4)\} \qquad f_2(e_2) = \{(a, 0.6), (b, 0.5)\}$$
$$f_3(e_1) = \{(a, 0.3), (b, 0.5)\} \qquad f_3(e_2) = \{(a, 0.3), (b, 0.2)\}.$$

Let $\tau = \{\tilde{0}_E, \tilde{1}_E, (f_1, E)\}$ *be a fuzzy soft topology on X, then* **FSTS** *(X,*τ *,E) is fuzzy soft connected between the* **FSS***s (* f_2*,E) and (* f_3*,E), but (* f_2*,E)* $\tilde{q}$ *(* f_3*,E).*

Theorem 3.6. *If a* **FSTS** *(X,*τ*,E) is neither fuzzy soft connected between (f,E) and (*g_0*,E) nor fuzzy soft connected between (f,E) and (*g_1*,E) then it is not fuzzy soft connected between (f,E) and (*g_0*,E)* $\cup$ *(*g_1*,E).*

Proof. *Since a* **FSTS** *(X,*τ*,E) is not fuzzy soft connected between (f ,E) and (*g_0*,E), there is a fuzzy soft clopen set (*h_0*,E) over X such that (f,E)* $\subset$ *(*h_0*,E) and (*h_0*,E)* $\tilde{q}$ *(*g_0*,E). Also since (X,*τ*,E) is not fuzzy soft connected between (f ,E) and (*g_1*,E) there exists a fuzzy soft clopen set (*h_1*,E) over X such that (f,E)* $\subset$ *(*h_1*,E) and (*g_1*,E)* $\tilde{q}$ *(*h_1*,E). Put (h,E) = (*h_0*,E)* $\cap$ *(*h_1*,E). Since any intersection of fuzzy soft closed sets is fuzzy soft closed, (h,E) is fuzzy soft closed. Again intersection of finite family of fuzzy soft open sets is fuzzy soft open, (h,E) is fuzzy soft open. Therefore (h,E) is fuzzy soft clopen set over X such that (f,E)* $\subset$ *(h,E) and (h,E)* $\tilde{q}$*((*g_0*,E)* $\cup$ *(*g_1, E*)). If (h,E) q ((*g_0*,E)* $\cup$ *(*g_1, E*)) there exists x* $\in$ *X such that* $\mu^e_{(h,E)}$ *(x) +* ($\mu^e_{(g_0,E)} \cup \mu^e_{(g_1,E)}$)*(x)* $\succ$ *1. This implies that (h,E) q (*g_0*,E) or (h,E) q (*g_1*,E) a contradiction. Hence, (X,*τ*,E) is not fuzzy soft connected between (f,E) and* $(g_0, E) \cup (g_1, E)$. □

Theorem 3.7. *A* **FSTS** *(X,*τ*,E) is fuzzy soft connected if and only if it is fuzzy soft connected between every pair of its nonempty* **FSS***s.*

Proof. *Let (f,E) and (g,E) be a pair of nonempty* **FSS***s over X. Suppose (X,*τ*,E) is not fuzzy soft connected between (f,E) and (g,E). Then there is a fuzzy soft clopen set (h,E) over X such that (f,E)* $\subset$ *(h,E) and (g,E)* $\tilde{q}$ *(h,E). Since (f,E) and (g,E) are nonempty it follows that (h,E) is a nonempty fuzzy soft proper clopen set over X. Hence, (X,*τ*,E) is not fuzzy soft connected.*

Conversely, suppose that (X,τ*,E) is not fuzzy soft connected. Then there exists a nonempty proper* **FSS** *(h,E) over X which is both fuzzy soft open and fuzzy soft closed. Consequently, (X,*τ*,E) is not fuzzy soft connected between (h,E) and* $(h, E)^c$*. Thus, (X,*τ*,E) is not fuzzy soft connected between arbitrary pair of its nonempty* **FSS***s.*□

Remark 3.2. *If a* **FSTS** *(X,*τ*,E) is fuzzy soft connected between a pair of its* **FSS***s, then it is not necessarily that it is fuzzy soft connected between each pair of its* **FSS***s and so it is not necessarily fuzzy soft connected.*

Example 3.2. *Let X = $\{a,b\}$ be an universe set, E = $\{e_1, e_2\}$ be the set of parameter and the soft sets (f_1, E) , (f_2, E) , (f_3,E), (f_4,E), (f_5,E) and (f_6,E) over X are defined as follows:*

$$f_1(e_1) = \{(a, 0.3), (b, 0.4)\} \qquad f_1(e_2) = \{(a, 0.2), (b, 0.3)\}$$
$$f_2(e_1) = \{(a, 0.7), (b, 0.6)\} \qquad f_2(e_2) = \{(a, 0.8), (b, 0.7)\}$$
$$f_3(e_1) = \{(a, 0.3), (b, 0.1)\} \qquad f_3(e_2) = \{(a, 0.4), (b, 0.2)\}$$
$$f_4(e_1) = \{(a, 0.8), (b, 0.7)\} \qquad f_4(e_2) = \{(a, 0.9), (b, 0.8)\}$$
$$f_5(e_1) = \{(a, 0.2), (b, 0.1)\} \qquad f_5(e_2) = \{(a, 0.1), (b, 0.2)\}$$
$$f_6(e_1) = \{(a, 0.4), (b, 0.3)\} \qquad f_6(e_2) = \{(a, 0.5), (b, 0.4)\}.$$

Let τ = $\{\tilde{0}_E, \tilde{1}_E, (f_1, E), (f_2, E)\}$ be a fuzzy soft topology over X. Then the $\mathbf{FSTS}$ *(X, τ, E) is fuzzy soft connected between the* $\mathbf{FSS}$*s (f_3, E) and (f_4, E) but it is not fuzzy soft connected between (f_5,E) and (f_6,E). Also the* $\mathbf{FSTS}$ *(X, τ, E) is not fuzzy soft connected.*

Theorem 3.8. *Let (Y,τ_Y,E) be a fuzzy soft subspace of a* $\mathbf{FSTS}$ *(X,τ,E). If (Y,τ_Y,E) is fuzzy soft connected between the* $\mathbf{FSS}$*s (f ,E) and (g,E) over Y, then* $\mathbf{FSTS}$ *(X,τ,E) is fuzzy soft connected between (f ,E) and (g,E).*

Proof. *Suppose* $\mathbf{FSTS}$ *(X,τ,E) is not fuzzy soft connected between* $\mathbf{FSS}$*s (f,E) and (g,E),then there is fuzzy soft clopen set (h,E) over X such that (f,E) $\subset$ (h,E) and (h,E) $\tilde{q}$ (g,E). Then $\tilde{Y} \cap$ (h ,E) is fuzzy soft clopen over Y such that (f,E) $\subset$ (h,E) $\cap \tilde{Y}$ and $\{$(h,E) $\cap \tilde{Y}$ $\}$ $\tilde{q}$ (g ,E). Consequently, (Y,τ_Y,E) is not fuzzy soft connected between (f,E) and (g,E), a contradiction.*□

Theorem 3.9. *Let (Y,τ_Y,E) be a fuzzy soft clopen subspace of a* $\mathbf{FSTS}$ *(X,τ,E) and (f ,E), (g,E) $\subset \tilde{Y}$. If (X,τ,E) is fuzzy soft connected between (f,E) and (g,E) then (Y,τ_Y,E) is fuzzy soft connected between(f,E) and (g,E).*

Proof. *Suppose (Y,τ_Y,E) is not fuzzy soft connected between (f,E) and (g,E). Then there is fuzzy soft clopen set (h,E) of (Y,τ_Y,E) such that (f,E) $\subset$ (h,E) and (h,E) $\tilde{q}$ (g,E). Since, (Y,τ_Y,E) is fuzzy soft clopen in (X,τ,E), by Lemma 2.2 and Lemma 2.3 (h,E) is fuzzy soft clopen set of (X,τ,E) such that (f,E) $\subset$ (h,E) and (h,E) $\tilde{q}$ (g,E). Consequently, (X,τ,E) is not fuzzy soft connected between (f,E) and (g,E), a contradiction.*□

4 Fuzzy soft set-connected mappings

Definition 4.1. *A fuzzy soft mapping $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ is said to be fuzzy soft set-connected provided, if* $\mathbf{FSTS}(X, \tau, E)$ *is fuzzy soft connected between* $\mathbf{FSS}$*s (f,E) and (g,E) then fuzzy soft subspace $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ is fuzzy*

soft connected between $\varrho_{pu}(f,E)$ and $\varrho_{pu}(g,E)$ with respect to fuzzy soft relative topology.

Theorem 4.1. *A fuzzy soft mapping $\varrho_{pu} : (X,\tau,E) \to (Y,\vartheta,K)$ is fuzzy soft set-connected mapping if and only if $\varrho_{pu}^{-1}(F,K)$ is a fuzzy soft clopen set over X for any fuzzy soft clopen set (h,K) of ($\varrho_{pu}(X)$,$\vartheta_{\varrho_{pu}(X)}, K$).*

Proof. *Necessity : Let ϱ_{pu} be fuzzy soft set-connected mapping and (h,K) be fuzzy soft clopen set in $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$. Suppose $\varrho_{pu}^{-1}(h,K)$ is not fuzzy soft clopen in (X ,τ,E). Then (X ,τ,E) is fuzzy soft-connected between $\varrho_{pu}^{-1}(h,K)$ and $(\varrho_{pu}^{-1}(h,K))^c$. Therefore, $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ is fuzzy soft-connected between $\varrho_{pu}(\varrho_{pu}^{-1}(h,K))$ and $\varrho_{pu}((\varrho_{pu}^{-1}(h,K))^c)$ because ϱ_{pu} is fuzzy soft set-connected. But, $\varrho_{pu}(\varrho_{pu}^{-1}(h,K)) = (h,K) \cap (\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K) = (h,K)$ and $\varrho_{pu}((\varrho_{pu}^{-1}(h,K))^c = (h,K)^c$ imply that (h,K) is not fuzzy soft clopen in $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$, a contradiction. Hence, $\varrho_{pu}^{-1}(h,K)$ is fuzzy soft clopen in (X ,τ,E).*

Sufficiency : Let (X ,τ,E) be fuzzy soft-connected between (f,E) and (g,E). If $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ is not fuzzy soft-connected between $\varrho_{pu}(f,E)$ and $\varrho_{pu}(g,E)$ then there exists a fuzzy soft clopen set (h,K) in $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ such that $\varrho_{pu}(f,E) \subset (h,K) \subset (\varrho_{pu}(g,E))^c$. By hypothesis, $\varrho_{pu}^{-1}(h,K)$ is fuzzy soft clopen set over X and $(f,E) \subset \varrho_{pu}^{-1}(h,K) \subset (g,E)^c$. Therefore, (X ,$\tau$,E) is not fuzzy soft-connected between (f,E) and (g,E). This is a contradiction. Hence, ϱ_{pu} is fuzzy soft set-connected.□

Theorem 4.2. *Every fuzzy soft continuous mapping ϱ_{pu} : (X ,τ,E) $\to$ (Y,ϑ,K) is a fuzzy soft set-connected mapping.*

Proof. *It is obvious.*□

Remark 4.1. *The converse of Theorem 4.2 need not be true.*

Example 4.1. *Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and $Y = \{y_1, y_2\}$, $K = \{k_1, k_2\}$. The soft sets (f,E) and (g,K) defined as follows:*

$$f(e_1) = \{x_1 = 0.3, x_2 = 0\}, \qquad f(e_2) = \{x_1 = 0, x_2 = 0.4\}$$
$$g(k_1) = \{y_1 = 0.6, y_2 = 0\}, \qquad g(k_2) = \{y_1 = 0, y_2 = 0.5\}$$

Let τ = $\{\tilde{0}_E, \tilde{1}_E$,(f,E) $\}$ and υ = $\{\tilde{0}_K, \tilde{1}_K$, (g ,K) $\}$ are fuzzy soft topologies on X and Y respectively. Then the fuzzy soft mapping $\varrho_{pu} : (X,\tau,E) \to (Y,\upsilon,K)$ defined by u(x_1) = y_1 , u(x_2)= y_2 and p(e_1)= k_1, p(e_2) = k_2 is fuzzy soft set-connected but it is not fuzzy soft continuous, because fuzzy soft set (g,K) is fuzzy soft open set in Y not fuzzy soft open in X.

Theorem 4.3. *Every fuzzy soft mapping $\varrho_{pu} : (X,\tau,E) \to (Y,\vartheta,K)$ such that $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ is a fuzzy soft connected set is a fuzzy soft set-connected mapping.*

Proof. *Let $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ be fuzzy soft connected. Then by Lemma 3.1, no nonempty proper* **FSS** *of $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ which is fuzzy soft clopen. Hence , ϱ_{pu} is fuzzy soft set-connected.*□

Theorem 4.4. *Let $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ be a fuzzy soft set-connected mapping. If (X ,τ,E) is fuzzy soft connected set, then $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ is a fuzzy soft connected set of (Y,ϑ,K).*

Proof. *Suppose $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ is not fuzzy soft connected in (Y,ϑ,K), Then by Lemma 3.1, there is a nonempty proper fuzzy soft clopen set (h,K) of $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$. Since ϱ_{pu} is fuzzy soft set-connected, $\varrho_{pu}^{-1}(h, K)$ is a nonempty proper fuzzy soft clopen set over X. Consequently,(X ,τ,E) is not fuzzy soft connected.*□

Theorem 4.5. *Let $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ be a fuzzy soft set-connected mapping and (f,E) be a fuzzy soft set over X such that $\varrho_{pu}(f, E)$ is fuzzy soft clopen set of $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$. Then $\varrho_{pu}/(f, E) : (f, E) \to (Y, \vartheta, K)$ is fuzzy soft set-connected mapping.*

Proof: Let (f,E) be fuzzy soft connected between (g,E) and (h,E). Then by Theorem 3.8, (X, τ, E) is fuzzy soft connected between (g,E) and (h,E). Since ϱ_{pu} is fuzzy soft set-connected, $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ is fuzzy soft connected between $\varrho_{pu}(g, E)$ and $\varrho_{pu}(h, E)$. Now, since $\varrho_{pu}(f, E)$ is fuzzy soft clopen set of $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$, it follows by Theorem 3.9 that $\varrho_{pu}(f, E)$ is fuzzy soft connected between $\varrho_{pu}(g, E)$ and $\varrho_{pu}(h, E)$. This proves the theorem.□

Theorem 4.6. *Let $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ be a fuzzy soft set-connected surjection. Then for any fuzzy soft clopen set (h,K) of (Y, ϑ, K) is fuzzy soft connected if $\varrho_{pu}^{-1}(h, K)$ is fuzzy soft connected in (X ,τ,E). In particular, if (X ,τ,E) is fuzzy soft connected then (Y,ϑ,K) is fuzzy soft connected.*

Proof. *By Theorem 4.5 $\varrho_{pu}/\varrho_{pu}^{-1}(h, K) : \varrho_{pu}^{-1}(h, K) \to (Y, \vartheta, K)$ is fuzzy soft set-connected. And, since $\varrho_{pu}^{-1}(h, K)$ is fuzzy soft connected by Theorem 4.4,*

$\varrho_{pu}/\varrho_{pu}^{-1}(h, K)[\varrho_{pu}^{-1}(h, K)] = (h, K)$ is fuzzy soft connected.□

Theorem 4.7. *Let $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ be a fuzzy soft set connected fuzzy soft open surjection and $\varrho_{pu}^{-1}((y_k^{\alpha})_K)$ is fuzzy soft connected for each soft point $(y_k^{\alpha})_K$ of Y. Then for every fuzzy soft clopen set (h,K) of Y is fuzzy soft connected if and only if $\varrho_{pu}^{-1}((h, K))$ is fuzzy soft connected.*

Proof. *Necessity: Let (h,K) be a fuzzy soft clopen fuzzy soft connected set of Y. Suppose $\varrho_{pu}^{-1}(h, K)$ is not fuzzy soft connected in X. Then there are fuzzy soft open sets (f,E) and (g,E) of X such that $\varrho_{pu}^{-1}(h, K) \cap ((f, E) \cap (g, E)) = \phi$, $\varrho_{pu}^{-1}(h, K) = ((f, E) \cup (g, E))$ and $\varrho_{pu}^{-1}(h, K) \cap (f, E) \neq \phi \neq \varrho_{pu}^{-1}(h, K) \cap (g, E)$. Since, $\varrho_{pu}^{-1}((y_k^{\alpha})_K)$ is fuzzy soft connected either $\varrho_{pu}^{-1}((y_k^{\alpha})_K) \subset (f, E)$ or $\varrho_{pu}^{-1}((y_k^{\alpha})_K) \subset (g, E)$ for every fuzzy soft point $(y_k^{\alpha})_K \in (h, K)$. Therefore $(h, K) \cap \varrho_{pu}(f, E) \cap \varrho_{pu}(g, E) = \phi$. $(h, K) \subset \varrho_{pu}(f, E) \cup \varrho_{pu}(g, E)$ and $(h, K) \cap$*

$\varrho_{pu}(f,E) \neq \phi \neq (h,K) \cap \varrho_{pu}(g,E)$. *Since,* ϱ_{pu} *is fuzzy soft open mapping* $\varrho_{pu}(f,E)$ *and* $\varrho_{pu}(g,E)$ *are fuzzy soft open set over Y. Hence, (h,K) is not fuzzy soft connected.*

Sufficiency : Follows from Theorem 4.6.□

Theorem 4.8. *Let* $\varrho_{p_1u_1}$ *: (X ,*τ*,E)* $\rightarrow$ *(Y,*ϑ*,K) be a surjective fuzzy soft set-connected and* $\sigma_{p_2u_2}$ *:(Y,*ϑ*,K)* $\rightarrow$ *(Z,*η*,T) a fuzzy soft set-connected mapping. Then* $(\sigma_{p_2u_2} o \varrho_{p_1u_1})$ *: (X ,*τ*,E)* $\rightarrow$ *(Z,*η*,T) is fuzzy soft set-connected.*

Proof. *Let (h,T) be a fuzzy soft clopen set in* $\sigma_{p_2u_2}(Y)$*. Then* $\sigma^{-1}_{p_2u_2}$*(h,T) is fuzzy soft clopen over Y =* $\varrho_{p_1u_1}$ *(X) and so* $\varrho^{-1}_{p_1u_1}$ *(*$\sigma^{-1}_{p_2u_2}$*(h,T)) is fuzzy soft clopen in (X ,*τ*,E). Now* $(\sigma_{p_2u_2} o \varrho_{p_1u_1})$*(X) =* $\sigma_{p_2u_2}$ *(Y) and* $(\sigma_{p_2u_2} o \varrho_{p_1u_1})^{-1}$ *(h,T) =* $\varrho^{-1}_{p_1u_1}$ *(*$\sigma^{-1}_{p_2u_2}$*(h,T)) is fuzzy soft clopen in (X ,*τ*,E). Hence,* $(\sigma_{p_2u_2} o \varrho_{p_1u_1})$ *is fuzzy soft set connected.*□

Definition 4.2. *A fuzzy soft mapping* $\varrho_{pu} : (X,\tau,E) \rightarrow (Y,\vartheta,K)$ *is said to be fuzzy soft weakly continuous if for each fuzzy soft point* $(x^{\alpha}_e)_E \in X$ *and each fuzzy soft open set (g,K) over Y containing* $\varrho_{pu}((x^{\alpha}_e)_E)$*, there exists a fuzzy soft open set (f,E) over X containing* $(x^{\alpha}_e)_E$ *such that* ϱ_{pu}*(f,E)* $\subset$ *Cl(g,K).*

Theorem 4.9. *A soft mapping* $\varrho_{pu} : (X,\tau,E) \rightarrow (Y,\vartheta,K)$ *is fuzzy soft weakly continuous if and only if for each fuzzy soft open set (h,K) over Y,* $\varrho^{-1}_{pu}(h,K) \subset Int(\varrho^{-1}_{pu}(Cl(h,K)))$.

Proof. *Necessity : Let (h,K) be a fuzzy soft open set over Y and let* $(x^{\alpha}_e)_E \in \varrho^{-1}_{pu}(h,K)$ *then* $\varrho_{pu}((x^{\alpha}_e)_E) \in (h,K)$*. Then, there exists a fuzzy soft open set (f,E) over X such that* $(x^{\alpha}_e)_E \in (f,E)$ *and* $\varrho_{pu}(f,E) \subset Cl(h,K)$*. Hence,* $(x^{\alpha}_e)_E \in (f,E) \subset \varrho^{-1}_{pu}(Cl(h,K))$ *and* $(x^{\alpha}_e)_E \in Int(\varrho^{-1}_{pu}(Cl(h,K)))$ *since (f,E) is fuzzy soft open.*

Sufficiency : Let $(x^{\alpha}_e)_E \in X$ *and* $\varrho_{pu}((x_e)_E) \in (h,K)$*. Then* $(x_e)_E \in \varrho^{-1}_{pu}(h,K) \subset Int(\varrho^{-1}_{pu}(Cl(h,K)))$*. Let* $(f,E) = Int(\varrho^{-1}_{pu}(Cl(h,K)))$ *then (f,E) is fuzzy soft open set containing* $(x^{\alpha}_e)_E$ *and* $\varrho_{pu}(f,E) = \varrho_{pu}(Int(\varrho^{-1}_{pu}(Cl(h,K)))) \subset \varrho_{pu}(\varrho^{-1}_{pu}(Cl(h,K))) \subset Cl(h,K)$*. Hence,* ϱ_{pu} *is fuzzy soft weakly continuous.*□

Theorem 4.10. *If a* **FSTS** *space (X ,*τ*,E) is fuzzy soft connected and* $\varrho_{pu} : (X,\tau,E) \rightarrow (Y,\vartheta,K)$ *is a fuzzy soft weakly continuous surjection, then (Y,*ϑ*,K) is fuzzy soft connected.*

Proof . *Suppose* (Y,ϑ,K) *is not fuzzy soft connected. Then, there exist nonempty fuzzy soft open sets* (h_1,K) *and* (h_2,K) *in Y such that* $(h_1,K) \cap (h_2,K) = \phi$ *and* $(h_1,K) \cup (h_2,K) = \tilde{X}$*. Hence we have* $\varrho^{-1}_{pu}(h_1,K) \cap \varrho^{-1}_{pu}(h_2,K) = \phi$ *and* ϱ^{-1}_{pu}*(*h_1*,K)* $\cup$ ϱ^{-1}_{pu}*(*V_2*,K) =* $\tilde{X}$*. Since* ϱ_{pu} *is surjective,* ϱ^{-1}_{pu}*(*V_j*,K)* $\neq \phi$ *for j = 1, 2. By Theorem 4.9, we have* ϱ^{-1}_{pu}*(*h_j*,K)* $\subset$*Int (*ϱ^{-1}_{pu}*(Cl(*h_j*,K))) because* ϱ_{pu} *is fuzzy soft weakly continuous. Since (*h_j*,K) is fuzzy soft open and also fuzzy soft closed, we have* ϱ^{-1}_{pu}*(*h_j*,K)* $\subset$ *Int(* ϱ^{-1}_{pu}*(*h_j*,K. Hence,* ϱ^{-1}_{pu}*(*h_j*,K) is fuzzy soft open in X for*

j = 1,2. This implies that X is not fuzzy soft connected. This is contrary to the hypothesis that X is fuzzy soft connected. Hence, (Y,ϑ,K) is fuzzy soft connected.□

Theorem 4.11. *A fuzzy soft mapping $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ is fuzzy soft weakly continuous, then $Cl(\varrho_{pu}^{-1}(h, K)) \subset (\varrho_{pu}^{-1}(Cl(h, K))$ for each fuzzy soft open set (h,K) over Y.*

Proof. *Suppose there exists a fuzzy soft point $(x_e^{\alpha})_E \in Cl(\varrho_{pu}^{-1}(h, K)) - \varrho_{pu}^{-1}(Cl(h, K))$. Then $\varrho_{pu}((x_e)_E) \notin Cl(h, K)$. Hence there exists a fuzzy soft open set (g ,K) containing $\varrho_{pu}((x_e)_E)$ such that $(g, K) \cap (V, K) = \phi$. Since (h,K) is fuzzy soft open set over Y, we have $(h, K) \cap Cl(g, K) = \phi$. Since ϱ_{pu} is fuzzy soft weakly continuous ,there exists a fuzzy soft open set (f,E) over X containing $(x_e^{\alpha})_E$ such that $\varrho_{pu}(f, E) \subset Cl(g, K)$. Thus, we obtain $\varrho_{pu}(f, E) \cap (h, K) = \phi$. On the other hand, since $(x_e^{\alpha})_E \in Cl(\varrho_{pu}^{-1}(h, K))$, we have $(f, E) \cap \varrho_{pu}^{-1}(h, K) \neq \phi$ and hence, $\varrho_{pu}(f, E) \cap (h, K) \neq \phi$. Thus we have a contradiction. Hence $Cl(\varrho_{pu}^{-1}(h, K)) \subset (\varrho_{pu}^{-1}(Cl(h, K))$.*□

Theorem 4.12. *If a fuzzy soft surjection ϱ_{pu} : (X ,τ,E) $\to$ (Y,ϑ,K) is fuzzy soft weakly continuous ,then ϱ_{pu} is fuzzy soft set-connected.*

Proof. *Let (h,K) be any fuzzy soft clopen set over Y. Since (h,K) is fuzzy soft closed, We have Cl(h,K) = (h,K). Thus, by Theorem 4.9, we obtain $\varrho_{pu}^{-1}(h, K) \subset Int(\varrho_{pu}^{-1}(h, K))$. This shows that $\varrho_{pu}^{-1}(h, K)$ is fuzzy soft open set over X. Moreover , by Theorem 4.11, we obtain $Cl(\varrho_{pu}^{-1}(h, K)) \subset \varrho_{pu}^{-1}(h, K)$. This shows that $\varrho_{pu}^{-1}(h, K)$ is a fuzzy soft closed set over X. Since ϱ_{pu} is fuzzy soft surjection, by Theorem 4.1, we observe that ϱ_{pu} is a fuzzy soft set-connected mapping.*□

Remark 4.2. *The converse of Theorem 4.12 is not true.*

Example 4.2. *Let$X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and $Y = \{y_1, y_2\}$, $K = \{k_1, k_2\}$. The fuzzy soft sets (f,E)and (g,K) are defined as follows :*

$$f(e_1) = \{x_1 = 0.3, x_2 = 0\}, \qquad f(e_2) = \{x_1 = 0, x_2 = 0.4\}$$
$$g(k_1) = \{y_1 = 0.4, y_2 = 0\}, \qquad g(k_2) = \{y_1 = 0, y_2 = 0.5\}$$

Let τ = $\{\tilde{0_E}, \tilde{1_E}$,(h,E) $\}$ and υ = $\{\tilde{0_K}, \tilde{1_K}$, (G ,K) $\}$ are fuzzy soft topologies on X and Y respectively. Then fuzzy soft mapping $\varrho_{pu} : (X, \tau, E) \to (Y, \upsilon, K)$ defined by u(x_1) = u(x_2)=y_1 and p(e_1)= k_1 , p(e_2) = k_2 is fuzzy soft set-connected but it is not fuzzy soft weakly continuous.

Theorem 4.13. *Let (Y,ϑ,K) be an fuzzy soft extremally disconnected space .If a fuzzy soft mapping $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ is fuzzy soft set-connected, then ϱ_{pu} is fuzzy soft weakly continuous.*

Proof. *Let $(x_e^{\alpha})_E$ be a fuzzy soft point of X and (g,K) any fuzzy soft open set over Y containing $\varrho_{pu}((x_e^{\alpha})_E)$. Since (Y,$\vartheta$,K) is fuzzy soft extremally disconnected,*

Cl(g,K) is fuzzy soft clopen set over Y. Thus $Cl(g,K) \cap \varrho_{pu}(\tilde{X})$ *is fuzzy soft clopen set in the fuzzy soft subspace (*$\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K$*). Put* $\varrho^{-1}(Cl(g,K) \cap \varrho_{pu}(\tilde{X})) = (f,E)$*. Then, since* ϱ_{pu} *is fuzzy soft set-connected, it follows from Theorem 4.1 that (f,E) is fuzzy soft clopen set over X. Therefore, (f,E) is a fuzzy soft open set containing* $(x_e^{\alpha})_E$ *over X such that* $\varrho_{pu}(f,E) \subset Cl(g,K)$*. This implies that* ϱ_{pu} *is fuzzy soft weakly continuous.*□

Theorem 4.14. *Let* (Y, ϑ, K) *be a fuzzy soft extremally disconnected space. A fuzzy soft surjection mapping* $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ *is fuzzy soft set-connected if and only if* ϱ_{pu} *is fuzzy soft weakly continuous.*

Proof. *It follows from Theorem 4.12 and Theorem 4.13.*□

5 Conclusions

Connectedness is an important and major area of topology and it can give many relationships between other scientific areas and mathematical models. The notion of connectedness captures the idea of hanging-togetherness of image elements in an object by assigning a strength of connectedness to every possible path between every possible pair of image elements. This paper, introduces the notion of fuzzy soft connectedness between fuzzy soft sets in fuzzy soft topological spaces. It is shown that a fuzzy soft topological space is fuzzy soft connected if and only if it is fuzzy soft connected between every pair of its nonempty fuzzy soft sets. Further two new classes of fuzzy soft mappings called fuzzy soft set connected and soft weakly continuous have been introduced. It is shown that the class of fuzzy soft set connected (respt. fuzzy soft weakly continuous) mappings properly contains the class of all fuzzy soft continuous mappings. Several properties and characterizations of fuzzy soft set connected and fuzzy soft weakly continuous mappings have been studied. Hope that the concepts and results established in this paper will help researcher to enhance and promote the further study on fuzzy soft topology to carry out a general framework for the development of information systems.

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Anti-Adjacency Matrices of Certain Graphs Derived from Some Graph Operations

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Abstract

If we go through the literature, one can find many matrices that are derived for a given simple graph. The one among them is the anti-adjacency matrix which is given as follows; The anti-adjacency matrix of a simple undirected graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is an $n \times n$ matrix $B(G) = (b_{ij})$, where $b_{ij} = 0$ if there exists an edge between v_i and v_j and 1 otherwise. In this paper, we try to bring out an expression, which establishes a connection between the anti-adjacency matrices of the two graphs G_1 and G_2 and the anti-adjacency matrix of their tensor product, $G_1 \otimes G_2$. In addition, an expression for the anti-adjacency matrix of the disjunction of two graphs, $G_1 \vee G_2$, is obtained in a similar way. Finally, we obtain an expression for the anti-adjacency matrix for the generalized tensor product and generalized disjunction of two graphs. Adjacency and anti-adjacency matrices are square matrices that are used to represent a finite graph in graph theory and computer science. The matrix elements show whether a pair of vertices in the graph are adjacent or not.

Keywords: Anti-adjacency matrix, tensor product, disjunction, generalized tensor product and generalized disjunction.
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1 Introduction

The tensor product (conjunction) and disjunction of two graphs were first described in 1967 by Frank Harary and Gordon W. Wilcox [2]. The idea of the generalized tensor product of two graphs was first suggested in 2014 by U P Acharya and H S Mehta [5]. The concept of generalized disjunction of two graphs with respect to distance as a parameter was first suggested in 2020 by Manju V N and G Suresh Singh [4]. M. Edwina and K.A. Sugeng established the concept of anti-adjacency matrix for undirected graphs in 2017[1]. Let G be a simple undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. According to [1], the anti-adjacency matrix of G, denoted by $B(G)$ is the $n \times n$ matrix $B = (b_{ij})$, where $b_{ij} = 0$ if $v_i v_j \in E(G)$ or otherwise $b_{ij} = 1$. Some properties of anti-adjacency matrix are related to its characteristic polynomial and its eigenvalues. In chemical graph theory, the physical characteristics of chemical graphs can be obtained by using the eigenvalues of the anti-adjacency matrix of graphs. This motivates us to compute the anti-adjacency matrix of some graphs obtained by few graph theoretical operations.

In this paper, we first obtain an expression for the anti-adjacency matrix of tensor product and disjunction of two graphs. Next, we define the r^{th} stage anti-adjacency matrix of a simple undirected graph and then try to get an expression for the anti-adjacency matrix for the generalized tensor product and generalized disjunction of two graphs.

Definition 1.1. *[3] The distance between two vertices u and v in a connected graph G, denoted by $d_G(u, v)$ is the length of the shortest path between u and v in G.*

Definition 1.2. *[2] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The tensor product (or conjunction) of G_1 and G_2, denoted by $G_1 \otimes G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \otimes G_2$ if u is adjacent to u' in G_1 and v is adjacent to v' in G_2.*

Definition 1.3. *[5] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The m-tensor product of G_1 and G_2, denoted by $G_1 \otimes_m G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \otimes_m G_2$ if $d_{G_1}(u, u') = m$ and $d_{G_2}(v, v') = m$.*

It is clear that when $m = 1$, we get definition 1.2. When $m = 2$, we have the following

Definition 1.4. *[5] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The 2-tensor product of G_1 and G_2, denoted by $G_1 \otimes_2 G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \otimes_2 G_2$ if $d_{G_1}(u, u') = 2$ and $d_{G_2}(v, v') = 2$.*

Definition 1.5. *[2] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The disjunction of G_1 and G_2, denoted by $G_1 \vee G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \vee G_2$ if u is adjacent to u' in G_1 or v is adjacent to v' in G_2 or both*

Definition 1.6. *[4] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The $m-$ disjunction of G_1 and G_2, denoted by $G_1 \vee_m G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \vee_m G_2$ if $d_{G_1}(u, u') = m$ or $d_{G_2}(v, v') = m$ or both.*

It is clear that when $m = 1$, we get definition 1.5. When $m = 2$, we have

Definition 1.7. *[4] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The $2-$ disjunction of G_1 and G_2, denoted by $G_1 \vee_2 G_2$ is the graph with vertex set $V = V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent in $G_1 \vee_2 G_2$ if $d_{G_1}(u, u') = 2$ or $d_{G_2}(v, v') = 2$ or both.*

Definition 1.8. *[1] Let G be a simple undirected graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. The anti-adjacency matrix of G, denoted by $B(G)$ is the $n \times n$ matrix $B = [b_{ij}]$, where $b_{ij} = 0$ if $e = v_i v_j \in E(G)$ or otherwise $b_{ij} = 1$.*

Definition 1.9. *[2] Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be a $p \times q$ matrix, then their tensor product denoted by $A \otimes B$ is an $mp \times nq$ matrix and is defined as*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

All graphs under our consideration are simple, finite undirected connected graphs.

2 Anti-Adjacency Matrix of $m-$ Tensor Product of Graphs

In this section we study some properties of the anti-adjacency matrix of $G_1 \otimes G_2$ and generalize the result for m- tensor product of G_1 and G_2.

Theorem 2.1. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \otimes G_2$ is given by $B(G_1 \otimes G_2) = B(G_1) \otimes J_n \oplus J_m \otimes B(G_2) \oplus B(G_1) \otimes B(G_2)$, where J_n and J_m are the $n \times n$ and $m \times m$ matrices with every entry 1, $\oplus$ denotes addition modulo 2 and $\otimes$ denotes the tensor product.*

Proof. Let $V(G_1) = \{u_1, u_2, ..., u_m\}$ and $V(G_2) = \{v_1, v_2, ..., v_n\}$. Then consider $B(G_1 \otimes G_2)$,

$$B(G_1 \otimes G_2) = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} & \cdots & B_{1m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rs} & ... & B_{rm} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{ms} & \cdots & B_{mm} \end{bmatrix}$$

where

$$\mathrm{B}_{rs} = \begin{array}{c} \\ {\scriptstyle (u_r,v_1)} \\ \vdots \\ {\scriptstyle (u_r,v_n)} \end{array} \begin{array}{c} \begin{array}{cccc} {\scriptstyle (u_s,v_1)} & {\scriptstyle (u_s,v_2)} & {\scriptstyle ...} & {\scriptstyle (u_s,v_n)} \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & ... & P_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

Suppose $d_{G_1}(u_r, u_s) \neq 1$, then $P_{ij} = 1 \; \forall \; i$ and j. So, in this case $B_{rs} = J_n$.
Next suppose that $d_{G_1}(u_r, u_s) = 1$, then $P_{ij} = \begin{cases} 0 & \text{if } d_{G_2}(v_i, v_j) = 1 \\ 1 & \text{otherwise} \end{cases}$
Thus in this case, $P_{ij} = (i, j)^{th}$ entry of $B(G_2)$. Hence,

$$B_{rs} = \begin{cases} B(G_2) & \text{if } d_{G_1}(u_r, u_s) = 1 \\ J_n & \text{otherwise} \end{cases}$$

Assume, $B(G_1) = [b_{ij}]_{m \times m}$ and $B(G_2) = [c_{ij}]_{n \times n}$ and consider

$$B(G_1) \otimes J_n = \begin{bmatrix} b_{11}J_n & b_{12}J_n & \cdots & b_{1m}J_n \\ b_{21}J_n & b_{22}J_n & \cdots & b_{2m}J_n \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}J_n & b_{m2}J_n & \cdots & b_{mm}J_n \end{bmatrix}$$

Then the $(i, j)^{th}$ block of $B(G_1) \otimes J_n = [b_{ij}J_n]$ and

$$[b_{ij}J_n] = \begin{cases} (0)_{n \times n} & \text{if } b_{ij} = 0 \text{ (ie, if} d_{G_1}(u_i, u_j) = 1) \\ J_n & \text{otherwise} \end{cases}$$

Now,

$$J_m \otimes B(G_2) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \otimes B(G_2)$$
$$= \begin{bmatrix} B(G_2) & B(G_2) & \cdots & B(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ B(G_2) & B(G_2) & \cdots & B(G_2) \end{bmatrix}$$

So, the $(i,j)^{th}$ block of $J_m \otimes B(G_2)$ is $B(G_2)\ \forall\ i$ and j. Next consider,

$$B(G_1) \otimes B(G_2) = \begin{bmatrix} b_{11}B(G_2) & b_{12}B(G_2) & \cdots & b_{1m}B(G_2) \\ b_{21}B(G_2) & b_{22}B(G_2) & \cdots & b_{2m}B(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}B(G_2) & b_{m2}B(G_2) & \cdots & b_{mm}B(G_2) \end{bmatrix}$$

Hence, the $(i,j)^{th}$ block of $B(G_1) \otimes B(G_2)$ is $b_{ij}B(G_2)$ and

$$b_{ij}B(G_2) = \begin{cases} (0)_{n\times n} & \text{if } b_{ij} = 0 \text{ (ie, if } d_{G_1}(u_i, u_j) = 1) \\ B(G_2) & \text{if } b_{ij} = 1 \text{ (ie, if } d_{G_1}(u_i, u_j) \neq 1) \end{cases}$$

It follows that the $(i,j)^{th}$ block of $B(G_1) \otimes J_n \oplus J_m \otimes B(G_2) \oplus B(G_1) \otimes B(G_2)$

$$= \begin{cases} (0)_{n\times n} \oplus B(G_2) \oplus (0)_{n\times n} & \text{if } d_{G_1}(u_i, u_j) = 1 \\ J_n \oplus B(G_2) \oplus B(G_2) & \text{if } d_{G_1}(u_i, u_j) \neq 1 \end{cases}$$
$$= \begin{cases} B(G_2) & \text{if } d_{G_1}(u_i, u_j) = 1 \\ J_n & \text{if } d_{G_1}(u_i, u_j) \neq 1 \end{cases}$$

Therefore, the $(i,j)^{th}$ block of $B(G_1) \otimes J_n \oplus J_m \otimes B(G_2) \oplus B(G_1) \otimes B(G_2)$ is equal to the $(i,j)^{th}$ block of $B(G_1 \otimes G_2)$. Hence, $B(G_1 \otimes G_2) = B(G_1) \otimes J_n \oplus J_m \otimes B(G_2) \oplus B(G_1) \otimes B(G_2)$. □

Definition 2.1. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. The second stage anti-adjacency matrix of G, $B_2(G)$ is the $n \times n$ matrix $B_2(G) = [b_{ij}]$, where*

$$b_{ij} = \begin{cases} 0 & \textit{if } d_G(v_i, v_j) = 2 \\ 1 & \textit{otherwise} \end{cases}.$$

Note that $B_2(G)$ is real, square symmetric matrix.

Theorem 2.2. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \otimes_2 G_2$ is given by $B(G_1 \otimes_2 G_2) = B_2(G_1) \otimes J_n \oplus J_m \otimes B_2(G_2) \oplus B_2(G_1) \otimes B_2(G_2)$, where J_n and J_m are the $n \times n$ and $m \times m$ matrices with every entry 1, $\oplus$ denotes addition modulo 2 and $\otimes$ denotes the tensor product.*

Proof. Let $V(G_1) = \{u_1, u_2, ..., u_m\}$ and $V(G_2) = \{v_1, v_2, ..., v_n\}$. Then consider $B(G_1 \otimes_2 G_2)$,

$$B(G_1 \otimes_2 G_2) = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} & \cdots & B_{1m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rs} & ... & B_{rm} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{ms} & \cdots & B_{mm} \end{bmatrix}$$

where

$$\mathrm{B}_{rs} = \begin{array}{c} \\ {\scriptstyle (u_r,v_1)} \\ \vdots \\ {\scriptstyle (u_r,v_n)} \end{array} \begin{array}{c} \begin{array}{cccc} {\scriptstyle (u_s,v_1)} & {\scriptstyle (u_s,v_2)} & {\scriptstyle \cdots} & {\scriptstyle (u_s,v_n)} \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & ... & P_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

Suppose $d_{G_1}(u_r, u_s) \neq 2$, then $P_{ij} = 1 \; \forall \; i$ and j. So, in this case $B_{rs} = J_n$.

Next suppose that $d_{G_1}(u_r, u_s) = 2$, then $P_{ij} = \begin{cases} 0 & \text{if } d_{G_2}(v_i, v_j) = 2 \\ 1 & \text{otherwise} \end{cases}$

Thus in this case, $P_{ij} = (i, j)^{th}$ entry of $B_2(G_2)$. Hence,

$$B_{rs} = \begin{cases} B_2(G_2) & \text{if } d_{G_1}(u_r, u_s) = 2 \\ J_n & \text{otherwise} \end{cases}$$

Assume, $B_2(G_1) = [b_{ij}]_{m \times m}$ and $B_2(G_2) = [c_{ij}]_{n \times n}$ and consider

$$B_2(G_1) \otimes J_n = \begin{bmatrix} b_{11}J_n & b_{12}J_n & \cdots & b_{1m}J_n \\ b_{21}J_n & b_{22}J_n & \cdots & b_{2m}J_n \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}J_n & b_{m2}J_n & \cdots & b_{mm}J_n \end{bmatrix}$$

Then the $(i, j)^{th}$ block of $B_2(G_1) \otimes J_n = [b_{ij}J_n]$ and

$$[b_{ij}J_n] = \begin{cases} (0)_{n \times n} & \text{if } b_{ij} = 0 \text{ (ie, if } d_{G_1}(u_i, u_j) = 2) \\ J_n & \text{otherwise} \end{cases}$$

Now,

$$J_m \otimes B_2(G_2) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \otimes B_2(G_2)$$

$$= \begin{bmatrix} B_2(G_2) & B_2(G_2) & \cdots & B_2(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_2(G_2) & B_2(G_2) & \cdots & B_2(G_2) \end{bmatrix}$$

So, the $(i,j)^{th}$ block of $J_m \otimes B_2(G_2)$ is $B_2(G_2)\ \forall\ i$ and j. Next consider,

$$B_2(G_1) \otimes B_2(G_2) = \begin{bmatrix} b_{11}B_2(G_2) & b_{12}B_2(G_2) & \cdots & b_{1m}B_2(G_2) \\ b_{21}B_2(G_2) & b_{22}B_2(G_2) & \cdots & b_{2m}B_2(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}B_2(G_2) & b_{m2}B_2(G_2) & \cdots & b_{mm}B_2(G_2) \end{bmatrix}$$

Hence, the $(i,j)^{th}$ block of $B_2(G_1) \otimes B_2(G_2)$ is $b_{ij}B_2(G_2)$ and

$$b_{ij}B_2(G_2) = \begin{cases} (0)_{n\times n} & \text{if } b_{ij} = 0 \text{ (ie, if } d_{G_1}(u_i,u_j) = 2) \\ B_2(G_2) & \text{if } b_{ij} = 1 \text{ (ie, if } d_{G_1}(u_i,u_j) \neq 2) \end{cases}$$

It follows that the $(i,j)^{th}$ block of $B_2(G_1)\otimes J_n \oplus J_m \otimes B_2(G_2) \oplus B_2(G_1)\otimes B_2(G_2)$

$$= \begin{cases} (0)_{n\times n} \oplus B_2(G_2) \oplus (0)_{n\times n} & \text{if } d_{G_1}(u_i,u_j) = 2 \\ J_n \oplus B_2(G_2) \oplus B_2(G_2) & \text{if } d_{G_1}(u_i,u_j) \neq 2 \end{cases}$$

$$= \begin{cases} B_2(G_2) & \text{if } d_{G_1}(u_i,u_j) = 2 \\ J_n & \text{if } d_{G_1}(u_i,u_j) \neq 2 \end{cases}$$

Therefore, the $(i,j)^{th}$ block of $B_2(G_1) \otimes J_n \oplus J_m \otimes B_2(G_2) \oplus B_2(G_1) \otimes B_2(G_2)$ is equal to the $(i,j)^{th}$ block of $B(G_1 \otimes_2 G_2)$. Hence, $B(G_1 \otimes_2 G_2) = B_2(G_1) \otimes J_n \oplus J_m \otimes B_2(G_2) \oplus B_2(G_1) \otimes B_2(G_2)$. □

Definition 2.2. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. The r^{th} stage anti-adjacency matrix of G, $B_r(G)$ is the $n \times n$ matrix $B_r(G) = [b_{ij}]$, where*

$$b_{ij} = \begin{cases} 0 & \textit{if } d_G(v_i,v_j) = r \\ 1 & \textit{otherwise} \end{cases}.$$

Theorem 2.3. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \otimes_m G_2$ is given by $B(G_1 \otimes_m G_2) = B_m(G_1) \otimes J_n \oplus J_m \otimes B_m(G_2) \oplus B_m(G_1) \otimes B_m(G_2)$, where J_n and J_m are the $n \times n$ and $m \times m$ matrices with every entry 1, $\oplus$ denotes addition modulo 2, $\otimes$ denotes the tensor product and $B_m(G)$ is the m^{th} stage anti-adjacency matrix of G.*

Proof. Proof is same as the proof of Theorem 2.2. □

3 Anti-adjacency Matrix of $m-$ Disjunction of G_1 and G_2

In this section we study some properties of the anti-adjacency matrix of $G_1 \vee G_2$ and generalize the result for m- disjunction G_1 and G_2.

Theorem 3.1. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \vee G_2$ is given by $B(G_1 \vee G_2) = B(G_1) \otimes B(G_2)$, where $B(G)$ is the anti-adjacency matrix of G and $\otimes$ denotes the tensor product.*

Proof. Let $V(G_1) = \{u_1, u_2, ..., u_m\}$ and $V(G_2) = \{v_1, v_2, ..., v_n\}$. Then consider $B(G_1 \vee G_2)$,

$$B(G_1 \vee G_2) = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} & \cdots & B_{1m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rs} & ... & B_{rm} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{ms} & \cdots & B_{mm} \end{bmatrix}$$

where

$$\mathrm{B}_{rs} = \begin{array}{c} \\ {\scriptstyle (u_r,v_1)} \\ \vdots \\ {\scriptstyle (u_r,v_n)} \end{array} \begin{array}{c} \begin{array}{cccc} {\scriptstyle (u_s,v_1)} & {\scriptstyle (u_s,v_2)} & {\scriptstyle \cdots} & {\scriptstyle (u_s,v_n)} \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & ... & P_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

Suppose $d_{G_1}(u_r, u_s) = 1$, then $P_{ij} = 0 \; \forall \; i$ and j. So, in this case B_{rs} is the zero matrix of order $n \times n$.
Next suppose that $d_{G_1}(u_r, u_s) \neq 1$, then

$$P_{ij} = \begin{cases} 0 & \text{if } d_{G_2}(v_i, v_j) = 1 \\ 1 & \text{otherwise} \end{cases}$$

Thus in this case, $P_{ij} = (i, j)^{th}$ entry of $B(G_2)$. Hence,

$$B_{rs} = \begin{cases} B(G_2) & \text{if } d_{G_1}(u_r, u_s) \neq 1 \\ (0)_{n \times n} & \text{otherwise} \end{cases}$$

Assume, $B(G_1) = [b_{ij}]_{m\times m}$ and $B(G_2) = [c_{ij}]_{n\times n}$ and consider

$$B(G_1) \otimes B(G_2) = \begin{bmatrix} b_{11}B(G_2) & b_{12}B(G_2) & \cdots & b_{1m}B(G_2) \\ b_{21}B(G_2) & b_{22}B(G_2) & \cdots & b_{2m}B(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}B(G_2) & b_{m2}B(G_2) & \cdots & b_{mm}B(G_2) \end{bmatrix}$$

Then the $(i,j)^{th}$ block of $B(G_1) \otimes B(G_2) = [b_{ij}B(G_2)]$ and

$$[b_{ij}B(G_2)] = \begin{cases} B(G_2) & \text{if } b_{ij} = 1\text{(ie, if } d_{G_1}(u_i, u_j) \neq 1) \\ (0)_{n\times n} & \text{if } b_{ij} = 0 \text{ (ie, if } d_{G_1}(u_i, u_j) = 1) \end{cases}$$

Hence, the $(i,j)^{th}$ block of $B(G_1)\otimes B(G_2)$ is equal to the $(i,j)^{th}$ block of $B(G_1 \vee G_2)$.
Therefore, $B(G_1 \vee G_2) = B(G_1) \otimes B(G_2)$.

□

Theorem 3.2. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \vee_2 G_2$ is given by $B(G_1 \vee_2 G_2) = B_2(G_1) \otimes B_2(G_2)$, where $B_2(G)$ is the second stage anti-adjacency matrix of G and $\otimes$ denotes the tensor product.*

Proof. Let $V(G_1) = \{u_1, u_2, ..., u_m\}$ and $V(G_2) = \{v_1, v_2, ..., v_n\}$. Then consider $B(G_1 \vee_2 G_2)$,

$$B(G_1 \vee_2 G_2) = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} & \cdots & B_{1m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rs} & ... & B_{rm} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{ms} & \cdots & B_{mm} \end{bmatrix}$$

where

$$\mathbf{B}_{rs} = \begin{array}{c} \\ (u_r,v_1) \\ \vdots \\ (u_r,v_n) \end{array} \begin{array}{c} \begin{array}{cccc} (u_s,v_1) & (u_s,v_2) & ... & (u_s,v_n) \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & ... & P_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

Suppose $d_{G_1}(u_r, u_s) = 2$, then $P_{ij} = 0 \ \forall \ i$ and j. So, in this case B_{rs} is the zero matrix of order $n \times n$.
Next suppose that $d_{G_1}(u_r, u_s) \neq 2$, then

$$P_{ij} = \begin{cases} 0 & \text{if } d_{G_2}(v_i, v_j) = 2 \\ 1 & \text{otherwise} \end{cases}$$

Thus in this case, $P_{ij} = (i,j)^{th}$ entry of $B_2(G_2)$. Hence,

$$B_{rs} = \begin{cases} B_2(G_2) & \text{if } d_{G_1}(u_r, u_s) \neq 2 \\ (0)_{n\times n} & \text{otherwise} \end{cases}$$

Assume, $B_2(G_1) = [b_{ij}]_{m\times m}$ and $B_2(G_2) = [c_{ij}]_{n\times n}$ and consider

$$B_2(G_1) \otimes B_2(G_2) = \begin{bmatrix} b_{11}B_2(G_2) & b_{12}B_2(G_2) & \cdots & b_{1m}B_2(G_2) \\ b_{21}B_2(G_2) & b_{22}B_2(G_2) & \cdots & b_{2m}B_2(G_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}B_2(G_2) & b_{m2}B_2(G_2) & \cdots & b_{mm}B_2(G_2) \end{bmatrix}$$

Then the $(i,j)^{th}$ block of $B_2(G_1) \otimes B_2(G_2) = [b_{ij}B_2(G_2)]$ and

$$[b_{ij}B_2(G_2)] = \begin{cases} B_2(G_2) & \text{if } b_{ij} = 1 \text{(ie, if } d_{G_1}(u_i, u_j) \neq 2) \\ (0)_{n\times n} & \text{if } b_{ij} = 0 \text{ (ie, if } d_{G_1}(u_i, u_j) = 2) \end{cases}$$

Hence, the $(i,j)^{th}$ block of $B_2(G_1) \otimes B_2(G_2)$ is equal to the $(i,j)^{th}$ block of $B(G_1 \vee_2 G_2)$.
Therefore, $B(G_1 \vee_2 G_2) = B_2(G_1) \otimes B_2(G_2)$.

□

Theorem 3.3. *Let G_1 and G_2 be two graphs of order m and n respectively. Then, the anti-adjacency matrix of $G_1 \vee_m G_2$ is given by $B(G_1 \vee_m G_2) = B_m(G_1) \otimes B_m(G_2)$, where $B_m(G)$ is the m^{th} stage anti-adjacency matrix of G and $\otimes$ denotes the tensor product.*

Proof. The Proof is same as the proof of Theorem 3.2. □

4 Conclusions

In this paper, we found the relation between the anti-adjacency matrices $B(G_1 \otimes G_2)$, $B(G_1)$ and $B(G_2)$. Also, we obtained the relationship between $B(G_1 \vee G_2), B(G_1)$ and $B(G_2)$. Further, we introduced the concept of r^{th} stage anti-adjacency matrix of a simple undirected graph and then tried to find the anti-adjacency matrices of graphs derived from the graph operations generalized tensor product and generalized disjunction. We strongly believe that this work can be further extended to find the anti-adjacency matrices of graphs, derived from other graph theoretical operations such as composition, Cartesian product etc.

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Commutative Medial Near Ring

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Abstract

We deliberate a substructure of a near ring called mediality so as to create a new platform, on which many of the properties like commutativity, regular, idempotent and zero symmetric are applied with. It is shown that every medial near ring is commutative whenever cancellation law holds. Commutative medial near ring is left medial, right medial, reverse, reverse medial, reverse left medial and reverse right medial. It is proved that a non-trivial left medial regular near ring is reduced. Also a structure theorem has been obtained.
Keywords: Commutative; Medial; Nilpotent; Regular; Zero symmetric.
2020 AMS subject classifications: 16Y30.[1]

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1 Introduction

It is common to refer to algebra as the mathematical language. Algebra and Algorithm both have the same historical antecedent. Near rings are generalized rings: in a ring, we get a near ring if we ignore the commutativity of addition and one distributive law. Gunter Pilz [11] "Near Rings" is a significant compilation of work done in the field of near rings.

This paper mostly focuses on medial near rings. The theory of quasi groups has developed the identity $xyzt = xzyt$. In the year 1987, Silvia Pellegrini Manara [8] defined such near rings as medial near rings. Similar to the Strong IFP near rings, medial near rings feature a subdirect structure. He also attained a condition that is both required and sufficient for a near ring to be regular, medial and subdirectly irreducible. Two years later, G. Birkenmeier and H. Heatherly [6] discussed (left) near rings satisfying the medial identity. On mediality in semigroups several authors such as Attila Nagy [9], Fitore Abdullahu and Abdullah Zejnullahu [1], Muhammad Akram, Naveed Yaqoob and Madad [2] have contributed their work. In 2015, R. Perumal and P. Chinnaraj [10] introduced the concept of medial left bipotent seminear rings and discussed some of its properties. Also, a description of such a seminear ring has also been provided. Later, in the year 2019, A. O. Atagun, H. Kamaci, I. Tastekin and A. Sezgin [3] assumed that N is a near ring and P is an ideal of N and defined P-medial near rings with some properties. Furthermore, numerous examples are used to demonstrate the results.

Now this paper deals with some distinct near ring structures namely left medial, right medial, reverse, reverse medial, reverse left medial, reverse right medial, medial pseudo commutative, medial quasi weak commutative, medial cyclic commutative and medial quasi cyclic weak commutative. These near rings are defined and illustrated with some examples. Using these near rings, several results have been demonstrated.

$\aleph$ represents a right near ring $(\aleph, +, .)$ (That is a right near ring is a non-empty set $\aleph$ combined with two binary operations "+" and "." in such a way that (i) $(\aleph$,+) is a group, (ii) $(\aleph, .)$ is a semigroup, (iii) $(\eta_1 + \eta_2)\eta_3 = \eta_1\eta_3 + \eta_2\eta_3$ for all $\eta_1, \eta_2, \eta_3 \in \aleph$.) [11] in this paper, with at least two elements and '0' represents the identity element of the group $(\aleph, +)$ and we write kl for any two elements k, l of $\aleph$. Obviously $0\eta = 0$ for all $\eta \in \aleph$. If, additionaly, $\eta 0 = 0$ for all $\eta \in \aleph$ then we say that $\aleph$ is zero symmetric [11].

2 Preliminaries

Definition 2.1. *[3],[6],[8],[12]* $\aleph$ *is referred to as a medial near ring if* $klmn = kmln$ *for all* $k, l, m, n \in \aleph$.

Definition 2.2. *[13] A near ring $\aleph$ is referred to as a pseudo commutative near ring if $klm = mlk$ for all $k, l, m \in \aleph$.*

Definition 2.3. *[7] A near ring $\aleph$ is referred to as regular if for each $x \in \aleph$, there exists $y \in \aleph$ such that $x = xyx$.*

Definition 2.4. *[5] An element $b \in \aleph$ is referred to be nilpotent if there exists a positive integer k such that $b^k = 0$.*

Definition 2.5. *[5] An element $b \in \aleph$ is referred to be idempotent if $b^2 = b$.*

Definition 2.6. *[4] A near ring $\aleph$ is referred to as a P_k near ring if there exists a positive integer k such that $x^k\aleph = x\aleph x$ for all x in $\aleph$.*

3 Main Results

Definition 3.1. *A near ring $\aleph$ is referred to as Left Medial if $klmn = mkln$ for all $k, l, m, n \in \aleph$.*

Example 3.2. *Consider the near ring $(\aleph, +, .)$ where $(\aleph, +)$ represents the Klein's four group $\{0,k.l,m\}$ and '.' is defined as follows (scheme 14, p.408 of [11]).*

.	0	k	l	m
0	0	0	0	0
k	0	k	0	m
l	0	0	0	0
m	0	k	0	m

Table 1: Multiplication Table

which is a left medial near ring and medial near ring.

Definition 3.3. *A near ring $\aleph$ is referred to as Right Medial if $klmn = kmnl$ for all $k, l, m, n \in \aleph$.*

Example 3.4. *(i) Example (3.2) demonstrates that $\aleph$ is not a Right Medial near ring (Since $kkkm \neq kkmk$).*
(ii) The near ring $(\aleph, +, .)$ defined on the Klein's four group $\aleph = \{0, k.l, m\}$ and where multiplication is defined as follows (scheme 2, p.408 of [11]).

is a right medial near ring and medial near ring.

.	0	k	l	m
0	0	0	0	0
k	0	0	k	k
l	0	k	l	l
m	0	k	m	m

Table 2: Multiplication Table

Note 3.5. *In any near ring, left mediality and right mediality does not imply mediality.*

Definition 3.6. *A near ring $\aleph$ is referred to as Reverse if $klmn = nmlk$ for all $k, l, m, n \in \aleph$.*

Example 3.7. *(i) Example (3.2) demonstrates that $\aleph$ is not a reverse near ring (Since $kkkm \neq mkkk$).*
(ii) Example (3.4 (ii)) demonstrates that $\aleph$ is not a reverse near ring (Since $lllm \neq mlll$).

Definition 3.8. *A near ring $\aleph$ is referred to as Reverse Medial if $klmn = nlmk$ for all $k, l, m, n \in \aleph$.*

Example 3.9. *(i) Example (3.2) demonstrates that $\aleph$ is not a Reverse Medial near ring (Since $kkkm \neq mkkk$).*
(ii) Example (3.4 (ii)) demonstrates that $\aleph$ is not a reverse medial near ring (Since $mlll \neq lllm$).

Definition 3.10. *A near ring $\aleph$ is referred to as Reverse Left Medial if $klmn = nlkm$ for all $k, l, m, n \in \aleph$.*

Example 3.11. *(i) Example (3.2) demonstrates that $\aleph$ is not a reverse left medial near ring (Since $kkkm \neq mkkk$).*
(ii) Example (3.4 (ii)) demonstrates that $\aleph$ is not a reverse left medial near ring (Since $mlll \neq llml$).

Definition 3.12. *A near ring $\aleph$ is referred to as Reverse Right Medial if $klmn = lnmk$ for all $k, l, m, n \in \aleph$.*

Example 3.13. *(i) The near ring ($\aleph$,+,.) defined on the Klein's four group $\aleph = \{0, k, l, m\}$ where multiplication is defined as follows (scheme 4, p.408, [11]). which is a reverse right medial near ring and medial near ring.*
(ii) Example (3.2) demonstrates that $\aleph$ is not a reverse right medial near ring (Since $kkkm \neq kmkk$).
(iii) Example (3.4 (ii)) demonstrates that $\aleph$ is not a reverse right medial near ring (Since $lmll \neq mlll$).

.	0	k	l	m
0	0	0	0	0
k	0	0	k	k
l	0	k	m	l
m	0	k	l	m

Table 3: Multiplication Table

Definition 3.14. *A near ring $\aleph$ is referred to as Medial Pseudo Commutative if $klmn = knml$ for all $k, l, m, n \in \aleph$.*

Definition 3.15. *A near ring $\aleph$ is referred to as Medial Quasi Weak Commutative if $klmn = mlkn$ for all $k, l, m, n \in \aleph$.*

Definition 3.16. *A near ring $\aleph$ is referred to as Medial Cyclic Commutative if $klmn = lmkn$ for all $k, l, m, n \in \aleph$.*

Definition 3.17. *A near ring $\aleph$ is referred to as Medial Quasi Cyclic Weak Commutative if $klmn = knlm$ for all $k, l, m, n \in \aleph$.*

Proposition 3.18. *Every Medial near ring which satisfies cancellation law is commutative.*

Proof.
Assume $\aleph$ is a medial near ring. Then $klmn = kmln$ for all $k, l, m, n \in \aleph$. By left cancellation law we have $lmn = mln$. Again, by right cancellation law, we have $lm = ml$. Hence $\aleph$ is commutative. □

Proposition 3.19. *Let $\aleph$ be a medial near ring. If $\aleph$ is commutative, then*
(i) $\aleph$ is left medial.
(ii) $\aleph$ is right medial.
(iii) $\aleph$ is reverse.
(iv) $\aleph$ is reverse medial.
(v) $\aleph$ is reverse left medial.
(vi) $\aleph$ is reverse right medial.

Proof. Let $\aleph$ be a medial near ring. Then $klmn = kmln$ for all $k, l, m, n \in \aleph$.
(i) Now $klmn = kmln = (km)ln = (mk)ln = mkln$. That is $klmn = mkln$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is left medial.
(ii) Now $klmn = kmln = km(ln) = km(nl) = kmnl$. That is $klmn = kmnl$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is right medial.
(iii) Now $klmn = (kl)(mn) = (lk)(nm) = lknm = lnkm$ (Since $\aleph$ is Medial)

$= (ln)(km) = (nl)(mk) = nlmk = nmlk$ (Since ℵ is Medial) $= nmlk$. That is $klmn = nmlk$ for all $k, l, m, n \in \aleph$. Hence ℵ is reverse.
(iv) Now $klmn = kl(mn) = kl(nm) = klnm = knlm$ (Since ℵ is medial) $= (kn)lm = nklm = nlkm$ (Since ℵ is medial) $= nl(km) = nl(mk) = nlmk$. That is $klmn = nlmk$ for all $k, l, m, n \in \aleph$. Hence ℵ is reverse medial.
(v) Now $klmn = kl(mn) = kl(nm) = klnm = knlm$ (Since ℵ is Medial) $= (kn)lm = (nk)lm = nklm = nlkm$ (Since ℵ is Medial). That is $klmn = nlkm$ for all $k, l, m, n \in \aleph$. Hence ℵ is reverse left medial.
(vi) Now $klmn = (kl)mn = (lk)mn = lkmn = lmkn$ (Since ℵ is Medial) $= lm(kn) = lm(nk) = lmnk = lnmk$ (Since ℵ is Medial). That is $klmn = lnmk$ for all $k, l, m, n \in \aleph$. Hence ℵ is reverse right medial. □

Proposition 3.20. *Let ℵ be a medial near ring. If ℵ is left medial then*
(i) ℵ is medial quasi weak commutative
(ii) ℵ is medial cyclic commutative.

Proof.
(i) Since ℵ is left medial near ring, for all $k, l, m, n \in \aleph$, $klmn = mkln = mlkn$ (Since ℵ is medial). That is $klmn = mlkn$. Hence ℵ is medial quasi weak commutative.
(ii) Since ℵ is a medial near ring, for all $k, l, m, n \in \aleph$, $klmn = kmln = lmkn$ (Since ℵ is left medial). That is $klmn = lmkn$. Hence ℵ is medial cyclic commutative. □

Proposition 3.21. *Every left medial near ring with commutativity is medial.*

Proof. Let ℵ be a left medial near ring. Then for all $k, l, m, n \in \aleph$, $klmn = mkln = (mk)ln = (km)ln = kmln$. That is $klmn = kmln$ for all $k, l, m, n \in \aleph$. Hence ℵ is medial. □

Proposition 3.22. *Let ℵ be a medial quasi cyclic weak commutative near ring. If ℵ is right medial then ℵ is medial.*

Proof. Since ℵ is right medial, then $klmn = kmnl$ for all $k, l, m, n \in \aleph$. It is given that ℵ is medial quasi cyclic weak commutative, $klmn = knlm = kmln$ (Since ℵ is right medial). That is $klmn = kmln$ for all $k, l, m, n \in \aleph$. Hence ℵ is medial. □

Proposition 3.23. *Let ℵ be a medial quasi weak commutative near ring. If ℵ is left medial then ℵ is medial.*

Proof. Since ℵ is medial quasi weak commutative, $klmn = mlkn = kmln$ (Since ℵ is left medial) $k, l, m, n \in \aleph$. That is $klmn = kmln$ for all $k, l, m, n \in \aleph$. Hence ℵ is medial. □

Proposition 3.24. *If ℵ is a medial near ring with medial Quasi weak commutativity then*
(i) ℵ is left medial.
(ii) ℵ is medial cyclic commutative.

Proof. Since ℵ is medial, $klmn = kmln$ for all $k, l, m, n \in \aleph$.
(i) It is given that ℵ is medial quasi weak commutative, $klmn = mlkn = mkln$ (Since ℵ is medial) for all $k, l, m, n \in \aleph$. That is $klmn = mkln$ for all $k, l, m, n \in \aleph$. Hence ℵ is left medial.
(ii) Since ℵ is medial, $klmn = kmln = lmkn$ (Since ℵ is medial quasi weak commutative) for all $k, l, m, n \in \aleph$. That is $klmn = lmkn$ for all $k, l, m, n \in \aleph$. Hence ℵ is medial cyclic commutative. □

Proposition 3.25. *If ℵ is both medial quasi weak commutative and medial cyclic commutative near ring, then every left medial near ring is medial.*

Proof. Since ℵ is a left medial near ring, for all $k, l, m, n \in \aleph$, $klmn = mkln = lkmn$ (Since ℵ is medial quasi weak commutative) $= kmln$ (Since ℵ is medial cyclic commutative). That is $klmn = kmln$ for all $k, l, m, n \in \aleph$. Hence ℵ is medial. □

Proposition 3.26. *If ℵ is both left medial and right medial near ring then ℵ is commutative.*

Proof. Since ℵ is left medial near ring, $klmn = mkln$ for all $k, l, m, n \in \aleph$. Since ℵ is right medial near ring, $klmn = kmnl$ for all $k, l, m, n \in \aleph$. Let $a = cd$ and $b = ef$ for all $a, b, c, d, e, f \in \aleph$. Now $ab = cdef = ecdf = decf = dcfe = fdce = fced = efcd = ba$. That is $ab = ba$ for all $a, b \in \aleph$. Hence ℵ is commutative. □

Proposition 3.27. *Let ℵ be a reverse left medial near ring. If ℵ is a pseudo commutative near ring satisfying the left cancellation law, then ℵ is commutative.*

Proof. Let ℵ be a reverse left medial near ring. then $klmn = nlkm$ for all $k, l, m, n \in \aleph$. Since ℵ is pseudo commutative, $klm = mlk$ for all $k, l, m \in \aleph$. Now $klmn = nlkm = (nlk)m = (kln)m$. That is $klmn = klnm$ for all $k, l, m, n \in \aleph$. Since ℵ satisfies left cancellation law, $mn = nm$. Hence ℵ is commutative. □

Corollary 3.28. *Let ℵ be a pseudo commutative near ring.*
(i) If ℵ is a left medial near ring satisfying the right cancellation law, then ℵ is commutative.
(ii) If ℵ is a right medial near ring satisfying the left cancellation law, then ℵ is commutative.

Proof. Let $\aleph$ be a pseudo commutative near ring. Then $klm = mlk$ for all $k, l, m, \in \aleph$.
(i) Since $\aleph$ is a left medial near ring, for all $k, l, m, n \in \aleph$, $klmn = mkln = (mkl)n = (lkm)n = lkmn$. That is $klmn = lkmn$ for all $k, l, m, n \in \aleph$. Since $\aleph$ satisfies right cancellation law, $kl = lk$. Hence $\aleph$ is commutative.
(ii) Since $\aleph$ is right medial near ring, for all $k, l, m, n \in \aleph$, $klmn = kmnl = k(mnl) = k(lnm) = klnm$. That is $klmn = klnm$ for all $k, l, m, n \in \aleph$. Since $\aleph$ satisfies left cancellation law, $mn = nm$. Hence $\aleph$ is commutative.

Lemma 3.29. *A non-trivial left medial regular near ring is reduced.*

Proof. Let $\aleph$ be a left medial regular near ring. If $\eta \in \aleph$ is nilpotent then $\eta^k = 0$ for some positive integer k. Since $\aleph$ is regular, $\eta = \eta y \eta$ for some $y \in \aleph$. Hence $\eta y = (\eta y \eta)y = \eta\eta yy = \eta^2 y^2 = \eta(\eta y \eta)yy = \eta(\eta y \eta y)y = \eta(\eta\eta yy)y = \eta^3 y^3 ... = \eta^k y^k = 0y^k = 0$. Thus $\eta y = 0$. Hence $\eta = \eta y \eta = 0\eta = 0$. Hence $\aleph$ has no non-zero nilpotent elements. Hence $\aleph$ is reduced.

Note 3.30. *A non-trivial right medial regular near ring is reduced.*

Theorem 3.31. *Let $\aleph$ be both left medial and right medial near ring with regularity. If $\aleph$ is zero symmetric then $\aleph$ satisfies the following condition:*
(i) $ab = 0 \Longrightarrow ba = b0$ for every $a, b \in \aleph$.
(ii) $a\eta = a\eta a$ for every idempotent $a \in \aleph$ and for every $\eta \in \aleph$.
(iii) $\aleph$ is a P_1 near ring.

Proof. (i) Let $a, b \in \aleph$ such that $ab = 0$. Since $\aleph$ is regular, we have $a = axa$ and $b = byb$ for some $x, y \in \aleph$. Hence $ba = (byb)(axa) = b(ybax)a = b(yaxb)a = b(xyab)a = bxy(ab)a = bxy0 = bx0 = bx00 = b00x = b00 = b0$. Hence $ba = b0$.
(ii) Let $a \in \aleph$ be an idempotent element and let $\eta \in \aleph$. Now $(a\eta - a\eta a)a\eta = a\eta a\eta - a\eta a^2\eta = a\eta a\eta - a\eta a\eta = 0$. That is $(a\eta - a\eta a)a\eta = 0$. Hence by (i) $a\eta(a\eta - a\eta a) = a\eta 0 = a\eta 00 = a00\eta = a0$. That is $a\eta(a\eta - a\eta a) = a0$. Again since $(a\eta - a\eta a)a\eta a = a\eta a\eta a - a\eta a^2\eta a = a\eta a\eta a - a\eta a\eta a = 0$. Hence by (i) we have $a\eta a(a\eta - a\eta a) = a\eta a0 = aa0\eta = aa0 = a^20 = a0$. That is $a\eta a(a\eta - a\eta a) = a0$. Hence $(a\eta - a\eta a)^2 = (a\eta - a\eta a)(a\eta - a\eta a) = a\eta(a\eta - a\eta a) - a\eta a(a\eta - a\eta a) = a0 - a0 = 0$. That is $(a\eta - a\eta a)^2 = 0$. By lemma (3.29) and note (3.30) $a\eta - a\eta a = 0$. Hence $a\eta = a\eta a$.
(iii) Let $a \in \aleph$. Since $\aleph$ is regular,$a = axa$ for some $x \in \aleph$. Now $(xa)^2 = (xa)(xa) = x(axa) = xa$. That is xa is an idempotent. Now for any $\eta \in \aleph$, $a\eta = (axa)\eta = a[(xa)\eta] = a(xa)\eta(xa)$ (by (ii)) $= a(xa\eta x)a \in a\aleph a$. That is $a\eta \in a\aleph a$. Hence $a\aleph \subseteq a\aleph a$. Clearly $a\aleph a \subseteq a\aleph$. Thus $a\aleph = a\aleph a$. □

4 Conclusion

Near rings are a key idea in the field of ring theory, and in many fields of mathematics, computer science, and mathematical (theoretical) physics, research is currently engaged. They have extensive applications to the study of geometrical objects and topology and frequently have very direct linkages to other areas of algebra. This paper deals with medial near rings. For a near ring $\aleph$, we study the commutativity on medial near ring and introduced some definitions based on the substructures. Using these definitions we have proved some results to know more about the concepts of mediality in near rings and also structural theorem was further developed. In future, the same medial structures can be platformed in seminear ring and can be applied in cyrptography too. It is hoped that the ideas and findings presented in this work will encourage and improve future research on medial near rings and lead to its advancement in practical application.

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Application of $m-$ polar soft fuzzy bi-partite graph in residence selection process

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Abstract

An $m-$ polar fuzzy set and soft sets are two different tools for representing uncertainty and vagueness. An $m-$ polar soft fuzzy set is a mapping from parameter set to the $m-$ polar fuzzy subsets of universe. An $m-$ polar soft fuzzy set theory provides a parameterized point of view for uncertainty modeling and soft computing model. In this paper, we have introduced the notions of $m-$ polar soft fuzzy bi-partite graph, size and degree of $m-$ polar soft fuzzy bi-partite graph as well as an investigation on buying of residence by considering various parameter. People while buying residence have many options. So, to choose the best one, they have to consider many parameters. $m-$ polar soft fuzzy graph is one of the major area of graph theory, which finds solution to this problem.

Keywords: m − polar soft fuzzy bi-partite graph, m − polar soft fuzzy graph, size of m − polar soft fuzzy bi-partite graph.
2020 AMS subject classifications:$54B05, 54C05$. [1]

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1 Introduction

Ali et al. [2011] conducted a study on soft sets stimulation of the fuzzy sets and fuzzy soft sets. In 2014, Juanjuan Chen [2014] defined the $m-$ polar fuzzy graphs. Mathew Varkey and T. K. Shyla [2017] introduced bipartite and balanced fuzzy graph structures. The concept of fuzzy soft graph was presented by Mohinta Sumit and T. K. Samanta [2015]. D. A Molodtsov [1997] initiated the notion of the soft set theory. M. Akram and S. Nawaz [2016] presented new ideas known as fuzzy soft graphs with applications. A. Rosenfeld et al. [1975] developed the concept of fuzzy graphs in fuzzy sets and their applications. S. Ramkumar and R. Sridevi [2021] introduced their perception on proper $m-$ polar soft fuzzy graphs. S. Ramkumar and R. Sridevi [2023] presented some applications of Min-Max $m-$ polar soft fuzzy graphs in decision making. In S. Ramkumar and R. Sridevi [2023] introduced on improper $m-$polar soft fuzzy graph and it's applications for medical diagnosis in the current COVID-19 scenario. An $m-$ polar soft fuzzy graph is a useful mathematical tool for simulating the ambiguity of the real world. The $m-$ polar soft fuzzy set and the graph model are combined to create the commonly used $m-$ polar soft fuzzy graph. The newly integrated ideas of the soft sets and $m-$ polar fuzzy sets will lead to numerous prospective applications in the $m-$ polar soft fuzzy set theoretical domain by adding extra fuzziness in analysing, as an $m-$polar soft fuzzy sets are the most useful in practical applications. The current paper proposes the application of $m-$ polar soft fuzzy bi-partite graphs in residence selection process. This topic focuses on this problem so that we are able to choose the best one. In normal fuzzy set or crisp set, We can consider just one parameter. By considering one parameter, it is somewhat difficult to choose the best one. But in $m-$ polar soft fuzzy set, we are able to apply various facilities required as per the wish. To solve this problem, we consider $m-$ polar soft fuzzy bipartite graph with first bi-partition having exclusive amenities close to each type of residence and second bi-partition having exclusive in-built amenities of each type of residence. Here, the parameter represents specific condition near by residence in addition to the above conditions. We have considered the application for $m = 3$ attributes.

2 Preliminaries

Definition 2.1. *An $m-polar$ soft fuzzy graph $\widetilde{G}_{P,V} = (G^*, \widetilde{\rho}, \widetilde{\mu}, P)$ is a $4-$tuple such that*

(a) $G^* = (V, E)$ is a simple graph,

(b) P is a nonempty set of parameters,

(c) $\widetilde{\rho} : P \to F(V)\,(collection\ of\ all\ m-polar fuzzy\ subsets\ in\ V)$
$\mathfrak{e} \to \widetilde{\rho}(\mathfrak{e}) = \widetilde{\rho}_{\mathfrak{e}}(say)$
and $\widetilde{\rho}_{\mathfrak{e}} : V \to [0,1]^m$
$(x_1, x_2, \ldots, x_m) \mapsto \widetilde{\rho}_{\mathfrak{e}}\,(x_1, x_2, \ldots, x_m)$
$(\widetilde{\rho}, P)$ *is an* m *−polar soft fuzzy set over* V

(d) $\widetilde{\mu} : P \to F\,(V \times V)\,(collection\ of\ all\ m-polar fuzzy\ subsets\ in\ V \times V)$
$\mathfrak{e} \mapsto \widetilde{\mu}\,(\mathfrak{e}) = \widetilde{\mu}_{\mathfrak{e}}\,(say)$
and $\widetilde{\mu}_{\mathfrak{e}} : V \times V \to [0,1]^m$
$(x_1, x_2, \ldots, x_m) \mapsto \widetilde{\mu}_{\mathfrak{e}}\,(x_1, x_2, \ldots, x_m)$
$(\widetilde{\mu}, P)$ *is an* m *−polar soft fuzzy set over* E

(e) $(\widetilde{\rho}_e, \widetilde{\mu}_e)$ *is an* m *−polar fuzzy subgraph of* G^* *for all* $e \in P$. *That is,*

$$\widetilde{\mu}_e x_1(uv) \leq (\widetilde{\rho}_e x_1(u) \wedge \widetilde{\rho}_e x_1(v))$$
$$\widetilde{\mu}_e x_2(uv) \leq (\widetilde{\rho}_e x_2(u) \wedge \widetilde{\rho}_e x_2(v))$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\widetilde{\mu}_e x_m(uv) \leq (\widetilde{\rho}_e x_m(u) \wedge \widetilde{\rho}_e x_m(v))$$

for all $e \in P$ *and* $u, v \in V$. *The* m *−polar fuzzy graph* $(\widetilde{\rho}_e, \widetilde{\mu}_e)$ *is denoted by* $\widetilde{H}_{P,V}(e)$ *for convenience.*

Definition 2.2. *A fuzzy graph* $G = (\mu, \rho)$ *is said to be a fuzzy Bi-partite graph. if the vertex set* V *is partitioned into two disjoint union of two vertex sets* V_1 *and* V_2 *such that for all* $x, y \in V_1$ *or for all* $x, y \in V_2$ *,* $\rho(x, y) = \frac{1}{2}[\mu(x) \wedge \mu(y)]$. *This fuzzy Bi-partite graph can be denoted by* $G(V_1, V_2, \mu, \rho)$.

3 $m-$ polar soft fuzzy bi-partite graph

Definition 3.1. *An* $m-$ *polar soft fuzzy graph* $\widetilde{G}_{P,V} = (G^*, \widetilde{\rho}, \widetilde{\mu}, P)$ *is said to be an* $m-$*polar soft fuzzy bi-partite graph if the vertex set* V *is partitioned into two disjoint vertex pair* V_1 *and* V_2

$$\widetilde{\mu}_e x_1(uv) \leq (\widetilde{\rho}_e x_1(u) \wedge \widetilde{\rho}_e x_1(v))$$
$$\widetilde{\mu}_e x_2(uv) \leq (\widetilde{\rho}_e x_2(u) \wedge \widetilde{\rho}_e x_2(v))$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\widetilde{\mu}_e x_m(uv) \leq (\widetilde{\rho}_e x_m(u) \wedge \widetilde{\rho}_e x_m(v))$$

in $\widetilde{H}_{P,V}(e)$ $\forall e \in P$ *and for all* $u \in V_1, v \in V_2$.

Definition 3.2. *The Size of an $m-$polar soft fuzzy bi-partite graph is defined as* $S(\widetilde{G}_{P,V_1\cup V_2}) = \sum_{e\in P}\sum_{(x_1,x_2,\ldots,x_m)uv\in V_1\cup V_2}\widetilde{\mu}_e(x_1,x_2,\ldots,x_m)(uv)$.

Example 3.1. *Consider a $3-$ polar soft fuzzy graph $\widetilde{G}_{P,V}$, where $V = V_1 \cup V_2 = \{a_1,a_2,a_3,b_1,b_2,b_3\}$ and $P = \{e_1,e_2\}$ and $E = \{a_1b_1, a_1b_2, a_2b_1, a_1b_3, a_2b_2, a_2b_3, a_3b_3\}$.*

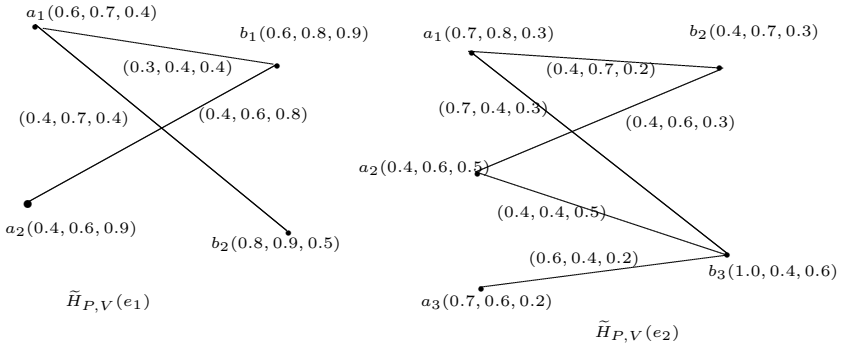

Figure 1: $3-$polar fuzzy bi-partite graph

In Figure 1. The size of $3-$polar fuzzy bi-partite graph $\widetilde{H}_{P,V}(e)$ is
$S(\widetilde{H}_{P,V}(e_1)) = \sum_{(x_1,x_2,x_3)ab\in V_1\cup V_2}\widetilde{\mu}_e(x_1,x_2,x_3)(ab) = (1.1,1.7,1.6)$,
$S(\widetilde{H}_{P,V}(e_2)) = \sum_{(x_1,x_2,x_3)ab\in V_1\cup V_2}\widetilde{\mu}_e(x_1,x_2,x_3)(ab) = (2.5,2.5,1.5)$.

The size of $3-$polar soft fuzzy bi-partite graph $\widetilde{G}_{P,V}$ is
$S(\widetilde{G}_{P,V_1\cup V_2}) = \sum_{e\in P}\sum_{(x_1,x_2,x_3)ab\in V_1\cup V_2}\widetilde{\mu}_e(x_1,x_2,x_3)(ab) = (3.6,4.2,3.1)$.

The degree of the vertices
$d_{\widetilde{G}_{P,V_1\cup V_2}}(a_1) = (1.8,2.2,1.3), d_{\widetilde{G}_{P,V_1\cup V_2}}(a_2) = (1.2,1.6,1.6)$,
$d_{\widetilde{G}_{P,V_1\cup V_2}}(a_3) = (0.6,0.4,0.2), d_{\widetilde{G}_{P,V_1\cup V_2}}(b_1) = (0.7,1.0,1.2)$,
$d_{\widetilde{G}_{P,V_1\cup V_2}}(b_2) = (1.2,2.0,0.9), d_{\widetilde{G}_{P,V_1\cup V_2}}(b_3) = (1.7,1.2,1.0)$.

4 Application of $3-$ polar soft fuzzy bi-partite graph in residence selection process

In the evolution of human life, the all-time dream of people is to own a residence, which is a symbol of status for some and it's a basic need for others. Primarily, people show an interest to buy a residence with a proper DTCP approval. Then, people also prefer eco-friendly environment, closely built neighbourhood, and good water facilities are added to a perfect residence selection. People are also cautious about buying their residence on a land surface not very close to well or lakes.

Let $V = V_1 \cup V_2 = \{V_1 : (a_1, a_2, a_3, a_4, a_5), V_2 : (b_1, b_2, b_3, b_4, b_5)\}$ be set of all two disjoint vertices and $P = \{e_1, e_2, e_3, e_4, e_5\}$ are parameterized set where

Exclusive amenities close by each types of residence $a_1, a_2, \ldots, a_5$ are given as follows:

a_1 - Airport, Motor Racing Track, Railway Station,

a_2 - Hospital, Top Professional Educational Institution, IT Park,

a_3 - Auto mobile hubs, IT Park, Railway Station,

a_4 - Temples, park, EB office,

a_5 - Motel Highway, Sipcot, Airport.

The exclusive in -built amenities of each type of residences $b_1, b_2, \ldots, b_5$ are as follows:

b_1 - Spill AC provision for all bedrooms, TV and Telephone points in all bed room, Kitchen and work area,

b_2 - Pressure checked plumbing, drainage line, 4 bed rooms,

b_3 - 2 bed rooms and separate toilet, living and dining room, car parking,

b_4 - Electronic surveillance, Two high speed elevators, 4 bed rooms,

b_5 - Swimming pool, 4 bed rooms, car parking.

and the parameters are

e_1-Residence closer to Industries and built with 3 bed rooms and modular Kitchen,

e_2-Residence closer to Educational Institution and built with living room and car parking,

e_3- Residence closer to get away with living room, car parking,

e_4-Residence closer to Highway and built with Swimming pool, 4 bed rooms,

e_5 -Residence closer to Airport and built with 2 bed rooms, kids area.

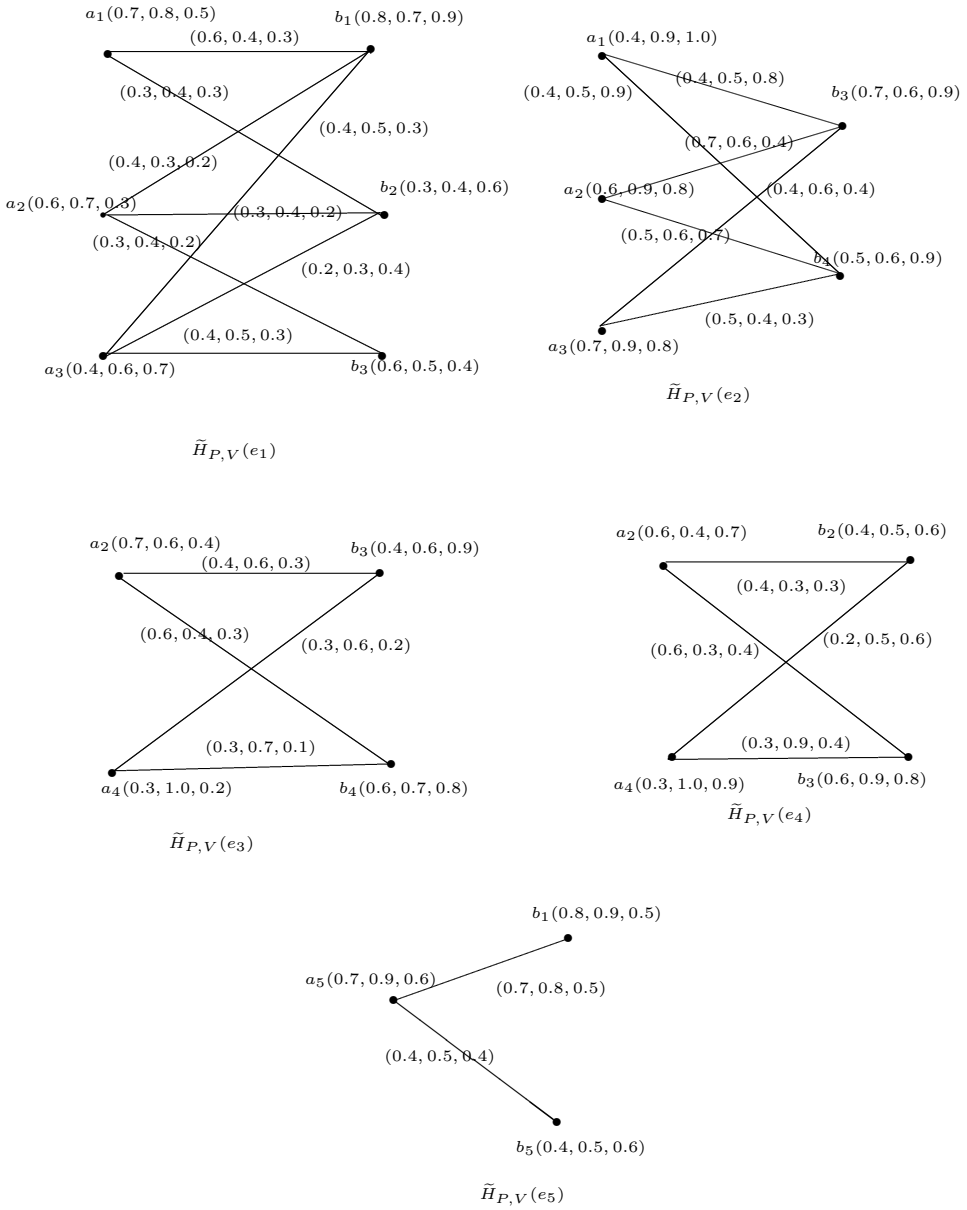

Figure 2 : $3-$polar fuzzy bi-partite graph

In Figure 2. The size of the each parameterized graph is calculated as

$$S(\widetilde{H}_{P,V}(e_1)) = \sum_{(x_1,x_2,x_3)ab \in V_1 \cup V_2} \widetilde{\mu}_e(x_1, x_2, x_3)(ab) = (2.9, 3.2, 2.2),$$
$$S(\widetilde{H}_{P,V}(e_2)) = \sum_{(x_1,x_2,x_3)ab \in V_1 \cup V_2} \widetilde{\mu}_e(x_1, x_2, x_3)(ab) = (2.9, 3.2, 3.5),$$

$$S(\widetilde{H}_{P,V}(e_3)) = \sum_{(x_1,x_2,x_3)ab\in V_1\cup V_2} \widetilde{\mu}_e(x_1,x_2,x_3)(ab) = (1.6, 2.3, 0.9),$$
$$S(\widetilde{H}_{P,V}(e_4)) = \sum_{(x_1,x_2,x_3)ab\in V_1\cup V_2} \widetilde{\mu}_e(x_1,x_2,x_3)(ab) = (1.5, 2.0, 1.7),$$
$$S(\widetilde{H}_{P,V}(e_5)) = \sum_{(x_1,x_2,x_3)ab\in V_1\cup V_2} \widetilde{\mu}_e(x_1,x_2,x_3)(ab) = (1.1, 1.3, 0.9).$$

From the above calculation, it is explicit that most of the people want their residence nearby Educational Institution, Hospital and built with bed room, living room and car parking.

To find the most preferable residence closer to various amenities, we are going to calculate degrees of each vertex.

$d_{\widetilde{H}_{P,V}(e)(a)}$	a_1	a_2	a_3	a_4	a_5
e_1	(0.9,0.8,0.6)	(1.0,1.1,0.6)	(1.0,1.3,1.0)	(0.0,0.0,0.0)	(0.0,0.0,0.0)
e_2	(0.8,1.0,1.7)	(1.2,1.2,1.1)	(0.9,1.0,0.7)	(0.0,0.0,0.0)	(0.0,0.0,0.0)
e_3	(0.0,0.0,0.0)	(1.0,1.0,0.6)	(0.0,0.0,0.0)	(0.6,1.3,0.3)	(0.0,0.0,0.0)
e_4	(0.0,0.0,0.0)	(1.0,0.6,0.7)	(0.0,0.0,0.0)	(0.5,1.4,1.0)	(0.0,0.0,0.0)
e_5	(0.0,0.0,0.0)	(0.0,0.0,0.0)	(0.0,0.0,0.0)	(0.0,0.0,0.0)	(1.1,1.3,0.9)
$d_{\widetilde{G}_{P,V}}(a)$	(1.7,1.8,2.3)	(4.2,3.9,3.0)	(1.9,2.3,1.7)	(1.1,2.7,1.3)	(1.1,1.3,0.9)

Table 1: Degree of each vertex according to closer by amenities in each type

$d_{\widetilde{H}_{P,V}(e)(b)}$	b_1	b_2	b_3	b_4	b_5
e_1	(1.4,1.2,0.8)	(0.8,1.1,0.9)	(0.7,0.9,0.5)	(0.0,0.0,0.0)	(0.0,0.0,0.0)
e_2	(0.0,0.0,0.0)	(0.0,0.0,0.0)	(1.5,1.7,1.6)	(1.4,1.5,1.9)	(0.0,0.0,0.0)
e_3	(0.0,0.0,0.0)	(0.0,0.0,0.0)	(0.7,1.2,0.5)	(0.9,1.1,0.4)	(0.0,0.0,0.0)
e_4	(0.0,0.0,0.0)	(0.6,0.8,0.9)	(0.9,1.2,0.8)	(0.0,0.0,0.0)	(0.0,0.0,0.0)
e_5	(0.7,0.8,0.5)	(0.0,0.0,0.0)	(0.0,0.0,0.0)	(0.0,0.0,0.0)	(0.4,0.5,0.4)
$d_{\widetilde{G}_{P,V}}(b)$	(2.1,2.0,1.3)	(1.4,1.9,1.8)	(3.8,5.0,3.4)	(2.3,2.6,2.3)	(0.4,0.5,0.4)

Table 2: Degree of each vertex according to in - built amenities

In above table, we have considered five different parameters and their corresponding exclusive amenities closer by each type of residence (table 1) and exclusive in-built amenities (table 2). The size of each parameterized graph is calculated and we have concluded the best parameter of buying house from the highest size.

Regarding closer by amenities: When the vertex a_2 dominates the other vertex, it is inferred that Educational institutions, Hospitals and IT parks are preferred over the rest.

In the case of residence: The vertex b_3 dominates the other qualities, the residence with 2 bed rooms and separated toilet, dining room and car parking is mostly preferred.

From the above calculation, we have revealed that most of the people want their residence nearby Educational Institutions, Hospitals and built with bed room, living room and car parking and also inferred that Educational institutions, Hospitals and IT parks are preferred over the rest and the residence with 2 bed rooms and separated toilet, dining room and car parking is mostly preferred.

5 Discussion and Conclusion

Bipartite graph is one of the important concepts in graph theory, which has wide applications since it's origination. Soft fuzzy graph is an interesting area in the field of research that most people are concentrating. S. Ramkumar and R. Sridevi [2021] introduced the concept of $m-$ polar soft fuzzy graph. In this article, Buying of own house is merely a challenge for every human being. So we are in need of considering many parameters to buy a new house as we are buying once in life time. This is the motivation behind this study. We consider m-polar soft fuzzy graph with various parameters and with the help of size of the graph we have chosen which one is the best. The parameters may vary from person to person. We have consider just five parameters according to our survey. It may also vary in rural and urban areas. In future, we have an idea to focus on any one of areas with accurate survey. Which would help us to conclude the best way of buying own house.

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A perishable inventory optimization model for two commodities in multi-echelon system

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Abstract

In this article, we delve into the quest for an optimal solution within the realm of a practical inventory system, which serves as an integral component of a larger Multiechelon inventory system. Our study focuses on a warehouse (WH) that houses a single Distribution Center (DC) and a sole retailer (R) tasked with managing two distinct yet interconnected perishable products. This setup is modeled as a three-stage continuous review inventory system.

At the retailer node, we implement an (s, S) inventory system, accommodating Poisson demand and exponentially distributed lead times. It's important to note that the primary products are susceptible to perishing exclusively at the retailer node, characterized by a rate ζ, whereas the sub-products are non-perishable in nature. Deliveries to retailers occur in packs of $Q_i(= S_i - s_i)$ units, originating from the distribution center (DC) with immediate replenishment capabilities sourced from the well-stocked warehouse (WH).

Throughout our analysis, we rigorously maintain steady-state probability distributions and behavioral insights. Furthermore, we diligently collect and analyze performance measurements for the system under steady-state conditions. Our findings are showcased through numerical examples, shedding light on the practical implications of our inventory management approach.

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1 Introduction

Inventory management is a critical aspect of efficient business operations. Whether dealing with a single item at a single location or a complex, multi-item, multi-location scenario, the proper control of inventory is central to a company's success. This article explores the dynamic landscape of inventory management, from its fundamental principles to its practical applications in a world where maintaining stocks of goods is commonplace.

In the business world, maintaining an inventory, or a stockpile of products, is essential to meet future demand and ensure timely deliveries to customers. The ultimate goal of inventory theory is to provide management with strategies and rules to minimize the costs associated with maintaining inventory while meeting customer demands effectively. Effective inventory management can translate into substantial cost savings for businesses. At its core, inventory theory seeks to answer two pivotal questions: When should an order be placed for a product? And how large should each order be? The amalgamation of responses to these questions constitutes an inventory policy.

In certain business scenarios, companies handle not just one product but a pair of different yet interrelated products. These product pairs, such as tires and tubes, pens and refills, printers and cartridges, or insulin vials and injections, share a common manufacturing unit and exhibit distinct demand patterns. Initially, customers may need to purchase the main product alongside the subproduct, but as they become familiar with the offerings, they may choose to acquire the subproduct separately. Typically, there is a higher demand for the subproduct compared to the main product, making it essential to manage these variations effectively.

Perishable goods, with their finite or specified lifespans, present unique challenges in inventory management. Real-world examples include items like milk, blood, drugs, food, vegetables, and certain chemicals, all of which have limited shelf lives. Handling such perishable inventory systems requires meticulous planning and management. These goods, once expired, not only occupy valuable storage space but can also affect neighboring items. Hence, it is crucial to remove items that have exceeded their shelf life at specific intervals.

Inventory selection plays a pivotal role in the realm of Multi-echelon management, where inventory exists in diverse forms, including perishable goods, semi-

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finished products, or packaged items, at different points in the supply chain. It can also be in transit between various levels or stations as a work in progress. Given that inventory holdings can account for a substantial portion, ranging from 20 % to 40 %, of their overall value, effective inventory management is indispensable in Multi-echelon operations.

The various inventory systems (s, S) for individual products have been very extensively investigated in the past. The primary quantitative inventory survey began with the study of Harris [1915]. Clark and Scarf [1960] first proposed multi-level inventory. They analyzed Nesheron's pipeline framework without considering partial measures. Subsequent advances in the two-stage model can be found in He et al. [1998]. Axsater [1990] proposed an approximate model of inventory structure in SC. One of the oldest papers in the field of continuous review multi-echelon inventory system is a basic and seminal paper written by Sherbrooke [1998]. A Complete review was provided by Benita [1998]. The Multi-echelon concept grow largely out of two-stage multi-echelon inventory models, and it is important to note that considerable research in this area is based on the classic work of Clark and Scarf [1960] . A continuous review perishable inventory system at Service Facilities was studied by Elango [2001]. A continuous review (s, S) policy with positive lead times in two-echelon Multi-echelon in single item was considered by Krishnan [2007].

For an inventory system with continuous review, Krishnamoorthy et al. [1994] addressed the inventory problem of two commodities. Kalpakam and Arivarignan [1993] analyzed inventory models with multiple items and renewal requirements under a common replenishment policy. Anbazhagan and Arivarignan [2000] analyzed two inventory systems under different ordering policies. Yadavalli et al. [2004] analyzed the model using a common order policy and different order quantities. Bakthavachalam et al. [2012] addressed stochastic perishable inventory control with partial backorders. Kumaresan et al. [2014] explored perishable inventory systems with (s, Q) policies. Kartheeswari and Bakthavachalam [2017] analyzed integrated inventory control systems with partial backlogging. These works collectively advance the understanding of optimizing perishable inventory management in supply chains.

This article is structured as follows: Section 2 presents a comprehensive mathematical model of the inventory management problem, along with crucial notations used throughout our exploration. In Section 3, we delve into both transient and steady-state analyses of the inventory system. Section 4 offers insights into the operational characteristics of the system, while Section 5 focuses on the cost analysis associated with inventory operations. Finally, we conclude our exploration in Section 6.

2 The Mathematical Model

The multi-echelon system under consideration in this article can be described as follows: It comprises a serial multi-echelon system composed of a warehouse facility (WH), a distribution center (DC), and a retailer (R), handling two distinct yet interconnected products. Initially, the finished product is dispatched from WH to DC following a $(0, M)$ replenishment policy. Subsequently, the product is transported to R, and the (s_i, S_i) policy $(i = 1, 2)$ is applied. Demand at the retailer node follows independent Poisson distributions with rates λ_A, λ_B, and λ_C for the main product (A), sub-product (B), or both products.

If demand occurs during the stock period and cannot be met, it is considered a lost sale. The main product is subject to perishability at a rate ξ, but only while it is at the retailer nodes. In contrast, sub-products are assumed to be non-perishable. Supplies to the manufacturer, delivered in packets of Q items, follow an exponential time interval with a parameter $\mu(> 0)$. Replenishment of items in the form of packets, from WH to DC, occurs instantaneously. In this model, the maximum inventory level at the R node is denoted as S_i, with a corresponding reorder level of $s_i(i = 1, 2)$. The ordering quantity is defined as $Q = S_i - s_i(i = 1, 2)$, and the maximum inventory level at DC_i is $M_i(M_i = nQ_i)$.

Parameters	Particulars
$[R]_{ij}$	The element /sub matrix at (i,j)th position of R.
0	Zero matrix
I	Identity matrix.
e	A column vector of ones of appropriate dimension.
k_D	Fixed ordering cost, regardless of order size at DC nodes.
k_A	The average ordering cost for main product at retailer node.
K_B	The average ordering cost for sub product at retailer node.
h_D	The holding cost per unit of item per unit time at DC nodes.
h_A	The average holding cost per unit of item per unit time for main product at retailer.
h_B	The average holding cost per unit of item per unit time for sub product at retailer.
g_A	The average unit shortage cost for main product at retailer nodes.
g_B	The average unit shortage cost for sub product at retailer nodes.

Table 1: Notation and indice

$$\sum_{i=1}^{k} a_i = a_1 + a_2 + ... + a_k \tag{1}$$

$$\sum_{i=0}^{nQ} a_i = 0 + Q + 2Q. + ... + nQ \tag{2}$$

3 Analysis

Let $I - i(t), (i = 1, 2, 3)$ represent the amounts of main product (A) and sub product (B) that are currently in stock at Retailer and DC, respectively, at time t^+. Using the input and output process assumptions,

$I(t) = \{ (I_i(t), : t \geq 0 \}, (i = 1, 2, 3)$

is a state-space Markov process.

$$E = \begin{cases} (i, k, m)/i = S_1, (S_1 - 1), ..., s_1, (s_1 - 1), ..., 2, 1, 0; \\ \qquad\qquad k = S_2, (S_2 - 1), ..., s_2, (s_2 - 1) \\ m = nQ, (n-1)Q, ..., Q \end{cases} \tag{3}$$

It follows that $\{I(t), t \geq 0\}$ is an irreducible Markov process with state space E and is an ergodic process because E is finite and all of its states are recurrent non-null. As a result, the limiting distribution is real and unaffected by the initial condition.

Theorem 3.1. *The vector process $I(t) : t \geq 0$ where $I(t) = (I_0(t), I_1(t))$ for $t \geq 0$ is a continuous time Markov Chain with state space*

$$E = \{(j, q)/j = 0, 1, 2..., S; q = Q, 2Q, ..., nQ\}$$

Proof. The stochastic process $\{I(t) : t \geq 0\}$ has a discrete state space with order relation $' \leq'$ that $(j, q) \leq (k, r)$ if and only if $j \leq k$ and $q \leq r$.To prove that $I(t) : t \geq 0$ is a Markov chain ,first we do a transformation for state space E to E' such that $(j, q) \longrightarrow j + q \in E'$,where $E' = \{Q, Q + 1, ..., Q + S, ..., nQ + S\}$.

Now we may realize that $\{I(t) : t \geq 0\}$ is a stochastic process with discrete state space E'.

The joint distribution of random variables $\{I(t_1), I(t_2), ..., I(t_n)\}$ and $\{I(t_1 + \tau), I(t_2 + \tau), ..., I(t_n + \tau)\}$ with $\tau > 0$ (an arbitrary real number) are equal.

In particular the conditional probability

$$P_r\{I_n = k|I_{n-1} = j, I_{n-2} = i, ...I_0 = 1\} = P_r\{I_n = k|I_{n-1} = j\}$$

due to the single step transition of states in E.

Hence $\{I(t) : t \geq 0\}$ is a continuous time Markov Chain.

□

The following parameters can be used to determine the infinitesimal generator of this process $R = (a(i, k, m : j, l, n))_{(i,k,m),(j,l,n) \in E}$.

- When a demand for the primary product (A) arrives at the retailer node, the Markov process undergoes a state transition from (i, k, m) to $(i-1, k, m)$, with an intensity of transition of λ_A or $\xi > 0$.
- When a demand for subproduct(B) arrives at the retailer node, the Markov process transitions from the state (i, k, m) to $(i, k-1, m)$ with transition intensity λ_B.
- When a demand for both product(A and B) arrives at the retailer node, the Markov process transitions from the state (i, k, m) to $(i-1, k-1, m)$ with transition intensity λ_C.
- Replenishment of inventory at retailer node makes a state transition in the Markov process from (i, k, m) to $(i+Q, k, m-Q)$ or (i, k, m) to $(i, k+Q, m-Q)$, with intensity of transition μ .

Given by is the infinitesimal generator R.

$$R = \begin{pmatrix} A & B & 0 & 0 & 0 & \cdots & 0 \\ 0 & A & B & 0 & 0 & \cdots & 0 \\ 0 & 0 & A & B & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B & 0 & 0 & 0 & 0 & \cdots & A \end{pmatrix} \tag{4}$$

It is possible to express the entries of the matrix R as

$$[A]_{m\times n} = \begin{cases} A_1 \quad if \quad m = n, & m = S_1, S_1-1, \ldots s_1. + 1 \\ A_2 \quad if \quad n = m+1, & m = S_1, S_1-1, \ldots s_1. + 1 \\ A_3 \quad if \quad m = n & m = s_1, s_1 - 1, \ldots 1 \\ A_4 \quad if \quad n = m+1 & m = s_1, s_1 - 1, \ldots 1 \\ E_1 \quad if \quad m = 4 & m = 0 \end{cases} \tag{5}$$

$$B = \begin{cases} M_1 \quad if & m = n, & m = S, S-1, \ldots 0 \\ M_2 \quad if & n = m+1, & m = s, s-1, \ldots 0 \\ 0 \quad otherwise & & \end{cases} \tag{6}$$

Then, A and B's submatrices are provided by

$$[A_1] = \begin{cases} \lambda_B & if \quad n = m+1 & m = S, S-1, \ldots 1 \\ -(\lambda_A + \lambda_B + \lambda_C + i\xi) & if \quad m = n_1 & m = S, S-1, \ldots s+1 \\ -(\lambda_A + \lambda_B + \lambda_C + \mu) & if \quad m = n_1 & m = s, \ldots 1 \\ -(\lambda_A + \mu) & if \quad m = 4 & m = 0 \\ 0 & otherwise & \end{cases} \tag{7}$$

$$[A_2] = \begin{cases} \lambda_A + i\xi & if\ m = n_1 \quad m = S, S-1,...0 \\ \lambda_C & if\ n = m+1 \quad m = S, S-1, ...1 \\ 0 & otherwise \end{cases} \tag{8}$$

$$[A_3] = \begin{cases} \lambda_B & if \quad n = m+1 \quad m = S, ...1 \\ -(\lambda_A + \lambda_B + \lambda_C + i\xi) & if \quad m = n_1 \quad m = S, S-1, ...s+1 \\ -(\lambda_A + \lambda_B + \lambda_C + 2\mu + i\xi) & if \quad m = n_1 \quad m = s, s-1...1 \\ -(\lambda_A + 2\mu + i\xi) & if \quad m = 4 \quad m = 0 \\ 0 & otherwise \end{cases} \tag{9}$$

$$[A_4] = \begin{cases} \lambda_B & if \quad n = m+1 \quad m = S, ...1 \\ -(\lambda_B + \mu) & if \quad m = n_1 \quad m = S, S-1,...s+1 \\ -(\lambda_B + 2\mu) & if \quad m = n \quad m = s, s-1...1 \\ -(2\mu) & if \quad m = 4 \quad m = 0 \\ 0 & otherwise \end{cases} \tag{10}$$

$$[B_1] = \begin{cases} \mu & if \quad n = m+Q \quad m = s, s-1, , ...0 \\ 0 & otherwise \end{cases} \tag{11}$$

$$[B_2] = \begin{cases} \mu & if \quad m = n \quad m = S, S-1, , ..., 1, 0 \\ 0 & otherwise \end{cases} \tag{12}$$

3.1 Transient analysis

Determine the function of the transient probability.

$$\begin{aligned} p_{i,\,k,\,m}\,(j,\ l,\ n\ :\ t) &= p_r\{(I_1\,(t)\,,\ I_2\,(t)\,,\ I_3\,(t)) \\ &= (j,\ l,\ n)\ |\ (I_1\,(0)\,,\ I_2\,(0)\,,\ I_3\,(0)) = (i,\ k,\ m)\} \end{aligned} \tag{13}$$

The transient matrix for $t \in 0$ is of the form $P\,(t) = (p_{i,\,k,\,m}\,(j,\ l,\ n\ :\ t))_{(i,\,k,\,m)(j,\,l,\,n)\in E}$ satisfies the Kolmogorov- forward equation

$$P'(t) = P(t).R \tag{14}$$

Where R is the infinitesimal generator of the process $\{I(t), t \geq 0\}$

The solutions to the equation (14) can be written out in the manner of

$$P(t) = P(0)e^{Rt} = e^{Rt}(t) = P(0)e^{At}\, e^{At} \tag{15}$$

Where the matrix

$$e^{At}\, e^{Rt} \;=\; I \;+ \sum_{n=1}^{\infty} \frac{R^n\, t^n}{n} \tag{16}$$

Suppose that all of the Eigen values for R are unique. Then, we obtain $R = HDH^{-1}$ from the spectral theorem of matrices.

Where H is a non-singular matrix (created using R's right eigenvectors) and D is a diagonal matrix using R's Eigen values as its diagonal elements, respectively. Considering that 0 is an Eigen value of R and that $d_i \neq 0,\ i = 1, 2, \ldots m$ are the other separate Eigen values,

$$D \;=\; \begin{pmatrix} 0 & 0 & \ldots & \ldots & 0 \\ 0 & \mathrm{d}_1 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & d_m \end{pmatrix} \tag{17}$$

We than have

$$D^n = \begin{pmatrix} 0 & 0 & \ldots & \ldots & 0 \\ 0 & d_1^n & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & d_m^n \end{pmatrix} \tag{18}$$

and

$$R^n \;=\; HD^nH^{-1} \tag{19}$$

We have

$$P(t) \;=\; I + \sum_{n=1}^{\infty} \frac{(HD^nH^{-1})}{n!} \;=\; H\left\{I + \sum_{n=1}^{\infty} \frac{D^n t^n}{n!}\right\} H^{-1} \;=\; He^{Dt}H^{-1} \tag{20}$$

where

$$\overset{Dt}{e} = \begin{pmatrix} 1 & 0 & ... & ... & 0 \\ 0 & e^{d_1 t} & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & e^{d_m t} \end{pmatrix} \tag{21}$$

The explicit solution to the matrix $p(t)$ is provided on the right-hand side of the equation above. It should be noted that a canonical representation of $R = SZS^{-1}$ exists even in the broader case when the Eigen values of R are not necessarily distinct.

3.2 Steady state analysis

The construction of the infinitesimal matrix R demonstrates the finite and irreducible nature of the state space E of the Markov chain $I(t) : t \geq 0$. Let the inventory level procedure' limiting distribution be determined by

$$\mathrm{P}_j^q = \lim_{t\to\infty} \Pr\left\{(I_0(t), I_1(t)) = (j,q)\right\}_{(j,q)\in E} \tag{22}$$

where P_j^q is the steady state probability for the system be in state (j,q), (Cinlar [5]).

Let $P = \left(P^{nQ}, P^{(n-1)Q}, P^{(n-2)Q}...P^{Q}\right)$ denote the steady state probability distribution where $P^q = \left(P_S^q, P_{S-1}^q....P_0^q\ P_{-1}^q...\ P_{-b}^q\right)$ for the system under consideration. For each (j,q), P_j^q can be obtained by solving the matrix equation $PA = 0$ together with normalizing condition $\sum_{j,q} P_j^q = 1$ Assuming $P_Q^Q = a$, we Obtain the steady state probability

$$\overset{iQ}{P} = \text{(-1)}^{\text{k}}\ \text{a}\ (BA)^k, i = 1,2,...n; k = n-i+1. \tag{23}$$

Where

$$\text{a} = \text{e}^1 \left[\sum_{i=0}^{\text{n-1}} \text{(-1)}^{\text{i}}\ \left(BA^{-1}\right)^i\right]^{-1} \tag{24}$$

4 Performance measures

Consider the r_R and r_D events, which represent reorders at the nodes of Retailer and Distributor, respectively. Keep in mind that the r_D event occurs anytime the inventory level at the DC_i node reaches 0, whereas the r_R event occurs whenever the inventory level at the retailer nodes reaches the s_1 or s_2 reorder level.

4.1 Mean inventory level

Let ID_1, ID_2 stand for the expected inventory level at the distribution centre for the commodities 1 and 2, while IR_1 and IR_2 stand for the expected inventory level in the steady state at retailer node for the commodities 1 and 2, respectively. When defining them,

$$I_A = \sum_{m=Q}^{nQ} \sum_{j=0}^{S} i \sum_{i=0}^{S} i.P^m{}_{i,j} \tag{25}$$

$$I_B = \sum_{m=Q}^{nQ} \sum_{i=0}^{S} j \sum_{j=0}^{S} j.P^m{}_{i,j} \tag{26}$$

$$I_D = \sum_{i=S}^{S} \sum_{j=0}^{S} \sum_{m=Q}^{nQ} \sum_{i=0}^{S1} m.P^m{}_{i,j} \tag{27}$$

4.2 Mean reorder rate

For the commodities 1 and 2, the mean reorder rate at retailer nodes and distributors is provided by

$$R_A = \sum_{m=Q}^{nQ} \sum_{j=o}^{S} \lambda_A.P^m{}_{s+1,j} \tag{28}$$

$$R_B = \sum_{m=Q}^{nQ} \sum_{i=o}^{S} \lambda_B.P^m{}_{i,s+1} \tag{29}$$

$$R_D = \sum_{i=0}^{S} \sum_{j=0}^{S} \mu.P^Q{}_{i,j} \tag{30}$$

4.3 Mean shortage rate

Only at the retailer node does a shortfall exist, and the retailer's shortage rate for commodities 1 and 2 is shown by the symbols S_A and S_B, which are given by

$$S_A = \sum_{m=Q}^{nQ} \sum_{j=0}^{S} \lambda_A . {P^m}_{0,j} \tag{31}$$

$$S_B = \sum_{m=Q}^{nQ} \sum_{i=0}^{S} \lambda_B . {P^m}_{i,0} \tag{32}$$

$$S_C = \sum_{m=Q}^{nQ} \sum_{i=0}^{S} \lambda_C . {P^m}_{i,0} + \sum_{m=Q}^{nQ} \sum_{j=0}^{S} \lambda_C . {P^m}_{0,j} \tag{33}$$

5 Cost Analysis

The minimization of the steady state total projected cost per time is taken into account as we assess the cost structure for the offered models in this section. The model's long-term anticipated cost rate is specified as

$$TC(s,Q) = h_A\, I_A + h_B\, I_B + h_D\, I_D + k_A R_A + k_B\, R_B + k_D \\ R_D + g_A S_A + g_B S_B + g_C S_C \tag{34}$$

Although we haven't analytically demonstrated the cost function's convexity, our experience with a large number of numerical instances suggests that it is convex in s for fixed Q. It turned out to be an increasing function of s in some instances. In order to find the best values for $s*$, we therefore used a numerical search technique and got the best $n*$. Our calculation of $C(s,Q)$ revealed a convex structure for the same for a large number of factors. As a result, we used a numerical search method to get the best value s_i* for each $S_i (i = 1, 2)$.

6 Numerical Example and Sensitivity Analysis

This section discusses the challenge of reducing the long-term total predicted cost per unit of time given the following cost structure. The following numerical example might be used to show the results from the steady state scenarios.

s	Cost	Q
2	23.75207	10
3	24.86943	9
4	23.21998	8
5	20.49645*	7

Table 2: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=12 and M=36

The table 2 shows the total expected cost rate as a function of the reorder level (s) and the order quantity (Q), where S=12 and M=36. The minimum cost is 20.49645, which occurs when s=5 and Q=7. This is indicated by the asterisk (*) in the table.

The figure 1 shows a graphical representation of the data in the table 2. The x-axis represents the reorder level (s) and the y-axis represents the total expected cost rate.

Figure 1: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=12 and M=36

s	Cost	Q
2	36.35414	16
3	36.25592	15
4	35.15921	14
5	36.85936	13
6	34.99194	12
7	33.321*	11
8	34.85312	10

Table 3: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=18 and M=54

Figure 2: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=18 and M=54

The table 3 shows the total expected cost rate as a function of the reorder level (s) and the order quantity (Q), where S=18 and M=54. The minimum cost is achieved when s=7 and Q=11, with a cost of 33.321. This suggests that the optimum reorder level is 7 and the optimum order quantity is 11.

The figure 2 shows the table 3 data as a line graph.

s	Cost	Q
2	49.29698	22
3	50.9338	21
4	50.26087	20
5	48.62867	19
6	51.2149	18
7	49.0414	17
8	46.91525	16
9	44.84975*	15
10	46.5718	14
11	45.83538	13

Table 4: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=24 and M=72

From the table 4 we observe that the optimum reorder level is $s = 9$ and the respective minimum cost is indicated by the symbol $*$ in the table.

The figure 3 shows graphical representation of the table 4 data.

Figure 3: Total Expected Cost Rate Analysis (TC(s, Q)) in a Multi-Echelon System with S=24 and M=72

$\lambda_1 \backslash \lambda_2$	1	1.2	1.4	1.6	1.8	2
1	45.27567	45.5311	45.76223	45.96867	46.15064	46.30876
1.2	45.29586	45.55532	45.7905	46.00094	46.1868	46.34868
1.4	45.31815	45.58147	45.82056	46.03488	46.22453	46.39008
1.6	45.3425	45.60952	45.85239	46.07046	46.2638	46.43292
1.8	45.36889	45.63946	45.88596	46.10767	46.30457	46.47717
2	45.39727	45.67124	45.92124	46.14645	46.34683	46.5228

Table 5: Comparison of λ_1 & λ_2

It is observed that from the Table 5, the demand rate λ_1 & λ_2 increases then the total cost are also increased. Hence the demand rate is one of the key parameter of the system. The table 5 data is depicted in Figure 4

Figure 4: Comparison of λ_1 & $\lambda 1_2$

$h_A \backslash h_B$	0.1	0.2	0.3	0.4	0.5
0.1	21.6443	22.79427	23.94425	25.09423	26.24421
0.2	23.13187	24.28185	25.43183	26.58181	27.73179
0.3	24.61945	25.76943	26.91941	28.06939	29.21937
0.4	26.10703	27.25701	28.40698	29.55696	30.70694
0.5	27.5946	28.74458	29.89456	31.04454	32.19452

Table 6: Comparison of h_A & h_B

From the table 6, we observed that when h_A & h_B increases, the total cost also increases. The data from Table 6 is visualized in Figure 5.

Figure 5: Comparison of h_A & h_B

$h_A \backslash h_D$	0.1	0.2	0.3	0.4	0.5
0.1	11.24444	14.99439	18.74433	22.49427	26.24421
0.2	12.73202	16.48196	20.2319	23.98185	27.73179
0.3	14.2196	17.96954	21.71948	25.46942	29.21937
0.4	15.70718	19.45712	23.20706	26.957	30.70694
0.5	17.19475	20.94469	24.69464	28.44458	32.19452

Table 7: Comparison of h_A & h_D

From the findings in Table 7, it is evident that as both h_A & h_D increase, there is a simultaneous increase in the total cost. This observation underscores the direct relationship between these variables, which is graphically illustrated in Figure 6.

Figure 6: Comparison of h_A & h_D

$h_B \backslash h_D$	0.1	0.2	0.3	0.4	0.5
0.1	27.47061	31.22055	34.97049	38.72043	42.47038
0.2	28.62059	32.37053	36.12047	39.87041	43.62036
0.3	29.77057	33.52051	37.27045	41.02039	44.77033
0.4	30.92055	34.67049	38.42043	42.17037	45.92031
0.5	32.07052	35.82047	39.57041	43.32035	47.07029

Table 8: Comparison of h_B & h_D

The data in Table 8 underscores the positive correlation between higher values of h_B & h_D and an increase in the total cost, as also visually represented in Figure 7.

Figure 7: Comparison of h_B & h_D

$k_A \backslash k_B$	0.1	0.2	0.3	0.4	0.5
0.1	47.03705	47.04203	47.04702	47.05201	47.05699
0.2	47.04369	47.04868	47.05367	47.05866	47.06364
0.3	47.05034	47.05533	47.06032	47.0653	47.07029
0.4	47.05699	47.06198	47.06697	47.07195	47.07694
0.5	47.06364	47.06863	47.07362	47.0786	47.08359

Table 9: Comparison of k_A & k_B

Our analysis of Table 9 conclusively reveals that as k_A & k_B increase, there is a concurrent increase in total costs, as illustrated in Figure 8.

Figure 8: Comparison of k_A & k_B

$k_A \backslash k_D$	0.1	0.2	0.3	0.4	0.5
0.1	47.0272	47.03465	47.04209	47.04954	47.05699
0.2	47.03385	47.04129	47.04874	47.05619	47.06364
0.3	47.04049	47.04794	47.05539	47.06284	47.07029
0.4	47.04714	47.05459	47.06204	47.06949	47.07694
0.5	47.05379	47.06124	47.06869	47.07614	47.08359

Table 10: Comparison of k_A & k_D

From Table 10, it is evident that an escalation in both k_A & k_D results in a noticeable rise in total costs, a pattern also reflected in Figure 9.

Figure 9: Comparison of k_A & k_D

$k_B \backslash k_D$	0.1	0.2	0.3	0.4	0.5
0.1	47.02055	47.028	47.03545	47.04289	47.05034
0.2	47.02553	47.03298	47.04043	47.04788	47.05533
0.3	47.03052	47.03797	47.04542	47.05287	47.06032
0.4	47.03551	47.04296	47.05041	47.05786	47.0653
0.5	47.04049	47.04794	47.05539	47.06284	47.07029

Table 11: Comparison of k_B & k_D

These findings, as presented in Table 11 and visually confirmed in Figure 10, highlight the positive correlation between k_B & k_D and the resulting increase in total costs.

Figure 10: Comparison of k_B & k_D

S_A/S_B	0.1	0.2	0.3	0.4	0.5
0.1	47.03143	47.05087	47.07032	47.08976	47.1092
0.2	47.03142	47.05086	47.0703	47.08975	47.10919
0.3	47.03141	47.05085	47.07029	47.08974	47.10918
0.4	47.03139	47.05084	47.07028	47.08972	47.10917
0.5	47.03138	47.05082	47.07027	47.08971	47.10915

Table 12: Comparison of S_A & S_B

The findings in Table 12 clearly demonstrate that higher values of S_A & S_B coincide with elevated total costs. This positive correlation aligns with the visual representation in Figure 11.

Figure 11: Comparison of S_A & S_B

$S_A \backslash S_D$	0.1	0.2	0.3	0.4	0.5
0.1	46.99258	47.03145	47.07032	47.10919	47.14806
0.2	46.99257	47.03143	47.0703	47.10917	47.14804
0.3	46.99255	47.03142	47.07029	47.10916	47.14803
0.4	46.99254	47.03141	47.07028	47.10915	47.14802
0.5	46.99253	47.0314	47.07027	47.10914	47.14801

Table 13: Comparison of S_A & S_D

Table 13 underscores the direct impact of increasing S_A & S_D on total cost, with Figure 12 providing a visual confirmation of this relationship.

Figure 12: Comparison of S_A & S_D

$S_B \backslash S_D$	0.1	0.2	0.3	0.4	0.5
0.1	46.95367	46.99254	47.03141	47.07028	47.10914
0.2	46.97311	47.01198	47.05085	47.08972	47.12859
0.3	46.99255	47.03142	47.07029	47.10916	47.14803
0.4	47.012	47.05087	47.08974	47.1286	47.16747
0.5	47.03144	47.07031	47.10918	47.14805	47.18692

Table 14: Comparison of S_B & S_D

In accordance with the data in Table 14, we find that higher values of S_B & S_D are associated with increased total costs, a relationship that is graphically corroborated in Figure 13.

Figure 13: Comparison of S_B & S_D

$h_A \backslash k_A$	0.1	0.2	0.3	0.4	0.5
0.1	26.23091	26.23756	26.24421	26.25086	26.25751
0.2	27.71849	27.72514	27.73179	27.73844	27.74509
0.3	29.20607	29.21272	29.21937	29.22601	29.23266
0.4	30.69364	30.70029	30.70694	30.71359	30.72024
0.5	32.18122	32.18787	32.19452	32.20117	32.20782

Table 15: Comparison of h_A & k_A

Our examination of Table 15 reveals a consistent pattern: as h_A & k_A increase, total cost follows suit, confirming the graphical representation in Figure 14.

Figure 14: Comparison of h_A & k_A

$h_B \backslash k_B$	0.1	0.2	0.3	0.4	0.5
0.1	42.45043	42.45542	42.4604	42.46539	42.47038
0.2	43.60041	43.60539	43.61038	43.61537	43.62036
0.3	44.75039	44.75537	44.76036	44.76535	44.77033
0.4	45.90037	45.90535	45.91034	45.91533	45.92031
0.5	47.05034	47.05533	47.06032	47.0653	47.07029

Table 16: Comparison of h_B & k_B

Based on the data presented in Table 16, it is evident that as the values of h_B & k_B increase, there is a concurrent increase in the total cost. This relationship is visually depicted in Figure 15.

Figure 15: Comparison of h_B & k_B

$h_D \backslash k_D$	0.1	0.2	0.3	0.4	0.5
0.1	32.04073	32.04818	32.05563	32.06308	32.07052
0.2	35.79067	35.79812	35.80557	35.81302	35.82047
0.3	39.54061	39.54806	39.55551	39.56296	39.57041
0.4	43.29055	43.298	43.30545	43.3129	43.32035
0.5	47.04049	47.04794	47.05539	47.06284	47.07029

Table 17: Comparison of h_D & k_D

From the tabulated data in Table 17, it is apparent that as both h_D & k_D increase, there is a proportional increase in the total cost, as visually represented in Figure 16.

Figure 16: Comparison of h_D & k_D

$h_A \backslash S_A$	0.1	0.2	0.3	0.4	0.5
0.1	26.24424	26.24422	26.24421	26.2442	26.24419
0.2	27.73181	27.7318	27.73179	27.73178	27.73176
0.3	29.21939	29.21938	29.21937	29.21935	29.21934
0.4	30.70697	30.70696	30.70694	30.70693	30.70692
0.5	32.19454	32.19453	32.19452	32.19451	32.19449

Table 18: Comparison of h_A & S_A

In line with the information presented in Table 18, it is evident that when h_A & S_A are elevated, the total cost also experiences an increase, as graphically portrayed in Figure 17.

Figure 17: Comparison of h_A & S_A

$h_B \backslash S_B$	0.1	0.2	0.3	0.4	0.5
0.1	42.43149	42.45093	42.47038	42.48982	42.50926
0.2	43.58147	43.60091	43.62036	43.6398	43.65924
0.3	44.73145	44.75089	44.77033	44.78978	44.80922
0.4	45.88143	45.90087	45.92031	45.93976	45.9592
0.5	47.03141	47.05085	47.07029	47.08974	47.10918

Table 19: Comparison of h_B & S_B

Our observations from Table 19 reveal that an increase in both h_B & S_B directly corresponds to a rise in total cost, as graphically depicted in Figure 19.

Figure 18: Comparison of h_B & S_B

$h_D \backslash S_D$	0.1	0.2	0.3	0.4	0.5
0.1	31.99279	32.03165	32.07052	32.10939	32.14826
0.2	35.74273	35.7816	35.82047	35.85934	35.89821
0.3	39.49267	39.53154	39.57041	39.60928	39.64815
0.4	43.24261	43.28148	43.32035	43.35922	43.39809
0.5	46.99255	47.03142	47.07029	47.10916	47.14803

Table 20: Comparison of h_D & S_D

Analyzing the data from Table 20, it becomes evident that heightened values of both h_D & S_D are directly associated with an increase in the total cost, as depicted graphically in Figure 19.

Figure 19: Comparison of h_D & S_D

In our research on the "A Multi-Echelon System Perishable Inventory Optimization Model for Two Commodities," we conducted a comprehensive analysis

of the Total Expected Cost Rate (TC) with varying system parameters. Specifically, we examined scenarios with different values of S (the maximum inventory level) and M (the maximum demand rate).

Our findings indicate that the demand rate (λ_1 and λ_2) is a critical factor affecting the total cost. When demand rate increases, the total cost also rises. This observation underscores the importance of demand forecasting and effective inventory management strategies to mitigate increased costs.

Additionally, we investigated the impact of various cost-related parameters, including ordering costs (k_A, k_B, k_D), holding costs (h_A, h_B, h_D). In each case, an increase in these costs was associated with a corresponding increase in the total cost, emphasizing the need for cost-efficient procurement and inventory control practices.

Furthermore, we analyzed the relationships between these parameters and inventory levels (S_A, S_B, S_D) at the retailer nodes, highlighting that higher inventory levels are associated with increased total costs.

Lastly, our study explored the interplay between holding costs (h_A, h_B, h_D) and ordering costs (k_A, k_B, k_D), indicating that simultaneous increases in these parameters led to higher total costs.

7 Concluding Remarks

In this comprehensive research endeavor, we delved into the intricacies of a continuous review inventory control system within the context of a supply chain that involves two distinct yet interrelated commodities. Our primary objectives were twofold: first, to investigate the dynamics of this complex inventory control system, and second, to evaluate its performance using various measures and metrics. One of the significant achievements of this study was the determination of the steady-state probability distribution of system states. This analysis provides invaluable insights into the long-term behavior and stability of the inventory control system. By understanding the probabilities associated with different inventory states, we can make informed decisions to optimize inventory levels, reduce costs, and enhance overall supply chain efficiency.

Furthermore, we rigorously assessed the performance of the inventory control system. Measures such as inventory turnover, fill rate, order lead time, and other key performance indicators were examined. These measures serve as critical benchmarks for evaluating the effectiveness of the system in meeting supply chain goals and customer demands.

As we look to the future, our research will not stagnate; instead, it will evolve to address even more complex scenarios. We recognize that the real-world supply chain landscape is multifaceted, and our model must adapt accordingly. To

that end, our forthcoming work will extend the current model to accommodate perishable products. Perishable goods introduce a new layer of complexity due to their limited shelf life, and optimizing their inventory control is paramount in minimizing waste and ensuring product availability.

Additionally, we intend to delve into the realm of partial backlogging. This aspect of inventory management involves allowing for partial fulfillment of backlogged orders, a common occurrence in many supply chains. Incorporating partial backlogging into our model will provide a more realistic representation of inventory systems, ensuring that our research remains practical and applicable in real-world scenarios.

In conclusion, this paper marks a significant step forward in the study of continuous review inventory control systems within supply chains. Our analysis of two interconnected commodities, the derivation of steady-state probability distributions, and the assessment of system performance have laid a solid foundation for future advancements. As we embark on the path ahead, we remain committed to enhancing our model to better address the evolving challenges and complexities of supply chain management.

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Nano δI-closed Sets in Nano Ideal Topological Spaces

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Abstract

This article investigates the notion of N-δI-closed sets and its properties in ideal nano topological space using the closure operator $Ncl_{\delta I}(S)=\{x\in \mathbf{U}/\ Nint(Ncl^{*}(W))\cap S\neq \emptyset$, for each $W\in W_N(x)\}$. We establish $Ncl_{\delta I}(S)$ is the intersection of all N-δI-closed supersets of S and $Nint_{\delta I}(S)$ is the union of all N-δI-open subsets of S in this space. Apart from that N-δI-closed sets occurs between the class of Nδ-closed sets and N-closed sets. Moreover, we explore the concepts of N-δI-interior, N-δI-Exterior, N-δI-Border, and N-δI-Frontier and discuss its properties.

Keywords: nano ideal topological space; nano δI-open sets; nano δI-closed sets.

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1. Introduction and Preliminaries

A non-empty sub collection $\mathcal{J}$ of $P(\mathbf{U})$ is called ideal [2] if it satisfies: (i) $S \in \mathcal{J}$ and $T \subset S \Rightarrow T \in \mathcal{J}$, and (ii) $S \in \mathcal{J}$ and $T \in \mathcal{J} \Rightarrow S \cup T \in \mathcal{J}$. In [1], Jankovic and Hamlet introduced and studied the local function and the kuratowski closure operator cl*(.) in ideal topological spaces. In [3], nano closed set, nano interior, and nano closure in nano topological spaces were defined and researched by Lellis Thivagar. In [4], Lellis Thivagar and Richard introduced the lower (resp., upper) approximation and border as $L_{(\mathcal{R})}(X) = \cup\{\mathcal{R}(x): \mathcal{R}(x) \subset X, x \in \mathbf{U}\}$, where $\mathcal{R}(x)$ denotes the equivalence class determined by $x \in \mathbf{U}$ (resp., $U_{(\mathcal{R})}(X) = \cup\{\mathcal{R}(x): \mathcal{R}(x) \cap X \neq \phi, x \in \mathbf{U}\}$) and $B_{(\mathcal{R})}(X) = U_{(\mathcal{R})}(X) - L_{(\mathcal{R})}(X)$ of $X \subset \mathbf{U}$ using the equivalence relation $\mathcal{R}$. In [10], Carmel discussed some weak nano open sets. In [5, 6, 7], Parimala et al. introduced nano ideal topological spaces or ideal nano topological spaces, N-local function which is defined as: $S_N^*(\mathcal{J}, \mathbb{N}) = \{x \in \mathbf{U}: W \cap S \notin \mathcal{J}$, for every $W \in W_N(x)\}$ where $W_N(x) = \{W / x \in W$ and $W \in \mathcal{J}\}$ and the closure operator n-cl*(.) defined as: $n\text{-cl}^*(S) = S \cup S_N^*$, for $S \subset \mathbf{U}$ and investigated its some basic properties. Parimala et al. [8] presented the concept of Nδ-open sets. A set S is known as Nδ-open set if $n\text{-cl}_\delta(S) = S$, where $n\text{-cl}_\delta(S) = \cup\{x \in \mathbf{U}: n\text{-int}(n\text{-cl}(W)) \cap S \neq \phi$, W is n-open and $x \in W\}$. Premkumar and Rameshpandi [9] introduced R-NI-open sets. A set S is known as R-NI-open [9] (resp., nano regular open [3]) set if $n\text{-int}(n\text{-cl}^*(S)) = S$ (resp., $n\text{-int}(n\text{-cl}(S) = S)$). The complement of an R-NI-open (resp., nano regular open) set is R-NI-closed (resp. nano regular closed). Yuksel, Acikgoz, and Noiri [11] introduced and studied the notion of δ-I-open set in an ideal topological space. By this motivation here we focus on nano δI-closed sets in nano ideal topological spaces.

Property 1.1. [3] If $(\mathbf{U}, \mathcal{R})$ is an approximation space and $W, T \subseteq \mathbf{U}$. Then,

(i) $L_{(\mathcal{R})}(W) \subseteq W \subseteq U_{(\mathcal{R})}(W)$.

(ii) $L_{(\mathcal{R})}(\phi) = U_{(\mathcal{R})}(\phi) = \phi$.

(iii) $L_{(\mathcal{R})}(\mathbf{U}) = U_{(\mathcal{R})}(\mathbf{U}) = \mathbf{U}$.

(iv) $U_{(\mathcal{R})}(W \cup Z) = U_{(\mathcal{R})}(X) \cup U_{(\mathcal{R})}(Z)$.

(v) $U_{(\mathcal{R})}(W \cap Z) \subseteq U_{(\mathcal{R})}(W) \cap U_{(\mathcal{R})}(Y)$.

(vi) $L_{(\mathcal{R})}(W \cup Z) \supseteq L_{(\mathcal{R})}(W) \cup L_{(\mathcal{R})}(Z)$.

(vii) $L_{(\mathcal{R})}(W \cap Z) = L_{(\mathcal{R})}(W) \cap L_{(\mathcal{R})}(Z)$.

(viii) $L_{(\mathcal{R})}(W) \subseteq L_{(\mathcal{R})}(Z)$ and $U_{(\mathcal{R})}(W) \subseteq U_{(\mathcal{R})}(W)$ if $W \subseteq Z$.

(ix) $U_{(\mathcal{R})}(W^c) = [L_{(\mathcal{R})}(W)]^c$ and $L_{(\mathcal{R})}(W^c) = [U_{(\mathcal{R})}(W)]^c$.

(x) $U_{(\mathcal{R})}[U_{(\mathcal{R})}(W)] = L_{(\mathcal{R})}[U_{(\mathcal{R})}(W)] = U_{(\mathcal{R})}(W)$.

(xi) $L_{(\mathcal{R})}[L_{(\mathcal{R})}(W)] = U_{(\mathcal{R})}[L_{(\mathcal{R})}(W)] = U_{(\mathcal{R})}(W)$.

Definition 1.2. [3] Let $\mathbf{U}$ be the universal set and $\mathscr{R}$ be an equivalence relation defined on $\mathbf{U}$. $T_{(\mathscr{R})}(W)=\{\mathbf{U}, \phi, L_{(\mathscr{R})}(W), U_{(\mathscr{R})}(W), B_{(\mathscr{R})}(W)\}$ where $W\subseteq\mathbf{U}$, Then $T_{(\mathscr{R})}(W)$ by property 1.1 satisfies the given below axioms:

(i) $\mathbf{U}$ and $\phi\in T_{(\mathscr{R})}(W)$.
(ii) Arbitrary union of members of $T_{(\mathscr{R})}(W)$ is in $T_{(\mathscr{R})}(W)$.
(iii) Finite intersection of members of $T_{(\mathscr{R})}(W)$ is in $T_{(\mathscr{R})}(W)$.

Then, $T_{(\mathscr{R})}(W)$ is a topology on $\mathbf{U}$ called the nano topology on $\mathbf{U}$ with respect to W. The pair $(\mathbf{U}, T_{(\mathscr{R})}(W))$ is known as nano topological space (simply, NTS). The members of $T_{(\mathscr{R})}(W)$ are known as nano open sets (briefly, N-open sets) and their complements are N-closed sets. We denote, $T_{(\mathscr{R})}(W)=\mathbb{N}$ and so $(\mathbf{U}, \mathbb{N})$ is the nano topological space.

The nano topology $(\mathbf{U}, \mathbb{N})$ with ideal $\mathscr{J}$ forms the triplet $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ called nano ideal topological space or ideal nano topological space [5]. We use ideal nano topological space for convenience in understanding. We simply mention an ideal nano topological space by INTS. Also, for $S\subseteq\mathbf{U}$ we denote the nano interior (resp., nano closure, nano δ-closure, nano *-closure) of S by Nint(S) (resp., Ncl(S), $Ncl_{\delta}(S)$, Ncl*(S)).

2. N-δI-closed set

Definition 2.1. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be an INTS. For $S\subseteq\mathbf{U}$, $Ncl_{\delta I}(S)=\{x\in\mathbf{U} / Nint(Ncl^{*}(W)) \cap S \neq \emptyset$, for each $W\in W_N(x)\}$ where $W_N(x) =\{W / x \in W$ and $W \in \mathbb{N}\}$. If $Ncl_{\delta I}(S) = S$ then S is known as a nano δI-closed set and $\mathbf{U}$ – S is nano δI-open set.

Rest of the paper nano δI-closed (resp., nano δI-open) sets are simply denoted by N-δI-closed (resp., N-δI-open) and the N-δI-open sets collection is mentioned by $N\text{-}T_{\delta I}$. Generally, Nδ-open sets are N-δI-open and N-δI-open sets are N-open sets. But their reverse implications are not always hold as in the forthcoming Example. Let $\mathbf{U}$={e, f, g, h}, X={g}, $\mathbf{U}/\mathscr{R}$={{e}, {f}, {g, h}}, $\mathbb{N}=\{\mathbf{U}, \emptyset, \{g, h\}\}$ and $I=\{\emptyset, \{g\}\}$. Then, $n\text{-}T_{\delta I}$ $=\{\mathbf{U}, \emptyset\}$. If $I =\{\emptyset, \{g\}, \{h\}, \{g, h\}\}$ then $N\delta C(\mathbf{U})=\{\mathbf{U}, \emptyset\}$ and $n\text{-}T_{\delta I} =\{U, \emptyset, \{g, h\}\}$.

Theorem 2.2. Let S and T be subsets of an INTS $(\mathbf{U}, \mathbb{N}, \mathscr{J})$. Then,
(i) Nint (Ncl*(S)) is R-NI-open.
(ii) If S and T are R-NI-open, then S ∩ T is R-NI-open.
(iii) N-Regular open sets are R-NI-open.
(iv) R-NI-open sets are N-δI-open.
(v) If S is an N-δI-open set, then S = ∪{W/W is R-NI-open}.

Proof.(i)Let W = Nint(Ncl*(S)), $S \subseteq \mathbf{U}$. Then, Nint(Ncl*(W)) = Nint(Ncl*(Nint(Ncl*(S)) ⊆ Nint(Ncl*(S)) = W = Nint(W) ⊆ Nint(Ncl*(S)).
(ii) The proof from the fact that Nint(Ncl*(S ∩ T)) ⊆ Nint(Ncl*(S) ∩ Ncl*(T)).

(iii) The proof is from the fact that, $Ncl^*(S)) \subseteq Ncl(S)$.
(iv) Let S be an R-NI-open set. Then, for every $x \in S$, $(S^C) \cap S = \phi$. Hence, $x \notin Ncl_{\delta I}(S^C)$. Always, $S \subseteq Ncl_{\delta I}(S)$. Hence, $Ncl_{\delta I}(S^C) = S^C$.
(v) Since S is an N-δI-open set, $N\text{-}cl_{\delta I}(S^C) = S^C$. For every $x \in S$, $x \notin Ncl_{\delta I}(S^C)$. Therefore, there exists a $W \in W_N(x)$ such that $Nint(Ncl^*(W)) \cap (S^C) = \emptyset$. Therefore, $x \in Nint(Ncl^*(W)) \subseteq S$ and hence $S = \cup \{Nint(Ncl^*(W)) / x \in S\}$. Hence, by (i) proof completes.

Theorem 2.3. Let S and T be subsets of an INTS $(\mathbf{U}, \mathbb{N}, \mathscr{J})$. Then,
(i) $S \subseteq Ncl_{\delta I}(S)$.
(ii) If $S \subseteq T$, then $Ncl_{\delta I}(S) \subseteq Ncl_{\delta I}(T)$.
(iii) $Ncl_{\delta I}(S) = \cap \{H \subseteq \mathbf{U} / S \subseteq H, H \text{ is an N-}\delta\text{I-closed set}\}$.
(iv) If S_j is an N-δI-closed subset of $\mathbf{U}$ for every $j \in J$, then $\cap \{S_j / j \in J\}$ is N-δI-closed.
(v) $N\text{-}cl_{\delta I}(S)$ is N-δI-closed.

Proof. (i) Let $x \in S$. Then, $W \cap S \neq \emptyset$ for all $W \in W_N(x)$. But, $W \cap S \subseteq Nint(Ncl^*(W)) \cap S$. Therefore, $x \in Ncl_{\delta I}(S)$.
(ii)Let $x \in Ncl_{\delta I}(S)$. Suppose that, $x \notin Ncl_{\delta I}(T)$. Then there is a $W \in W_N(x)$ with $Nint(Ncl^*(W) \cap T = \emptyset$. Therefore, $Nint(Ncl^*(W) \cap S = \emptyset$. A contradiction.
(iii) Let $x \in Ncl_{\delta I}(S)$. $H \subseteq S$, H is N-δI-closed. Then, $\emptyset \neq Nint(Ncl^*(W)) \cap S \subseteq Nint(Ncl(W)) \cap H$. Therefore, $x \in Ncl_{\delta I}(H)$ and hence $x \in H$. Conversely, suppose that $x \notin Ncl_{\delta I}(S)$. Then there is a $W \in W_N(x)$ for which $Nint(Ncl^*(W)) \cap S = \emptyset$. By Theorem 2.2, $\mathbf{U} - (Nint(Ncl^*(W)))$ is a N-δI-closed set contains S. $x \notin \mathbf{U} - (Nint(Ncl^*(W)))$. Therefore, $x \notin \cap \{H \subseteq \mathbf{U} / S \subseteq H \text{ and H is N-}\delta\text{I-closed}\}$.
(iv) Since $Ncl_{\delta I}(\cap_{j \in J} S_j) \subseteq Ncl_{\delta I}(S_j) = S_j$ for each j. Therefore, $Ncl_{\delta I}(\cap_{j \in J} S_j) \subseteq \cap_{j \in J} S_j$. Hence, we have, $Ncl_{\delta I}(\cap_{j \in J} S_j) = \cap_{j \in J} S_j$, by(i).
(v) The proof is from (iii) and (iv).

Definition 2.4. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be an INTS and S a subset of $X \subseteq \mathbf{U}$. A point $x \in \mathbf{U}$ is said to be an N-δI-Limit point of S if for every $W \in N\text{-}T_{\delta I}$ containing x, $W \cap (S - \{x\} \neq \emptyset$. The set of all N-δI-Limit points of S is called an N-δI-Derived set of S and is simply mentioned by $N\text{-}D_{\delta I}(S)$.

Theorem 2.5. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be an INTS and $X \subseteq \mathbf{U}$. Then for S, $T \subseteq X$,
(i) $N\text{-}D_{\delta I}(\emptyset) = \emptyset$.
(ii) If $S \subseteq T$, then $N\text{-}D_{\delta I}(S) \subseteq N\text{-}D_{\delta I}(T)$.
(iii) $N\text{-}D_{\delta I}(S) \cup N\text{-}D_{\delta I}(T) \subseteq N\text{-}D_{\delta I}(S \cup T)$.
(iv) $N\text{-}D_{\delta I}(S \cap T) \subseteq N\text{-}D_{\delta I}(S) \cap N\text{-}D_{\delta I}(T)$.
(v) $N\text{-}D_{\delta I}(N\text{-}D_{\delta I}(S)) - S \subseteq N\text{-}D_{\delta I}(S)$.
(vi) $N\text{-}D_{\delta I}(S \cup N\text{-}D_{\delta I}(S)) \subseteq S \cup N\text{-}D_{\delta I}(S)$.

Proof. (i) Let $x \in \mathbf{U}$. Then for every $W \in N\text{-}T_{\delta I}$ containing x, $W \cap (\emptyset \cap \{x\}^C) = \emptyset$. Hence, $N\text{-}D_{\delta I}(\emptyset) = \emptyset$.

(ii) Let $x \in N\text{-}D_{\delta I}(S)$. Then, $W \cap (S - \{x\}) \neq \emptyset$ for every $W \in N\text{-}T_{\delta I}$ containing x. Since $S \subseteq T$, $G \cap (T - \{x\}) \neq \emptyset$. This implies that $x \in N\text{-}D_{\delta I}(T)$.
(iii) Since S and T are the subsets of $S \cup T$, $N\text{-}D_{\delta I}(S) \subseteq N\text{-}D_{\delta I}(S \cup T)$ and $N\text{-}D_{\delta I}(T) \subseteq N\text{-}D_{\delta I}(S \cup T)$ by (ii). Hence, $N\text{-}D_{\delta I}(S) \cup N\text{-}D_{\delta I}(T) \subseteq N\text{-}D_{\delta I}(S \cup T)$.
(iv) Since $S \cap T$ is the subset of both S and T, again by (ii) the proof completes.
(v) Let $x \in N\text{-}D_{\delta I}(N\text{-}D_{\delta I}(S)) - S$ and $W \in N\text{-}T_{\delta I}$ containing x. Then, $W \cap (N\text{-}D_{\delta I}(S) - \{x\}) \neq \emptyset$. Let $u \in W \cap (N\text{-}D_{\delta I}(S) - \{x\})$. Then, $u \in W$ and $u \in N\text{-}D_{\delta I}(S)$. Then, $W \cap (S - \{u\}) \neq \emptyset$. Let $v \in W \cap (S - \{u\})$. Then $v \neq x$ for $v \in S$ and $x \notin S$. Therefore, $W \cap (S - \{x\}) \neq \emptyset$ and so $x \in N\text{-}D_{\delta I}(S)$.
(vi) Let $x \in N\text{-}D_{\delta I}(S \cup N\text{-}D_{\delta I}(S))$. If $x \in S$ then the proof is clear. If $x \in N\text{-}D_{\delta I}(S \cup N\text{-}D_{\delta I}(S)) - S$, then for $W \in N\text{-}T_{\delta I}$ containing x, $W \cap (S \cup N\text{-}D_{\delta I}(S) - \{x\}) \neq \emptyset$. Thus, $W \cap (S - \{x\}) \neq \emptyset$ or $W \cap (N\text{-}D_{\delta I}(S) - \{x\}) \neq \emptyset$. Also, by (iv) $W \cap (S - \{x\}) \neq \emptyset$. Therefore, $x \in N\text{-}D_{\delta I}(S)$. Hence, $N\text{-}D_{\delta I}(S \cup N\text{-}D_{\delta I}(S)) \subseteq S \cup N\text{-}D_{\delta I}(S)$.

Theorem 2.6. Let S be any subset of an INTS $(\mathbf{U}, \mathbb{N}, \mathscr{J})$. Then $Ncl_{\delta I}(S) = S \cup N\text{-}D_{\delta I}(S)$.

Proof. Since $N\text{-}D_{\delta I}(S) \subseteq Ncl_{\delta I}(S)$, $S \cup N\text{-}D_{\delta I}(S) \subseteq Ncl_{\delta I}(S)$. Let $x \in Ncl_{\delta I}(S)$. If $x \in S$, then the proof is clear. Suppose $x \notin S$, then $W \cap (S - \{x\}) \neq \emptyset$, for every $W \in N\text{-}T_{\delta I}$ containing x. Therefore, $x \in N\text{-}D_{\delta I}(S)$.

Definition 2.7. A set M contained in $\mathbf{U}$ is said to be a nano δI-Neighbourhood (briefly, N-δI-Nbd) of a member $x \in \mathbf{U}$ if there exists a $W \in N\text{-}T_{\delta I}$ containing x such that $x \in W \subseteq M$. The set of all N-δI-Nbd of $x \in \mathbf{U}$ is called nano δI-Neighbourhood system of x (briefly, N-δI-NbdS(x)).

Theorem 2.8. Arbitrary union of a family of N-δI-Nbd of a point $x \in \mathbf{U}$ is a N-δI-Nbd of x.

Proof. Let $\{S_\alpha : \alpha \in J\}$ be an arbitrary family of N-δI-Nbd of $x \in \mathbf{U}$. Then, S_α is N-δI-Nbd of x for every α. Therefore, there is a $W \in N\text{-}T_{\delta I}$ containing x with $x \in W \subseteq S_\alpha$. But for every $\alpha \in J$, $S_\alpha \subseteq \cup S_\alpha$. Therefore, $x \in G \subseteq \cup S_\alpha$. Hence, $\cup S_\alpha$ is a N-δI-Nbd of x.

Theorem 2.9. For any $x \in \mathbf{U}$, N-δI-NbdS(x) satisfies,
(i) $N\text{-}\delta I\text{-}NbdS(x) \neq \emptyset$.
(ii) If a subset $S \in N\text{-}\delta I\text{-}NbdS(x)$ then $x \in S$.
(iii) If $S \in N\text{-}\delta I\text{-}NbdS(x)$ and $S \subseteq M$ then $M \in N\text{-}\delta I\text{-}NbdS(x)$.

Proof. (i) For each $x \in W$ and $W \in N\text{-}T_{\delta I}$, $x \in W \subseteq W$. Hence, W is a N-δI-Nbd(x) and so $W \in N\text{-}\delta I\text{-}NbdS(x)$. Therefore, $N\text{-}\delta I\text{-}NbdS(x) \neq \emptyset$.
(ii) Since $S \in N\text{-}\delta I\text{-}NbdS(x)$, S is a N-δI-Nbd(x). Then there exist a $W \in N\text{-}T_{\delta I}$ such that $x \in W \subseteq S$.

(iii) Since $S \in$ N-δI-NbdS(x), there exist a $W \in$ N-$T_{\delta I}$ such that, $x \in W \subseteq S \subseteq M$. Hence, $M \in$ N-δI-NbdS(x).

Theorem 2.10. Let the arbitrary union of a family of N-δI-open sets is N-δI-open and $S \subseteq \mathbf{U}$. Then, S is N-δI-open iff S is N-δI-Nbd of each its points.

Proof. Let $S \in$ N-$T_{\delta I}$ containing x. Then S is N-δI-Nbd(x) for every $x \in S$. Conversely, let S be N-δI-Nbd of each $x \in S$. Then for every $x \in S$, there exist a $W_x \in$ N-$T_{\delta I}$ containing x such that $x \in W_x \subseteq S$. If $x \in S$ then there is atleast one $W_x \in$ N-$T_{\delta I}$ containing x such that $x \in W_x \subseteq \cup_{x \in S} W_x$. Hence, $S \subseteq \cup_{x \in S} W_x$. Again, $y \in \cup_{x \in S} W_x$ implies $y \in W_x$ for some $x \in S$. Then $y \in S$. Therefore, $\cup_{x \in S} W_x \subseteq S$. Hence, $S = \cup_{x \in S} W_x$. As each $W_x \in$ N-$T_{\delta I}$, $S \in$ N-$T_{\delta I}$.

Theorem 2.11. Let the family of N-δI-open subsets of an INTS $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ is closed under finite intersection. If S is N-δI-closed and $x \in S^C$ then there exists an N-δI-Nbd W of x such that $W \cap S = \emptyset$.

Proof. Let S be an N-δI-closed set. Then by Theorem 2.10, S^C is N-δI-Nbd of every one of its points. Let $x \in S^C$. Therefore, there exists a $W \in$ N-$T_{\delta I}$ such that $x \in W \subseteq S^C$. Hence, $W \cap S = \emptyset$.

Definition 2.12. A point $x \in \mathbf{U}$ is known as N-δI-interior point of S if there exists $W \in$ N-$T_{\delta I}$ containing x such that $W \subseteq S$. The collection of all N-δI-interior point of S is called N-δI-interior of S and is mentioned by $\text{Nint}_{\delta I}(S)$.

Theorem 2.13. For a subset S of an INTS $(\mathbf{U}, \mathbb{N}, \mathscr{J})$, $\text{Nint}_{\delta I}(S) = \cup \{W \setminus W \in \text{N-}T_{\delta I} \text{ and } W \subseteq S\}$.

Proof. Let $x \in \text{Nint}_{\delta I}(S)$. Then, x is the N-δI-interior point of S. Therefore, there is a $W \in$ N-$T_{\delta I}$ and $x \in W \subseteq S$. Therefore, $x \in \cup \{W \setminus W \in \text{N-}T_{\delta I} \text{ and } W \subseteq S\}$. On the other hand, suppose, $x \in \cup \{W \setminus W \in \text{N-}T_{\delta I} \text{ and } W \subseteq S\}$. Then, $x \in W$ and $W \subseteq S$ for some $W \in$ N-$T_{\delta I}$. Hence, x is the N-δI-interior point of S.

Theorem 2.14. Let S and T be subsets of an INTS $(\mathbf{U}, \mathbb{N}, \mathscr{J})$. Then,

(i) N-δI-interior of S is the biggest N-δI-open subset in S.
(ii) S is a N-δI-open iff $S = \text{Nint}_{\delta I}(S)$.
(iii) $\text{Nint}_{\delta I}(\emptyset) = \emptyset$, and $\text{Nint}_{\delta I}(\mathbf{U}) = \mathbf{U}$.
(iv) If $S \subseteq T$ then $\text{Nint}_{\delta I}(S) \subseteq \text{Nint}_{\delta I}(T)$.
(v) $\text{Nint}_{\delta I}(S) \cup \text{Nint}_{\delta I}(T) \subseteq \text{Nint}_{\delta I}(S \cup T)$.
(vi) $\text{Nint}_{\delta I}(S \cap T) \subseteq \text{Nint}_{\delta I}(S) \cap \text{Nint}_{\delta I}(T)$.
(vii) $\text{Nint}_{\delta I}(\text{Nint}_{\delta I}(S)) = \text{Nint}_{\delta I}(S)$.
(viii) $\text{Nint}_{\delta I}(S) = S - \text{N-D}_{\delta I}(\mathbf{U} - S)$.
(ix) $\mathbf{U} - \text{Nint}_{\delta I}(S) = \text{Ncl}_{\delta I}(\mathbf{U} - S)$.

Proof. (i) Let $W \subseteq S$ and W be any N-δI-open subset of S and $x \in W$. Then $x \in W \subseteq S$. Since $W \in$ N-$T_{\delta I}$, x is the N-δI-interior point of S. Therefore, for $x \in W$ implies that $x \in$ $Nint_{\delta I}(S)$. This implies that, every N-δI-open subset of S is in $Nint_{\delta I}(S)$. Therefore, $Nint_{\delta I}(S)$ is the biggest N-δI-open subset in S.
(ii) Necessity: Let $S \in$ N-$T_{\delta I}$. Since $S \subseteq S$, S is the biggest N-δI-open subset of S. Hence the proof completes by (i).
Sufficiency: Let $S = Nint_{\delta I}(S)$. Then proof is clear again by (i).
(iii) Always, $\mathbf{U}, \emptyset \in$ N-$T_{\delta I}$. Therefore, $Nint_{\delta I}(\mathbf{U}) = \mathbf{U}$ and $Nint_{\delta I}(\emptyset) = \emptyset$, by (ii).
(iv) Let $S \subseteq T$. Then for $x \in Nint_{\delta I}(S)$, there exist a $W \in$ N-$T_{\delta I}$ containing x such that $x \in W \subseteq S$. Therefore, $x \in W \subseteq T$. Hence, $x \in Nint_{\delta I}(T)$.
(v) Since $S \cup T$ contains both S and T by (iv), $Nint_{\delta I}(S) \subseteq Nint_{\delta I}(S \cup T)$ and $Nint_{\delta I}(T) \subseteq Nint_{\delta I}(S \cup T)$. Therefore, $Nint_{\delta I}(S) \cup Nint_{\delta I}(T) \subseteq Nint_{\delta I}(S \cup T)$.
(vi) The proof is clear by (iv).
(vii) By (ii) and (i), the proof is clear.
(viii) Let $x \in S - (N\text{-}D_{\delta I}(\mathbf{U} - S))$. Then $x \in S$ and $x \notin N\text{-}D_{\delta I}(\mathbf{U} - S)$. Therefore, x is not a N-δI-Limit point of $\mathbf{U} - S$. Therefore, there exist a $W \in$ N-$T_{\delta I}$ with $x \in W$, $W \cap (\mathbf{U} - S) = \emptyset$ and so $W \subseteq S$. Therefore, $x \in W \subseteq S$ gives $x \in Nint_{\delta I}(S)$. For other side, let $x \in Nint_{\delta I}(S)$. Then $x \in Nint_{\delta I}(S) \subseteq S$. But $Nint_{\delta I}(S)$ is the biggest N-δI-open set and $x \in Nint_{\delta I}(S)$, and not containing the points of $(\mathbf{U} - S)$. Therefore, x is not an N-δI-limit point of $(\mathbf{U} - S)$ and so $x \in S - (N\text{-}D_{\delta I}(U - S))$. Hence, $Nint_{\delta I}(S) \subseteq S - (N\text{-}D_{\delta I}(U - S))$.
(ix) $\mathbf{U} - Nint_{\delta I}(S) = \mathbf{U} - (S - N\text{-}D_{\delta I}(S^C)) = \mathbf{U} - (S \cap (\mathbf{U} - N\text{-}D_{\delta I}(S^C))) = (S^C) \cup N\text{-}D_{\delta I}(S^C) = Ncl_{\delta I}(S^C)$, by (viii) and by Theorem 2.6.

Theorem 2.15. Let S and T be subsets of an INTS $(\mathbf{U}, \mathbb{N}, \mathscr{J})$. Then,
(i) $Nint_{\delta I}(\mathbf{U} - S) \subseteq \mathbf{U} - Nint(S)$.
(ii) $Nint_{\delta I}(S - T) \subseteq Nint_{\delta I}(S) - Nint_{\delta I}(T)$.

Proof. (i) Let $x \in Nint_{\delta I}(\mathbf{U} - S)$. Then $x \notin S$ and hence $x \notin Nint_{\delta I}(S)$. Therefore, $x \in \mathbf{U} - Nint_{\delta I}(S)$.
(ii) $Nint_{\delta I}(S - T) = Nint_{\delta I}(S \cap (\mathbf{U} - T)) \subseteq Nint_{\delta I}(S) \cap Nint_{\delta I}(\mathbf{U} - T) \subseteq Nint_{\delta I}(S) \cap (\mathbf{U} - Nint_{\delta I}(T)$ $= Nint_{\delta I}(S) - Nint_{\delta I}(T)$, by (i).

3. Nano δI-Exterior (resp. Border and Frontier)

Definition 3.1. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be an INTS and $S \subseteq \mathbf{U}$. Then,
(i) the nano δI-Exterior of S (briefly, N-$Ext_{\delta I}(S)$) is defined as N-$Ext_{\delta I}(S) = Nint_{\delta I}(\mathbf{U} - S)$.
(ii) the nano δI-Border of S (briefly, N-$Br_{\delta I}(S)$) is defined as N-$Br_{\delta I}(S) = S - Nint_{\delta I}(S)$.
(iii) the nano δI-Frontier of S (briefly, N-$Fr_{\delta I}(S)$) is defined as N-$Fr_{\delta I}(S) = Ncl_{\delta I}(S) - Nint_{\delta I}(S)$.

Theorem 3.2. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be an INTS and $S, T \subseteq \mathbf{U}$. Then,
(i) N-$Ext_{\delta I}(\emptyset) = \mathbf{U}$ and N-$Ext_{\delta I}(\mathbf{U}) = \emptyset$.

(ii) $N\text{-}Ext_{\delta I}(S)$ is N-δI-open set.
(iii) $N\text{-}Ext_{\delta I}(S) = \mathbf{U} - Ncl_{\delta I}(S)$.
(iv) If $S \subseteq T$ then $N\text{-}Ext_{\delta I}(T) \subseteq N\text{-}Ext_{\delta I}(S)$.
(v) $N\text{-}Ext_{\delta I}(N\text{-}Ext_{\delta I}(S)) = Nint_{\delta I}(Ncl_{\delta I}(S))$.
(vi) $N\text{-}Ext_{\delta I}(S \cup T) \subseteq N\text{-}Ext_{\delta I}(S) \cup N\text{-}Ext_{\delta I}(T)$.
(vii) $N\text{-}Ext_{\delta I}(S) \cap N\text{-}Ext_{\delta I}(T) \subseteq N\text{-}Ext_{\delta I}(S \cap T)$.
(viii) $N\text{-}Ext_{\delta I}(S) = N\text{-}Ext_{\delta I}(\mathbf{U} - N\text{-}Ext_{\delta I}(S))$.
(ix) $Nint_{\delta I}(S) \subseteq N\text{-}Ext_{\delta I}(N\text{-}Ext_{\delta I}(S))$.
(x) $S \cap N\text{-}Ext_{\delta I}(S) = \emptyset$.
(xi) $N\text{-}Ext_{\delta I}(S) \subseteq \mathbf{U} - S$.

Proof. (i) clear by Definition.
(ii) The proof is obvious.
(iii) $N\text{-}Ext_{\delta I}(S) = Nint_{\delta I}(\mathbf{U} - S) = \mathbf{U} - Ncl_{\delta I}(S)$.
(iv) Since $S \subseteq T$, $Nint_{\delta I}(\mathbf{U} - T) \subseteq Nint_{\delta I}(\mathbf{U} - S)$. Hence, the proof is clear by Definition.
(v) $N\text{-}Ext_{\delta I}(N\text{-}Ext_{\delta I}(S)) = N\text{-}Ext_{\delta I}(Nint_{\delta I}(\mathbf{U} - S)) = N\text{-}Ext_{\delta I}(\mathbf{U} - Ncl_{\delta I}(S)) = Nint_{\delta I}(\mathbf{U} - (\mathbf{U} - Ncl_{\delta I}(S))) = Nint_{\delta I}(Ncl_{\delta I}(S))$.
(vi) by (iv) proof completes.
(vii) again, by (iv) proof is clear.
(viiii) $N\text{-}Ext_{\delta I}(\mathbf{U} - N\text{-}Ext_{\delta I}(S)) = Nint_{\delta I}(\mathbf{U} - (\mathbf{U} - N\text{-}Ext_{\delta I}(S))) = Nint_{\delta I}(\mathbf{U} - Nint_{\delta I}(S)) = Nint_{\delta I}(\mathbf{U} - S) = N\text{-}Ext_{\delta I}(S)$.
(ix) Since $S \subseteq Ncl_{\delta I}(S)$, $Nint_{\delta I}(S) \subseteq Nint_{\delta I}(Ncl_{\delta I}(S)) = Nint_{\delta I}(\mathbf{U} - Nint_{\delta I}(\mathbf{U} - S) = N\text{-}Ext_{\delta I}(Nint_{\delta I}(\mathbf{U} - S)) = N\text{-}Ext_{\delta I}(N\text{-}Ext_{\delta I}(S))$.
(x) $S \cap N\text{-}Ext_{\delta I}(S) = S \cap Nint_{\delta I}(\mathbf{U} - S) \subseteq S \cap (\mathbf{U} - S) = \emptyset$.
(xi) $N\text{-}Ext_{\delta I}(S) = Nint_{\delta I}(\mathbf{U} - S) \subseteq U - S$.

Theorem 3.3. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be a INTS and $S \subseteq \mathbf{U}$. Then S is an N-δI-open if and only if $N\text{-}Br_{\delta I}(S) = \emptyset$.

Proof. Let $S \in N\text{-}T_{\delta I}$. Then $Nint_{\delta I}(S) = S$. Hence, $N\text{-}Br_{\delta I}(S) = \emptyset$. Conversely, suppose $N\text{-}Br_{\delta I}(S) = \emptyset$. Then, $S - Nint_{\delta I}(S) = \emptyset$.

Theorem 3.4. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be a INTS and $S, T \subseteq \mathbf{U}$. Then the following holds true.
(i) $N\text{-}Br_{\delta I}((Nint_{\delta I}(S)) = \emptyset$.
(ii) $Nint_{\delta I}(N\text{-}Br_{\delta I}(S)) = \emptyset$.
(iii) $N\text{-}Br_{\delta I}(N\text{-}Br_{\delta I}(S)) = N\text{-}Br_{\delta I}(S)$.
(iv) $N\text{-}int_{\delta I}(S) \cap N\text{-}Br_{\delta I}(S) = \emptyset$.

Proof. (i) $N\text{-}Br_{\delta I}(Nint_{\delta I}(S)) = Nint_{\delta I}(S) - Nint_{\delta I}(Nint_{\delta I}(S)) = Nint_{\delta I}(S) - Nint_{\delta I}(S) = \emptyset$.
(ii) $Nint_{\delta I}(N\text{-}Br_{\delta I}(S)) = Nint_{\delta I}(S - Nint_{\delta I}(S)) = Nint_{\delta I}(S \cap (\mathbf{U} - Nint_{\delta I}(S))) \subseteq Nint_{\delta I}(S) \cap Nint_{\delta I}(\mathbf{U} - Nint_{\delta I}(S)) = \emptyset$ since, $Nint_{\delta I}(S) \subseteq S$.
(iii) $N\text{-}Br_{\delta I}(N\text{-}Br_{\delta I}(S)) = N\text{-}Br_{\delta I}(S) - Nint_{\delta I}(N\text{-}Br_{\delta I}(S)) = N\text{-}Br_{\delta I}(S)$, by (ii).
(iv) $Nint_{\delta I}(S) \cap N\text{-}Br_{\delta I}(S) = Nint_{\delta I}(S) \cap (S - Nint_{\delta I}(S)) = Nint_{\delta I}(S) \cap (S \cap (\mathbf{U} - Nint_{\delta I}(S))) = Nint_{\delta I}(S) \cap \emptyset = \emptyset$.

Theorem 3.5. Let S and T be two subsets of a INTS ($\mathbf{U}$, $\mathbb{N}$, $\mathscr{J}$). Then,
(i) $Ncl_{\delta I}(S) = Nint_{\delta I}(S) \cup N\text{-}Fr_{\delta I}(S)$.
(ii) $Nint_{\delta I}(S) \cap N\text{-}Fr_{\delta I}(S) = \emptyset$.
(iii) $N\text{-}Fr_{\delta I}(Nint_{\delta I}(S)) \subseteq N\text{-}Fr_{\delta I}(S)$.
(iv) $N\text{-}Br_{\delta I}(S) \subseteq N\text{-}Fr_{\delta I}(S)$.

Proof. (i) $Nint_{\delta I}(S) \cup N\text{-}Fr_{\delta I}(S) = Nint_{\delta I}(S) \cup (Ncl_{\delta I}(S) - Nint_{\delta I}(S)) = Ncl_{\delta I}(S)$.
(ii) $Nint_{\delta I}(S) \cap N\text{-}Fr_{\delta I}(S) = Nint_{\delta I}(S) \cap (Ncl_{\delta I}(S) - Nint_{\delta I}(S)) = \emptyset$.
(iii) $N\text{-}Fr_{\delta I}(Nint_{\delta I}(S)) = Ncl_{\delta I}(Nint_{\delta I}(S)) - Nint_{\delta I}(Nint_{\delta I}(S)) \subseteq Ncl_{\delta I}(S) - Nint_{\delta I}(S) = N\text{-}Fr_{\delta I}(S)$.
(iv) $N\text{-}Br_{\delta I}(S) = S - Nint_{\delta I}(S) = S \cap (\mathbf{U} - Nint_{\delta I}(S)) \in Ncl_{\delta I}(S) \cap (\mathbf{U} - Nint_{\delta I}(S)) = Ncl_{\delta I}(S) - Nint_{\delta I}(S) = N\text{-}Fr_{\delta I}(S)$.

4 Discussion and Conclusion

The introduction of nano topology created a new space in topological research. Many of the weaker and stronger forms of open sets were studied in this space. In this article we studied N-δI-closed sets in ideal nano topological spaces. The closure operator discussed is N-δI-closed as it is the intersection of all N-δI-closed set contains $S \subseteq \mathbf{U}$. Therefore it is the smallest N-δI-closed set contains S. We proved $Ncl_{\delta I}(S)$ is the union of S and its N-δI-Derived set. It is also proved that the arbitrary union of a family of N-δI-Nbd of a point $x \in \mathbf{U}$ is a N-δI-Nbd of x. Similarly the $Nint_{\delta I}(S)$ is N-δI-open as it is the union of all N-δI-open subsets of S so that it is the largest N-δI-open subset of S. It is deduced that N-δI-closed sets occurs between Nδ-closed sets and N-closed sets and counter example have been shown to evident the reverse implication do not imply both. Moreover, we introduced the concepts of N-δI-interior, N-δI-Exterior, N-δI-Border, and N-δI-Frontier and several necessary properties of all these are investigated. Thus this concept can be used to investigate many concepts of topological spaces and we also have views on the continuation of this work.

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Weighted Sharing of Meromorphic Functions Concerning Certain Type of Linear Difference Polynomials

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Abstract

In this research article, with the help of Nevanlinna theory we study the uniqueness problems of transcendental meromorphic functions having finite order in the complex plane $\mathbb{C}$, of the form is given by $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)$ where $L(z,\phi) = \sum_{j=1}^{d} a_j\phi(z+c_j)$ which share a non-zero polynomial $p(z)$ with finite weight. By considering the concept of weighted sharing introduced by I. Lahiri (Complex Variables and Elliptic equations,2001,241-253), we investigate difference polynomials for the cases $(0,2),(0,1),(0,0)$. Our new findings extends and generalizes some classical results of Sujoy Majumder[11]. Some examples have been exhibited which are relevant to the content of the paper.

Keywords: Meromorphic functions, Linear difference polynomial, weighted sharing, uniqueness, non-zero polynomial.
2020 AMS subject classifications: 30D35 [1]

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1 Introduction

The Nevanlinna theory is a branch of complex analysis that has seen extensive research work, which mainly deals with the study of distribution of solutions of the equation $f(z) = a$ in a disc $|z| \leq r$, where $f(z)$ is an meromorphic(entire) function in the complex plane $\mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$. In recent years many researchers have been interested in value distribution of meromorphic functions. The proofs in this paper uses the Nevanlinna theory and one can refer(W. K Hayman[4], C. C Yang and H. X. Yi[13]) for the standard notations and definitions.

Let ϕ and ψ be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ and $k \in \mathbb{Z}^+ \cup \infty$, the set $E(a, \phi) = \{z : \phi(z) - a = 0\}$ denotes all those $a-$ points of ϕ, where each $a-$ point of ϕ with multiplicity k is counted k times in the set and the set $\overline{E}(a, \phi) = \{z : \phi(z) - a = 0\}$ denotes all those $a-$ points of ϕ, where the multiplicities are ignored. If $\phi(z) - a$ and $\psi(z) - a$ assumes the same zeros with the same multiplicities, then we say that $\phi(z)$ and $\psi(z)$ share the value a CM (Counting Multiplicity) and we have $E(a, \phi) = E(a, \psi)$. Suppose if $\phi(z) - a$ and $\psi(z) - a$ assumes the same zeros ignoring the multiplicities, then we say that $\phi(z)$ and $\psi(z)$ share the value a IM (Ignoring Multiplicity) and we will have $\overline{E}(a, \phi) = \overline{E}(a, \psi)$.

In general, for a meromorphic function $\phi(z)$, the quantity $m(r, \phi)$ denotes the proximity function of $\phi(z)$, while $N(r, \phi)$ denotes the counting function of poles $\phi(z)$ whose multiplicities are taken into account (respectively $\overline{N}(r, \phi)$ denotes the reduced counting function when multiplicities are ignored). The quantity $N(r, a; \phi)$ denotes the counting function of $a-$ points of $\phi(z)$ whose multiplicities are taken into account(respectively $\overline{N}(r, a; \phi)$ denotes the reduced counting function when multiplicities are ignored). The notation $N(r, a; \phi| = 1)$ denotes the counting function of simple $a-$ points of ϕ and the notation $N(r, a; \phi| \geq 2)$ denotes the counting function of those $a-$ points of ϕ whose multiplicities are atleast 2(respectively $\overline{N}(r, a; \phi| = 1)$ and $\overline{N}(r, a; \phi| \geq 2)$ denotes the reduced counting functions).

Suppose ϕ and ψ share 1 IM z_0 is a zero of $\phi(z) - 1$ of order s and also a zero $\psi(z) - 1$ of order t, then $\overline{N}_L(r, 1; \phi)$ counts those 1-points of $\phi(z)$ and $\psi(z)$ where $s > t$, $\overline{N}^{1)}_E(r, 1; \phi)$ counts those 1-points of $\phi(z)$ and $\psi(z)$ where $s = t = 1$, $\overline{N}^{(2}_E(r, 1; \phi)$ counts those 1-points of $\phi(z)$ and $\psi(z)$ where $s = t \geq 2$, $\overline{N}_{\phi>2}(r, 1; \psi)$ counts those 1-points of $\phi(z)$ and $\psi(z)$ where $s > t = 2$. It is to be noted that each point in these counting functions are counted only once. Similarly $\overline{N}_L(r, 1; \psi)$, $\overline{N}^{(2}_E(r, 1; \psi)$, $\overline{N}_{\psi>2}(r, 1; \phi)$ are defined.

The Nevanlinna characteristic function of a meromorphic function ϕ plays a very important role in the value distribution theory and it is denoted by $T(r, \phi)$. We have $T(r, \phi) = m(r, \phi) + N(r, \phi)$, which clearly shows that $T(r, \phi)$ is non-

negetive. A meromorphic function $\alpha(z)$ is called a small function with respect to $\phi(z)$, if $T(r,\alpha) = S(r,\phi)$, where $S(r,\phi)$ any quantity satisfying $S(r,\phi) = o\{T(r,\phi)\}$ as $r \to \infty$ possibly outside a set of finite linear measure.

2 Definitions and Theorems

Definition 2.1. *[6] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a,\phi)$ the set of all $a-$ points of ϕ where an a- points of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,\phi) = E_k(a,\psi)$, we say that ϕ, ψ share the value a with weight k.*

Definition 2.2. *[13] Let $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer k, we denote by*
(i) $N_{k)}(r, \frac{1}{\phi-a})$ the counting function of $a-$ points of f with multiplicity $\leq k$.
(ii) $N_{(k}(r, \frac{1}{\phi-a})$ the counting function of $a-$ points of ϕ with multiplicity $\geq k$.
Similarly the reduced counting function $\overline{N}_{k)}(r, \frac{1}{\phi-a})$ and $\overline{N}_{(k}(r, \frac{1}{\phi-a})$ are defined.

Definition 2.3. *[6] Let $p \in \mathbb{N} \cup \{\infty\}$. We denote by $N_p(r,a;\phi)$ the counting function of $a-$ points of ϕ, where an a- point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then*

$N_p(r,a;\phi) = \overline{N}(r,a,\phi) + \overline{N}(r,a,\phi| \geq 2) + ... + \overline{N}(r,a,\phi| \geq p)$. *Clearly* $N_1(r,a;\phi) = \overline{N}(r,a,\phi)$.

Definition 2.4. *[15] Let $\phi(z)$ be a meromorphic function, Linear difference polynomial $L(z,\phi)$ defined as follows.*
$L(z,\phi) = a_1(z)\phi(z+c_1) + a_2(z)\phi(z+c_2) + a_3(z)\phi(z+c_3).....a_d(z)\phi(z+c_d)$ *be a linear difference polynomial, where $a_1(z), a_2(z)...a_d(z)$ are non-zero small functions relative to $\phi(z)$, where $c_1, c_2, c_3...c_d$ are complex constants.*

In 2011, K. Liu, X. L, Liu and T. B Cao [9] studied the uniqueness of the difference monomials and obtained the following results.

Theorem 2.1. *[9] Let ϕ and ψ be two transcendental meromorphic functions with finite order. Suppose $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$. If $n \geq 14$, $\phi^n(z)\phi(z+c)$ and $\psi^n(z)\psi(z+c)$ share 1 CM, then $\phi(z) \equiv t\psi(z)$ or $\phi(z)\psi(z) \equiv t$, where $t^{n+1} = 1$.*

Theorem 2.2. *[9] Let ϕ and ψ be two transcendental meromorphic functions with finite order. Suppose $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$. If $n \geq 26$, $\phi^n(z)\phi(z+c)$ and $\psi^n(z)\psi(z+c)$ share 1 IM, then $\phi(z) \equiv t\psi(z)$ or $\phi(z)\psi(z) \equiv t$, where $t^{n+1} = 1$.*

In 2015, Y. Liu, J. P. Wang and F. H. Liu [10] improved Theorems 2.1 and 2.2 obtained the following results.

Theorem 2.3. *[10] Let $c \in \mathbb{C}\backslash\{0\}$ and let $\phi(z)$ and $\psi(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 14), k(\geq 3)$ be two positive integers. If $E_k(1, \phi^n(z)\phi(z+c)) = E_k(1, \psi^n(z)\psi(z+c))$, then $\phi(z) \equiv t_1\psi(z)$ or $\phi(z)\psi(z) \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem 2.4. *[10] Let $c \in \mathbb{C}\backslash\{0\}$ and let $\phi(z)$ and $\psi(z)$ be two transcendental meromorphic functions with finite order and $n(\geq 16)$ be a positive integer. If $E_2(1, \phi^n(z)\phi(z+c)) = E_2(1, \psi^n(z)\psi(z+c))$, then $\phi(z) \equiv t_1\psi(z)$ or $\phi(z)\psi(z) \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem 2.5. *[10] Let $c \in \mathbb{C}\backslash\{0\}$ and let $\phi(z)$ and $\psi(z)$ be two transcendental meromorphic functions with finite order, and $n(\geq 22)$ be a positive integers. If $E_1(1, \phi^n(z)\phi(z+c)) = E_1(1, \psi^n(z)\psi(z+c))$, then $\phi(z) \equiv t_1\psi(z)$ or $\phi(z)\psi(z) \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In 2017, Sujoy Majumder [11] replaced the sharing value 1 by a nonzero polynomial $p(z)$ in 2.3,2.4 and 2.5 and obtained the following results.

Theorem 2.6. *[11] Let ϕ and ψ be two transcendental meromorphic functions with finite order. Suppose $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 14$. Let $p(\not\equiv 0)$ be a polynomial such that $deg(p) < (n-1)/2$. If $\phi^n(z)\phi(z+c) - p(z)$ and $\psi^n(z)\psi(z+c) - p(z)$ share (0,2), then one of the following two cases holds:*

1. *$\phi(z) \equiv t\psi(z)$ for some constant t such that $t^{n+1} = 1$,*
2. *$\phi(z)\psi(z) \equiv t$, where $p(z)$ reduces to a non-zero constant c and t is a constant such that $t^{n+1} = c^2$.*

Theorem 2.7. *[11] Let ϕ and ψ be two transcendental meromorphic functions with finite order. Suppose $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 16$. Let $p(\not\equiv 0)$ be a polynomial such that $deg(p) < (n-1)/2$. If $\phi^n(z)\phi(z+c) - p(z)$ and $\psi^n(z)\psi(z+c) - p(z)$ share (0,1). Then the conclusion of Theorem 2.6 holds.*

Theorem 2.8. *[11] Let ϕ and ψ be two transcendental meromorphic functions with finite order. Suppose $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 26$. Let $p(\not\equiv 0)$ be a polynomial such that $deg(p) < (n-1)/2$. If $\phi^n(z)\phi(z+c) - p(z)$ and $\psi^n(z)\psi(z+c) - p(z)$ share (0,0). Then the conclusion of Theorem 2.6 holds.*

It is quite natural to ask the following question.

Question 1. Suppose $L(z, \phi) = \sum_{j=1}^{d} a_j\phi(z+c_j)$ is a linear difference polynomial of finite order of meromorphic function ϕ with $a_1(z), a_2(z)...a_d(z)$ are nonzero small functions relative to $\phi(z)$, where $c_1, c_2, c_3...c_d$ are complex constants,

then what can we say about relation between two finite order non-constant meromorphic functions ϕ and ψ, if their linear difference polynomials $\phi^n(z)\sum_{j=1}^{d} a_j \phi(z+c_j)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)$ share $p(z)$?

In this paper, we have attempted to answer the above question successfully.

Following are the main results of our paper.

3 Main Results

Theorem 3.1. *Let ϕ and ψ be two transcendental meromorphic functions of finite order and n be a positive integer such that $n > 4d+5$. Suppose that $c_j \in \mathbb{C}\backslash\{0\}$ for $j = 1,2,3...d$. Let $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j) - p(z)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j) - p(z)$ share $(0,2)$ where p be a nonzero polynomial such that $deg(p) < (n-d)/2$, then one of the following two cases holds:*

1. *$\phi(z) \equiv t\psi(z)$ for some constant t such that $t^{n+d} = 1$,*
2. *$\phi(z)\psi(z) \equiv t$, where $p(z)$ reduces to a nonzero constant c and t is a constant such that $t^{n+d} = c^2$.*

Theorem 3.2. *Let ϕ and ψ be two transcendental meromorphic functions of finite order and n be a positive integer such that $n > \frac{9d+11}{2}$. Suppose that $c_j \in \mathbb{C}\backslash\{0\}$ for $j = 1,2,3...d$. Let $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j) - p(z)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j) - p(z)$ share $(0,1)$ where p be a nonzero polynomial such that $deg(p) < (n-d)/2$, then the conclusion of Theorem 3.1 holds.*

Theorem 3.3. *Let ϕ and ψ be two transcendental meromorphic functions of finite order and n be a positive integer such that $n > 6d+8$. Suppose that $c_j \in \mathbb{C}\backslash\{0\}$ for $j = 1,2,3...d$. Let $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j) - p(z)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j) - p(z)$ share $(0,0)$ where p be a nonzero polynomial such that $deg(p) < (n-d)/2$, then the conclusion of Theorem 3.1 holds.*

Remark 3.1. *Theorems 3.1 - 3.3 extends and generalizes of Theorems 2.6 - 2.8 respectively.*

Example 1. Let $\phi(z) = sinz$, $\psi(z) = cosz$. Choose $c = 2\pi, a_j = 1$ then $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)$ share $p(z)$. Clearly we get $\phi \equiv t\psi$ for a constant t.

Example 2. Let $\phi(z) = (7z-3)^2(z+1)e^{(z-1)^3}$, $\psi(z) = (7z-3)^2(z+1)e^{-(z-1)^3}$. Choose $c_j = 1, a_j = 1$ then $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)$ and $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)$ share $p(z)$. Clearly we get $\phi \not\equiv t\psi$ for a constant t.

4 Auxiliary Lemmas

In this section, we present some necessary Lemmas.
Denote H by the following function.

$$H = \Big(\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1}\Big) - \Big(\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1}\Big). \tag{1}$$

Lemma 4.1. *[12] Let ϕ be a non-constant meromorphic function and let $a_1, a_2, ...a_n$ be finite complex numbers, $a_n \not\equiv 0$, $a_{n-1}, ..., a_0$ be meromorphic functions such that $T(r, a_i) = S(r, f)$ for $i = 0, 1, 2...n$ Then*

$$T(r, a_n\phi^n + a_{n-1}\phi^{n-1} + ... + a_1\phi + a_0) = nT(r, \phi) + S(r, \phi).$$

Lemma 4.2. *[3] Let ϕ be a non-constant meromorphic function of finite order ρ and let $c \in \mathbb{C}\backslash\{0\}$ be fixed. Then for each $\epsilon > 0$, we have*

$$m\left(r, \frac{\phi(z+c)}{\phi(z)}\right) + m\left(r, \frac{\phi(z)}{\phi(z+c)}\right) = O\left(r^{\rho-1+\epsilon}\right) = S(r, \phi).$$

The following lemma is a slight modification of the original version.

Lemma 4.3. *[5] Let ϕ be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$\begin{aligned} N(r, \infty; \phi(z+c)) &\le N(r, \infty; \phi) + S(r, \phi) \\ N(r, 0; \phi(z+c)) &\le N(r, 0; \phi) + S(r, \phi) \\ \overline{N}(r, \infty; \phi(z+c)) &\le \overline{N}(r, \infty; \phi) + S(r, \phi) \\ \overline{N}(r, 0; \phi(z+c)) &\le \overline{N}(r, 0; \phi) + S(r, \phi). \end{aligned}$$

Lemma 4.4. *[3] Let ϕ be a transcendental meromorphic function of finite order, $c \in \mathbb{C}\backslash\{0\}$ be fixed then*

$$T(r, \phi(z+c) = T(r, \phi) + S(r, \phi).$$

Lemma 4.5. *[14] Let ϕ and ψ be two non-constant meromorphic functions. Then*

$$N(r, \infty; \frac{\phi}{\psi}) - N(r, \infty; \frac{\psi}{\phi}) = N(r, \infty; \phi) + N(r, 0; \psi) - N(r, \infty; \psi) - N(r, 0; \phi).$$

Lemma 4.6. *Let ϕ be a transcendental meromorphic function of finite order and let $\Phi = \phi^n(z)\sum_{j=1}^{d} a_j\phi(z + c_j)$, where n is positive integer. Then*

$$(n - d)T(r, \phi) \le T(r, \Phi) + S(r, \phi).$$

Proof. By Lemmas 4.1,4.2 and First Fundamental Theorem, we obtain

$$\begin{aligned}(n+d)T(r,\phi) &= T(r,\phi^{n+d}) + S(r,\phi),\\ &\leq T\left(r, \frac{\phi^d(z)\Phi}{\sum_{j=1}^{d} a_j\phi(z+c_j)}\right) + S(r,\phi),\\ &\leq T(r,\Phi) + T\left(r, \frac{\phi^d(z)}{\sum_{j=1}^{d} a_j\phi(z+c_j)}\right) + S(r,\phi),\\ &\leq T(r,\Phi) + N\left(r, \frac{\sum_{j=1}^{d} a_j\phi(z+c_j)}{\phi^d(z)}\right) + S(r,\phi),\\ &\leq T(r,\Phi) + 2dT(r,\phi) + S(r,\phi),\\ (n-d)T(r,\phi) &\leq T(r,\Phi) + S(r,\phi).\end{aligned}$$

This completes the proof of Lemma 4.6.

Lemma 4.7. *Let ϕ and ψ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$ such that $n > d$. Let $p(z)$ be a nonzero polynomial such that $deg(p) < (n-d)/2$. Then*

1. *if $deg(p) \geq 1$, then $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j).\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j) \not\equiv p^2(z)$;*

2. *if $p(z) = c \in \mathbb{C}\backslash\{0\}$, then the relation $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\psi^n(z)$ $\sum_{j=1}^{d} a_j\psi(z+c_j) \equiv p^2(z)$, always implies $\phi.\psi = t$, where t is a constant such that $t^{n+d} = c^2$.*

Proof. Suppose,

$$\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right)\left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right) \equiv p^2(z). \tag{2}$$

Let $h_1 = \phi\psi$ then by equation 2, we have

$$h_1^n(z) = \frac{p^2(z)}{\sum_{j=1}^{d} h_1 a_j(z+c_j)} \tag{3}$$

We now consider the following two cases,
Case 1. *Suppose h_1 is a transcendental meromorphic function. Now by Lemmas*

4.1, 4.2, 4.3 we get

$$nT(r,h_1) = T(r,h_1^n) + S(r,h_1) = T\left(r, \frac{p^2(z)}{\sum_{j=1}^{d} h_1 a_j(z+c_j)}\right) + S(r,h_1),$$
$$\leq N\left(r,0;\sum_{j=1}^{d} a_j\phi(z+c_j)\right) + N\left(r,0;\sum_{j=1}^{d} a_j\psi(z+c_j)\right)$$
$$+ S(r,h_1),$$
$$\leq d(T(r,\phi)+T(r,\psi)) + S(r,h_1).$$

which is a contradiction.
Case 2. *Suppose h_1 is a rational function . Let*

$$h_1 = \frac{h_2}{h_3}, \tag{4}$$

where h_2 and h_3 are two non-zero relatively prime polynomials. By equation 4, we have

$$T(r,h_1) = max\{deg(h_2), deg(h_3)\}logr + O(1). \tag{5}$$

Now by equations 3-5, we have

$$n.\, max\{deg(h_2), deg(h_3)\}logr = T(r,h_1^n) + O(1) \leq d(T(r,\phi)+T(r,\psi))$$
$$+ 2T(r,p) + O(1). \tag{6}$$

We see that $max\{deg(h_2), deg(h_3)\} \geq 1$. Now by equation 6, we deduce that $(n-d)/2 \leq deg(p)$, which contradicts our assumption that $deg(p) < (n-d)/2$. Hence h_1 must be a non-zero constant. Let

$$h_1 = t \in \mathbb{C}\backslash\{0\}. \tag{7}$$

Now when $deg(p) \geq 1$, by equations 3 and 7, we arrive at a contradiction.Then

$$\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right)\left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right) \not\equiv p^2(z).$$

Suppose $p(z) = c \in \mathbb{C}\backslash\{0\}$. So by equation 3 we see that $h_1^{n+d} \equiv c^2$. By equation 7 we get $t^{n+d} \equiv c^2$.
This completes the proof of Lemma 4.7.

Lemma 4.8. *Let ϕ and ψ be two transcendental meromorphic functions of finite order, $c \in \mathbb{C}\backslash\{0\}$ be finite complex constant such that for $(j = 1, 2...d)$ and n be*

a positive integer such that $n \geq 4d+3$. *Let* $\Phi(z) = \left(\frac{\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)}{p(z)}\right)$ *and* $\Psi(z) = \left(\frac{\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)}{p(z)}\right)$, *where* $p(z)$ *is non-zero polynomial and* $H \equiv 0$, *then one of the following conclusions occur.*

(i) $\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right)\left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right) \equiv p^2(z)$, *where* $\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j) - p(z)$ *and* $\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j) - p(z)$ *share 0 CM,*

(ii) $\phi(z) = t\psi(z)$ *for a constant* t *with* t^{n+d}*=1.*

Proof. By equation 1, $H \equiv 0$ *on integration we get*

$$\frac{1}{\Phi - 1} = \frac{B\Psi + A - B}{\Psi - 1}, \tag{8}$$

where A,B *are constants and* $A \neq 0$. *From equation 8 it is clear that* Φ *and* Ψ *share* $(1,\infty)$. *We now consider following cases.*
Case 1. *Let* $B \neq 0$ *and* $A \neq B$. *If* $B = -1$, *then from equation 8 we have*

$$\Phi = \frac{-A}{\Psi - A - 1}.$$

Therefore $\overline{N}(r, A+1; \Psi) = \overline{N}(r, \Phi) \leq N(r, 0; p) \leq T(r, p) = S(r, \psi)$. *So in view of Lemma 4.6 and Second Fundamental Theorem, we get*

$$\begin{aligned}(n-d)T(r,\psi) &\leq T\left(r, \frac{\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)}{p(z)}\right) + S(r.\psi)\\ &\leq T(r,\Psi) + S(r.\psi),\\ &\leq \overline{N}(r,\infty;\Psi) + \overline{N}(r,0;\Psi) + \overline{N}(r,A+1;\Psi) + S(r,\psi),\\ &\leq 2(d+1)T(r,\psi) + S(r,\psi).\end{aligned}$$

which contradicts $n > 3d+2$.
If $B \neq -1$, *from equation 8 we obtain that*

$$\Phi - \left(1 + \frac{1}{B}\right) = \frac{-A}{B^2\left(\Psi + \frac{A-B}{B}\right)}$$

$$\overline{N}\left(r, \frac{B-A}{B}; \Psi\right) = S(r,\psi).$$

Using the Lemma 4.6 and the same argument as used in the case when $B = -1$ we can get a contradiction.

Case 2. *Let $B \neq 0$ and $A = B$. If $B = -1$, then from equation 8, we have*

$$\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right)\left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right) \equiv p^2(z).$$

when $\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right) - p(z)$ and $\left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right) - p(z)$ share 0 CM. If $B \neq -1$, from equation 8 we have

$$\frac{1}{\Phi} = \frac{B\Psi}{(1+B)\Psi - 1}.$$

$$\overline{N}\left(r, \frac{1}{1+B}; \Psi\right) = \overline{N}(r, 0; \Phi) + S(r.\phi).$$

So in view of Lemmas 4.2, 4.6 and the Second Fundamental Theorem, we get

$$\begin{aligned}(n-d)T(r,\psi) &\leq T(r,\Psi) + S(r,\psi),\\ &\leq \overline{N}(r,\infty;\Psi) + \overline{N}(r,0;\Psi) + \overline{N}(r,(1+B);\Psi) + S(r,\psi),\\ &\leq 2(d+1)T(r,\psi) + (1+d)T(r,\phi) + S(r,\phi) + S(r,\psi),\\ (n-d-3d-3)T(r,\psi) &\leq S(r,\psi).\end{aligned}$$

which is contradiction, since $n > 4d + 3$.

Case 3. *Let $B = 0$. From equation 8 we obtain*

$$\Phi = \frac{\Psi + A - 1}{A} \tag{9}$$

If $A \neq 1$, then from equation 9, we obtain $\overline{N}(r, 1-A; \Psi) = \overline{N}(r, 0; \Phi)$.
We can similarly deduce a contradiction as in Case 2. Therefore $A = 1$ and from equation 9, we obtain $\Phi(z) = \Psi(z)$. That is

$$\left(\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)\right) \equiv \left(\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)\right). \tag{10}$$

Let $h = \frac{\phi}{\psi}$, and then substituting $\phi = \psi h$ in equation 10 we deduce

$$h^{n+1} = \frac{\phi}{\psi\sum_{j=1}^{d} ha_j(z+c_j)}, \tag{11}$$

If h is not a constant , then we have

$$(n+1)T(r,h) \leq T\left(r, \frac{\phi}{\sum_{j=1}^{d} a_j \phi(z+c_j)}\right) + T\left(r, \frac{\sum_{j=1}^{d} a_j \psi(z+c_j)}{\psi}\right) + S(r,\phi) + S(r,\psi),$$

where $T(r,h)$=$T(r, \frac{\phi}{\psi}) = T(r,\phi) + T(r,\psi) + S(r,\phi) + S(r,\psi)$. we obtain $(n-d)[T(r,\phi) + T(r,\psi)] \leq S(r,\phi) + S(r,\psi)$, which is impossible. Therefore h is a constant , then substituting $\phi = \psi h$ in equation 10, we have $h^{n+d} \equiv 1$. Therefore $\phi = \psi t$, where t is a constant with $t^{n+d} = 1$.

This completes the proof of Lemma 4.8

Lemma 4.9. *[7] If $N(r,0;\phi^{(k)}|\phi \neq 0)$ denotes the counting function of those zeros of $\phi^{(k)}(z)$ is counted according to its multiplicity, then $N(r,0;\phi^{(k)}|\phi \neq 0) \leq k\overline{N}(r,\infty;\phi) + N(r,0;\phi| < k) + k\overline{N}(r,0;\phi| \geq k) + S(r,\phi)$.*

Lemma 4.10. *[1] If ϕ and ψ be two non-constant meromorphic functions such that they share $(1,1)$. Then $2\overline{N}_L(r,1;\phi) + 2\overline{N}_L(r,1;\psi) + \overline{N}_E^{(2}(r,1;\phi) - \overline{N}_{f>2}(r,1;\psi) \leq N(r,1;\psi) - \overline{N}(r,1;\psi)$.*

Lemma 4.11. *[2] Let ϕ and ψ share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;\psi) \leq \frac{1}{2}\overline{N}(r,0;\phi) + \frac{1}{2}\overline{N}(r,\infty;\phi) - \frac{1}{2}N_0(r,0;\phi') + S(r,\phi),$$

,

where $N_0(r,0;\phi')$ is the counting function of those zeros of ϕ' which are not the zeros of $\phi(\phi-1)$.

Lemma 4.12. *[2] If ϕ and ψ share $(1,0)$. Then*

1. *$\overline{N}_{\phi>1}(r,1;\psi) \leq \overline{N}(r,0;\phi) + \overline{N}(r,\infty;\phi) - N_0(r,0;\phi') + S(r,\phi)$;*
2. *$\overline{N}_{\psi>1}(r,1;\phi) \leq \overline{N}(r,0;\psi) + \overline{N}(r,\infty;\psi) - N_0(r,0;\psi') + S(r,\phi)$.*

Lemma 4.13. *[2] Let ϕ and ψ share $(1,0)$. Then*

$$\overline{N}_L(r,1;\phi) \leq \overline{N}(r,0;\phi) + \overline{N}(r,\infty;\phi) + S(r,\phi).$$

Lemma 4.14. *[2] Let ϕ and ψ be two non-constant meromorphic functions sharing $(1,0)$. Then*

$$\overline{N}_L(r,1;\phi) + 2\overline{N}_L(r,1;\psi) + \overline{N}_E^{(2}(r,1;\phi) - \overline{N}_{\phi>1}(r,1;\psi) - \overline{N}_{\psi>1}(r,1;\phi) \leq N(r,1;\psi) - \overline{N}(r,1;\psi).$$

5 Proof of Main results

Proof of Theorem 3.1.
Let $\Phi(z) = \left(\frac{\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)}{p(z)}\right)$ and $\Psi(z) = \left(\frac{\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)}{p(z)}\right)$, it follows that Φ and Ψ share $(1,2)$ except the zeros of $p(z)$.
Case 1. Let $H \not\equiv 0$ from 1, we obtain

$$N(r,\infty;H) \leq \overline{N}_*(r,1;\Phi,\Psi) + \overline{N}(r,0;\Phi| \geq 2) + N(r,0;\Psi| \geq 2) + \overline{N}_0(r,0;\Phi^{'}) + \overline{N}_0(r,0;\Psi^{'}). \quad (12)$$

where $\overline{N}_0(r,0;\Phi^{'})$ is the reduced counting function of those zeros of $\Phi^{'}$ which are not zeros of $\Phi(\Phi-1)$ and $\overline{N}_0(r,0;\Psi^{'})$ is similarly defined. Let z_0 be a simple zero of $\Phi-1$ such that $p(z_0)\neq 0$. Then z_0 is a simple zero of $\Psi-1$ and a zero of H. So

$$N(r,1;\Phi|=1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,\phi) + S(r,\psi). \quad (13)$$

From equations 12 and 13 we get

$$\overline{N}(r,1;\Phi) \leq \overline{N}(r,0;\Phi| \geq 2) + \overline{N}(r,0;\Psi| \geq 2) + \overline{N}_*(r,1;\Phi,\Psi) + \overline{N}(r,1;\Phi| \geq 2) + \overline{N}_0(r,0;\Phi^{'}) + \overline{N}_0(r,0;\Psi^{'}) + S(r,\phi) + S(r,\psi). \quad (14)$$

Now in view of Lemma 4.9 we get

$$\overline{N}_0(r,0;\Psi^{'}) + \overline{N}(r,1;\Phi| \geq 2) + \overline{N}_*(r,1;\Phi,\Psi) \leq \overline{N}_0(r,0;\Psi^{'}) + \overline{N}(r,1;\Phi| \geq 2) +$$

$$\overline{N}(r,1;\Phi| \geq 3) \leq N(r,0;\Psi^{'}|\Psi \neq 0) \leq \overline{N}(r,0;\Psi) + S(r,\psi). \quad (15)$$

Hence using equations 14, 15 and Lemmas 4.2, 4.6 we get from Second fundamental theorem that

$$\begin{aligned}(n-d)T(r,\phi) &\leq T(r,\Phi) + S(r,\phi)\\ &\leq \overline{N}(r,\infty;\Phi) + \overline{N}(r,0;\Phi) + \overline{N}(r,1;\Phi) - N_0(r,0;\Phi^{'}) + S(r,\phi),\\ &\leq \overline{N}(r,\infty;\Phi) + N_2(r,0;\Phi) + N_2(r,0;\Psi) + S(r,\phi) + S(r,\psi),\\ (n-d)T(r,\phi) &\leq (3+2d)T(r,\phi) + (2+d)T(r,\psi) + S(r,\phi) + S(r,\psi). \quad (16)\end{aligned}$$

Similarly, we can obtain

$$(n-d)T(r,\psi) \leq (3+2d)T(r,\psi) + (2+d)T(r,\phi) + S(r,\phi) + S(r,\psi). \quad (17)$$

Combining equations 16 and 17 we see that

$$(n-d-5-3d) \leq S(r,\phi) + S(r,\psi) \quad (18)$$

Since $n > 4d + 5$ equation 18 leads to a contadiction.
Case 2. Let $H \equiv 0$. Then the theorem follows from Lemmas 4.7 and 4.8.
This completes the proof of Theorem 3.1.

Proof of Theorem 3.2.
Let $\Phi(z) = \left(\frac{\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)}{p(z)}\right)$ and $\Psi(z) = \left(\frac{\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)}{p(z)}\right)$, it follows that Φ and Ψ share $(1,1)$ except the zeros of $p(z)$. We now consider the following two cases.
Case 1. $H \not\equiv 0$.
Using Lemmas 4.6 , 4.8 , 4.10 and equations 12 and 13 we get

$$\begin{aligned}
\overline{N}(r,1;\Phi) &\leq N(r,1;\Phi|=1) + \overline{N}_L(r,1;\Phi) + \overline{N}_L(r,1;\Psi) + \overline{N}_E^{(2}(r,1;\Phi)\\
&\leq \overline{N}(r,0;\Phi|\geq 2) + \overline{N}(r,0;\Psi|\geq 2) + \overline{N}_*(r,1;\Phi,\Psi) + \overline{N}_L(r,1;\Phi)\\
&\quad + \overline{N}_L(r,1;\Psi) + \overline{N}_E^{(2}(r,1;\Phi) + \overline{N}_0(r,0;\Phi') + \overline{N}_0(r,0;\Psi')\\
&\quad + S(r,\phi) + S(r,\psi),\\
&\leq \overline{N}(r,0;\Phi|\geq 2) + \overline{N}(r,0;\Psi|\geq 2) + 2\overline{N}_L(r,1;\Phi) + 2\overline{N}_L(r,1;\Psi)\\
&\quad + \overline{N}_E^{(2}(r,1;\Phi) + \overline{N}_0(r,0;\Phi') + \overline{N}_0(r,0;\Psi') + S(r,\phi) + S(r,\psi)\\
&\leq \overline{N}(r,0;\Phi|\geq 2) + \overline{N}(r,0;\Psi|\geq 2) + \overline{N}_{\Phi>2}(r,1;\Psi) + N(r,1;\Psi)\\
&\quad - \overline{N}(r,1;\Psi) + \overline{N}_0(r,0;\Phi') + \overline{N}_0(r,0;\Psi') + S(r,\phi) + S(r,\psi)\\
&\leq \overline{N}(r,0;\Phi|\geq 2) + \frac{1}{2}\overline{N}(r,0;\Phi) + \overline{N}(r,0;\Psi|\geq 2) +\\
&\quad + N(r,0;\Psi'|\Psi\neq 0) + \overline{N}_0(r,0;\Phi') + S(r,\phi) + S(r,\psi)\\
\overline{N}(r,1;\Phi) &\leq \overline{N}(r,0;\Phi|\geq 2) + \frac{1}{2}\overline{N}(r,0;\Phi) + N_2(r,0;\Psi) + \overline{N}_0(r,0;\Phi') \quad (19)\\
&\quad + S(r,\phi) + S(r,\psi).
\end{aligned}$$

Hence by using equation 19, Lemmas 4.2 and 4.6 we get from Second Fundamental Theorem that

$$\begin{aligned}
(n-d)T(r,\phi) &\leq T(r,\Phi) + S(r,\phi),\\
&\leq \overline{N}(r,\infty;\Phi) + \overline{N}(r,0;\Phi) + \overline{N}(r,1;\Phi) - N_0(r,0;\Phi') + S(r,\phi),\\
&\leq \overline{N}(r,\infty;\Phi) + \frac{1}{2}\overline{N}(r,0;\Phi) + N_2(r,0;\Phi) + N_2(r,0;\Psi)\\
&\quad + S(r,\phi) + S(r,\psi),\\
(n-d)T(r,\phi) &\leq \frac{5d+7}{2}T(r,\phi) + (2+d)T(r,\psi) + S(r,\phi) + S(r,\psi). \qquad (20)
\end{aligned}$$

Similarly,

$$(n-d)T(r,\psi) \leq \frac{5d+7}{2}T(r,\psi) + (2+d)T(r,\phi) + S(r,\phi) + S(r,\psi). \quad (21)$$

Combining equations 20 and 21 we get

$$\left(n - \frac{9d+11}{2}\right)[T(r,\phi) + T(r,\psi)] \leq S(r,\phi) + S(r,\psi) \quad (22)$$

Since $n > \left(\frac{9d+11}{2}\right)$ equation 22 leads to a contradiction.
Case 2. Let $H \equiv 0$. Then the theorem follows from Lemmas 4.7 and 4.8.
This completes the proof of Theorem 3.2.
Proof of Theorem 3.3.
Let $\Phi(z) = \left(\frac{\phi^n(z)\sum_{j=1}^{d} a_j\phi(z+c_j)}{p(z)}\right)$ and $\Psi(z) = \left(\frac{\psi^n(z)\sum_{j=1}^{d} a_j\psi(z+c_j)}{p(z)}\right)$, it follows that Φ and Ψ share $(1,0)$ except the zeros of $p(z)$.
Here equation 13 changes to

$$N_E^{1)}(r,1;\Phi) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,\Phi) + S(r,\Psi) \quad (23)$$

Taking Lemmas 4.9, 4.12, 4.13, 4.14 and equations 13 and 23 we get

$$\begin{aligned}
\overline{N}(r,1;\Phi) &\leq N_E^{1)}(r,1;\Phi) + \overline{N}_L(r,1;\Phi) + \overline{N}_L(r,1;\Psi) + N_E^{(2}(r,1;\Phi) \\
&\leq \overline{N}(r,0;\Phi| \geq 2) + \overline{N}(r,0;\Psi| \geq 2) + \overline{N}_*(r,1;\Phi,\Psi) + \overline{N}_L(r,1;\Phi) \\
&\quad + \overline{N}_L(r,1;\Psi) + N_E^{(2}(r,1;\Phi) + \overline{N}_0(r,0;\Phi') + \overline{N}_0(r,0;\Psi') \\
&\quad + S(r,\phi) + S(r,\psi) \\
&\leq \overline{N}(r,0;\Phi| \geq 2) + \overline{N}(r,0;\Psi| \geq 2) + 2\overline{N}_L(r,1;\Phi) + 2\overline{N}_L(r,1;\Psi) \\
&\quad + \overline{N}_E^{(2}(r,1;\Phi) + \overline{N}_0(r,0;\Phi') + \overline{N}_0(r,0;\Psi') + S(r,\phi) + S(r,\psi) \\
&\leq \overline{N}(r,0;\Phi| \geq 2) + \overline{N}(r,0;\Psi| \geq 2) + \overline{N}_{\Phi>1}(r,1;\Psi) + \overline{N}_{\Psi>1}(r,1;\Phi) \\
&\quad + \overline{N}_L(r,1;\Phi) + N(r,1;\Psi) - \overline{N}(r,1;\Psi) + \overline{N}_0(r,0;\Phi') + \overline{N}_0(r,0;\Psi') \\
&\quad + S(r,\phi) + S(r,\psi) \\
&\leq N_2(r,0;\Phi) + \overline{N}(r,0;\Phi) + N_2(r,0;\Psi) + N(r,0;\Psi'|\Psi \neq 0) + \overline{N}_0(r,0;\Phi') \\
&\quad + S(r,\phi) + S(r,\psi) \\
\overline{N}(r,1;\Phi) &\leq N_2(r,0;\Phi) + \overline{N}(r,0;\Phi) + N_2(r,0;\Psi) + \overline{N}(r,0;\Psi) + \overline{N}_0(r,0;\Phi') \\
&\quad + S(r,\phi) + S(r,\psi). \quad (24)
\end{aligned}$$

Hence taking equation 24, Lemmas 4.2 and 4.6, we get from Second Fundamental

Theorem that

$$\begin{aligned}(n-d)T(r,\phi) &\leq T(r,\Phi)+S(r,\phi),\\ &\leq \overline{N}(r,\infty;\Phi)+\overline{N}(r,0;\Phi)+\overline{N}(r,1;\Phi)-N_0(r,0;\Phi')+S(r,\phi),\\ &\leq \overline{N}(r,\infty;\Phi)+2N_2(r,0;\Phi)+N_2(r,0;\Psi)+\overline{N}(r,0;\Psi)\\ &\quad +S(r,\phi)+S(r,\psi),\\ (n-d)T(r,\phi) &\leq (3d+5)T(r,\phi)+(2d+3)T(r,\psi)+S(r,\phi)+S(r,\psi). \quad (25)\end{aligned}$$

Similarly, we obtain

$$(n-d)T(r,\psi) \leq (3d+5)T(r,\psi)+(2d+3)T(r,\phi)+S(r,\phi)+S(r,\psi). \quad (26)$$

Combining equations 25 and 26, we get

$$(n-d-5d-8)[T(r,\phi)+T(r,\psi)] \leq S(r,\phi)+S(r,\psi). \quad (27)$$

Since $n > 6d+8$ equation 27 leads to contradiction.
Case 2. Let $H \equiv 0$. Then the theorem follows from Lemmas 4.7 and 4.8.
This completes the proof of Theorem 3.3.

6 Discussion and Conclusions

Nevanlinna theory is a powerful tool in complex analysis, with wide range of applications in diverse areas of mathematics. Its insights into the behavior of meromorphic functions, the distribution of values and the properties of solutions of differential equations have profound implications for various fields of study. By utilizing Nevanlinna's theory, mathematicians can gain a deeper understanding of complex functions and their behavior, leading to advancements in mathematical theory like in signal processing, communication networks, design of filters and controllers for systems and some practical applications.

In this article using the Nevanlinna theory, we investigate the value distribution and uniqueness of linear difference polynomials of the type $\phi^n \sum_{j=1}^{d} a_j \phi(z+c_j)$ and $\psi^n \sum_{j=1}^{d} a_j \psi(z+c_j)$ transcendental meromorphic functions of finite order. Also, by using the concept of weighted sharing introduced by Indrajit Lahiri, we have studied uniqueness problem of linear difference polynomials sharing a non-zero polynomial $p(z)$ with finite weight. Our findings extends and generalizes some previous results of Theorems 2.6, 2.7 and 2.8 respectively.

Continuing further research, we can pose the following open questions:
Open Problems.
1. Is the condition for n sharp in Theorems 3.1,3.2 and 3.3 ?

2. Can $\sum_{j=1}^{d} a_j\phi(z+c_j)$ is replaced by $\Delta_c^n\phi$ in Theorems 3.1,3.2 and 3.3, where $\Delta_c^n\phi = \sum_{r=0}^{n-1}(-1)^r\binom{n}{r}\phi(z+(n-r)c)$?

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Distance Based Topological Indices of Double graphs and Strong Double graphs

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Abstract

Topological index is a numerical representation of structure of graph. They are mainly classified as Distance and Degree based topological indices. In this article Distance based topological indices of Double graphs and Strong Double graphs are calculated. Let G be a graph of order n with the vertex set $V(G)$ containing vertices $v_1, v_2, ..., v_n$. Double graph of graph G is constructed by taking two copies of G in which a vertex v_i in one copy is adjacent to a vertex v_j in the another copy if v_i and v_j are adjacent in G. Strong Double graph is a double graph in which a vertex v_i in one copy is adjacent to a vertex v_j in the another copy if $i = j$.
Keywords: Topological Indices; Double graphs; Strong Double graphs
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1 Introduction

Numerical representation of structure of a graphs is known as topological indices. They are prominently subdivided into Degree and Distance based topological indices. H. Wiener in 1947 introduced the Wiener index to study the variation of boiling points according to the structure of alkanes known as paraffin [13]. Zagreb indices are also most significant molecular descriptor introduced in 1972 by Gutman and Trinajstić [3, 4]. Many topological indices were introduced. Some of the indices are Hyper-Wiener index, reciprocal complementary Wiener index, Status connectivity indices. Here simple, connected, finite, undirected graphs are considered. For undefined terminologies refer [5]. The total number of edges connected to a vertex u in G is Degree of a vertex u in G and is represented by $d_G(u)$. The length of the shortest path between the vertices u and v in G is distance of u and v and is represented by $d_G(u, v)$. For details on indices see [1, 3, 4, 6, 7, 8, 13]. The Wiener index of G is,

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u, v).$$

First Zagreb index $M_1(G)$ is,

$$M_1(G) = \sum_{u \in V(G)} (d_G(u))^2.$$

Second Zagreb index $M_2(G)$ is,

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

In 1993, Randić proposed the Hyper-Wiener index of G is [12],

$$WW(D[G]) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u, v) + (d_G(u, v))^2). \quad (1)$$

The reciprocal complementary Wiener index of a graph G is introduced by O. Ivanciuc and is defined as [7, 8],

$$RCW(G) = \sum_{1 \leqslant i \leqslant j \leqslant n} \frac{1}{1 + diam(G) - d_{[G]}(u, v)}. \quad (2)$$

The largest distance between any pair of vertices in G is Diameter of G and is represented by diam(G). The sum of the distances from the vertex $u \in V(G)$ to all other vertices in V(G) is known as Status [6] of a vertex u and is represented by $\sigma(u)$

$$\sigma(u) = \sum_{u \in V(G)} d(u, v). \tag{3}$$

H. S. Ramane and A. S. Yalnaik introduced the Status connectivity indices [11]. First Status connectivity index $S_1(G)$ of G is,

$$S_1(G) = \sum_{uv \in E(G)} (\sigma(u) + \sigma(v)). \tag{4}$$

Second Status connectivity index $S_2(G)$ of G is,

$$S_2(G) = \sum_{uv \in E(G)} (\sigma(u)\sigma(v)). \tag{5}$$

The double graph of G is defined by Munarini et al. [10]. It is represented as D[G]. Let G be a graph containing $v_1, v_2, \ldots, v_n$ as vertices. Double graph of G is constructed by considering two copies of G in which a vertex v_i in first copy is adjacent to a vertex v_j in the second copy if v_i and v_j are adjacent in G.
Strong Double graph is a double graph in which a vertex v_i in first copy is adjacent to a vertex v_j in the second copy if $i = j$. It is denoted as SD[G] [2]. The

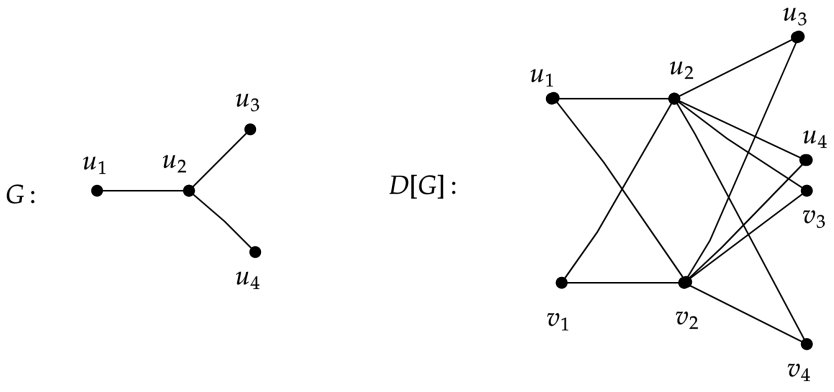

Figure 1: Double graph of $K_{1,3}$

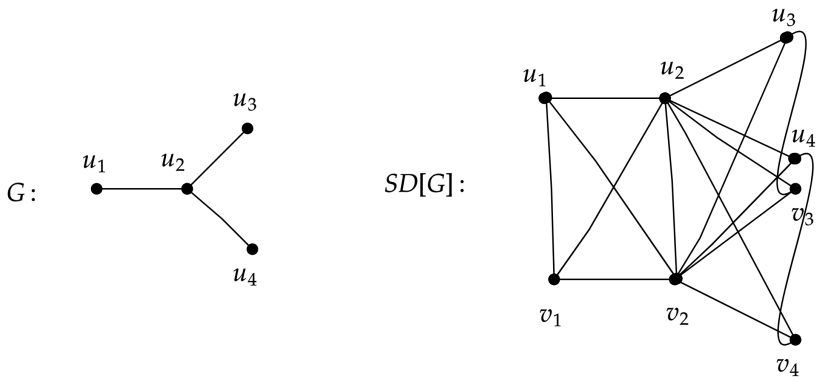

Figure 2: Strong Double graph of $K_{1,3}$

following are used for calculating topological indices.

Remark 1: [9] The Wiener index of Double graph is,

$$W(D[G]) \;=\; 4W[G] + 2n. \tag{6}$$

Remark 2: [1] The Wiener index of Strong Double graph is,

$$W(SD[G]) \;=\; 4W[G] + n. \tag{7}$$

Remark 3:

$$\sum_{1 \leqslant i \leqslant j \leqslant n} \frac{1}{1 + diam(D[G]) - d_{D[G]}(u_i, v_i)} \;=\; \frac{n}{diam(D[G]) - 1}$$

Remark 4:

$$\sum_{1 \leqslant i \leqslant j \leqslant n} \frac{1}{1 + diam(SD[G]) - d_{SD[G]}(u_i, v_i)} \;=\; \frac{n}{diam(SD[G]) - 1}$$

Remark 5:

$$\sum_{u_i, v_i \in V(D(G))} (d_G(u_i, v_i))^2 \;=\; 8n$$

2 Main results

Here, distance based topological indices of Double and Strong Double graphs are established. let G_1 and G_2 be two same copies of a graph G. They are of order n and size m. G_1 contains $u_1, u_2, ..., u_n$ as vertices and G_2 contains $v_1, v_2, ..., v_n$ as vertices.

Theorem 2.1. *Hyper Wiener index of Double graph is*

$$WW(D[G]) = 4WW(G) + 6n.$$

Proof. By definition (1) we have,

$$\begin{aligned} WW(D[G]) &= \frac{1}{2}\sum_{u,v\in V(D[G])}(d_{D[G]}(u,v) + (d_{D[G]}(u,v))^2) \\ &= \frac{1}{2}\sum_{u,v\in V(D[G])}(d_{D[G]}(u,v)) + \frac{1}{2}\sum_{u,v\in V(D[G])}(d_{D[G]}(u,v))^2. \end{aligned}$$

From remark(1) we get,

$$WW(D[G]) = [4W(G) + 2n] + \frac{1}{2}\sum_{u,v\in V(D[G])}(d_{D[G]}(u,v))^2. \tag{8}$$

Consider,

$$\begin{aligned} \frac{1}{2}\sum_{u,v\in V(D[G])}(d_{D[G]}(u,v))^2 = \frac{1}{2}[&\sum_{u_i,u_j\in V(D(G))}(d_G(u_i,u_j))^2 + \\ &\sum_{v_i,v_j\in V(D(G))}(d_G(v_i,v_j))^2 + \\ &\sum_{u_i,v_j\in V(D(G))}(d_G(u_i,v_j))^2 + \\ &\sum_{v_i,u_j\in V(D(G))}(d_G(v_i,u_j))^2 + 8n] \end{aligned} \tag{9}$$

Using (9) in (8) we get,

$$\begin{aligned}
WW(D[G]) &= [4W(G)+2n] + \frac{1}{2}[\sum_{u_i,u_j\in V(D(G))} (d_G(u_i,u_j))^2] \\
&\quad +[4W(G)+2n] + \frac{1}{2}[\sum_{v_i,v_j\in V(D(G))} (d_G(v_i,v_j))^2] \\
&\quad +[4W(G)+2n] + \frac{1}{2}[\sum_{u_i,v_j\in V(D(G))} (d_G(u_i,v_j))^2] \\
&\quad +[4W(G)+2n] + \frac{1}{2}[\sum_{v_i,u_j\in V(D(G))} (d_G(v_i,u_j))^2 + 8n] \\
WW(D[G]) &= WW(G)+WW(G)+WW(G)+WW(G)+6n \\
WW(D[G]) &= 4WW(G)+6n.
\end{aligned}$$

□

Theorem 2.2. *The reciprocal complementary Wiener index of a Double graph is*

$$RCW(D[G]) = 4RCW(G) + \frac{n}{diam(D[G])-1}.$$

Proof. By definition (2) we have,

$$RCW(D[G]) = \sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{1+diam(D[G])-d_{D[G]}(u,v)}$$

From remark (3) we have,

$$\begin{aligned}
&= \sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{1+diam(D[G])-d_G(u_i,u_j)} + \\
&\quad \sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{1+diam(D[G])-d_G(v_i,v_j)} + \\
&\quad 2\sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{1+diam(D[G])-d_G(u_i,v_j)} + \frac{n}{1+diam(D[G])-2} \\
&= \frac{4RCW(D[G])(diam(D[G])-1)+n}{diam(D[G])-1} \\
&= 4RCW(G) + \frac{n}{diam(D[G])-1}.
\end{aligned}$$

□

Theorem 2.3. *The reciprocal complementary Wiener index of a Strong Double graph is*

$$RCW(SD[G]) \;=\; 4RCW(G) + \frac{n}{diam(SD[G])}.$$

Proof. By definition (2) we have,

$$RCW(SD[G]) \;=\; \sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{1 + diam(SD[G]) - d_{SD[G]}(u,v)}$$

From remark (4) we have,

$$\begin{aligned}
&= \sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{1 + diam(SD[G]) - d_G(u_i,u_j)} + \\
&\quad \sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{1 + diam(SD[G]) - d_G(v_i,v_j)} + \\
&\quad 2\sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{1 + diam(SD[G]) - d_G(u_i,v_j)} + \frac{n}{1 + diam(SD[G]) - 1} \\
&= \frac{4RCW(SD[G])(diam(SD[G])) + n}{diam(SD[G])} \\
&= 4RCW(G) + \frac{n}{diam(SD[G])}.
\end{aligned}$$

□

Theorem 2.4. *The first Status connectivity index of Double graph is*

$$S_1(D[G]) \;=\; 8S_1(G) + 16m.$$

Proof. By using definition of Double graph we have,

$$\sigma(u_i) = \sigma(v_i). \tag{10}$$

$$\begin{aligned}
\sigma(u_n) &= \sum_{i=1}^{n}(d(u_n,u_i) + d(u_n,v_i)) + \sum_{i=n} d(u_n,v_n) \\
&= 2\sum_{i=1}^{n}(d(u_n,u_i)) + 2. \qquad (11)
\end{aligned}$$

$$\begin{aligned}\sigma(v_n) &= \sum_{i=1}^{n}(d(v_n,v_i)+d(v_n,u_i))+\sum_{i=n}d(v_n,u_n)\\ &= 2\sum_{i=1}^{n}(d(v_n,v_i))+2. \qquad (12)\end{aligned}$$

By definitions (3) and (4) we have,

$$\begin{aligned}S_1(D[G]) &= \sum_{uv\in E(D[G])}(\sigma(u)+\sigma(v))\\ &= \sum_{u_iu_j\in E(D[G])}(\sigma(u_i)+\sigma(u_j))+\sum_{v_iv_j\in E(D[G])}(\sigma(v_i)+\sigma(v_j))+\\ &\quad \sum_{u_iv_j\in E(D[G])}(\sigma(u_i)+\sigma(v_j))+\sum_{v_iu_j\in E(D[G])}(\sigma(v_i)+\sigma(u_j)).\end{aligned} \qquad (13)$$

Consider first part of equation (13) using (10), (11), (12) we have,

$$\begin{aligned}\sum_{u_iu_j\in E(D[G])}(\sigma(u_i)+\sigma(u_j)) &= \sum_{u_iu_j\in E(D[G])}[2[\sum_{i\neq j}(d(u_i,u_j))+\\ &\quad \sum_{i\neq j}(d(u_j,u_i))]]+4\\ &= \sum_{u_iu_j\in E(D[G])}[2\sum_{i\neq j}(d(u_i,u_j))+2]+\\ &\quad [2\sum_{i\neq j}(d(u_j,u_i))+2]\\ &= 2[(\sigma(u_i)+\sigma(u_j))]+4\\ &= 2S_1(G)+4m.\end{aligned} \qquad (14)$$

Using equation (14) in equation (13) we get,

$$\begin{aligned}S_1(D[G]) &= (2S_1(G)+4m)+(2S_1(G)+4m)+(2S_1(G)+4m)+\\ &\quad (2S_1(G)+4m)\\ &= 8S_1(G)+16m.\end{aligned}$$

□

Theorem 2.5. *The first Status connectivity index of Strong Double graph SD[G] is*

$$S_1(SD[G]) = 8S_1(G)+8W(G)+2n+8m.$$

Proof. By using definition of Strong Double graph we have,

$$\sigma(u_i) = \sigma(v_i). \tag{15}$$

$$\begin{aligned}\sigma(u_n) &= \sum_{i=1}^{n}(d(u_n,u_i)+d(u_n,v_i))+\sum_{i=n} d(u_n,v_n)\\ &= 2\sum_{i=1}^{n}(d(u_n,u_i))+1. \end{aligned}\tag{16}$$

$$\begin{aligned}\sigma(v_n) &= \sum_{i=1}^{n}(d(v_n,v_i)+d(v_n,u_i))+\sum_{i=n} d(v_n,u_n)\\ &= 2\sum_{i=1}^{n}(d(v_n,v_i))+1. \end{aligned}\tag{17}$$

By definitions (3) and (4) we have,

$$\begin{aligned}S_1(SD[G]) &= \sum_{uv\in E(SD[G])}(\sigma(u)+\sigma(v))\\ &= \sum_{u_iu_j\in E(SD[G])}(\sigma(u_i)+\sigma(u_j))+\sum_{v_iv_j\in E(SD[G])}(\sigma(v_i)+\sigma(v_j))+\\ &\quad \sum_{u_iv_j\in E(SD[G])}(\sigma(u_i)+\sigma(v_j))+\sum_{v_iu_j\in E(SD[G])}(\sigma(v_i)+\sigma(u_j)).+\\ &\quad \sum_{u_iv_i\in E(SD[G])}(\sigma(u_i)+\sigma(v_i)).\end{aligned}\tag{18}$$

Consider first part of equation (18) using (15), (16), (17) we have,

$$\begin{aligned}\sum_{u_iu_j\in E(SD[G])}(\sigma(u_i)+\sigma(u_j)) &= \sum_{u_iu_j\in E(SD[G])}[2\sum_{i\neq j}(d(u_i,u_j))+1]+\\ &\quad [2\sum_{i\neq j}(d(u_j,u_j))+1]\\ &= \sum_{u_iu_j\in E(SD[G])}[2[\sum_{i\neq j}(d(u_i,u_j))+\\ &\quad \sum_{i\neq j}(d(u_j,u_i))]]+2\\ &= 2[(\sigma(u_i)+\sigma(u_j))]+2m\\ &= 2S_1(G)+2m.\end{aligned}\tag{19}$$

Using equation(19) in equation (18) and from Remark 2 we have,

$$\begin{aligned} S_1(SD[G]) &= 4(2S_1(G)+2m)+2[W(SD(G))] \\ &= 8S_1(G)+8m+2[4W(G)+n] \\ &= 8S_1(G)+8W(G)+2n+8m. \end{aligned}$$

□

Theorem 2.6. *The second Status connectivity index of Double graph is*

$$S_2(D[G]) = 16S_2(G)+16S_1(G)+16m.$$

Proof. By using definition of Double graph we have,

$$\sigma(u_i)=\sigma(u_j). \tag{20}$$

$$\begin{aligned} \sigma(u_n) &= \sum_{i=1}^{n}(d(u_n,u_i)+d(u_n,v_i))+\sum_{i=n} d(u_n,v_n) \\ &= 2\sum_{i=1}^{n}(d(u_n,u_i))+2. \end{aligned} \tag{21}$$

$$\begin{aligned} \sigma(v_n) &= \sum_{i=1}^{n}(d(v_n,u_i)+d(v_n,v_i))+\sum_{i=n} d(v_n,u_n) \\ &= 2\sum_{i=1}^{n}(d(v_n,v_i))+2. \end{aligned} \tag{22}$$

By definitions (3) and (5) we have,

$$\begin{aligned} S_2(D[G]) &= \sum_{uv\in E(D[G])}(\sigma(u)\sigma(v)) \\ &= \sum_{u_iu_j\in E(D[G])}(\sigma(u_i)\sigma(u_j))+\sum_{v_iv_j\in E(D[G])}(\sigma(v_i)\sigma(v_j))+ \\ &\quad \sum_{u_iv_j\in E(D[G])}(\sigma(u_i)\sigma(v_j))+\sum_{v_iu_j\in E(D[G])}(\sigma(v_i)\sigma(u_j)). \end{aligned} \tag{23}$$

Consider first part of (23) using (20), (21), (22) we have,

$$\begin{aligned}\sum_{u_iu_j\in E(D[G])}(\sigma(u_i)\sigma(u_j)) &= \sum_{u_iu_j\in E(D[G])}[2\sum_{i\neq j}(d(u_i,u_j)+2][2\sum_{i\neq j}(d(u_j,u_i)+2] \\ &= \sum_{u_iu_j\in E(D[G])}[2\sum_{i\neq j}(d(u_i,u_j))2\sum_{i\neq j}(d(u_j,u_i))] + \\ &\quad 4\sum_{i\neq j}(d(u_i,u_j))+4\sum_{i\neq j}(d(u_j,u_i))+4 \\ &= 4[(\sigma(u_i)\sigma(u_j))]+4[\sigma(u_i)+\sigma(u_j)]+4m \\ &= 4S_2(G)+4S_1(G)+4m. \qquad (24)\end{aligned}$$

By using (24) in (23) we have,

$$\begin{aligned}S_2(D[G]) &= (4S_2(G)+4S_1(G)+4m)4 \\ &= 16S_2(G)+16S_1(G)+16m.\end{aligned}$$

□

Theorem 2.7. *The second Status connectivity index of Strong Double graph is*

$$S_2(SD[G]) = 16S_2(G)+8S_1(G)+4m+\sum_{u_i\in V(SD[G])}(\sigma(u_i))^2.$$

Proof. By definition Strong Double graph we have,

$$\sigma(u_i)=\sigma(v_i). \qquad (25)$$

$$\begin{aligned}\sigma(u_n) &= \sum_{i=1}^{n}(d(u_n,u_i)+d(u_n,v_i))+\sum_{i=n}d(u_n,v_n) \\ &= 2\sum_{i=1}^{n}(d(u_n,u_i))+2. \qquad (26)\end{aligned}$$

$$\begin{aligned}\sigma(v_n) &= \sum_{i=1}^{n}(d(v_n,v_i)+d(v_n,u_i))+\sum_{i=n}d(v_n,u_n) \\ &= 2\sum_{i=1}^{n}(d(v_n,v_i))+2. \qquad (27)\end{aligned}$$

By using definitions (3) and (5) we have,

$$\begin{aligned} S_2(SD[G]) &= \sum_{uv \in E(SD[G])} (\sigma(u)\sigma(v)) \\ &= \sum_{u_iu_j \in E(SD[G])} (\sigma(u_i)\sigma(u_j)) + \sum_{v_iv_j \in E(SD[G])} (\sigma(v_i)\sigma(v_j)) + \\ &\quad \sum_{u_iv_j \in E(SD[G])} (\sigma(u_i)\sigma(v_j)) + \sum_{v_iu_j \in E(SD[G])} (\sigma(v_i)\sigma(u_j)) + \\ &\quad \sum_{u_iv_i \in E(SD[G])} (\sigma(u_i)\sigma(v_i)). \end{aligned} \tag{28}$$

From (25), (26), (27), (28) we have,

$$S_2(SD[G]) = 16S_2(G) + 8S_1(G) + 4m + \sum_{u_i \in V(SD[G])} (\sigma(u_i))^2.$$

□

3 Application with calculation examples

Distance based topological indices has applications in Chemistry, Biology, Material science, Computer science.
Calculation examples:

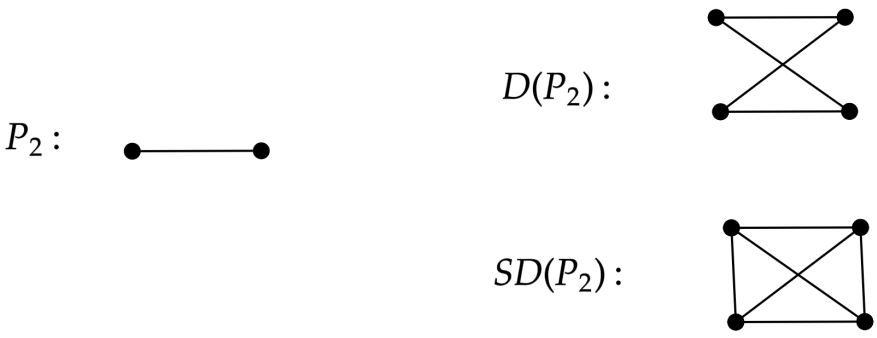

Figure 3: Double and Strong Double graph of P_2

Let $G = P_2$.
1) From Theorem 2.1,
Hyper Wiener index of Double graph is,

$$WW(D[G]) = 4WW(G) + 6n.$$

Calculation:
By definition of Hyper Wiener index,
$WW(P_2) = 2$, $n = 2$.
From Equation 1,

$$\begin{aligned} WW(D[P_2]) &= \frac{1}{2}\sum_{u,v\in V(D[P_2])}(d_{D[P_2]}(u,v)) + \frac{1}{2}\sum_{u,v\in V(D[P_2])}(d_{D[P_2]}(u,v))^2. \\ &= \frac{1}{2}[1+1+2+1+2+1+1+1+2+1+2+1] + \frac{1}{2}[1+ \\ &\quad 1+4+1+4+1+1+1+4+1+4+1] \\ &= \frac{1}{2}[16] + \frac{1}{2}[24] \\ &= 20. \end{aligned} \tag{29}$$

From Theorem 2.1,

$$\begin{aligned} WW(D[P_2]) &= 4WW(P_2) + 6n. \\ &= 4(2) + 6(2) \\ &= 8 + 12 \\ &= 20. \end{aligned} \tag{30}$$

Hence, from equations (29) and (30) values are equal.

2) From Theorem 2.2,
The reciprocal complementary Wiener index of a Double graph is,

$$RCW(D[G]) = 4RCW(G) + \frac{n}{diam(D[G]) - 1}.$$

Calculation:
By definition of reciprocal complementary Wiener index,
$RCW(P_2) = 1$, $diam(D[P_2]) = 2$.

From equation 2,

$$\begin{aligned} RCW(D[P_2]) &= \sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{1+diam(D[P_2])-d_{D[P_2]}(u,v)} \\ &= \frac{1}{1+2-1}+\frac{1}{1+2-1}+\frac{1}{1+2-2}+\frac{1}{1+2-1}+ \\ &\quad \frac{1}{1+2-2}+\frac{1}{1+2-1} \\ &= \frac{1}{2}+\frac{1}{2}+1+\frac{1}{2}+1+\frac{1}{2} \\ &= 4. \end{aligned} \tag{31}$$

From Theorem 2.2,

$$\begin{aligned} RCW(D[P_2]) &= 4RCW(P_2)+\frac{n}{diam(D[P_2])-1}. \\ &= 4.[1/2]+\frac{2}{2-1} \\ &= 4. \end{aligned} \tag{32}$$

Hence, from equations (31) and (32) values are equal.

3) From Theorem 2.3,
The reciprocal complementary Wiener index of a Strong Double graph is,

$$RCW(SD[G]) = 4RCW(G)+\frac{n}{diam(D[G])}.$$

Calculation:
By definition of reciprocal complementary Wiener index,
$RCW(P_2)=1$, $diam(SD[P_2])=1$.
From equation 2,

$$\begin{aligned} RCW(SD[P_2]) &= \sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{1+diam(SD[P_2])-d_{SD[P_2]}(u,v)} \\ &= \frac{1}{1+1-1}+\frac{1}{1+1-1}+\frac{1}{1+1-1}+\frac{1}{1+1-1}+ \\ &\quad \frac{1}{1+1-1}+\frac{1}{1+1-1} \\ &= 6. \end{aligned} \tag{33}$$

From Theorem 2.3,

$$\begin{aligned} RCW(SD[P_2]) &= 4RCW(P_2) + \frac{n}{diam(D[P_2])}. \\ &= 4(1) + 2 \\ &= 6. \end{aligned} \tag{34}$$

Hence, from equations (33) and (34) values are equal.

4) From Theorems 2.4 and 2.5,
The first Status connectivity index of Double graph is,

$$S_1(D[G]) = 8S_1(G) + 16m.$$

The first Status connectivity index of Strong Double graph SD[G] is,

$$S_1(SD[G]) = 8S_1(G) + 8W(G) + 2n + 8m.$$

Calculation:
By definition of first Status connectivity index,
$S_1(P_2) = 2$.
For Double graph $\sigma(u_i) = 4$.
For Strong Double graph $\sigma(u_i) = 3$.
From equations 3 and 4,

$$\begin{aligned} S_1(D[P_2]) &= \sum_{uv \in E(D[P_2])} (\sigma(u) + \sigma(v)) \\ &= [4+4] + [4+4] + [4+4] + [4+4] \\ &= 32. \end{aligned} \tag{35}$$

By definition of first Status connectivity index,

$$\begin{aligned} S_1(SD[P_2]) &= \sum_{uv \in E(SD[P_2])} (\sigma(u) + \sigma(v)) \\ &= [3+3] + [3+3] + [3+3] + [3+3] + \\ &\quad [3+3] + [3+3] \\ &= 36. \end{aligned} \tag{36}$$

From Theorem 2.4,

$$\begin{aligned} S_1(D[P_2]) &= 8S_1(P_2) + 16m. \\ &= 8(2) + 16(1) \\ &= 32 \end{aligned} \tag{37}$$

From Theorem 2.5,

$$\begin{aligned} S_1(SD[P_2]) &= 8S_1(P_2) + 8W(P_2) + 2n + 8m. \\ &= 8(2) + 8(1) + 2(2) + 8(1) \\ &= 36. \end{aligned} \tag{38}$$

Hence, from equations (35) and (37) values are equal and from equations (36) and (38) values are equal.

5) From Theorems 2.6 and 2.7,
The second Status connectivity index of Double graph is,

$$S_2(D[G]) = 16S_2(G) + 16S_1(G) + 16m.$$

The second Status connectivity index of Strong Double graph SD[G] is,

$$S_2(SD[G]) = 16S_2(G) + 8S_1(G) + 4m + \sum_{u_i \in V(SD[G])} (\sigma(u_i))^2.$$

Calculation:
By definition of second Status connectivity index,
$S_2(P_2) = 1$.
For Double graph $\sigma(u_i) = 4$.
For Strong Double graph $\sigma(u_i) = 3$.
From equations 3 and 5,

$$\begin{aligned} S_2(D[P_2]) &= \sum_{uv \in E(D[P_2])} (\sigma(u)\sigma(v)) \\ &= [4(4)] + [4(4)] + [4(4)] + [4(4)] \\ &= 64. \end{aligned} \tag{39}$$

By definition of second Status connectivity index,

$$\begin{aligned} S_2(SD[P_2]) &= \sum_{uv \in E(SD[P_2])} (\sigma(u)\sigma(v)) \\ &= [3(3)] + [3(3)] + [3(3)] + [3(3)] + [3(3)] + [3(3)] \\ &= 54. \end{aligned} \tag{40}$$

From Theorem 2.6,

$$\begin{aligned} S_2(D[P_2]) &= 16S_2(P_2) + 16S_1(G) + 16m. \\ &= 16(2) + 16(1) + 16(1) \\ &= 64 \end{aligned} \tag{41}$$

From Theorem 2.7,

$$\begin{aligned} S_2(SD[P_2]) &= 16S_2(P_2) + 8S_1(P_2) + 4m + \sum_{u_i \in V(SD[P_2])} (\sigma(u_i))^2. \\ &= 16(1) + 16(2) + 4(1) + 2 \\ &= 54. \end{aligned} \tag{42}$$

Hence, from equations (39) and (41) values are equal and from equations (40) and (42) values are equal.

4 Discussion

H. Wiener in 1947 introduced the Wiener index to study the variation of boiling points according to the structure of alkanes known as paraffin [13]. After that, many topological indices were introduced. As topological indices has huge applications in Chemistry, calculating topological indices is useful to know about molecules in Chemistry. In this article, distance based topological indices of Double graphs and Strong Double graphs is calculated. In the section Application with calculation examples for the Distance based topological indices is calculated by definition and also it is calculated for all the Theorems from 2.1 to 2.7. Both the calculations are giving same values. By following the calculation techniques of this article other topological indices for different graphs can be calculated.

5 Conclusions

In this article the Hyper Wiener index of Double graph is calculated and also reciprocal complementary Wiener index of Double graph and Strong Double graph is calculated. Further results on first Status and second Status connectivity indices of Double graph and Strong Double graph is computed.

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Results on Relatively Prime Domination Number of Vertex Switching of Some Graphs

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C. Jayasekaran†

Abstract

If a set $S \subseteq V$ has at least two members and every pair of vertices u and v is such that $(d(u), d(v)) = 1$, then it is said to be a relatively prime dominating set. The relatively prime domination number, represented by $\gamma_{rpd}(G)$, is the lowest cardinality of a relatively prime dominating set. The switching of a finite undirected graph by a subset is defined as the graph $G^{\sigma}(V, E')$, which is obtained from G by removing all edges between σ and its complement $V - \sigma$ and adding as edges all non-edges between σ and $V - \sigma$. In this paper, we compute the relatively prime domination number of vertex switching of cycle type graphs like David Star Graph, Helm Graph, Friendship Graph and Book Graph .

Keywords: Dominating set, relatively prime dominating set, vertex switching.
2020 AMS subject classifications:05C69.[1]

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1 Introduction

In the realm of graph theory, a finite undirected graph without loops and multiple edges is typically denoted as $G = (V, E)$, where G's order and size are represented by the symbols $|V|(p)$ and $|E|(q)$, respectively. This foundational terminology is commonly cited in literature, with references to Harary[2] for graph theoretical concepts and Haynes [3] for domination-related terminology.

One fundamental concept in graph theory is the notion of domination sets. In a subset $V - S$ of vertices V, each vertex is considered adjacent to at least one vertex in set S if S functions as a dominating set in graph G. The smallest cardinality of such a dominating set in G is referred to as the domination number, denoted as $\gamma(G)$. This concept traces its origins to the works of Berge [2] and Ore [9], paving the way for a plethora of other domination-related graph metrics.

In our study, we consider nontrivial graphs and introduce the concept of relatively prime dominating sets. Specifically, if a set S is dominant, and every pair of its vertices u and v share a greatest common divisor of 1 (i.e) $((d(u), d(v)) = 1)$, then the set is termed relatively prime. The corresponding parameter, denoted as $\gamma_{rpd}(G)$ [5], represents the minimum cardinality of a relatively prime dominating set. Furthermore, our previous research [6, 7] introduced the notion of a substantially prime dominating polynomial.

Building on this foundation, we delve into the concept of switching in graphs, originally introduced by Lint and Sidel [8]. By switching vertices within the graph, we obtain a transformed graph $G^{\sigma}(V, E^{'})$, where edges between σ and its complement $V - \sigma$ are removed, and all non-edges between σ and $V - \sigma$ are added. This transformation, often referred to as vertex switching when $\sigma = v$ and G^{v} is used in place of G[4], plays a crucial role in our investigation.

In this paper, we venture into a novel direction by introducing the concept of relatively prime dominating sets (RPDS) in nontrivial graphs. The idea of RPDS emerges from the observation that in some scenarios, it is desirable for the vertices in a dominating set to be relatively prime with respect to their degrees. This concept not only enriches our understanding of domination in graphs but also has intriguing implications in various practical scenarios, such as resource allocation in networks.

Furthermore, our prior research introduced the substantially prime dominating polynomial, which offers an interesting connection between algebraic and combinatorial aspects of domination. This polynomial provides a powerful tool for

analyzing the structure of dominating sets in graphs.

In the following sections of this paper, we narrow our focus to a specific class of graphs, the cycle-type graphs, which include well-known examples such as the David Star Graph, Helm Graph, Friendship Graph, and Book Graph. By studying these cycle-type graphs, we aim to uncover patterns and properties related to relatively prime domination numbers resulting from vertex switching. Our primary objective is to provide a comprehensive understanding of how the interplay of vertices and the unique characteristics of these graphs affect domination, shedding light on the broader implications of our findings.

2 Definition and examples

Definition 2.1. *For a finite undirected graph G(V, E) and $v \in V$, the vertex switching of G by v is the graph G^v which is obtained from G by removing all edges incident to v and adding edges which are not adjacent to v.*

Example 2.1. *The graphs C_5 and C_5^v are given in figures 2. 1 and 2. 2, respectively. Clearly, $\{u, x\}$ is a minimal relatively prime dominating set of C_5^v and hence $\gamma_{rpd}(C_5^v)$ = 2.*

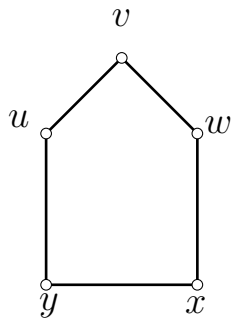


Fig.2.1.C_5

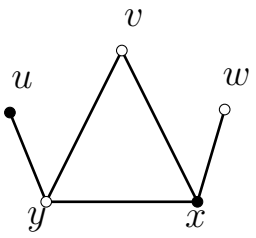


Fig.2.2.C_5^v

Example 2.2. *Consider the graph G given in figure 2. 3. The graph G^{v_1} is given in figure 2. 4. Clearly, $\{v_1, v_2, v_4\}$ is a dominating set of G^{v_1}. Also $(d(v_1), d(v_2))$ = (3, 1) = 1; $(d(v_1), d(v_4))$ = (3, 2) = 1 and $(d(v_2), d(v_4))$ = (1, 2) = 1. By definition, $\{v_1, v_2, v_4\}$ is a relatively prime dominating set of G^{v_1}. Also $\{v_1, v_2, v_4\}$ is a minimal dominating set with this property and hence $\gamma_{rpd}(G^{v_1})$ = 3. But $\gamma(G^{v_1})$ = 2, since $\{v_3, v_6\}$ is the minimal dominating set.*

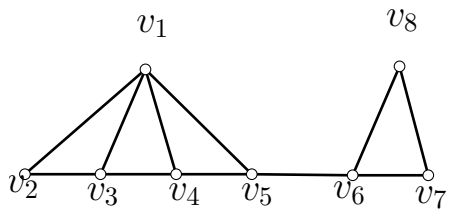


Fig.2.3.G

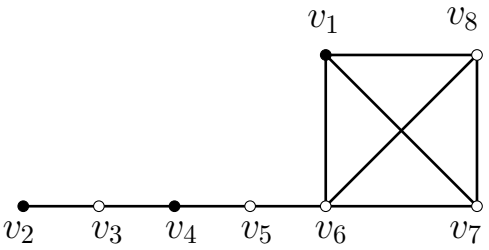


Fig.2.4.G^{v_1}

Definition 2.2. [10] Let $G = (V, E)$ be a graph and let x, y, z be three variables taking values + or -. The transformation graph G^{xyz} is the graph having $V(G) \cup E(G)$ as the vertex set, and for $\alpha, \beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xyz} if and only if one of the following holds:

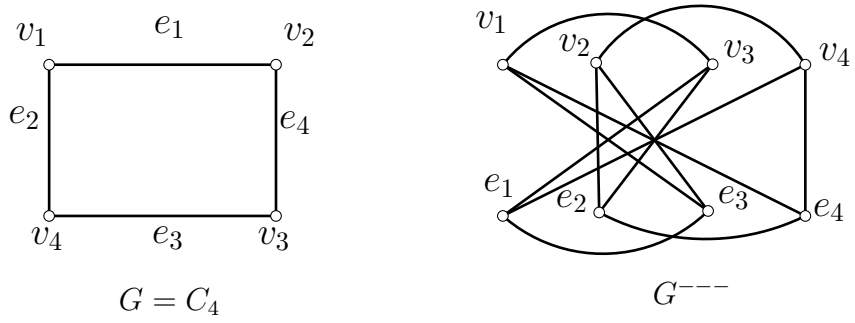
(i) For $\alpha, \beta \in V(G)$, α and β are adjacent in G if x = + ; α and β are not adjacent in G if $x = -$.

(ii) For $\alpha, \beta \in E(G)$, α and β are adjacent in G if y = + ; α and β are not adjacent in G if $y = -$.

(iii) For $\alpha \in V(G), \beta \in E(G)$, α and β are incident in G if z = + ; α and β are not incident in G if $z = -$.

Thus, we may obtain eight kinds of transformation graphs, in which G^{+++} is the total graph of G, and G^{---} is its complement. Also, G^{--+}, G^{-+-} and G^{-++} are the complements of G^{++-}, G^{+-+} and G^{+--} respectively.

Example 2.3. *The graph $G = C_4$ and G^{---} are given in figure 2. 5.*



$Fig.2.5$

Definition 2.3. *By adding a pendent edge at each node of the cycle, the wheel graph can be converted into the Helm graph H_n.*

Definition 2.4. *By combining n duplicates of the cycle graph C_3 with a common vertex, the friendship graph F_n can be created.*

Definition 2.5. *The Cartesian product of a star and a single edge is the book graph B_m. The definition of the $m-$book graph is the graph Cartesian product S_{m+1} times P_2, where $S_m + 1$ is a star graph and P_2 is a path graph on two vertices.*

Theorem 2.1. [5] $\gamma_{rpd}(G) = 0$ if a graph G has an isolated vertex and each dominating set contains at least one vertex with degree > 1.

Result 2.1. [5]For any graph G with more than one isolated vertex, $\gamma_{rpd}(G)$= 0.

Theorem 2.2. [5]For a complete bipartite graph $K_{m,n}$, $\gamma_{rpd}(K_{m,n})$= 2 if and only if (m, n) = 1.

Theorem 2.3. [5] $\gamma_{rpd}(K_m \cup K_n)$ = $\begin{cases} 2 \; if \; and \; only \; if \; (m-1, n-1) = 1 \\ 0 \; otherwise \end{cases}$.

Theorem 2.4. [5]If $G = nK_2 \cup K_1$, then $\gamma_{rpd}(G) = n + 1$.

Notation : We use the symbols $d(u)$ and $d'(u)$ to denote the degree of a vertex u in G and G^v, respectively.

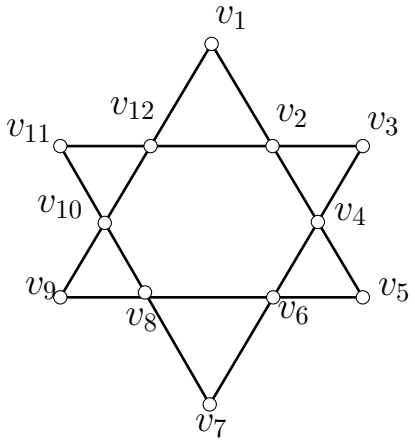
3 Main Results

Theorem 3.1. *Let G be the David Star graph and let v be any vertex of G. Then,* $\gamma_{rpd}(G^v) = \begin{cases} 3 \; if \; d(v) = 4 \\ 0 \; if \; d(v) = 2 \end{cases}$.

Proof. Label the vertices of G having degree 2 as $v_1, v_3, ..., v_{11}$ and degree 4 as $v_2, v_4, ..., v_{12}$ such that $v_{2j}v_{2j+2}, v_2v_{12}, v_iv_{i+1}$ and v_1v_{12} are edges in $G, 1 \leq j \leq 5, 1 \leq i \leq 11$. The graph G is given in figure 3.1.
Case 1. $d_G(v) = 2$

In this case v is $v_i, i = 2n + 1, 0 \leq n \leq 5$. Clearly, $G^{v_1} \cong G^{v_3} \cong ... \cong G^{v_{11}}$. Let v be v_1. The graph G^{v_1} is given in figure 3.2. In G, v_1 is adjacent to v_2 and v_{12}. This implies v_1 is adjacent to all the vertices except v_2 and v_{12} in G^{v_1}. Also v_2 and v_{12} are adjacent. Hence either $\{v_1, v_2\}$ or $\{v_1, v_{12}\}$ is a minimal dominating set of G^{v_1} and $d'(v_1)$ = 9, $d'(v_2)$ = 3 and $d'(v_{12})$ = 3. Now, $(d'(v_1), d'(v_2))$ = (9, 3) = 3 and $(d'(v_1), d'(v_{12}))$ = (9, 3) = 3. This implies that, neither $\{v_1, v_2\}$ nor $\{v_1, v_{12}\}$ is a minimal relativly prime dominating set of G^{v_1} and hence $\gamma_{rpd}(G^{v_1})$ is either 0 or greater than 2. Any dominating set that has more than two vertices either

has two vertices with degrees 3 and 9 or at least a pair of vertices of the same degree, making it a non-relatively prime dominating set. Hence, $\gamma_{rpd}(G^{v_1}) = 0$.



$Fig.3.1.\ G$

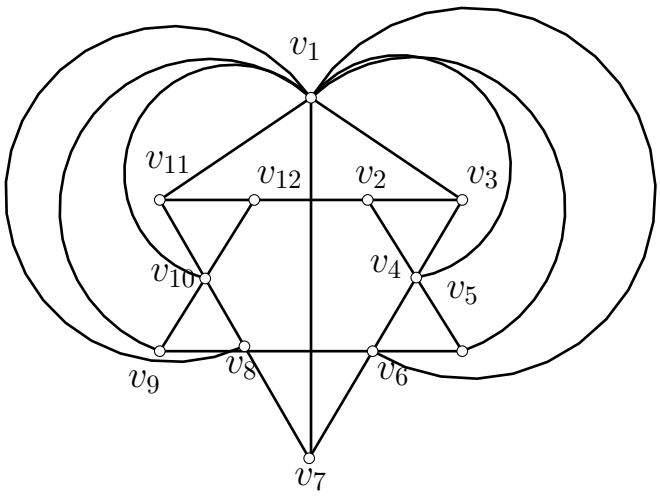


Fig.3.2. G^{v_1}

Case 2. $d_G(v) = 4$

Here v is $v_i, i = 2n+2, 0 \leq n \leq 5$. Clearly, $G^{v_2} \cong G^{v_4} \cong ... \cong G^{v_{12}}$. Let v be v_2. The graph G^{v_2} is given in figure 3.3. In G, v_2 is adjacent to v_1, v_3, v_4 and v_{12} only and hence v_2 is adjacent to all vertices except v_1, v_3, v_4 and v_{12} in G^{v_2}. Also v_1 is adjacent to v_{12} and v_3 is adjacent to v_4. Therefore, $\{v_1, v_2, v_3\}$ is a minimal dominating set of G^{v_2}. Clearly, $d'(v_1) = d'(v_3)$ = 1, $d'(v_2)$ = 7 and $(d'(v_1), d'(v_2)) = (d'(v_2), d'(v_3))$ = $(d'(v_1), d'(v_3))$ = 1. This implies that, $\{v_1, v_2, v_3\}$ is a minimal relatively prime dominating set of G^{v_2} and thereby $\gamma_{rpd}(G^{v_2})$ = 3.

The theorem follows from cases 1 and 2. □

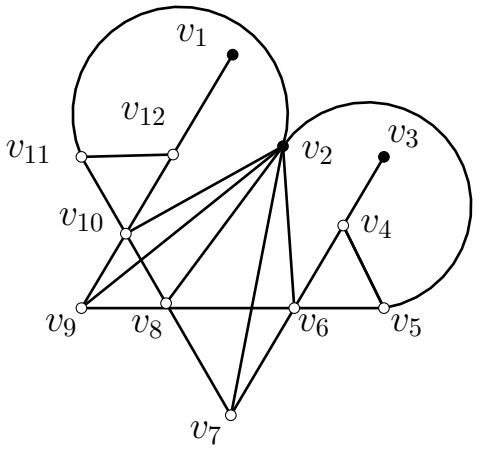


Fig.3.3. G^{v_2}

Theorem 3.2. *For the helm graph* H_n,

$$\gamma_{rpd}(H_n^v) = \begin{cases} 2 \; if \; v \; is \; an \; end \; vertex \; of \; H_n \; and \; 2n \neq 3r+1, 5r+1 \\ \quad where \; r \geq 1 \; is \; odd \; and \; n \neq 3r-1, r \geq 1 \\ 0 \; otherwise \end{cases} .$$

Proof. Let u be centre and $v_1, v_2, ..., v_n$ be vertices of the outer cycle of H_n. Let u_i be the vertex attached with $v_i, 1 \leq i \leq n$. The resultant graph G is H_n with $V(G) = \{u, v_i, u_j / 1 \leq i, j \leq n\}$ and $E(G) = \{uv_i, uv_n, v_iv_{i+1}, v_1v_n, v_iu_i, v_nu_n / 1 \leq i \leq n-1\}$. We consider three cases.

Case 1. v is the central vertex of H_n

Here v is u. The graph H_4 ios given in figure3 3.4 and the graph H_4^u.

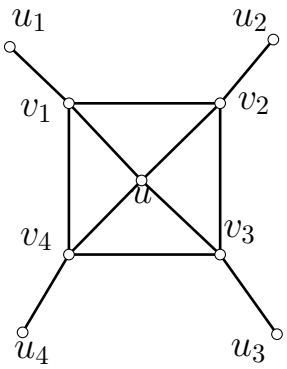


Fig.3.4. H_4

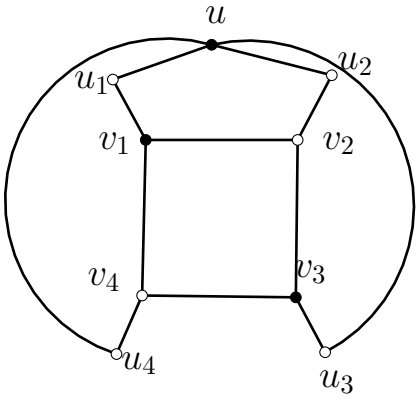


Fig.3.5. H_4^u

In H_n, u is adjacent to $v_i, 1 \leq i \leq n$ and non-adjacent to $u_i, 1 \leq i \leq n$. Hence u is adjacent to u_i and non-adjacent to v_i in H_n^u, $1 \leq i \leq n$. Clearly, $d'(u) = n$, $d'(u_i) = 2$, $d'(v_i) = 3$, $1 \leq i \leq n$. Now, if n is odd, then $\{u, v_1, v_3, ..., v_{n-2}\}$

is a minima1 dominating set of H_n^u and if n is even, then $\{u, v_1, v_3, ..., v_{n-1}\}$ is a minima1 dominating set of H_n^u. Clearly, the minimal dominating sets contain at least two vertices having equal degrees. Hence neither $\{u, v_1, v_4, ..., v_{n-2}\}$ nor $\{u, v_1, v_4, ..., v_{n-1}\}$ is a minimal relatively prime dominating set of H_n^u. Hence $\gamma_{rpd}(H_n^u)$ = 0.

Case 2. v is a vertex of the outer cycle of H_n

In this case v is v_i, $1 \leq i \leq n$. Now $H_n^{v_i}$ is a disconnected graph with the isolated vertex u_i and all other vertices have degree > 1. By Theorem 2.9, $\gamma_{rpd}(H_n^{v_i})$ = 0.

Case 3. v is an end vertex of H_n

Here v is u_i, $1 \leq i \leq n$. Let v be u_1. The graph $H_4^{u_1}$ is given in figure 3.6.

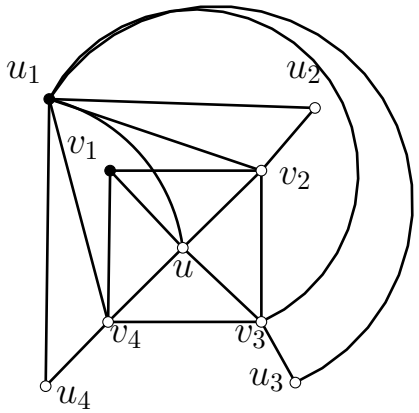


Fig.3.6.$H_4^{u_1}$

In H_n, u_1 is adjacent to v_1 and hence u_1 is adjacent to all vertices except v_1 in $H_n^{u_1}$. Therefore, $\{u_1, v_1\}$ is a minimal dominating set of $H_n^{u_1}$. Also v_1 is adjacent to u, v_2 and v_n in $H_n^{u_1}$. Hence $\{u_1, u\}, \{u_1, v_2\}$ and $\{u_1, v_n\}$ are also a minimal dominating sets of $H_n^{u_1}$. Clearly, $d'(u_1) = 2n-1$, $d'(v_1)$ = 3, $d'(u) = n+1$, $d'(v_i)$ = 5, $2 \leq i \leq n$, $d'(u_i) = 2, 2 \leq i \leq n$. Therefore, $(d'(u_1), d'(v_1)) = (2n-1, 3)$ = 1 if $2n - 1 \neq 3r$. This implies that $2n \neq 3r + 1$. Since $2n$ is even, r must be odd. Therefore, $(2n - 1, 3) = 1$ if $2n \neq 3r + 1$ where $r \geq 1$ is odd. This implies that $\{u_1, v_1\}$ is a minimal relatively prime dominating set of $H_n^{u_1}$ and hence $\gamma_{rpd}(H_n^{u_1})$ = 2 when $2n \neq 3r+1$ and $r \geq 1$ is odd. Also, $(d'(u_1), d'(v_2))$ = $(d'(u_1), d'(v_n))$ = $(2n - 1, 5)$ = 1 if $2n - 1 \neq 5r$. This implies that $2n \neq 5r + 1$. Since $2n$ is even, r must be odd. Therefore, $(2n - 1, 5)$ = 1 if $2n \neq 5r + 1$ where $r \geq 1$ is odd. This implies that $\{u_1, v_2\}$ and $\{u_1, v_n\}$ are minimal relatively prime dominating sets of $H_n^{u_1}$ and hence $\gamma_{rpd}(H_n^{u_1})$ = 2 when $2n \neq 5r + 1$ and $r \geq 1$ is odd. Now $(d'(u_1), d'(u)) = (2n - 1, n + 1)$ = 1 if $n \neq 3r - 1, r \geq 1$. Therefore, $\{u_1, u\}$ is a minimal relatively prime dominating set of $H_n^{u_1}$ if $n \neq 3r - 1$ and so $\gamma_{rpd}(H_n^{u_1})$

= 2.

The theorem follows from cases 1, 2 and 3. □

Theorem 3.3. *Let G =* P_m^{+++} *and v be an end vertex of* P_m*, then* $\gamma_{rpd}(G^v)$ *=2.*

Proof. Let $v_1v_2...v_m$ be the path $P_m, m \geq 2$ and let $e_i = v_ivi + 1$ be the edges of $P_m, 1 \leq i \leq m-1$. Then $V(P_m^{+++})$ = $\{v_1, v_2, ..., v_m, e_1, e_2, ..., e_{m-1}\}$ and hence G has $2m - 1$ vertices. We consider two cases.

Case 1. $m = 2$

In this case v is either v_1 or v_2 and G is K_3. Clearly, $G^v = K_1 \cup K_2$ where K_1 is v. Clearly $\{v, e_1\}$ is a minimal relatively prime dominating set of G^v and hence by Theorem 2. 12, $\gamma_{rpd}(G^v)$= 2.

Case 2. $m \geq 3$

Here v is either v_1 or v_n. Without loss of generality, let v be v_1. In G, v_1 is adjacent to v_2 and e_1 and hence v_1 is non-adjacent to v_2 and e_1 in G^v. Also v_2 is adjacent to e_1 in G^v. Therefore, $\{v_1, v_2\}$ is a minimal dominating set of G^v. In G^v, $d'(v_1) = 2m - 4 = 2(m - 2)$ is even and $d'(v_2)$ = 3 is odd and hence $(d'(v_1), d'(v_2))$ = 1. Thus $\{v_1, v_2\}$ is a minimal relatively prime dominating set of G^v and hence $\gamma_{rpd}(G^v)$ = 2. □

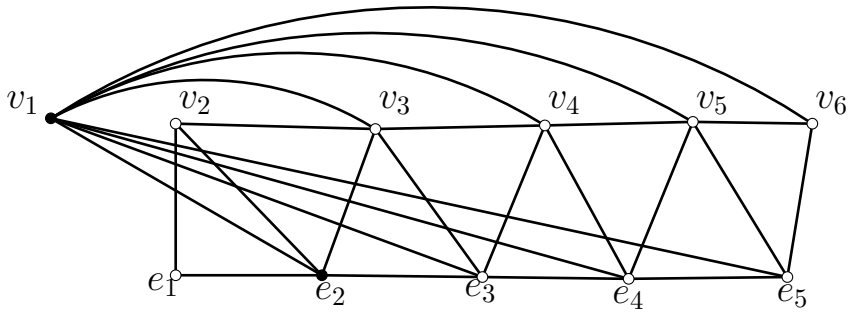


Fig.3.7.G^v where $G = P_6^{+++}$

Theorem 3.4. *Let G be a friendship graph* F_n *and let v be any vertex of G. Then,*

$$\gamma_{rpd}(F_n^v) = \begin{cases} n+1 \; if \; v \; is \; the \; central \; vertex \; of \; F_n \\ 2 \; otherwise \end{cases}.$$

Proof. Let v_{i1}, v_{i2}, v_{i3} be the vertices of i^{th} copy of cycle C_3. Identify the vertices $v_{i3}, 1 \leq i \leq n$ and denote it by u. The resultant graph G is the friendship graph F_n with vertex set $V(F_n)$ = $\{v_{i1}, v_{i2}, u/1 \leq i \leq n\}$ and edge set $E(G) = \{uv_{i1}, uv_{i2}, v_{i1}v_{i2}/1 \leq i \leq n\}$. Clearly, $d(v_{i1}) = d(v_{i2}) = 2, 1 \leq i \leq n$ and $d(u) = 2n$. The graphs $F_4, F_4^{v_{11}}$ and F_4^u are given in figures 3.8, 3.9 and 3.10 respectively.

Case 1. v is the central vertex of F_n

Here v is u and $F_n^u = nK_2 \cup K_1$. By Theorem 2. 13, $\gamma_{rpd}(F_n^u)$= $n+1$.

Case 2. v is not the central vertex of F_n

Here v is $v_{ij}, 1 \leq i \leq n, j = 1, 2$. Let v be v_{11}. In F_n, v_{11} is adjacent to both v_{12} and u and hence v_{11} is adjacent to every vertices except v_{12} and u in $F_n^{v_{11}}$. Since u and v_{12} are adjacent in $F_n^{v_{11}}$, $\{u, v_{11}\}$ is a minimal dominating set of $F_n^{v_{11}}$. Clearly, $d'(u) = 2n-1, d'(v_{11}) = 2n-2$ and $(d'(u), d'(v_{11})) = (2n-1, 2n-2) = 1$. Thus $\{u, v_{11}\}$ is a minimal relatively prime dominating set of $F_n^{v_{11}}$ and hence $\gamma_{rpd}(F_n^v)$= 2.

The theorem follows from cases 1 and 2. □

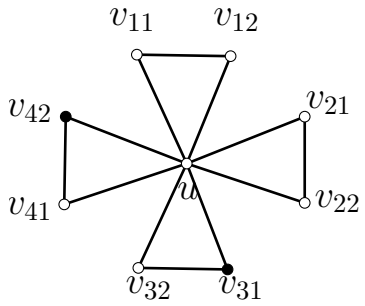


Fig.3.8.F_4

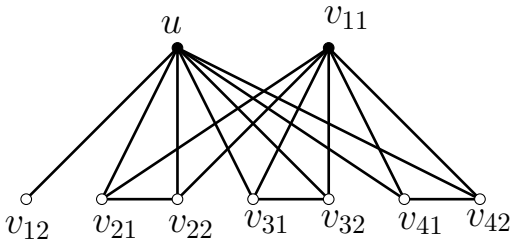


Fig.3.9.$F_4^{v_{11}}$

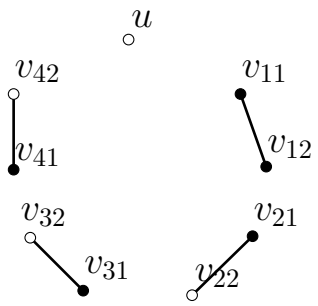


Fig.3.10.F_4^u

Theorem 3.5. *Let G be the book graph $B_m, m \geq 2$ and v be any vertex of G.*

(i) If $d(v)$ = 2 and $m \equiv 3(mod\ 5)$, then $\gamma_{rpd}(G^v)$= 3.

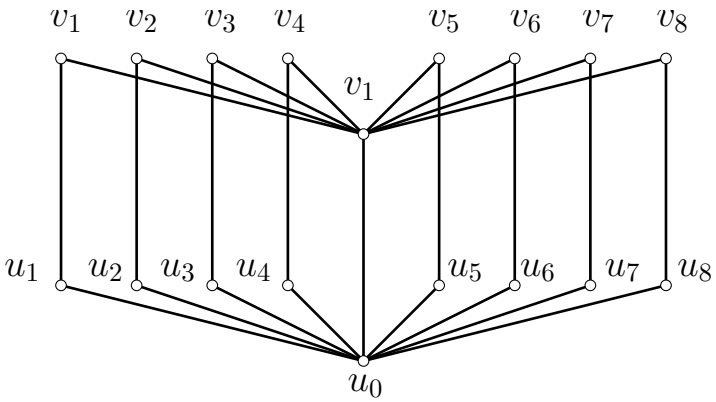
(ii) If $d(v)$ = 2 and $m \not\equiv 3(mod\ 5)$, then $\gamma_{rpd}(G^v)$= 2.

(iii) If $d(v) = m$, then $\gamma_{rpd}(G^v)$ = m.

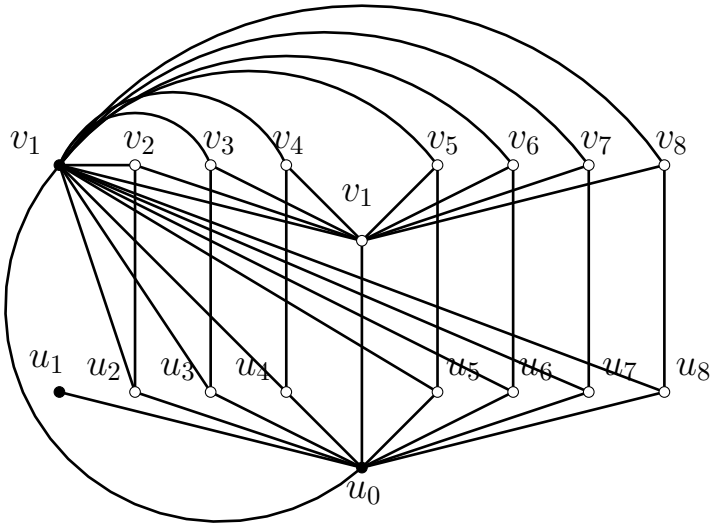
Proof. Let $v_0, v_1, ..., v_m$ and $u_0, u_1, ..., u_m$ be the two copies of star $K_{1,n}$ with central vertices v_0 and u_0 respectively. Join u_i with v_i for all $i, 1 \leq i \leq m$. The resultant graph G is B_m with vertex set $V(G) = \{v_0, u_0, v_i, u_i / 1 \leq i \leq m\}$ and edge set $E(G) = \{u_0v_0, u_iv_i, v_0v_i, u_0u_i / 1 \leq i \leq m\}$. Then G has $2m + 2$ vertices, $3m + 1$ edges, $d(v) = 2$ if $v \in \{u_i, v_i / 1 \leq i \leq m\}$ and $d(v) = m$ if $v \in \{u_0, v_0\}$. The graphs $G = B_8, G^{v_1}$ and G^{v_0} are given in figures 3.11, 3.12 and 3.13 respectively.

Case 1. $d(v)$ = 2

In this case v is either v_i or $u_i, 1 \leq i \leq m$. Clearly, $G^{v_i} \cong G^{u_i}$. Let v be v_1. In G, v_1 is adjacent to only v_0 and u_1 and hence v_1 is adjacent to all vertices except u_1 and v_0 in G^{v_1}. But u_0 is adjacent to both u_1 and v_0 in G^{v_1}. Clearly, $\{v_1, u_0\}$ is the minimal dominating set of G^{v_1} and hence $\gamma(G^{v_1})$= 2. Now, $d'(v_1) = 2m-1, d'(u_0) = m+2$ and $(d'(v_1), d'(u_0)) = (2n-1, n+2)$. If n is even, then $2m - 1$ is odd and so $(2m - 1, m + 2)$ = 1. This implies that $\gamma_{rpd}(G^{v_1})$= 2. If n is odd, then both $2n - 1$ and $n + 2$ are odd. Also $2m - 1 > m + 2$ for $m > 3$ and $2m - 1 = m + 2$ for $m = 3$. If $m \not\equiv 3(mod\ 5)$ which implies that $m = 3 + 5k + c$ where $1 \leq c \leq 4$. Then $n + 2 = 5(k + 1) + c$ and $2n - 1 = 5(2k + 1) + 2c$. Since $c \not\equiv 0(mod\ 5), 2c \not\equiv 0(mod\ 5)$ and so $(2m - 1, m + 2)$ = 1 and hence $\gamma_{rpd}(G^{v_1})$= 2. Suppose $n \equiv 3(mod\ 5)$, then $(d'(v_1), d'(u_0)) = (2m - 1, m + 2) = (2(5r + 3) - 1, (5r + 3) + 2) = (10r + 5, 5r + 5)$ = 5$\not\equiv$ 1. This implies that $\{v_1, u_0\}$ is not a relatively prime dominating set of G^{v_1}. Consider the dominating set $\{v_1, u_1, v_0\}$ of G^{v_1} in which $d'(v_1) = 2m-1, d'(u_1) = 1, d'(v_0) = m$. Clearly, $(d'(u_1), d'(v_1)) = (d'(u_1), d'(v_0)) = (d'(v_1), d'(v_0))$ = 1 if $m \equiv 3(mod\ 5)$. This implies that $\{v_1, u_1, v_0\}$ is a minimal relatively prime dominating set of G^{v_1} and hence $\gamma_{rpd}(G^{v_1})$= 3.



$Fig.3.11. G = B_8$

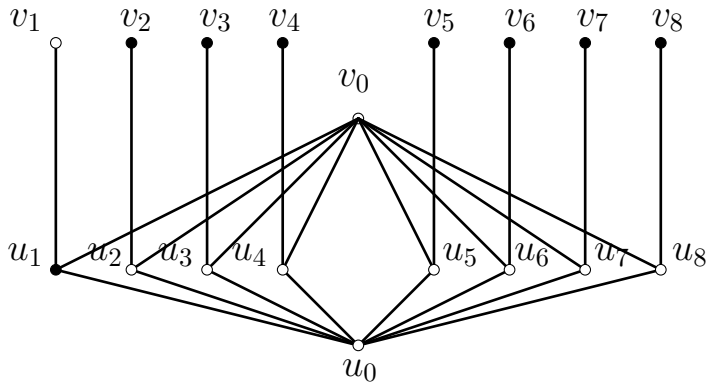


$Fig.3.12.G^{v_1}$

Case 2. $d_G(v) = m$

Here v is either u_0 or v_0. Clearly, $G^{v_0} \cong G^{u_0}$. A minimal dominating set of G^{v_0} is $\{u_1, v_2, ..., v_m\}$. Clearly, $d'(u_1)$= 3 and $d'(v_i)$ = 1, $1 \leq i \leq m$. This implies that $(d'(u_1), d'(v_i))$ = 1 and so $\{u_1, v_2, ..., v_m\}$ is a minimal relatively prime dominating set of G^{v_0}. Hence $\gamma_{rpd}(G^{v_0}) = m$.

The theorem follows from cases 1 and 2. □



$Fig.3.13.\ G^{v_0}$

Conclusion

As a result, we have demonstrated how to identify the relatively prime domina-

tion number of vertex switching graphs in this study. Additionally, we determine the relatively prime domination number for vertex switching in cycle type graphs such the David Star, Helm, Friendship, and Book graphs.

Acknowledgement

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Stochastic Perishable Inventory System Analysis With Two Types Of Customers In The Supply Chain

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Abstract

In the supply chain model being examined in this study, there are two different client expectations. A warehouse, a single distribution centre, a single retailer, and the handling of a single perishable product make up the system. Retailer node (lower echelon) assumes a $A(s, S)$ type inventory system with Poisson demand and exponentially dispersed lead times, and distribution centre assumes a $(0, kQ)$ policy (middle echelon). Demands that happen during the times when there is no stock are taken to be lost sales. The merchandise is sent from the distribution centre to the stores in packs of $Q(= S - s)$ items. It is believed that only the retailer nodes with rate γ experience item perishability. The upper echelon (warehouse) replaces the distribution center's stock with exponentially distributed lead times due to its ample supply. Demands arrive at the retailer node in two categories: (i) normal customers; and (ii) priority customers, with arrival rates of α and β. It is possible to obtain the metrics of system performance as well as the steady state probability distribution of system states. The proposed paradigm is demonstrated with numerical examples.

Keywords: Supply chain management, Perishable Inventory System, Markov process, Perishable inventory Optimization. [1]

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1 Introduction to Inventory Control System

For a healthy business, the inventory system should be updated daily or on a regular basis. These updates also make the system portable. What is in your factory, office, or retail space serves as the basis of your inventory system. This gives notice of goods for sale or allows you to create products to readily meet customer demands. The stock of goods, commodities, or other economic resources that are kept on hand or set aside for future use in manufacturing or satisfying demand can be widely referred to as inventory. When products are needed, the inventory system alerts the user to buy or supply them. To deal with circumstances where demand, lead time, or both fluctuate, an inventory control system was designed.

The study of supply chain management (SCM) started in the late 1980s and has acquired a growing amount of interest from both companies and researchers over the following 3 decades. The term supply chain management has several different definitions. The Stanford Global Supply Chain Management Forum's (1999) director, Hau Lee, provides the following clear and concise definition on the forum's website:

The management of resources, information, and financial flows in a network made up of suppliers, manufacturers, distributors, and customers is the focus of supply chain management. From this description, it is clear that SCM affects service and financial enterprises in addition to manufacturing businesses. We can even assert that practically all businesses will experience SCM issues, and that SCM decisions will have an impact on how the entire organisation runs. More and more businesses are recognising the value of SCM and making efforts to enhance the efficiency of their supply chains.

A supply chain can be described as an integrated process where a number of different corporate entities (suppliers, distributors, and retailers) collaborate to (1) obtain raw materials, (2) convert them into useful products, (3) produce and convey the finished goods to retailers. The movement of commodities through this network requires an effective transportation and communication system. The forward movement of products and materials and the backward flow of data and money are the two traditional characteristics of the supply chain.

One of the key elements of supply chain management is inventory control. Stock control models are usually always stochastic optimization problems with either expected costs, projected profits, or hazards as the objectives. In reality, a merchant might seek the best option that minimises expected costs or maximises expected profits with the least amount of chance of deviating from the goal.

Perishable commodities are those that have a fixed or predetermined lifespan beyond which they are deemed useless and cannot be used to satisfy demand. Because in real life products like milk, blood, drugs, food, vegetables, and some chemicals actually have defined life spans after which they would perish, planning

and controlling perishable inventory systems is crucial. The existence of these kind products after their lifetime will not only consume space of the store, but also harm the lifetime (damage) of the nearby objects. At specific times, the things that have degraded (exponentially degraded) may be taken out of the stock.

As a result of recent technology improvements and innovations in the sector of telecommunications and transportation, the supply chain management (SCM) industry is growing. This SCM performs a very significant function in each and every field. Retailers of goods such as food, medicine, and household appliances have opened locations all across the country in order to better service their clients.

Benita [1998] performed an exhaustive review. Yet, the performance, design, and analysis of the supply chain as a whole are receiving more and more attention. This type of investigation was started in 1977 by the Strategic Planning and Modeling (SPaM) team of HP (Hawlett Packard). From a practical standpoint, a number of changes in the manufacturing environment?including rising manufacturing costs, declining manufacturing base resources, shortened product life cycles, levelling of the playing field in manufacturing planning, inventory-driven costs (IDC) associated with distribution, and the globalisation of market economics leads to the development of the supply chain concept. The supply chain concept emerged from two-stage multi-echelon inventory models in manufacturing research, and it is significant to note that much of the research in this field is based on the seminal work of Clark and Scarf [1960].

Federgruen [1993] compiled a thorough analysis of this development. Significant advancements in two-echelon models can be found in Ming and Jewkes [2000], Axaster [1993], and Nahimas [1982]. Krishnan.K. and Elango.C. 2005 studied a continuous review (s, S) strategy with favourable lead times in a two-echelon supply chain. In supply chains with partial backorders, stochastic inventory control systems were thought to exist by Bakthavachalam et al. [2012]. Inventory policy for perishable inventory system with (s, Q) policy in supply chain is introduced by Kumaresan et al. [2014]. In the supply chain, an integrated inventory control system is being built by Kartheeswari and Bakthavachalam [2017]. Multi-Echelon Inventory Optimization for Fresh Goods in Supply Chains is being considered by Zhang et al. [2021].

The other sections of the study are structured as follows: section 2 describes the model formulation, section 3, and section 4 performs steady state analyses for our novel model. The system's operational features are covered in Section 4, and the cost analysis for the operation is covered in Section 5. A few numerical examples are provided in section 6, and section 7 concludes the paper.

2 The Model Description

The supply chain inventory control system taken into account in this paper is defined as follows: A perishable product is sent from the warehouse to the distribution centre, which follows the $(0, kQ)$ policy and from the distribution centre to the retailer, who follows the (s, S) policy. A Poisson distribution with rate $\alpha > 0$ and Priority demand $(1 + 1)$ with rate $\beta > 0$ describe the normal demand at the retailer node, respectively. online shopping plays a very essential part in the industry nowadays, they are provided priority and some discounts(1+1 Offer) by distinguishing themselves as the most important customer of their firm in order to attract customers. An exponential lead time with a parameter of $\mu > 0$ is used to administer the supply to the store in packets of $Q(= S - s)$ items. Demands that happen during the times when there is no stock are taken to be lost sales. The maximum inventory levels nQ and S in our model are constant, whereas the retailer's reorder level s varies so that $S - s = Q$. As stated in the definition above, both nodes' on-hand inventory amounts are determined by random processes. It is believed that only the retailer nodes with rate γ experience item perishability.

3 Analysis

Let $IR(t)$ and $ID(t)$ denote, respectively, the retailer node and distribution center's (DC) on-hand inventory levels at time t^+. Given the assumptions on the input and output process, $I(t) = (IR(t), ID(t)) : t \geq 0$ is a Markov process with state space
$E = \{(i, j)/i = S, S-1 \ldots, s, s-1, \ldots, 3, 2, 1, 0.$ and $j = nQ, (n-1)Q, \ldots Q\}$.

Theorem 3.1. *The vector process $I(t) : t \geq 0$ where $I(t) = (I_0(t), I_1(t))$ for $t \geq 0$ is a continuous time Markov Chain with state space*

$$E = \{(j, q)/j = 0, 1, 2..., S; q = Q, 2Q, ..., nQ\}$$

Proof. The stochastic process $\{I(t) : t \geq 0\}$ has a discrete state space with order relation $' \leq'$ that $(j, q) \leq (k, r)$ if and only if $j \leq k$ and $q \leq r$.To prove that $I(t) : t \geq 0$ is a Markov chain ,first we do a transformation for state space E to E' such that $(j, q) \longrightarrow j + q \in E'$,where $E' = \{Q, Q + 1, ..., Q + S, ..., nQ + S\}$.

Now we may realize that $\{I(t) : t \geq 0\}$ is a stochastic process with discrete state space E'.

The joint distribution of random variables $\{I(t_1), I(t_2), ..., I(t_n)\}$ and $\{I(t_1 + \tau), I(t_2 + \tau), ..., I(t_n + \tau)\}$ with $\tau > 0$ (an arbitrary real number) are equal.

In particular the conditional probability

$$P_r\{I_n = k|I_{n-1} = j, I_{n-2} = i, ...I_0 = 1\} = P_r\{I_n = k|I_{n-1} = j\}$$

due to the single step transition of states in E.
Hence $\{I(t) : t \geq 0\}$ is a continuous time Markov Chain.

□

The following inputs can be used to get this process' infinitesimal generator, R.

1. The Markov Process transitions from (i, j) to $(i - 1, j)$ with an intensity of transition of $\alpha > 0$ in response to the arrival of a normal demand or perish at retailer node.

2. When a Priority demand or perish arrives at the retailer node, the Markov Process transitions from (i, j) to $(i - 2, j)$ with an intensity of transition of $\beta > 0$.

3. Inventory replenishment from a destruction centre to a retailer changes from (i, j) to $(i + Q, j - Q)$ and from (i, Q) to $(i + Q, nQ)$ with a transition rate of $\mu(> 0)$.

The infinitesimal generator R is defined as follows:

$$R = \begin{array}{c} nQ \\ (n-1)Q \\ (n-2)Q \\ \cdots \\ Q \end{array} \begin{bmatrix} A & B & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & A & B & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & A & B & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & B \end{bmatrix} \tag{1}$$

As follows is a way to express the entries of the matrix R.

$$[R]_{p \times q} = \begin{cases} A \text{ if p } = \text{ q, q } = \text{ nQ , (n-1)Q,(n-2)Q...2Q} \\ A \text{ if p } = \text{ Q} \\ B \text{ if p } = \text{ q+Q, q = (n-1)Q,(n-2)Q...2Q,Q} \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

$$[A]_{m \times n} = \begin{cases} \alpha + i\gamma & \text{n=m+1} & \text{m=S,S-1,....,s+1,s,s-1,...3,2,1.} \\ \beta + i\gamma & \text{n=m+2} & \text{m=S,S-1,....,s+1,s,s-1,...3,2.} \\ -(\alpha + \beta + i\gamma) & \text{n=m} & \text{m=S,S-1,....,s+1} \\ -(\alpha + \beta + \mu + i\gamma) & \text{n=m} & m = \text{s,s-1,...3,2.} \\ -(\alpha + \mu + i\gamma) & \text{n=m} & \text{m=1.} \\ -\mu & \text{n=m} & \text{m=0.} \\ 0 & \text{otherwise} & \end{cases} \tag{3}$$

$$[\mathrm{B}]_{\mathrm{m\ x\ n}} = \begin{cases} \mu & \text{if} \quad \mathrm{m} = \mathrm{n} + \mathrm{Q}\,,\ j = \mathrm{s,s\text{-}1,\ldots 1,0.} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

3.1 Steady state analysis

The state space E of the Markov process, $\{I(t); t \geq 0\}$, is finite and irreducible, as shown by the structure of the infinitesimal matrix R. Allowing for the limiting probability Distribution of the inventory level process to be

$\Pi_i^j = \lim_{t\to\infty} \Pr\{(IR(t), ID(t) = (i,j)\}$, where$\Pi_i^j$ is the steady state probability that the system be in state (i,j), (Cinlar Hadley and Whitin [1963]).

Let $\Pi = \left(\Pi^{kQ}, \Pi^{(k-1)Q}, \Pi^{(k-2)Q}, \ldots, \Pi^{Q}.\right)$ be the probability vector indicating the steady state probability distribution for the system under investigation. By resolving the matrix equation $\Pi R = 0$, it is possible to get Π_j^q for each (i,j). Hence, the steady state balance equations are

$$\Pi^{kQ} A + \sum_{\mathrm{i=0}}^{\mathrm{k\text{-}1}} \Pi^{iQ} B = 0 \tag{5}$$

$$\Pi^{Q}\, B + \Pi^{kQ}\, \mathrm{A} = 0 \tag{6}$$

furthermore, the normalising equation

$$\sum_{(i,j)\in E} \Pi_j^q = 1. \tag{7}$$

4 System performance measures

4.1 Mean Inventory Levels

The steady-state mean inventory level at the retailer node is represented by the overline symbol $\overline{IR}$ and is calculated as follows:

$$\overline{IR} = \sum_{j=Q}^{nQ} * \left(\sum_{i=0}^{S} i.\Pi_i^j \right) \tag{8}$$

The Distribution Center's average inventory level is indicated by the symbol $\overline{ID}$ and is provided by

$$\overline{ID} = \sum_{i=0}^{S} \left(\sum_{j=Q}^{nQ\,*} j.\Pi_i^j \right). \tag{9}$$

Algorithm 1

1. Recursively determine the matrix $D_n, 0 \leq n \leq N$ by using

$$D_0 = A_0$$

$$D_n = A_n + B_n(-D_{n-1}^{-1})\, C, \quad n = 1, 2, ...K$$

2. the system's solution

$$\Pi^{<N>} D_N = 0$$

3. Calculate the vector iteratively $\Pi^{<n>}, \quad n = N-1, ..., 0$ using

$$\Pi^{<n>} = \Pi^{<n+1>} B_{n+1}(-D_n^{-1}), n = n-1, ..., 0$$

4. Renormalizing the vector Π, using $\Pi e = 1$.

4.2 Mean Reorder Rates

The mean reorder rate at the retailer node is represented by R_r and is calculated as follows:

$$R_r = (\alpha + \beta + \gamma) \sum_{j=Q}^{nQ} {}^*\Pi_{s+1}^j + (\beta + \gamma) \sum_{j=Q}^{nQ} {}^*\Pi_{s+2}^j \tag{10}$$

The Distribution Center's mean reorder rate is shown by the symbol R_d and is calculated as follows:

$$R_d = \mu \sum_{j=0}^{s} \Pi_j^Q \tag{11}$$

4.3 Mean Reorder Rates

The mean shortage rate at the retailer node is shown by T and is calculated as follows:

$$T = (\alpha + \beta) \sum_{j=Q}^{nQ} {}^*\Pi_0^j + \beta \sum_{j=Q}^{nQ} {}^*\Pi_1^j \tag{12}$$

5 Cost Analysis

The cost structure for the suggested model is imposed in this part, and it is evaluated according to the criterion of minimising the steady state (long run) total

predicted cost per unit of time. Let k_r and k_d be setup costs, and h_r and h_d be holding costs per unit item per unit time at the retailer and distribution centre, respectively. The cost of the scarcity, which only occurs at retailer nodes, is shown by the symbol S_r. The estimated long-term cost rate $C(s, Q)$ is provided by

$$C(s, Q) = h_r\overline{IR} + h_d\overline{IR} + k_r R_r + k_d R_d + s_r T \tag{13}$$

6 Numerical Example and Sensitivity Analysis

This section discusses the challenge of reducing the long-term total predicted cost per unit of time given the following cost structure. The following numerical example might be used to show the results from the steady state scenarios. In the example that follows, it is presummated that $S = 12, n = 3, \alpha = 5, \beta = 1.75, \mu = 1.25, h_r = 0.025, h_d = 0.075, k_r = 4.5, k_d = 5.5, T = 0.25$ For various reorder levels, the total anticipated cost rate is provided by

s	Cost	Q
2	248.5732765	28
3	261.6411365	27
4	252.9833491	26
5	262.6257595	25
6	252.5650419	24
7	259.6892737	23
8	248.8257261	22
9	253.7464519	21
10	257.2118433	20
11	244.9712722	19
12	246.2623777	18
13	246.0741916	17
14	244.3801994*	16

Table 1: Expected Cost Rate Analysis (C(s, Q)) in a Perishable Inventory System with S=30 and M=300

In Table 1, we present the total expected cost rate for different combinations of reorder level (s) and order quantity (Q) when S=30 and M=300. The minimum cost is achieved when s=14 and Q=16, with a cost of 240.3801994. This suggests that the optimum reorder level is 14, and the optimum order quantity is 16.

Figure 1: Expected Cost Rate Analysis (C(s, Q)) in a Perishable Inventory System with S=30 and M=300

Figure 1 visually represents the data from Table 1 as a line graph. The x-axis represents the reorder level (s), and the y-axis represents the total expected cost rate. Each line on the graph corresponds to a different order quantity (Q). It is evident from the graph that there is a clear minimum point, indicating the combination of s=14 and Q=16, which results in the lowest total expected cost rate.

This graphical representation reinforces the findings from Table 1, highlighting the importance of selecting the right combination of reorder level and order quantity to minimize the total expected cost in the stochastic perishable inventory system with two types of customers in the supply chain. The optimal values of s=14 and Q=16 should be considered for cost-efficient inventory management in this system.

s	Cost	Q
2	280.9923	48
3	277.6517	47
4	273.0369	46
5	267.6404	45
6	261.8845	44
7	256.0216	43
8	281.6608	42
9	275.125	41
10	268.6639	40
11	262.2744	39
12	255.9477*	38
13	277.4151	37

s	Cost	Q
14	270.435	36
15	263.4905	35
16	256.575	34
17	274.4263	33
18	266.8035	32
19	259.1963	31
20	274.0963	30
21	265.761	29
22	257.4307	28
23	269.3426	27
24	260.235	26

Table 2: Expected Cost Rate Analysis (C(s, Q)) in a Perishable Inventory System with S=50 and M=300

The results presented in Table 2 demonstrate the relationship between the total expected cost rate, reorder level (s), and order quantity (Q) in the context of the stochastic perishable inventory system with two types of customers. Notably, with fixed values of S=50 and M=300, the minimum cost of 255.9477 is attained when employing an optimal reorder level of 12 and an optimal order quantity of 38. This signifies that in the studied supply chain, the most cost-efficient strategy involves maintaining a reorder level of 12 units and placing orders of 38 units.

Figure 2: Expected Cost Rate Analysis (C(s, Q)) in a Perishable Inventory System with S=50 and M=300

Figure 2 visually represents the data from Table 2 through a line graph, reinforcing the observation that the total expected cost rate is minimized at the specified reorder level and order quantity. The graph provides a clear visualization of the cost-saving benefits associated with this optimal inventory management strategy.

s	Cost	Q	s	Cost	Q	s	Cost	Q
2	498.3686	98	18	483.9871	82	34	447.7754	66
3	497.8622	97	19	478.5274	81	35	441.7263	65
4	494.7201	96	20	473.0933	80	36	435.6802	64
5	489.9647	95	21	467.6803	79	37	429.6367	63
6	484.4736	94	22	462.2851	78	38	470.0873	62
7	478.7646	93	23	456.905	77	39	463.2988	61
8	473.0843	92	24	451.5376	76	40	456.5122	60
9	467.5287	91	25	446.1812	75	41	449.7275	59
10	462.1217	90	26	440.8343	74	42	442.9445	58
11	456.8565	89	27	435.4956	73	43	436.1631	57
12	451.7155	88	28	430.1641	72	44	429.3829*	56
13	446.6787	87	29	478.0778	71	45	463.8472	55
14	441.7277	86	30	472.0084	70	46	456.3177	54
15	436.8465	85	31	465.9439	69	47	448.7853	53
16	432.022	84	32	459.8838	68	48	441.243	52
17	489.4777	83	33	453.8278	67	49	433.6725	51

Table 3: Expected Cost Rate Analysis (C(s, Q)) in a Perishable Inventory System with S=100 and M=500

Figure 3: Expected Cost Rate Analysis (C(s, Q)) in a Perishable Inventory System with S=100 and M=500

Table 3 presents the total expected cost rate in the stochastic perishable inventory system with two customer types. With S=100 and M=500, the optimal reorder level (s=44) and order quantity (Q=56) result in the lowest cost of 429.3829. Figure 3 visualizes this data in a line graph.

h_r/h_d	0.1	0.2	0.3	0.4	0.5
0.1	41.94327	67.14327	92.34327	117.5433	142.7433
0.2	52.08175	77.28175	102.4817	127.6817	152.8817
0.3	62.22022	87.42022	112.6202	137.8202	163.0202
0.4	72.3587	97.5587	122.7587	147.9587	173.1587
0.5	82.49717	107.6972	132.8972	158.0972	183.2972

Table 4: Comparison of h_r & h_d

The results, as presented in Table 4 and illustrated in Figure 4, indicate that as both h_r & h_d increase, there is a noticeable increase in the total cost. This suggests a direct relationship between higher reliability and demand levels and increased operational costs in the perishable inventory system.

Figure 4: Comparison of h_r & h_d

k_r/k_d	0.1	0.2	0.3	0.4	0.5
0.1	435.2638	435.2676	435.2714	435.2753	435.2791
0.2	435.257	435.2608	435.2647	435.2685	435.2723
0.3	435.2502	435.254	435.2579	435.2617	435.2655
0.4	435.2434	435.2472	435.2511	435.2549	435.2588
0.5	435.2366	435.2404	435.2443	435.2481	435.252

Table 5: Comparison of k_r & k_d

It was observed that as both k_r & k_d increased, the total cost exhibited a consistent decrease trend, as illustrated in Figure 5, which corresponds to the data presented in Table 5.

Figure 5: Comparison of k_r & k_d

α/β	0.1	0.2	0.3	0.4	0.5
0.1	428.989	429.1205	429.252	429.3835	429.5151
0.2	429.1199	429.2514	429.3829	429.5144	429.646
0.3	429.2508	429.3823	429.5138	429.6453	429.7769
0.4	429.3817	429.5132	429.6447	429.7762	429.9077
0.5	429.5126	429.6441	429.7756	429.9071	430.0386

Table 6: Comparison of α & β

Table 6. clearly demonstrates that as both α & β increase, the total cost also increases. This relationship is further illustrated in Figure 6, where the data from Table 6. is graphically represented, reaffirming the positive correlation between α, β, and total cost.

Figure 6: Comparison of α & β

α/μ	0.1	0.2	0.3	0.4	0.5
0.1	428.76	429.004	429.252	429.5043	429.7608
0.2	428.8903	429.1345	429.3829	429.6355	429.8924
0.3	429.0205	429.2651	429.5138	429.7667	430.024
0.4	429.1508	429.3957	429.6447	429.898	430.1556
0.5	429.2811	429.5263	429.7756	430.0292	430.2872

Table 7: Comparison of α & μ

The cost value increases as the values of α & μ increase, according to Table 7.

Table 7 and Figure 7 show that the total expected cost rate increases as the arrival rate (α) and deterioration rate (μ) increase. This is because a higher α and μ mean that the retailer needs to hold a higher inventory level to meet demand and to prevent perishable items from deteriorating, which increases the holding costs.

Figure 7: Comparison of α & μ

7 Conclusion

In this research paper, our primary focus is on analyzing a complex supply chain scenario that involves two distinct client categories. Within this supply chain, we implement a continuous review inventory control system to efficiently manage inventory levels. Our investigation aims to shed light on various aspects of this system, ranging from its overall performance metrics to the steady-state probability distribution of different system states.

To provide a comprehensive understanding of the subject matter, we begin by outlining the key elements of our study. We delve into the intricacies of this supply chain to extract valuable insights and metrics that can assist in optimizing inventory management for businesses facing similar challenges.

One crucial aspect of our study is to quantify the system's performance. We will employ a variety of performance indicators and metrics to evaluate how effectively the supply chain functions under different conditions. These measurements will provide valuable insights into the efficiency, reliability, and responsiveness of the inventory control system.

Additionally, we will delve into the steady-state probability distribution of system states. This analysis will help us gain a deeper understanding of the system's behavior over time. By examining the probability distribution, we can identify potential bottlenecks, vulnerabilities, or areas for improvement within the supply chain.

As part of our research, we will also provide practical demonstrations of the suggested model. These numerical examples will illustrate the real-world applicability of our approach and offer valuable insights for practitioners and decision-makers in the field of supply chain management.

However, our research does not stop here. We have plans for further expansion and refinement of our model. In the future, we intend to incorporate perishable goods into the equation, taking into account their unique characteristics and handling requirements. This expansion will enable us to address the complexities associated with managing goods that have limited shelf lives.

Furthermore, we will consider partial backlogging as a crucial factor in our extended model. This addition will allow us to tackle situations where clients may not receive their orders in full, and how this partial backlog impacts inventory decisions and overall system performance.

Looking even further ahead, we have a vision to encompass a broader range of products in our analytical framework. This will involve integrating both perishable and non-perishable products into the same supply chain, a scenario that presents its own set of challenges and opportunities. By doing so, we aim to provide a more holistic and versatile approach to inventory management within complex supply chains.

In summary, our research journey begins with a detailed analysis of a supply chain involving two client categories and a continuous review inventory control system. Through empirical measurement, probability analysis, and practical examples, we aim to provide valuable insights for supply chain practitioners. Moreover, our commitment to continuous improvement drives us to expand our model to incorporate perishable goods, partial backlogging, and a wider product range, thereby enhancing its applicability and relevance in the ever-evolving world of supply chain management.

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Compatible mappings and its variants satisfying generalized $(\psi, \phi)-$weak contraction

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Abstract

Banach contraction principle behaves as a mathematical tool to solve various practical problems arising during mathematical formulation of many theoretical problems. In present work, the existence of a unique common fixed point for pairs of minimal commutative mappings is discussed, which satisfy a generalized $(\psi, \phi)-$weak contraction involving cubic terms of distance functions. Examples are given in support of the obtained results and as an application the existence of solution of system of certain functional equations arising in dynamic programming is discussed.

Keywords: generalized (ψ, ϕ)-weak contraction; compatible mappings; minimal commutative mappings, functional equations.
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1 Introduction

Fixed point theory is an important tool in nonlinear analysis. Its importance lies in finding solutions of many problems of applied sciences, engineering and economics. Banach contraction principle [2] is the versatile result of fixed point theory which ensures the existence and uniqueness of a fixed point for every contraction mapping defined on a complete metric space. It is based on iteration process.

Over a hundred years, researchers have been making efforts to extend, generalize and improve the Banach fixed point theorem in various directions. Jungck [15] was the first to prove a common fixed point theorem for a pair of commuting mappings. This theorem opened the door for researchers to generalize the Banach contraction principle for pair/pairs of mappings. Sessa [28] made an attempt to relax commutative condition of mappings to weak commutative condition .

Further, Jungck [16] weakened the notion of commutative/weak commutative mappings to compatible mappings. In 1993, Jungck, Murthy and Cho [17] generalized the notion of compatible mappings to compatible mappings of type (A). The process of generalizing the concept of compatible mappings is still going on. Pathak and Khan [23], Pathak *et al.* [24, 25], Rohen and Singh [27], Singh and Singh [30] and Jha *et al.* [13] weakened this concept of compatible mappings to compatible mappings of type (B), type (P), type (C), type (R), type (E) and type (K) respectively. One can call the variants of compatible mappings as minimal commutative mappings.

In 1971, Ćirić [8] generalized Banach contraction principle for a self map defined on a metric space $(\mathcal{X}, d)$ (say) satisfying the following condition:

$$d(Tu, Tv) \leq k \max\{d(u, v), d(u, Tu), d(v, Tv), \frac{1}{2}[d(u, Tv) + d(v, Tu)]\}, \quad (1)$$

$0 < k < 1$.

In 2005, Singh and Jain [29] proved a fixed point theorem for pairs of compatible mappings along with weakly compatible mappings satisfying the contractive condition of type (1) as follows.

Theorem 1.1. *[29] Let $(\mathcal{X}, d)$ be a complete metric space and let f, g, L, M, S and T be self mappings on $\mathcal{X}$ such that*

(H_1) *$L(\mathcal{X}) \subset ST(\mathcal{X})$, $M(\mathcal{X}) \subset fg(\mathcal{X})$;*

(H_2) *$ST = TS$, $Lg = gL$, $fg = gf$, $MT = TM$;*

(H_3) *either fg or L is continuous;*

(H_4) *the pair (M, ST) is weakly compatible and the pair (L, fg) is compatible;*

(H_5) *for all* $u, v \in \mathcal{X}$ *and for some* k *,* $0 < k < 1$*,*

$$d(Lu, Mv) \leq k \max\{d(Lu, fgv), d(Mv, STv), d(fgu, STv), \frac{1}{2}[d(Lu, STv) + d(Lv, fgu)]\}.$$

Then L, M, S, T, f *and* g *have a unique common fixed point.*

In 1984, Khan *et al.* [20] gave the idea of altering distance/ control function as follows. An altering distance is an increasing and continuous function $\phi : [0, \infty) \to [0, \infty)$ vanishing only at zero.

Many researchers used the notion of control function having various properties to generalize Banach contraction principle. In this direction, Boyd and Wong [7] introduced ϕ contraction of the form $d(Tu, Tv) \leq \phi(d(u, v))$ for all $u, v \in \mathcal{X}$, where T is a self mapping defined on a complete metric space $\mathcal{X}$ and $\phi : [0, \infty) \to [0, \infty)$ is an upper semi continuous function from right such that $0 \leq \phi(t) < t$ for all $t > 0$. In 1997, Alber and Guerre-Delabriere [1] introduced $\phi-$weak contraction to generalize ϕ contraction in Hilbert spaces.Further, Rhoades [26] extended the result of [1] in the setting of complete metric space using the following contraction.

A self mapping T of a complete metric space $(\mathcal{X}, d)$ is said to be a $\phi-$ weak contraction if for each $u, v \in \mathcal{X}$, there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) > 0$, for all $t > 0$ and $\phi(0) = 0$ such that

$$d(Tu, Tv) \leq d(u, v) - \phi(d(u, v)). \tag{2}$$

Dutta and Chaudhary [9] proved fixed point theorem for a self map satisfying $(\psi, \phi)-$weak contractive condition as follows.

Theorem 1.2. *Let* T *be a self mapping of a complete metric space* $\mathcal{X}$ *satisfying*

$$\psi(d(Tu, Tv)) \leq \psi(d(u, v)) - \phi(d(u, v)),$$

for all $u, v \in \mathcal{X}$ *and for some* ψ, ϕ*, where* $\phi, \psi : [0, +\infty) \to [0, +\infty)$ *such that* ϕ *is a lower semi continuous function and* $\phi^{-1}(0) = 0$ *and* ψ *is a non-decreasing continuous function with* $\psi^{-1}(0) = 0$*, then* T *has a unique fixed point in* $\mathcal{X}$*.*

In 2013, Murthy and Prasad [21] proved a fixed point theorem for a map satisfying a weak contraction involving cubic terms of distance functions.

Theorem 1.3. *[21] Let* T *be a self mapping of a complete metric space* $(\mathcal{X}, d)$ *and* $\phi : [0, \infty) \to [0, \infty)$ *be a continuous function with* $\phi(0) = 0$ *and* $\phi(t) > 0$*,*

for each $t > 0$ *such that*

$$[1 + pd(u,v)]d^2(Tu,Tv) \leq p \max \Big\{ \frac{1}{2}[d^2(u,Tu)d(v,Tv) + d(u,Tu)d^2(v,Tv)], \\ d(u,Tu)d(u,Tv)d(v,Tu), \\ d(u,Tv)d(v,Tu)d(v,Tv) \Big\} \\ + m(u,v) - \phi(m(u,v)), \tag{3}$$

where

$$m(u,v) = \max \Big\{ d^2(u,v), d(u,Tu)d(v,Tv), d(u,Tv)d(v,Tu), \\ \frac{1}{2}[d(u,Tu)d(u,Tv) + d(v,Tu)d(v,Tv)] \Big\}, \tag{4}$$

p *is a non negative real number. Then, map* T *has a unique fixed point in* $\mathcal{X}$.

In 2022, Kavita and Kumar [19] introduced a generalized $(\psi, \phi)-$weak contraction involving cubic terms of distance functions and generalized the weak contraction (3).

Motivated by the result of Singh and Jain [29], we establish the existence and uniqueness of a fixed point for pairs of compatible mappings and variants of compatible mappings (type (A), type (B), type (C), type (P), type (R), type (K), type (E)) satisfying the generalized $(\psi, \phi)-$weak contraction involving cubic terms of distance functions. These results generalize the results of Jain *et al.*[10, 11, 12], Jung *et al.*[14], Kang *et al.*[18], Murthy and Prasad [21], Pathak *et al.* [24] and various results presented in the literature.

2 Preliminaries

First, we recall some definitions which will be needed in the sequel.

Definition 2.1. *Let* $(\mathcal{X}, d)$ *be a metric space. A pair* (S, T) *of self mappings defined on* $\mathcal{X}$ *is said to be*

(i) compatible [16] if and only if

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0,$$

(ii) compatible of type (A) *[17] if*

$$\lim_{n\to\infty} d(SSu_n, TSu_n) = 0 \text{ and } \lim_{n\to\infty} d(TTu_n, STu_n) = 0,$$

(iii) compatible of type (P) *[23] if*

$$\lim_{n\to\infty} d(SSu_n, TTu_n) = 0,$$

(iv) compatible of type (B) *[23] if*

$$\lim_{n\to\infty} d(STu_n, TTu_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(STu_n, Sz) + \lim_{n\to\infty} d(Sz, SSu_n)],$$

and

$$\lim_{n\to\infty} d(TSu_n, SSu_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(TSu_n, Tz) + \lim_{n\to\infty} d(Tz, TTu_n)],$$

(v) compatible of type (C) *[25] if*

$$\lim_{n\to\infty} d(STu_n, TTu_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(STu_n, Sz) + \lim_{n\to\infty} d(Sz, SSu_n) + \lim_{n\to\infty} d(Sz, TTu_n)]$$

and

$$\lim_{n\to\infty} d(TSu_n, SSu_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(TSu_n, Tz) + \lim_{n\to\infty} d(Tz, TTu_n) + \lim_{n\to\infty} d(Tz, SSu_n)],$$

(vi) compatible of type (R) *[27] if*

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0$$

and

$$\lim_{n\to\infty} d(SSu_n, TTu_n) = 0,$$

(vii) compatible of type (E) *[30] if*

$$\lim_{n\to\infty} SSu_n = \lim_{n\to\infty} STu_n = Tz$$

and

$$\lim_{n\to\infty} TTu_n = \lim_{n\to\infty} TSu_n = Sz,$$

(viii) compatible of type (K) *[13] if*

$$\lim_{n\to\infty} SSu_n = Tz \text{ and } \lim_{n\to\infty} TTu_n = Sz,$$

whenever $\{u_n\}$ *is a sequence in* $\mathcal{X}$ *such that* $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$*, for some* $z \in \mathcal{X}$.

Now, we highlight the relationship among various types of compatible mappings.

Remark 2.1. *[6] Notions of compatible mappings and its variants are independent to each other. Although, all types of compatible mappings are equivalent to each other when one consider continuity of all mappings.*

Remark 2.2. *Notion of compatible mappings of type* (R) *is a combination of the notion of compatible mappings and compatible mappings of type* (P)*, but it is stronger than compatible mappings and compatible mappings of type* (P)*, see ([27, Example 1-2]).*

Remark 2.3. *If* $Sz = Tz$*, then compatible of type* (E) *implies compatible, compatible of type* (A)*, type* (B)*, type* (C) *and type* (P)*, however the converse may not true (see [31, Example 2.4]).*

Remark 2.4. *If* $Sz \neq Tz$*, then compatible of type* (E) *is neither compatible nor compatible of type* (A)*,type* (C)*, type* (P)*, (see [31, Example 2.3]).*

In 1998, Pant [22] introduced the notion of reciprocal continuous mappings as follows.

Definition 2.2. *[22] A pair* (S, T) *of self mappings of a metric space* $(\mathcal{X}, d)$ *is said to be reciprocal continuous, if* $\lim_{n\to\infty} STu_n = Sz$ *and* $\lim_{n\to\infty} TSu_n = Tz$*, whenever* $\{u_n\}$ *is a sequence in* $\mathcal{X}$ *such that* $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$*, for some* $z \in \mathcal{X}$.

Remark 2.5. *It is clear that a pair of continuous self mappings is reciprocal continuous, but the converse may not true (see [22]).*

In 2011, Singh and Singh [31] split the concept of compatible mappings of type (E) to the concept of $S-$compatible mappings of type (E) and $T-$ compatible mappings of type (E) and further, split the notion of reciprocal continuous to the notion of $S-$reciprocal continuous and $T-$reciprocal continuous.

Definition 2.3. *[31] Let* $(\mathcal{X}, d)$ *be a metric space and* $S, T : \mathcal{X} \to \mathcal{X}$ *be two mappings. The pair* (S, T) *is said to be*

(i) $S-$compatible of type (E), if $\lim_{n\to\infty} SSu_n = \lim_{n\to\infty} STu_n = Tz$,

(ii) $T-$compatible type (E), if $\lim_{n\to\infty} TTu_n = \lim_{n\to\infty} TSu_n = Sz$,

(iii) $S-$ reciprocal continuous, if $\lim_{n\to\infty} STu_n = Sz$,

(iv) $T-$ reciprocal continuous, if $\lim_{n\to\infty} TSu_n = Tz$,

whenever $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in \mathcal{X}$,

Remark 2.6. *Compatible of type (E) implies both $S-$compatible of type (E) and $T-$compatible of type (E), however, the converse may not true, see the example given below.*

Example 2.1. *Let $\mathcal{X} = [0,5]$ and d be a usual metric. Let $S, T : \mathcal{X} \to \mathcal{X}$ be two mappings defined as $Su = 5, Tu = 1$, for $u \in [0, \frac{5}{2}] - \{\frac{5}{4}\}$, $Su = 0, Tu = 5$, for $u = \frac{5}{4}$ and $Su = \frac{5-u}{2}, Tu = \frac{u}{2}$, for $u \in (\frac{5}{2}, 5]$. Clearly, S and T are not continuous at $u = \frac{5}{2}, \frac{5}{4}$. Let us assume that $u_n \to \frac{5}{2}$, $u_n > \frac{5}{2}$, for all n. Then, $Su_n = \frac{5-u_n}{2} \to \frac{5}{4} = t$ and $Tu_n = \frac{u_n}{2} \to \frac{5}{4} = t$. Therefore, we have $SSu_n = S(\frac{5-u_n}{2}) = 5 \to 5$, $STu_n = S(\frac{u_n}{2}) = 5 \to 5, Tt = 5$ and $TTu_n = T(\frac{u_n}{2}) = 1 \to 1$, $TSu_n = T(\frac{5-u_n}{2}) = 1 \to 1$, $St = 0$. Thus, the pair (S,T) is $S-$compatible of type (E), but not compatible of type (E).*

Remark 2.7. *The reciprocal continuity of the pair (S,T) implies both $S-$reciprocal continuity and $T-$ reciprocal continuity, however, the converse may not true, see example given below.*

Example 2.2. *Let $\mathcal{X} = [0,5]$ and d be a usual metric. Let $S, T : \mathcal{X} \to \mathcal{X}$ be two mappings defined as $Su = 5, Tu = 0$ for $u \in [0, \frac{5}{2})$ and $Su = 5 - u, Tu = u$, for $u \in [\frac{5}{2}, 5]$. Let $\{u_n\}$ be a sequence in $\mathcal{X}$ such that $u_n \to \frac{5}{2}$, $u > \frac{5}{2}$, for all n. Then $Su_n = 5 - u_n \to \frac{5}{2}$, $Tu_n = u_n \to \frac{5}{2} = t$, $STu_n = S(u_n) = 5 - u_n \to \frac{5}{2}$, $St = \frac{5}{2}$ and $TSu_n = T(5 - u_n) = 0 \to 0$, $Tt = \frac{5}{2}$. It follows that $\lim_{n\to\infty} STu_n = \frac{5}{2} = St$ and $\lim_{n\to\infty} TSu_n = 0 \neq Tt = \frac{5}{2}$. Therefore, the pair (S,T) is $S-$reciprocal continuous, but it is neither $T-$reciprocal continuous nor reciprocal continuous.*

Now, we present some prepositions which are useful for our work.
Let$(\mathcal{X}, d)$ be a metric space and (S,T) be a pair of self mappings of $\mathcal{X}$.

Proposition 2.1. *[17] Suppose the pair (S,T) is compatible of type (A). If either S or T is continuous, then (S,T) is compatible.*

Proposition 2.2. *[16, 27] Suppose the pair (S,T) is compatible or compatible of type (R). If*

(i) $Sz = Tz$, *then* $STz = SSz = TTz = TSz$, *for some* $z \in \mathcal{X}$.

(ii) $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, *for some* $z \in \mathcal{X}$, *then*

- *(a)* $\lim_{n\to\infty} TSu_n = Sz$, *if* S *is continuous at* z,
- *(b)* $\lim_{n\to\infty} STu_n = Tz$, *if* T *is continuous at* z,
- *(c)* $STz = TSz$ *and* $Sz = Tz$, *if* S *and* T *are continuous at* z.

Proposition 2.3. *[23, 24, 25] Suppose the pair (S,T) is compatible of type (B) or type (C) or type (P) on $\mathcal{X}$. If for some z in $\mathcal{X}$,*

(i) $Sz = Tz$, *then* $STz = SSz = TTz = TSz$.

(ii) $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, *then*

- *(a)* $\lim_{n\to\infty} TTu_n = Sz$, *if* S *is continuous at* z;
- *(b)* $\lim_{n\to\infty} SSu_n = Tz$, *if* T *is continuous at* z;
- *(c)* $STz = TSz$ *and* $Sz = Tz$, *if* S *and* T *are continuous at* z.

Proposition 2.4. *[31] Let $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in \mathcal{X}$. If one of the following conditions is satisfied:*

(i) the pair (S,T) is $S-$compatible of type (E) and $S-$ reciprocally continuous,

(ii) the pair (S,T) is $T-$compatible of type (E) and $T-$ reciprocally continuous,

Then (a) $Sz = Tz$ and (b) if there exists $t \in \mathcal{X}$ such that $St = Tt = z$, then $STt = TSt$.

3 Main Results

In this section, we establish the existence and uniqueness of fixed point for pairs of minimal commutative mappings satisfying generalized $(\psi,\phi)-$weak contraction involving cubic terms of distance functions, where $\psi \in \Psi$ and $\phi \in \Phi$,

Ψ is a collection of all non decreasing, upper semi continuous (in each coordinate variables) functions $\psi : [0,\infty)^4 \to [0,\infty)$ such that

$$\max\{\psi(t,t,0,0), \psi(0,0,0,t), \psi(0,0,t,0), \psi(t,t,t,t)\} \leq t, \text{ for each } t > 0.$$

and Φ is a collection of all continuous functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) > 0$ for each $t > 0$ and $\phi(0) = 0$.

Let $(\mathcal{X}, d)$ be a metric space and f, g, L, M, S and T be self mappings defined on $\mathcal{X}$ satisfying the following conditions:

(I) $f(\mathcal{X}) \subset LM(\mathcal{X})$ and $g(\mathcal{X}) \subset ST(\mathcal{X})$,

(II) $LM = ML$, $ST = TS$, $fT = Tf$ and $gM = Mg$,

(III) one of f, g, LM and ST is continuous,

(IV) for all $u, v \in \mathcal{X}$, there exists a function $\psi \in \Psi$, a function $\phi \in \Phi$ and a real number $p \geq 0$ such that

$$\begin{aligned}[1 + pd(STu, LMv)]d^2(fu, gv) \leq p\psi\Big(&d^2(STu, fu)d(LMv, gv),\\ &d(STu, fu)d^2(LMv, gv),\\ &d(STu, fu)d(STu, gv)d(LMv, fu),\\ &d(STu, gv)d(LMv, fu)d(LMv, gv)\Big)\\ &+m(STu, LMv) - \phi(m(STu, LMv)),\end{aligned} \tag{5}$$

where

$$\begin{aligned}m(STu, LMv) = \max\Big\{&d^2(STu, LMv),\\ &d(STu, fu)d(LMv, gv),\\ &d(STu, gv)d(LMv, fu),\\ &\frac{1}{2}[d(STu, fu)d(STu, gv) + d(LMv, fu)d(LMv, gv)]\Big\}.\end{aligned} \tag{6}$$

Let $u_0 \in \mathcal{X}$ be arbitrary point. Using condition (I), one can find $u_1, u_2 \in \mathcal{X}$ such that $fu_0 = LMu_1 = v_0$ and $gu_1 = STu_2 = v_1$. Continuing in this manner, one can construct sequences such that

$$v_{2n} = fu_{2n} = LMu_{2n+1} \quad \text{and} \quad v_{2n+1} = gu_{2n+1} = STu_{2n+2}, \tag{7}$$

for each $n = 0, 1, 2, 3 \ldots$

First, we prove a fixed point theorem for pairs of compatible mappings.

Theorem 3.1. *Let f, g, L, M, S and T be self mappings defined on a complete metric space $\mathcal{X}$ satisfying the conditions (I)-(IV). If pairs (f, ST) and (g, LM) are compatible, then f, g, L, M, S and T have a unique common fixed point in $\mathcal{X}$.*

Proof. In the view of [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is a complete metric space, so the sequence $\{v_n\}$ converges to a point, say, $z \in \mathcal{X}$, as $n \to \infty$. Consequently, the sub sequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point $z \in \mathcal{X}$.

Case (i) If f is continuous, then $\{ffu_{2n}\}$ and $\{f(ST)u_{2n}\}$ converges to fz as $n \to \infty$. Since the pairs (f, ST) is compatible on $\mathcal{X}$, it follows from the Proposition 2.2(ii) that $\{(ST)fu_{2n}\}$ converges to fz as $n \to \infty$.

Step 1. We claim that $z = fz$.Taking $u = fu_{2n}$ and $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we have

$$[1 + pd(fz, z)]d^2(fz, z) \leq p\psi(0, 0, 0, 0) + m(fz, z) - \phi(m(fz, z)),$$

where

$$\begin{aligned} m(fz, z) = \max\Big\{ & d^2(fz, z), d(fz, z)d(z, z), d(fz, z)d(z, fz), \\ & \frac{1}{2}[d(fz, fz)d(fz, z) + d(z, fz)d(z, z)]\Big\} \\ = {} & d^2(fz,z). \end{aligned}$$

On simplifying, we have $d^2(fz, z) = 0$, i.e., $fz = z$. Since $f(\mathcal{X}) \subset LM(\mathcal{X})$, there exists a point $w \in \mathcal{X}$ such that $z = fz = LMw$.

Step 2. Taking $u = fu_{2n}$ and $v = w$ in (5) and (6) and letting $n \to \infty$, we get

$$\begin{aligned} [1 + pd(z, z)]d^2(z, gw) \leq p\psi\Bigg(& d^2(z, z)d(z, gw), d(z, z)d^2(z, gw), \\ & d(z, z)d(z, gw)d(z, z), d(z, gw)d(z, z)d(z, gw)\Bigg) \\ & + m(z, z) - \phi(m(z, z)), \end{aligned}$$

where

$$\begin{aligned} m(z, z) = \max\Bigg\{ & d^2(z, z), d(z, z)d(z, gw), d(z, gw)d(z, z), \\ & \frac{1}{2}\big[d(z, z)d(z, gw) + d(z, z)d(z, gw)\big]\Bigg\} = 0. \end{aligned}$$

Using the property of ϕ, ψ in the above inequality, we have $d^2(z, gw) = 0$, i.e., $gw = z$. Since the pair (g, LM) is compatible on $\mathcal{X}$ and $LMw = gw = z$, therefore, by Proposition 2.2(i), we have $(LM)gw = g(LM)w$ and hence, $LMz = (LM)gw = g(LM)w = gz$.

Step 3. Taking $u = u_{2n}$ and $v = z$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(z, gz)]d^2(z, gz) \leq p\psi(0, 0, 0, 0) + m(z, gz) - \phi(m(z, gz)),$$

where

$$m(z, gz) = \max\Big\{d^2(z, gz), d(z, z)d(LMz, gz), d(z, gz)d(gz, z), \\ \frac{1}{2}[d(z, z)d(z, gz) + d(LMz, z)d(gz, gz)]\Big\} = d^2(z, gz).$$

On solving, we get $z = gz$. Since $g(\mathcal{X}) \subset ST(\mathcal{X})$, therefore, for this z there exists a point $x \in \mathcal{X}$ such that $z = gz = STx$.

Step 4. We claim that $z = fx$. For this, putting $u = x$ and $v = z$ in (5) and (6), we get

$$[1 + pd(STx, LMz)]d^2(fx, gz) \leq p\psi\Bigg(d^2(STx, fx)d(LMz, gz), \\ d(STx, fx)d^2(LMz, gz), \\ d(STx, fx)d(STx, gz)d(LMz, fx), \\ d(STx, gz)d(LMz, fx)d(LMz, gz)\Bigg) \\ +m(STx, LMz) - \phi\big(m(STx, LMz)\big),$$

where

$$m(STx, LMz) = \max\Big\{d^2(STx, LMz), d(STx, fx)d(LMz, gz), \\ d(STx, gz)d(LMz, fx), \frac{1}{2}\big[d(STx, fx)d(STx, gz) \\ + d(LMz, fx)d(LMz, gz)\big]\Big\} = 0.$$

After simplification, we conclude that $d^2(fx, z) = 0$. This gives that $fx = z$.

Since (f, ST) is compatible on $\mathcal{X}$ and $fx = STx = z$, therefore, by Proposition 2.2(i), we have $(ST)fx = f(ST)x$, i.e.,$STz = (ST)fx = f(ST)x = fz$.

Step 5. We claim that $Tz = z$. Substituting $u = Tz$, $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we have

$$[1 + pd((ST)Tz, z)]d^2(fTz, z) \leq p\psi(0, 0, 0, 0) + m((ST)Tz, z) - \phi(m((ST)Tz, z)),$$

where

$$m((ST)Tz, z) = \max\Big\{d^2((ST)Tz, z), d((ST)Tz, fTz)d(z, z), d((ST)Tz, z)d(z, fTz), \frac{1}{2}[d((ST)Tz, fTz)d((ST)Tz, z) + d(z, fTz)d(z, z)]\Big\} = d^2((ST)Tz, z).$$

Since $ST = TS$, $STz = z$, $fT = Tf$, $fz = z$, so $(ST)Tz = (TS)Tz = T(STz) = Tz$ and $fTz = Tfz = Tz$.

On Simplifying, the above inequality reduces to

$$pd^3(Tz, z) + \phi(d^2(Tz, z)) \leq 0,$$

which holds only for $d(Tz, z) = 0$, i.e., $Tz = z$. Also, $STz = z$ implies that $Sz = z$.

Step 6. Next we show that $Mz = z$. Substituting $u = u_{2n}$ $v = Mz$ in (5) and (6) and letting $n \to \infty$, we have

$$[1 + pd(z, (LM)Mz)]d^2(z, gMz) \leq p\psi(0, 0, 0, 0) + m(z, (LM)Mz) - \phi(m(z, (LM)Mz),$$

where

$$m(z, (LM)Mz) = \max\Big\{d^2(z, (LM)Mz), d(z, z)d((LM)Mz, gMz), d(z, gMz)d((LM)Mz, z), \frac{1}{2}[d(z, z)d(z, gMz) + d((LM)Mz, z)d((LM)Mz, gMz)]\Big\} = d^2(z, (LM)Mz).$$

Since $LM = ML$, $LMz = z$, $gM = Mg$, $gz = z$, so $(LM)Mz = (ML)Mz = M(LMz) = Mz$ and $gMz = Mgz = Mz$.

On simplifying, we get

$$pd^3(z, Mz) + \phi(d^2(z, Mz)) \leq 0,$$

which holds only for $d(z, Mz) = 0$, i.e., $Mz = z$. Also, $LMz = z$ implies that $Lz = z$. Thus, z is common fixed point of f, g, S, T, L and M.

Case (ii) If g is continuous, one can complete the proof on the similar lines of case (i).

Case(iii) If ST is continuous. Then $\{(ST)(ST)u_{2n}\}$ and $\{(ST)fu_{2n}\}$ converges to STz as $n \to \infty$. Since the mappings ST and f are compatible on $\mathcal{X}$, it follows from the Proposition 2.2(ii) that $\{f(ST)u_{2n}\}$ converges to STz as $n \to \infty$.

Next, we claim that $STz = z$. For this, substituting $u = STu_{2n}$ and $v = u_{2n+1}$ in equation (5) and inequality (6) and letting $n \to \infty$ with the property of ϕ and ψ, we get

$$[1 + pd(STz, z)]d^2(STz, z) \leq p\psi(0, 0, 0, 0) + m(STz, z) - \phi(m(STz, z)),$$

where

$$\begin{aligned} m(STz, z) = \max \Big\{ & d^2(STz, z), d(STz, STz)d(z, z), d(STz, z)d(z, STz), \\ & \frac{1}{2}[d(STz, STz)d(STz, z) + d(z, STz)d(z, z)] \Big\} \\ & = d^2(STz, z). \end{aligned}$$

Using the value of $m(STz, z)$ along with the property of ϕ and ψ, the above inequality reduces to $d^2(STz, z) = 0$, i.e., $STz = z$.

Next, we show that $fz = z$. Taking $u = z$ and $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we get

$$\begin{aligned} [1 + pd(STz, z)]d^2(fz, z) \leq p\psi\Big(& d^2(STz, fz)d(z, z), d(STz, fz)d^2(z, z), \\ & d(STz, Sz)d(STz, z)d(z, fz), \\ & d(STz, z)d(z, fz)d(z, z)\Big) \\ & + m(STz, z) - \phi\big(m(STz, z)\big), \end{aligned}$$

where

$$\begin{aligned} m(STz, z) = \max \Big\{ & d^2(STz, z), d(STz, fz)d(z, z), d(STz, z)d(z, fz), \\ & \frac{1}{2}\big[d(STz, fz)d(STz, z) + d(z, fz)d(z, z)\big] \Big\} = 0. \end{aligned}$$

After simplification, one gets $d^2(fz, z) = 0$. This implies that $fz = z$. Therefore, $STz = z = fz$. Since $f(\mathcal{X}) \subset LM(\mathcal{X})$, there exists a point $w \in \mathcal{X}$ such that $z = fz = LMw$.

Now, we claim that $gw = z$. Substituting $u = z$ and $v = w$ in (5) and (6), we have

$$\begin{aligned}[1 + pd(STz, LMw)]d^2(fz, gw) \leq p\psi\Big(&d^2(STz, fz)d(LMw, gw),\\ &d(STz, fz)d^2(LMw, gw),\\ &d(STz, fz)d(STz, gw)d(LMw, fz),\\ &d(STz, gw)d(LMw, fz)d(LMw, gw)\Big)\\ &+m(STz, LMw) - \phi\big(m(STz, LMw)\big),\end{aligned}$$

where

$$\begin{aligned}m(STz, LMw) = \max\Big\{&d^2(STz, LMw), d(STz, fz)d(LMw, gw),\\ &d(STz, gw)d(LMw, fz), \frac{1}{2}\big[d(STz, fz)d(STz, gw)\\ &+ d(LMw, fz)d(LMw, gw)\big]\Big\}\\ = \max\Big\{&d^2(z, z), d(z, z)d(z, gw), d(z, gw)d(z, z),\\ &\frac{1}{2}\big[d(z, z)d(z, gw) + d(z, z)d(z, gw)\big]\Big\} = 0.\end{aligned}$$

After simplification, we conclude that $gw = z$.

Since (g, LM) is a pair of compatible mappings in $\mathcal{X}$ and $LMw = z = gw$, therefore, by Proposition 2.2(i), we have $(LM)gw = g(LM)w$ and hence, $LMz = (LM)gw = g(LM)w = gz$.

Now, we claim that z is a fixed point of LM and g. For this, putting $u = v = z$ in (5) and (6), we have

$$\begin{aligned}[1 + pd(STz, LMz)]d^2(fz, gz) \leq p\psi\Big(&d^2(STz, fz)d(LMz, gz),\\ &d(STz, fz)d^2(LMz, gz),\\ &d(STz, Sz)d(STz, gz)d(LMz, fz),\\ &d(STz, gz)d(LMz, fz)d(LMz, gz)\Big)\\ &+m(STz, LMz) - \phi\big(m(STz, LMz)\big),\end{aligned}$$

where

$$m(STz, LMz) = \max\Big\{d^2(STz, LMz), d(STz, fz)d(LMz, gz), \\ d(STz, gz)d(LMz, fz), \\ \frac{1}{2}\big[d(STz, fz)d(STz, gz) \\ + d(LMz, fz)d(LMz, gz)\big]\Big\} = d^2(z, LMz).$$

On solving, the above inequality reduces to

$$[1 + pd(z, LMz)]d^2(z, LMz) \leq p\psi(0, 0, 0, 0) + d^2(z, LMz) - \phi(d^2(z, LMz)),$$

which gives $d^2(z, LMz) = 0$ and hence, $z = LMz = gz$.

Following steps 5 and 6 of case (i), we have $Tz = z$, $Mz = z$. Also, $STz = z$, $LMz = z$ imply that $Sz = z$ and $Lz = z$.

Case(iv) Suppose mapping LM is continuous. Following case (iii), one may obtain the desired result.

Uniqueness: Suppose $w_1 \neq w_2$ be two common fixed point of f, g, S, T, L and M. Putting $u = w_1$ and $v = w_2$ in (5) and (6) and solving, we get

$$[1 + pd(w_1, w_2)]d^2(w_1, w_2) \leq p\psi(0, 0, 0, 0) + m(w_1, w_2) - \phi\big(m(w_1, w_2)\big),$$

where

$$m(w_1, w_2) == \max\Big\{d^2(w_1, w_2), d(w_1, w_1)d(w_2, w_2), d(w_1, w_2)d(w_2, w_1), \\ \frac{1}{2}\big[d(w_1, w_1)d(w_1, w_2) + d(w_2, w_1)d(w_2, w_2)\big]\Big\} \\ = d^2(w_1, w_2).$$

Simplifying the above inequality, we get $d(w_1, w_2) = 0$, i.e., $w_1 = w_2$. Hence, mappings f, g, S, T, L and M have a unique common fixed point in $\mathcal{X}$. □

Now, we establish a common fixed point theorem for pairs of compatible mappings of type (A).

Theorem 3.2. *Let f, g,L,M, S and T be self mappings defined on a complete metric space $\mathcal{X}$ satisfying the conditions (I)-(IV). If pairs (f, ST) and (g, LM) are compatible of type (A), then f, g, L, M, S and T have a unique common fixed point in $\mathcal{X}$.*

Proof. Suppose f is continuous on $\mathcal{X}$ and pairs (f, ST) is compatible of type (A). By Proposition 2.1, (f, ST) is compatible. Then result follows easily from Theorem 3.1.

If g is assumed to be continuous, then by Proposition 2.1, (g, LM) being compatible of type (A) is compatible also. Then result follows from Theorem 3.1

Similarly, assuming ST or LM continuous, one can get the required result. □

Next, we establish a common fixed point theorem for pairs of compatible mappings of type (B).

Theorem 3.3. *Let f, g, L, M, S and T be self mappings defined on a complete metric space $\mathcal{X}$ satisfying the conditions (I)-(IV). If pairs (f, ST) and (g, LM) are compatible of type (B), then f, g, L, M, S and T have a unique common fixed point in $\mathcal{X}$.*

Proof. Following [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$. $\mathcal{X}$ being a complete metric space, the sequence $\{v_n\}$ converges to a point, say, $z \in \mathcal{X}$, as $n \to \infty$. Consequently, the sub sequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point $z \in \mathcal{X}$.

Suppose g is continuous. Then $\{ggu_{2n+1}\}$ and $\{g(LM)u_{2n+1}\}$ converges to gz as $n \to \infty$. Since the pair (g, LM) is compatible of type (B), it follows from the Proposition 2.3(ii) that $\{(LM)(LM)u_{2n+1}\}$ converges to gz as $n \to \infty$.

Step 1. Now, we prove that $gz = z$. For this, taking $u = u_{2n}$ and $y = LMu_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(z, gz)]d^2(z, gz) \leq p\psi(0, 0, 0, 0) + m(z, gz) - \phi(m(z, gz)),$$

where

$$m(z, gz) = \max\Big\{d^2(z, gz), d(z, z)d(gz, gz), d(z, gz)d(gz, z), \frac{1}{2}[d(z, z)d(z, gz) + d(gz, z)d(gz, gz)]\Big\} = d^2(z, gz).$$

Solving the above inequality, we get $p\,d^3(z, gz) + \phi(d^2(z, gz)) \leq 0$, which is possible only if $d(gz, z) = 0$. This implies that $gz = z$. Since $g(\mathcal{X}) \subset ST(\mathcal{X})$, there exists a point $w \in \mathcal{X}$ such that $z = gz = STw$.

Step 2. We claim that $fw = w$. For this, substituting $u = w$ and $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(z, z)]d^2(fw, z) \leq p\,\psi(0, 0, 0, 0) + m(z, z) - \phi\big(m(z, z)\big),$$

where

$$m(z, z) = \max\left\{d^2(z, z), d(z, fw)d(z, z), d(z, z)d(z, fw), \frac{1}{2}\big[d(z, fw)d(z, z) + d(z, fw)d(z, z)\big]\right\} = 0.$$

Simplifying the above inequality, we get $d(fw, z) = 0$ i.e., $fw = z$. Since (f, ST) is compatible of type (B) and $fw = STw$, therefore, by Proposition 2.3(i), $f(ST)w = (ST)fw$. Hence, $STz = (ST)fw = f(ST)w = fz$.

Step 3. Taking $u = z$ and $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(fz, z)]d^2(fz, z) \leq p\,\psi(0, 0, 0, 0) + m(fz, z) - \phi\big(m(fz, z)\big),$$

where

$$m(fz, z) = \max\Big\{d^2(fz, z), d(fz, fz)d(z, z), d(fz, z)d(z, fz), \frac{1}{2}\big[d(fz, fz)d(fz, z) + d(z, fz)d(fz, z)\big]\Big\} = d^2(fz, z).$$

After simplification, we get $p\,d^3(fz, z) + \phi(d^2(fz, z)) \leq 0$, which is possible only if $d(fz, z) = 0$, i.e., $fz = z$. Since $f(\mathcal{X}) \subset LM(\mathcal{X})$, there exists a point $x \in \mathcal{X}$ such that $z = fz = LMx$.

Step 4. We claim that $gx = z$. For this, putting $u = u_{2n}$ and $v = x$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(z, z)]d^2(z, gx) \leq p\,\psi(0, 0, 0, 0) + m(z, z) - \phi\big(m(z, z)\big),$$

where

$$m(z, z) = \max\left\{d^2(z, z), d(z, z)d(z, gx), d(z, gx)d(z, z), \frac{1}{2}\big[d(z, z)d(z, gx) + d(z, z)d(z, gx)\big]\right\} = 0.$$

Solving the above inequality, we conclude that $d(z, gx) = 0$, i.e., $gx = z$. Since the pair (g, LM) is compatible of type (B) in $\mathcal{X}$ and $gx = LMx$, therefore, by Proposition 2.3(i), we have $(LM)gx = g(LM)x$ and hence, $LMz = (LM)gx = g(LM)x = gz = z$.

Step 5. Following steps 5 and 6 of case (i) of Theorem 3.1, we have $Lz = Mz = Tz = Sz = z$. Uniqueness follows easily. Thus, z is a unique common fixed point of S, T, L, M, f and g.

Similarly, one can complete the proof taking the mapping f or ST or LM to be continuous. □

Now, we discuss the existence and uniqueness of a common fixed point for pairs of compatible mappings of type (C).

Theorem 3.4. *Let f, g, L, M, S and T be self mappings defined on a complete metric space $\mathcal{X}$ satisfying the conditions (I)-(IV). If pairs (f, ST) and (g, LM) are compatible of type (C), then f, g, L, M, S and T have a unique common fixed point in $\mathcal{X}$.*

Proof. Following [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$ and $(\mathcal{X}, d)$ being a complete metric space the sequence $\{v_n\}$ converges to a point, say, $z \in \mathcal{X}$, as $n \to \infty$. Consequently, the sub sequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point z.

Assume that g is continuous. Then sequences $\{ggu_{2n+1}\}$ and $\{g(LM)u_{2n+1}\}$ converges to gz, as $n \to \infty$. Since the pair (g, LM) is compatible of type (C), it follows from the Proposition 2.3(ii) that $\{(LM)(LM)u_{2n+1}\}$ converges to gz, as $n \to \infty$.

Following steps 1-5 of Theorem 3.3, z is a unique common fixed point of S, T, L, M, f and g. Similarly, one can complete the proof, assuming T or ST or LM to be continuous. □

Now, we investigate the existence of a fixed point theorem for pairs of compatible mappings of type(P) as follows.

Theorem 3.5. *Let f, g, S, T, L and M be self mappings of a complete metric space $(\mathcal{X}, d)$. If (f, ST) and (g, LM) are the pairs of compatible mappings of type (P) satisfying conditions (I)-(IV), then all the mappings f, g, S, T, L and M have a unique common fixed point in $\mathcal{X}$.*

Proof. In the view of [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$ and $(\mathcal{X}, d)$ being a complete metric space sequence $\{v_n\}$ converges to a point, say, $z \in \mathcal{X}$, as $n \to \infty$. Consequently, the sub sequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point z.

Assume that g is continuous. Then sequences $\{ggu_{2n+1}\}$ and $\{g(LM)u_{2n+1}\}$ converges to gz, as $n \to \infty$. Since the pair (g, LM) is compatible of type (P), it follows from the Proposition 2.3(ii) that $\{(LM)(LM)u_{2n+1}\}$ converges to gz, as $n \to \infty$.

Rest of the proof follows on steps 1-5 of Theorem 3.3 □

Now, we study the existence of a unique fixed point for pairs of compatible mappings of type(R) as follows.

Theorem 3.6. *Let f, g, S, T, L and M be self mappings of a complete metric space $(\mathcal{X}, d)$. If (f, ST) and (g, LM) are the pairs of compatible mappings of type (R) satisfying conditions (I)-(IV), then all the mappings f, g, S, T, L and M have a unique common fixed point in $\mathcal{X}$.*

Proof. Following [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$ and $(\mathcal{X}, d)$ is a complete metric space, so the sequence $\{v_n\}$ converges to a point, say, $z \in \mathcal{X}$, as $n \to \infty$. Consequently, the sub sequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point $z \in \mathcal{X}$.

Assume that f is continuous. Then sequences $\{ffu_{2n}\}$ and $\{f(ST)u_{2n}\}$ converges to fz, as $n \to \infty$. Since the pair (f, ST) is compatible of type (R), it follows from the Proposition 2.2(ii) that $\{(ST)fu_{2n}\}$ converges to fz, as $n \to \infty$.

In the view of steps 1-6 of case (i) of Theorem 3.1, we conclude that z is a unique common fixed point of S, T, L, M, f and g. Assuming either of g, ST, LM to be continuous, the proof follows from cases (ii)-(iv) of Theorem 3.1 □

Now, we prove fixed point theorem for pairs of compatible mappings of type (K) as well as reciprocal continuous mappings.

Theorem 3.7. *Let f, g, L, M, S and T be self mappings of a complete metric space $(\mathcal{X}, d)$ satisfying the conditions (I), (II) and (IV). Then all the mappings f, g, L, M, S and T have a unique common fixed point in $\mathcal{X}$, provided that (f, ST) and (g, LM) are the pairs of reciprocal continuous mappings and compatible mappings of type (K).*

Proof. Following proof of [19, Theorem 2.3], the sequence $\{v_n\}$, defined by (7), is a Cauchy sequence in $\mathcal{X}$. Since $(\mathcal{X}, d)$ is a complete metric space, so the sequence $\{v_n\} \to w \in \mathcal{X}$, as $n \to \infty$. Consequently, the subsequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point $w \in \mathcal{X}$.

Since the pair (f, ST) is compatible of type (K), $(ST)(ST)u_{2n} \to fw$, $ffu_{2n} \to STw$ as $n \to \infty$. Also reciprocal continuity of the pair (f, ST) implies that $(ST)fu_{2n} \to STw$ and $f(ST)u_{2n} \to fw$ as $n \to \infty$.

Also, the pair (g, LM) is compatible of type (K) and reciprocal continuous, therefore, $(LM)(LM)u_{2n+1} \to gw$, $ggu_{2n+1} \to LMw$, $(LM)gu_{2n+1} \to LMw$ and $g(LM)u_{2n+1} \to gw$ as $n \to \infty$.

Now, we claim that $LMw = STw$. Taking $u = fu_{2n}$, $v = gu_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(STw, LMw)]d^2(STw, LMw) \leq p\psi(0, 0, 0, 0) + m(STw, LMw) - \phi(m(STw, LMw)),$$

where

$$\begin{aligned} m(STw, LMw) = \max \Big\{ & d^2(STw, LMw), d(STw, STw)d(LMw, LMw), \\ & d(STw, LMw)d(LMw, STw), \\ & \frac{1}{2}[d(STw, STw)d(STw, LMw) \\ & + d(LMw, STw)d(LMw, LMw)] \Big\} \\ = & d^2(LMw, STw). \end{aligned}$$

Solving the above inequality, we get $d(LMw, STw) = 0$, which implies that $LMw = STw$. Next, we prove that $LMw = fw$. Letting $u = w$ and $v = gu_{2n+1}$ in (5) and (6) and taking the limit as $n \to \infty$, we get

$$\begin{aligned} [1 + pd(STw, LMw)]d^2(fw, LMw) \leq p\psi\big(& d^2(STw, fw)d(LMw, LMw), \\ & d(STw, fw)d^2(LMw, LMw), \\ & d(STw, fw)d(STw, LMw)d(LMw, fw), \\ & d(STw, LMw)d(LMw, fw)d(LMw, LMw)\big) \\ & + m(STw, LMw) - \phi\big(m(STw, LMw)\big), \end{aligned}$$

where

$$\begin{aligned} m(STw, LMw) = \max \Big\{ & d^2(STw, LMw), d(STw, fw)d(LMw, LMw), \\ & d(STw, LMw)d(LMw, fw), \\ & \frac{1}{2}\big[d(STw, fw)d(STw, LMw) \\ & + d(LMw, fw)d(LMw, LMw)\big] \Big\} = 0. \end{aligned}$$

Simplifying the above inequality, we get $d(fw, LMw) = 0$, i.e., $fw = LMw$. So $STw = LMw = fw$. Next, we claim that $fw = gw$. Putting $u = v = w$ in (5) and (6), we have

$$\begin{aligned} [1 + pd(STw, LMw)]d^2(fw, gw) \leq p\psi\Big(& d^2(STw, fw)d(LMw, gw), \\ & d(STw, fw)d^2(LMw, gw), \\ & d(STw, fw)d(STw, gw)d(LMw, fw), \\ & d(STw, gw)d(LMw, fw)d(LMw, gw)\Big) \\ & + m(STw, LMw) - \phi\big(m(STw, LMw)\big), \end{aligned}$$

where

$$m(STw, LMw) = \max\Big\{d^2(STw, LMw), d(STw, fw)d(LMw, gw), \\ d(STw, gw)d(LMw, fw), \\ \frac{1}{2}\big[d(STw, fw)d(STw, gw) \\ + d(LMw, fw)d(LMw, gw)\big]\Big\} = 0.$$

On solving, we get $d^2(fw, gw) \leq 0$, which is true for $fw = gw$.
Hence, $LMw = gw = STw = fw$, i.e., w is a coincidence point of f, g, ST and LM. Taking $u = u_{2n}$ and $v = w$ in (5) and (6) and letting $n \to \infty$, we get

$$[1 + pd(w, LMw)]d^2(w, gw) \leq p\psi(0, 0, 0, 0) + m(w, LMw) - \phi(m(w, LMw)),$$

where
$m(w, LMw) = \max\{d^2(w, LMw), 0, d(w, gw)d(LMw, w), 0\} = d^2(w, LMw)$.

After simplification, we get $d(w, gw) = 0$, i.e., $w = gw$. In the view of steps 5 and 6 of case (i) of Theorem 3.1, we have $Tw = w$, $Mw = w$. Using $STw = LMw = w$, we have $Sw = w$ and $Lw = w$. Therefore, w is a common fixed point of mappings f, g, S, T, L and M. The uniqueness follows easily. □

Now, we discuss the existence of a unique common fixed point of compatible mappings of type (E) .

Theorem 3.8. *Self mappings f, g, S, T, L and M of a complete metric space $(\mathcal{X}, d)$ satisfying conditions (I), (II) and (IV) have a unique common fixed point in $\mathcal{X}$, if the pairs (f, ST) and (g, LM) satisfy either of the following conditions:*

(a) (f, ST) is ST-compatible of type (E) and ST-reciprocal continuous, (g, LM) is LM-compatible of type (E) and $LM-$reciprocal continuous.

(b) (f, ST) is f-compatible of type (E) and f-reciprocal continuous, (g, LM) is g-compatible of type (E) and g-reciprocal continuous.

Proof. Following proof of [19, Theorem 2.3], the sequence $\{v_n\}$ defined by (7), is a Cauchy sequence in $\mathcal{X}$. Since $(\mathcal{X}, d)$ is a complete metric space, therefore, $\{v_n\}$ converges to a point $w \in \mathcal{X}$, as $n \to \infty$. Consequently, the subsequences $\{fu_{2n}\}$, $\{STu_{2n}\}$, $\{gu_{2n+1}\}$, and $\{LMu_{2n+1}\}$ also converges to the same point $w \in \mathcal{X}$.

Suppose that pair (f, ST) is $ST-$compatible of type (E) and $ST-$reciprocal continuous, therefore, by Proposition 2.4, $STw = fw$.

We claim that w is a fixed point of ST, i.e., $STw = w$. Letting $u = w$ and $v = u_{2n+1}$ in (5) and (6) and letting $n \to \infty$, we have

$$[1 + pd(STw, w)]d^2(fw, w) \leq p\psi(0, 0, 0, 0) + m(STw, w) - \phi(m(STw, w)),$$

where

$$m(STw, w) = \max\Big\{d^2(STw, w), d(STw, fw)d(w, w), d(STw, w)d(w, fw), \frac{1}{2}[d(STw, fw)d(STw, w) + d(w, fw)d(w, w)]\Big\} = d^2(STw, w).$$

Solving the above inequality, we get $pd^3(STw, w) + \phi(d^2(STw, w)) \leq 0$. This is true only if $d(STw, w) = 0$, which implies that $STw = w$. Therefore, we have $w = STw = fw$. Since $f(\mathcal{X}) \subset LM(\mathcal{X})$, there exists a point $u^* \in \mathcal{X}$ such that $fw = LMu^*$. Now, we claim that $gu^* = LMu^*$. Taking $u = w$, $v = u^*$ in (5) and (6), we get

$$[1 + pd(STw, LMu^*)]d^2(fw, gu^*) \leq p\psi(0, 0, 0, 0) + m(STw, LMu^*) - \phi(m(STw, LMu^*)),$$

where

$$m(STw, LMu^*) = \max\Big\{d^2(STw, LMu^*), d(STw, fw)d(LMu^*, gu^*), d(STw, gu^*)d(LMu^*, fw), \frac{1}{2}[d(STw, fw)d(STw, gu^*) + d(LMu^*, fw)d(LMu^*, gw)]\Big\} = 0.$$

Using the value of $m(STw, LMu^*)$ along with the property of ϕ and ψ, the above inequality reduces to $d^2(fw, gu^*) \leq 0$.This is true only if $d(fw, gu^*) = 0$, i.e., $gu^* = fw$. Hence, $gu^* = LMu^* = fw = STw = w$. Since the pair (g, LM) is $LM-$compatible of type (E) and $LM-$reciprocal continuous and $LMu^* = gu^*$, by Proposition 2.4, $LMw = (LM)gu^* = g(LM)u^* = gw$.
Now, we prove that w is a fixed point of LM. Putting $u = v = w$ in (5) and (6), we get

$$[1 + pd(w, LMw)]d^2(w, gw) \leq p\psi(0, 0, 0, 0) + m(w, LMw) - \phi(m(w, LMw)),$$

where

$$m(w, LMw) = \max\Big\{d^2(w, LMw), d(w, w)d(LMw, gw), d(w, gw)d(LMw, w), \frac{1}{2}[d(w, w)d(w, gw) + d(LMw, w)d(LMw, gw)]\Big\} = d^2(w, LMw).$$

On Simplifying, we get $d(w, LMw) = 0$, which implies that $w = LMw$. Thus, $w = STw = fw = LMw = gw$. In the view of steps 5 and 6 of case (i) of Theorem 3.1, we have $Tw = w$, $Mw = w$. Therefore, $STw = w$, $LMw = w$ imply that $Sw = w$ and $Lw = w$. Hence, w is common fixed point of mappings f, g, S, T, L and M. Uniqueness follows easily.

Similarly, one can complete the proof when the pairs (f, ST) and (g, LM) satisfy the condition (b). □

4 Consequences and Examples

Taking suitable mappings f, g, S, T, L and M one can derive corollaries involving two, three as well as four self mappings. As an example, one can deduce the following corollaries for four self mappings by taking $T = M = I_{\mathcal{X}}$(identity mappings of $\mathcal{X}$) in Theorems 3.1- 3.6, we have the following result for four mappings.

Corollary 4.1. *Let S, L, f and g be self mappings of a complete metric space $(\mathcal{X}, d)$ satisfying*

(i) $f(\mathcal{X}) \subset L(\mathcal{X})$ *and* $g(\mathcal{X}) \subset S(\mathcal{X})$,

(ii) one of S, L, f, g is continuous,

(iii) for all $u, v \in \mathcal{X}$, there exists a function $\psi \in \Psi$, a function $\phi \in \Phi$ and a real number $p \geq 0$ such that

$$[1 + pd(Su, Lv)]d^2(fu, gv) \leq p\psi\Big(d^2(Su, fu)d(Lv, gv), d(Su, fu)d^2(Lv, gv), d(Su, fu)d(Su, gv)d(Lv, fu), d(Su, gv)d(Lv, fu)d(Lv, gv)\Big) + m(Su, Lv) - \phi(m(Su, Lv)),$$

where

$$m(Su, Lv) = \max\Big\{d^2(Su, Lv), d(Su, fu)d(Lv, gv), d(Su, gv)d(Lv, fu), \frac{1}{2}[d(Su, fu)d(Su, gv) + d(Lv, fu)d(Lv, gv)]\Big\}.$$

If (S, f) and (L, g) are either of the followings

(a) compatible

(b) compatible of type (A)

(c) compatible of type (B)

(d) compatible of type (C)

(e) compatible of type (P)

(f) compatible of type (R).

Then S, L, f *and* g *have a unique common fixed point in* $\mathcal{X}$.

Taking $T = M = I_{\mathcal{X}}$(identity mappings of $\mathcal{X}$) in the Theorems 3.7-3.8, we have the following results.

Corollary 4.2. *Let* S, L, f *and* g *be self mappings of a complete metric space* $(\mathcal{X}, d)$ *satisfying conditions (i) and (iii) of Corollary 4.1. If* (S, f) *and* (L, g) *are compatible of type* (K) *and reciprocal continuous, then* S, L, f *and* g *have a unique common fixed point in* $\mathcal{X}$.

Corollary 4.3. *Self mappings* f, g, S *and* L *of a complete metric space* $(\mathcal{X}, d)$ *satisfying conditions (i) and (iii) of Corollary 4.1 have a unique common fixed point in* $\mathcal{X}$*, if the pairs* (S, f) *and* (L, g) *satisfy either of the following conditions:*

(a) (S, f) *is* S*-compatible of type* (E) *and* S*-reciprocal continuous,* (L, g) *is* L*-compatible of type* (E) *and* $L-$*reciprocal continuous.*

(b) (S, f) *is* f*-compatible of type* (E) *and* f*-reciprocal continuous,* (L, g) *is* g*-compatible of type* (E) *and* g*-reciprocal continuous.*

Taking $T = M = I_{\mathcal{X}}$(identity mappings of $\mathcal{X}$), $g = f$, $L = S$ in the Theorem 3.1- 3.6, we have the following result for two self mappings.

Corollary 4.4. *Let* $(\mathcal{X}, d)$ *be a complete metric space and* $S, f : \mathcal{X} \to \mathcal{X}$ *be two mappings satisfying the following conditions*

(C_1) $f(\mathcal{X}) \subset S(\mathcal{X})$,

(C_2) *for all* $u, v \in \mathcal{X}$*, there exists a real number* $p \geq 0$*, a function* $\psi \in \Psi$*, a function* $\phi \in \Phi$*, such that*

$$\begin{aligned}[1 + pd(Su, Sv)]d^2(fu, fv) \leq p\,\psi\Big(&d^2(Su, fu)d(Sv, fv),\\ &d(Su, fu)d^2(Sv, fv),\\ &d(Su, fu)d(Su, fv)d(Sv, fu),\\ &d(Su, fv)d(Sv, fu)d(Sv, fv)\Big)\\ &+ m(Su, Sv) - \phi(m(Su, Sv)),\end{aligned}$$

where

$$m(Su,Sv)=\max\Big\{d^2(Su,Sv),d(Su,fu)d(Sv,fv),d(Su,fv)d(Sv,fu),\\ \frac{1}{2}[d(Su,fu)d(Su,fv)+d(Sv,fu)d(Sv,fv)]\Big\},$$

(C_3) *either S or f is continuous.*

If S and f are compatible mappings or variants of compatible mappings (type (A) or type (B) or type (C) or type (P) or type (R)), then S and f have a unique common fixed point in $\mathcal{X}$.

Taking $T=M=I_{\mathcal{X}}$(identity mappings of $\mathcal{X}$), $g=f$, $L=S$ in the Theorems 3.7-3.8, we have the following results for two self mappings.

Corollary 4.5. *Let $(\mathcal{X},d)$ be a complete metric space. Suppose $f,S:\mathcal{X}\to\mathcal{X}$ are two mappings satisfying the conditions (C_1) and (C_2) of Corollary 4.4. If (f,S) is compatible of type (K) as well as reciprocal continuous, then f and S have a unique common fixed point in $\mathcal{X}$.*

Corollary 4.6. *Let f and S be self mappings of a complete metric space $(\mathcal{X},d)$ satisfying the conditions (C_1) and (C_2) of Corollary 4.4. If pair (f,S) satisfies either of the following conditions:*

(a) (f,S) is $f-$compatible of type (E) and $f-$reciprocal continuous;

(b) (f,S) is $S-$compatible of type (E) and $S-$reciprocal continuous.

Then f and S have a unique common fixed point in $\mathcal{X}$.

Now, we present examples in support of Corollaries 4.1 and 4.3.

Example 4.1. *Let $\mathcal{X}=[2,20]$ and d be a usual metric. Let f,g,S,L be self mappings of $\mathcal{X}$ defined by $Su=2$, $2\le u\le 5$, $Su=u-3$, $5<u\le 20$, $fu=6$, $2<u<5$, $fu=2$, $u=2$ or $5\le u\le 20$, $Lu=2$, $gu=2$, $u=2$, $Lu=6$, $gu=3$, $2<u\le 20$. Let $\phi:[0,\infty)\to[0,\infty)$ be a function defined by $\phi(t)=2t$ and $\psi:[0,\infty)^4\to[0,\infty)$ be a function defined by $\psi(w_1,w_2,w_3,w_4)=\max\{w_1,w_2,w_3,w_4\}$, $w_i\ge 0, i=1,2,3,4$. Consider a sequence $\{u_n\}$ with $u_n=2$, for each n. Clearly $\phi\in\Phi$ and $\psi\in\Psi$. Also, One can easily verified that all the conditions of the Corollary 4.1 (a) are satisfied and 2 is the unique common fixed point of f,g,S and L.*

Example 4.2. *Let $\mathcal{X} = [2, 20]$ and d be a usual metric. Let f, g, L and S be self mappings of $\mathcal{X}$ defined as $Su = 2$, $2 \leq u \leq 10$, $Su = u - 8$, $10 < u \leq 20$, $fu = 2$, $2 \leq u \leq 20$, $Lu = 2, gu = 2, u = 2$, $Lu = 6, gu = 3$, $2 < u \leq 20$. Here $g(\mathcal{X}) = \{2, 3\} \subset [2, 12] = S(\mathcal{X})$ and $f(\mathcal{X}) = \{2\} \subset \{2, 6\} = L(\mathcal{X})$. For the sequence $\{u_n\}$, where $u_n = 2$, for each n, pairs (S, f) and (L, g) are compatible of type (R). If we define a function $\phi : [0, \infty) \to [0, \infty)$ as $\phi(t) = 2t$, for each $t \geq 0$ and define a function $\psi : [0, \infty)^4 \to [0, \infty)$ as $\psi(w_1, w_2, w_3, w_4) = \max\{w_1, w_2, w_3, w_4\}$, $w_i \geq 0, i = 1, 2, 3, 4$ and take a real number $p \geq \frac{3}{2}$, then all the conditions of Corollary 4.1(f) are satisfied and 2 is the unique common fixed point of f, g, S and L.*

Example 4.3. *Let $\mathcal{X} = [0, 5]$ and d be a usual metric. Define $f, g, S, L : \mathcal{X} \to \mathcal{X}$ as $gu = fu = \frac{5+u}{2}$, $Lu = Su = \frac{5}{2} + u$, $0 \leq u < \frac{5}{2}$, $gu = fu = \frac{5}{2}$, $\frac{5}{2} \leq u \leq 5$, $Su = Lu = \frac{5}{2}$, $u = \frac{5}{2}$ and $Su = Lu = \frac{24}{5}$, $\frac{5}{2} < u \leq 5$. Clearly, $f(\mathcal{X}) = [\frac{5}{2}, \frac{15}{4}) = g(\mathcal{X})$ and $S(\mathcal{X}) = L(\mathcal{X}) = [\frac{5}{2}, 5)$. The mappings are not continuous at $u = \frac{5}{2}$. Let $\{u_n\}$ be a sequence in $\mathcal{X}$ such that $u_n \to 0$, $u_n > 0$, for all n. Then fu_n, $Su_n \to \frac{5}{2} = t$ and $ffu_n = S(\frac{5+u_n}{2}) \to \frac{5}{2}$, $fSu_n = f(\frac{5}{2} + u_n) \to \frac{5}{2}$, $SSu_n = S(\frac{5}{2} + u_n) \to \frac{24}{5}$ and $Sfu_n = S(\frac{5+u_n}{2}) \to \frac{24}{5}$. Also, we have $St = \frac{5}{2} = ft$. Thus $ffu_n, fSu_n \to \frac{5}{2} = St = S(\frac{5}{2})$ and $fSu_n \to \frac{5}{2} = ft = f(\frac{1}{2})$. Therefore, the pair (S, f) is $f-$compatible of type (E) and $f-$reciprocal continuous and the pair (L, g) is $g-$compatible of type (E) and $g-$reciprocal continuous. In particular, if we take $\psi(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$, where $t_i \geq 0, i = 1, 2, 3, 4$, $\phi(t) = 2t, t \geq 0$ and $p = \frac{3}{2}$, then all the conditions of Corollary 4.3 are satisfied and $\frac{5}{2}$ is a unique common fixed point of all the mappings f, g, S and L.*

5 Application

Let U, V denote Banach spaces, $\hat{S} \subset U$, $D \subset V$ are state space and decision space respectively. Let $\mathbb{R}$ denotes the set of all real numbers and $B(\hat{S}) = \{h : \hat{S} \to \mathbb{R}, h \text{ is bounded }\}$. Let $d(h, k) = \sup\{|h(u) - k(u)| : u \in \hat{S}\}$ for any $h, k \in B(\hat{S})$. Obviously, $(B(\hat{S}), d)$ is a complete metric space.

Bellman and Lee [5] gave the basic form of functional equation as follows:

$$g(u) = \operatorname*{opt}_{v} G(u, v, g(\tau(u, v))),$$

where $u \in \hat{S}$, $v \in D$, τ is the transformation process, $g(u)$ is the optimal return with initial state u and the opt denotes $\max$ or $\min$.

Now, we discuss the application of our result in finding a common solution of the following functional equations that are arising in dynamic programming (see

[3, 4, 5])

$$f_i(u) = \sup_{v\in D} \mathcal{F}_i(u, v, f_i(\tau(u, v))), u \in \hat{S} \tag{8}$$

$$g_i(u) = \sup_{v\in D} \mathcal{G}_i(u, v, g_i(\tau(u, v))), u \in \hat{S}, \tag{9}$$

where $\tau : \hat{S} \times D \to \hat{S}$ and $\mathcal{F}_i, \mathcal{G}_i : \hat{S} \times D \times \mathbb{R} \to \mathbb{R}, i = 1, 2.$

Theorem 5.1. *Let $\mathcal{F}_i, \mathcal{G}_i : \hat{S} \times D \times \mathbb{R} \to \mathbb{R}$, be bounded for $i \in \{1, 2\}$. Define the mappings $\mathcal{P}_i, \mathcal{Q}_i : B(\hat{S}) \to B(\hat{S})$ as follows*

$$\begin{aligned} \mathcal{P}_i h(u) &= \sup_{v\in D} \mathcal{F}_i(u, v, h(\tau(u, v))), \\ \mathcal{Q}_i k(u) &= \sup_{v\in D} \mathcal{G}_i(u, v, k(\tau(u, v))), \end{aligned} \tag{10}$$

for all $u \in \hat{S}$, $h, k \in B(\hat{S})$, $i = 1, 2$. Suppose that the following conditions hold:

(a) for all $u, t \in \hat{S}, v \in D$, $h, k \in B(\hat{S})$,

$$\begin{aligned} \big|\mathcal{F}_1(u, v, h(t)) - \mathcal{F}_2(u, v, k(t))\big|^2 \le \mathcal{M}^{-1}\Big(p\,\psi\big(&d^2(\mathcal{Q}_1 h, \mathcal{P}_1 h)d(\mathcal{Q}_2 k, \mathcal{P}_2 k), \\ &d(\mathcal{Q}_1 h, \mathcal{P}_1 h)d^2(\mathcal{Q}_2 k, \mathcal{P}_2 k), \\ &d(\mathcal{Q}_1 h, \mathcal{P}_1 h)d(\mathcal{Q}_1 h, \mathcal{P}_2 k)d(\mathcal{Q}_2 k, \mathcal{P}_1 h), \\ &d(\mathcal{Q}_1 h, \mathcal{P}_2 k)d(\mathcal{Q}_2 k, \mathcal{P}_1 h)d(\mathcal{Q}_2 k, \mathcal{P}_2 k)\big) \\ &+ m(\mathcal{Q}_1 h, \mathcal{Q}_2 k) - \phi(m(\mathcal{Q}_1 h, \mathcal{Q}_2 k))\Big), \end{aligned}$$

where

$$\begin{aligned} m(\mathcal{Q}_1 h, \mathcal{Q}_2 k) = \max\Big\{&d^2(\mathcal{Q}_1 h, \mathcal{Q}_2 k), d(\mathcal{Q}_1 h, \mathcal{P}_1 h)d(\mathcal{Q}_2 k, \mathcal{P}_2 k), \\ &d(\mathcal{Q}_1 h, \mathcal{P}_2 k)d(\mathcal{Q}_2 k, \mathcal{P}_1 h) \\ &\frac{1}{2}[d(\mathcal{Q}_1 h, \mathcal{P}_1 h)d(\mathcal{Q}_1 h, \mathcal{P}_2 k) \\ &+ d(\mathcal{Q}_2 k, \mathcal{P}_1 h)d(\mathcal{Q}_2 k, \mathcal{P}_2 k)]\Big\}, \end{aligned}$$

$\mathcal{M} = 1 + pd(\mathcal{Q}_1 h, \mathcal{Q}_2 k)$, $\phi \in \Phi$, $\psi \in \Psi$, p is a positive real number,

(b) either there exists $\mathcal{P}_i \in \{\mathcal{P}_1, \mathcal{P}_2\}$ such that for any sequence $\{k_n\}$ of $B(\hat{S})$ and $k \in B(\hat{S})$

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |k_n(u) - k(u)| = 0, \text{ implies } \lim_{n\to\infty} \sup_{u\in\hat{S}} |\mathcal{P}_i k_n(u) - \mathcal{P}_i k(u)| = 0$$

or there exists $\mathcal{Q}_i \in \{\mathcal{Q}_1, \mathcal{Q}_2\}$ *such that for any sequence* $\{k_n\}$ *of* $B(\hat{S})$ *and* $k \in B(\hat{S})$

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |k_n(u) - k(u)| = 0, \text{ implies } \lim_{n\to\infty} \sup_{u\in\hat{S}} |\mathcal{Q}_i k_n(u) - \mathcal{Q}_i k(u)| = 0;$$

(c) for any $h \in B(\hat{S})$*, there exists* $k_1, k_2 \in B(\hat{S})$ *such that*

$$\mathcal{P}_1 h(u) = \mathcal{Q}_2 k_1(u), \mathcal{P}_2 h(u) = \mathcal{Q}_1 k_2(u), u \in \hat{S};$$

(d) for any $i \in \{1, 2\}$, $\lim_{n\to\infty} \sup_{u\in\hat{S}} |\mathcal{Q}_i\mathcal{P}_i k_n(u) - \mathcal{P}_i\mathcal{Q}_i k_n(u)| = 0$, *whenever* $\{k_n\}$ *is a sequence of* $B(\hat{S})$ *such that* $\lim_{n\to\infty} \mathcal{Q}_i k_n = \lim_{n\to\infty} \mathcal{P}_i k_n = h$, *for some* $h \in B(\hat{S})$.

Then the system of functional equations (8) and (9) have a unique common solution in $B(\hat{S})$.

Proof. From conditions (b) -(d), $\mathcal{P}_i, \mathcal{Q}_i$ are self mappings of $B(\hat{S})$. One of $\mathcal{P}_i, \mathcal{Q}_i$ is continuous for $i \in \{1, 2\}$, $\mathcal{P}_1(B(\hat{S})) \subset \mathcal{Q}_2(B(\hat{S}))$ and $\mathcal{P}_2(B(\hat{S})) \subset \mathcal{Q}_1(B(\hat{S}))$ and the pairs of mappings $(\mathcal{P}_1, \mathcal{Q}_1)$ and $(\mathcal{P}_2, \mathcal{Q}_2)$ are compatible.

For $\eta > 0$, $u \in \hat{S}$ and $k_1, k_2 \in B(\hat{S})$, there exists $v_1, v_2 \in D$ such that

$$\mathcal{P}_i k_i(u) < \mathcal{F}_i(u, v_i, k_i(u_i)) + \eta, \tag{11}$$

where $u_i = \tau(u, v_i), i = 1, 2$. Also, we have

$$\mathcal{P}_1 k_1(u) \geq \mathcal{F}_1(u, v_2, k_1(u_2)), \tag{12}$$
$$\mathcal{P}_2 k_2(u) \geq \mathcal{F}_2(u, v_1, k_2(u_1)). \tag{13}$$

From (11),(13) and condition (a), we have

$$\begin{aligned}(\mathcal{P}_1 k_1(u) - \mathcal{P}_2 k_2(u))^2 &< (\mathcal{F}_1(u, v_1, k_1(u_1)) - \mathcal{F}_2(u, v_1, k_2(u_1)) + \eta)^2 \\ &= (\mathcal{F}_1(u, v_1, k_1(u_1)) - \mathcal{F}_2(u, v_1, k_2(u_1)))^2 + \xi, \\ &\leq \mathcal{M}^{-1}\Big(p\,\psi\big(d^2(\mathcal{Q}_1 k_1, \mathcal{P}_1 k_1) d(\mathcal{Q}_2 k_2, \mathcal{P}_2 k_2), \\ &\quad d(\mathcal{Q}_1 k_1, \mathcal{P}_1 k_1) d^2(\mathcal{Q}_2 k_2, \mathcal{P}_2 k_2), \\ &\quad d(\mathcal{Q}_1 k_1, \mathcal{P}_1 k_1) d(\mathcal{Q}_1 k_1, \mathcal{P}_2 k_2) d(\mathcal{Q}_2 k_2, \mathcal{P}_1 k_1), \\ &\quad d(\mathcal{Q}_1 k_1, \mathcal{P}_2 k_2) d(\mathcal{Q}_2 k_2, \mathcal{P}_1 k_1) d(\mathcal{Q}_2 k_2, \mathcal{P}_2 k_2)\big) + \\ &\quad m(\mathcal{Q}_1 k_1, \mathcal{Q}_2 k_2) - \phi(m(\mathcal{Q}_1 k_1, \mathcal{Q}_2 k_2))\Big) + \xi,\end{aligned} \tag{14}$$

where $\xi = \eta^2 + 2\eta(\mathcal{F}_1 - \mathcal{F}_2)$.

From (11), (12) and condition (a), we have

$$\begin{aligned}(\mathcal{P}_1k_1(u)-\mathcal{P}_2k_2(u))^2 &> (\mathcal{F}_1(u,v_2,k_1(u_2))-\mathcal{F}_2(u,v_2,k_2(u_2))-\eta)^2\\ &=(\mathcal{F}_1(u,v_1,k_1(u_2))-\mathcal{F}_2(u,v_1,k_2(u_2)))^2+\xi_1,\\ &\geq -\mathcal{M}^{-1}\Big(p\,\psi\big(d^2(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &\quad d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d^2(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &\quad d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1),\\ &\quad d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2)\big)+\\ &\quad m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2)-\phi(m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2))\Big)-\xi,\end{aligned}\tag{15}$$

where $\xi_1=\eta^2-2\eta(\mathcal{F}_1-\mathcal{F}2)<\xi$.

From (14) and (15), we obtain

$$\begin{aligned}|\mathcal{P}_1k_1(u)-\mathcal{P}_2k_2(u)|^2 \leq &\mathcal{M}^{-1}\Big(p\,\psi\big(d^2(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d^2(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2)\big)\\ &+m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2)-\phi(m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2))\Big)+\xi,\end{aligned}\tag{16}$$

As $\eta>0$ is arbitrary, so ξ is negligible and (16) is true for all $u\in\hat{S}$, taking supremum, we get

$$\begin{aligned}[1+pd(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2)]d^2(\mathcal{P}_1k_1,\mathcal{P}_2k_2) \leq\ & p\,\psi\big(d^2(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d^2(\mathcal{Q}_2k_2,\mathcal{P}_2k_2),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_1k_1)d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1),\\ &d(\mathcal{Q}_1k_1,\mathcal{P}_2k_2)d(\mathcal{Q}_2k_2,\mathcal{P}_1k_1)d(\mathcal{Q}_2k_2,\mathcal{P}_2k_2)\big)\\ &+m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2)-\phi(m(\mathcal{Q}_1k_1,\mathcal{Q}_2k_2)).\end{aligned}$$

All the hypotheses of Corollary 4.1 are satisfied and $\mathcal{P}_1,\mathcal{P}_2,\mathcal{Q}_1$ and $\mathcal{Q}_2$ corresponds to mappings f,g,S and L respectively. So, $\mathcal{P}_1,\mathcal{P}_2,\mathcal{Q}_1$ and $\mathcal{Q}_2$ have a unique common fixed point $k^*\in B(\hat{S})$, i.e., $k^*(u)$ is a unique common solution of the system of functional equations (8) and (9). □

6 Conclusion

In this paper, we have established common fixed point theorems for pairs of minimal commutative mappings. These results generalize the results of Jain *et*

al.[10, 11, 12], Jung *et al.*[14], Kang *et al.*[18], Murthy and Prasad [21], Pathak *et al.* [24] and various results presented in the literature. Our result is useful in studying the existence and uniqueness problems of certain functional equations arising in dynamic programming. This result can be extended for multivalued mappings and family of mappings.

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Analytical solution of time-fractional N-dimensional Black-Scholes equation using LHPM

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Abstract

A famous Black-Scholes differential equation is used for pricing options in financial world which represents financial derivatives more closely. Option is one of the crucial financial derivatives. Sawangtong P., Trachoo K., Sawangtong W. and Wiwattanapataphee B. obtained analytical solution of Black-Scholes equation with two assets in the Liouville-Caputo time-fractional derivative sense using Laplace homotopy perturbation method (LHPM). The aim of this paper is to derive solution of Liouville-Caputo time-fractional Black-Scholes equation with n assets using LHPM. Numerical results shows that our approach gives very accurate results and our formulas are quite close to the plain vanilla options.

Keywords: Financial derivatives, European options, n-dimensional Black-Scholes Equation; Liouville-Caputo Fractional Derivative, Laplace homotopy perturbation method.

2020 AMS subject classifications: Primary: 91G20. Secondary: 35Q91. [1]

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1 Introduction

An agreement between two parties or financial institutes is called financial derivatives. Options, Forward contacts, Futures, etc. are examples of financial derivatives. Financial derivatives are used to manage risk associated with the underlying asset significantly. Actually, financial derivatives are used to transfer risk to another trader or financial institution. Options are without a doubt the main component of a derivative that is frequently utilized in the financial market. Options will help to modified portfolios of the investors. As a result, the concept of option trading has continually evolved. A contract between two parties to buy or sell an underlying asset at a specific future date and price is known as option. The specific future date is called expiry date of option while specific price is called the striking price of option. There are mainly two types of options, namely call and put options. Call (Put) option is an option in which holder of the option has the right to buy (sell) the underlying asset at a specific future date and price. The cost paid by the holder of option is known as option price or premium. Options exist in two styles in the financial market, namely American and European. American option can be exercised before expiry date while European option is exercised at the expiry date only. In 1973, Fisher Black and Myron Scholes derived a model for pricing options for plain vanilla payoffs. This model was recognized worldwide when Black and Scholes awarded a Nobel prize in 1997. Consequently, lots of researcher worked with different payoffs Dedania and Ghevariya [2013a,b], Ghevariya [2020], R. J. Haber, P. J. Schönbucher and Wilmott [1999], Haug [2007], P. Wilmott, S. Howison and Dewynne [1993]. Note that differential equation derived by Black and Scholes known as Black-Scholes differential equation and Black-Scholes model is a solution of that equation with initial and boundary conditions. The solution of the Black-Scholes equation becomes the intrinsic aim of numerous researchers. As a result, the Black-Scholes equation has been solved using a variety of approaches to obtain approximate or closed-form solutions with a variety of payoffs such as projected differential transform method (PDTM) S. O. Edeki , O. O. Ugberbor and Owoloko [2015], S. O. Edeki , R. M. Jena, O. P. Ogundile and Chakraverty [2021], Ghevariya [2021, 2022b], binomial method P. Wilmott, S. Howison and Dewynne [2002], homotopy perturbation method (HPM) Ghevariya [2022a], Mellin transform Fadugba and Nwozo [2016], Ghevariya [2019], Panini and Srivastav [2004], Yoon [2014], etc. The Black-Scholes equation is a second order linear equation having parabolic nature.

It can be seen that fractional calculus plays an important role for finding analytical solution of partial differential equations with initial conditions Miller and Ross [2003], Podlubny [1999]. Fractional calculus has been used in various fields such as finance, physics, engineering, etc. Fractional differential equations and its applications attracted many researchers in the field of Black-Scholes theory dur-

ing the last four decades S. Kumar, D. Kumar and Singh [2014], Mehrdoust and Najafi [2017], Meng and Wang [2010], Song and Wang [2013], H. Zhang, F. Liu, I. Turner and Yang [2016]. Recently, various methods have been proposed to find solution of fractional differential equations namely, HPM, Adomian decomposition method (ADM), iteration method, Laplace homotopy perturbation method (LHPM), etc. LHPM is the powerful method to obtain explicit solution which will be transform fractional differential equations into algebraic equations and solving them we get solution using inverse Laplace transform. Many problems arise in real world with known physical interpretation. Liouville-Caputo fractional derivative is applicable to such problems because it has initial conditions same as the traditional differential equation. Let f be a real valued n-times differentiable function on $[a,t],\ t>a$ where $a\in\mathbb{R}$. Then for any $\alpha>0,\ \alpha\notin\{1,2,...,n\}$ and $n\in\mathbb{N}$ with $n-1<\alpha<n$, the Liouville-Caputo fractional derivative of f is defined as

$$ {}_aD_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-\eta)^{n-\alpha-1}f^{(n)}(\eta)d\eta. $$

Many researcher worked with multi-dimensional Black-Scholes equation S. J. Ghevariya, C. N. Patel and Fadugba [2022], Guillaue [2019], J. Kim, T. Kim, J. Jo, Y. Choi, S. Lee, H. Hwang, M. Yoo and Jeong [2016], Prathumwan and Trachoo [2020], P. Sawangtong, K. Trachoo, W. Sawangtong and Wiwattanapataphee [2018], K. Trachoo, W. Sawangtong and Sawangtong [2017]. In this paper, the LHPM has been used to solve n-dimensional Black-Scholes equation with Liouville-Caputo fractional derivative sense. Note that LHPM is a method that combines Laplace transform and homotopy perturbation method. Using a Liouville-Caputo fractional derivative, Sawangtong P., Trachoo K., Sawangtong W., and Wiwattanapataphee B. found an analytical solution to the Black-Scholes equation for two assets P. Sawangtong, K. Trachoo, W. Sawangtong and Wiwattanapataphee [2018]. For European call and put options, we will provide an analytical solution to the time-fractional Black-Scholes equation with n assets. This paper is summarized as follows. Section-2 deals with the n-dimensional Black-Scholes differential equation with Liouville-Caputo fractional derivative. In section-3, we shall discuss about time- fractional LHPM. Using LHPM, we will solve the time-fractional Black-Scholes differential equation analytically in Section 4. We compare these formulas with the solution of known n-dimensional Black-Scholes equation. The conclusion is covered in the final section.

2 N-Dimensional Black-Scholes Equation with Liouville-Caputo Fractional Derivative

The n-dimensional Black-Scholes equation Joonglee and Yongsik [2013] for European call option is given by

$$\frac{\partial \mathcal{C}}{\partial \tau} + \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 \mathcal{C}}{\partial S_i \partial S_j} + r\sum_{i=1}^{n} S_i \frac{\partial \mathcal{C}}{\partial S_i} - r\mathcal{C} = 0 \tag{1}$$

with initial condition $\mathcal{C}(S_1, S_2, ..., S_n, \tau) = 0$, if at least one $S_i = 0$ for $1 \leq i \leq n$ & boundary conditions $\mathcal{C}(S_1, S_2, ..., S_n, T) = \max\Big(\sum_{i=1}^{n} a_i S_i - K, 0\Big)$ and $\mathcal{C}(S_1, S_2, ..., S_n, \tau) = \sum_{i=1}^{n} a_i S_i - Ke^{-r(T-\tau)}$, if at least one $S_i \to \infty$ for $1 \leq i \leq n$, where $\mathcal{C}(S_1, S_2, ..., S_n, \tau)$ is the price of call option at time τ, S_i is the value of i^{th} underlying asset, σ_i is the volatility, a_i is portion of the i^{th} underlying asset, r is the risk free interest rate, K is the striking price and ρ_{ij} is the correlation of i^{th} and j^{th} underlying assets for $i \leq i, j \leq n$.

Replacing $\mathcal{C}$ by $\mathcal{P}$ in Equation (1), we get Black-Scholes differential equation for European put option with initial condition $\mathcal{P}(S_1, S_2, ..., S_n, \tau) = Ke^{-r(T-\tau)}$, if at least one $S_i = 0$ for $1 \leq i \leq n$ & boundary conditions $\mathcal{P}(S_1, S_2, ..., S_n, T) = \max\Big(K - \sum_{i=1}^{n} a_i S_i, 0\Big)$ and $\mathcal{P}(S_1, S_2, ..., S_n, \tau) = 0$, if at least one $S_i \to \infty$ for $1 \leq i \leq n$, where $\mathcal{P}(S_1, S_2, ..., S_n, \tau)$ is the price of put option at time τ.

Now for each $i = 1, 2, ..., n$, taking $x_i = \ln(S_i) - \big(r - \frac{1}{2}\sigma_i^2\big)\tau, t = T - \tau$ and $\mathcal{C}(x_1, x_2, ..., x_n, \tau) = e^{-r(T-\tau)}\ v(x_1, x_2, ..., x_n, \tau)$, the Equation (1) reduces to

$$\frac{\partial v}{\partial t} - \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} = 0, \quad (x_1, x_2, ..., x_n, t) \in \Re^n \times [0, T] \tag{2}$$

with initial and boundary conditions for call option transformed as

$$v(x_1, x_2, ..., x_n, 0) = \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big), \tag{3}$$

$$v(x_1, x_2, ..., x_n, t) = 0, \text{ if at least one } x_i \to -\infty \text{ and} \tag{4}$$

$$v(x_1, x_2, ..., x_n, t) = \sum_{i=1}^{n} \tilde{a}_i e^{x_i + \frac{1}{2}\sigma_i^2 t} - K, \text{ if at least one } x_i \to \infty, \tag{5}$$

where $\tilde{a}_i = a_i e^{(r - \frac{1}{2}\sigma_i^2)T}$. By replacing Liouville-Caputo fractional derivative in Equation (2), we get time-fractional Black-Scholes equation for call option with $\alpha \in (0, 1]$

$$D_t^{\alpha} v = \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}, \tag{6}$$

where initial and boundary conditions are given by Equations (3)-(5). Similarly, one can derive the time-fractional Black-Scholes equation for put option.

3 Introduction to Time Fractional LHPM

The HPM is introduced by He He [2003]. The basic idea about Laplace homotopy perturbation method (LHPM) was given by Khan and Wu Khan and Wu [2011]. Consider a general partial differential equation in n variables

$$G(v(x_1, x_2, ..., x_n, t)) - g(x_1, x_2, ..., x_n, t) = 0,\ t \in \Omega \tag{7}$$

with initial and boundary conditions are given by

$$v(x_1, x_2, ..., x_n, 0) = h(x_1, x_2, ..., x_n) \text{ and } B\Big(v, \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, ..., \frac{\partial v}{\partial x_n}, \frac{\partial v}{\partial t}\Big) = 0,$$

where $(x_1, x_2, ..., x_n) \in \Re^n, \Omega$ is the domain, B is the boundary operator, G is a differential operator, $v(x_1, x_2, ..., x_n, t)$ is to be determined and $g(x_1, x_2, ..., x_n, t)$ is a known analytic function. Further, G can be split into two parts as given below.

$$G(v(x_1, x_2, ..., x_n, t)) = D_t^{\alpha} v(x_1, x_2, ..., x_n, t) + N(v(x_1, x_2, ..., x_n, t)),$$

where D_t^{α} is the Liouville-Caputo fractional derivative with $\alpha \in (0, 1]$, N is the remaining part by taking out the simple part with first term. Hence, Equation (7) can be written as

$$D_t^{\alpha} v(x_1, x_2, ..., x_n, t) + N(v(x_1, x_2, ..., x_n, t)) = g(x_1, x_2, ..., x_n, t). \tag{8}$$

Now, applying the Laplace transform with respect to t on both sides of Equation (8), we get

$$\mathcal{L}[D_t^{\alpha} v(x_1, x_2, ..., x_n, t)] + \mathcal{L}[N(v(x_1, x_2, ..., x_n, t))] = \mathcal{L}[g(x_1, x_2, ..., x_n, t)].$$

Using Laplace transform of Liouville-Caputo fractional derivative Miller and Ross [2003] in the above Equation, we get

$$\begin{aligned} \mathcal{L}[v(x_1, x_2, ..., x_n, t)] =& s^{-\alpha} h(x_1, x_2, ..., x_n) - s^{-\alpha} \mathcal{L}[N(v(x_1, x_2, ..., x_n, t))] \quad (9) \\ &+ s^{-\alpha} \mathcal{L}[g(x_1, x_2, ..., x_n, t)]. \quad (10) \end{aligned}$$

Applying the inverse Laplace transform on both sides of Equation (9), we get

$$v(x_1, x_2, ..., x_n, t) = A(x_1, x_2, ..., x_n, t) - \mathcal{L}^{-1}[s^{-\alpha} \mathcal{L}[N(v(x_1, x_2, ..., x_n, t))]], \tag{11}$$

where $A(x_1, x_2, ..., x_n, t)$ can be find using the initial and boundary conditions given in Equation (7). Consider a homotopy of Equation (11), $u(t,p) : \Omega \times [0,1] \to \mathbb{R}$ satisfying

$$\begin{aligned} H(u,p) =&(1-p)[u-\tilde{u}_0] + p\big[(u-A) \\ &+ \mathcal{L}^{-1}\big[s^{-\alpha}\mathcal{L}\big[N(v(x_1,x_2,...,x_n,t;p))\big]\big]\big] = 0,\ t \in \Gamma, \end{aligned} \tag{12}$$

where Γ is boundary of the domain Ω, p is an embedding parameter and $\tilde{u_0}$ is the boundary condition given in Equation (7) which can be freely chosen. From Equation (12), we can see that

$$\begin{aligned} &H(u,0) = u - \tilde{u_0} = 0 \ and \\ &H(u,1) = u - A + \mathcal{L}^{-1}\big[s^{-\alpha}\mathcal{L}\big[N(v(x_1,x_2,...,x_n,t;p))\big]\big] = 0. \end{aligned} \tag{13}$$

The Equation (13) shows that when p changes from 0 to 1, u changes from $\tilde{u_0}$ to v. Hence, the approximate solution of Equation (7) can be assumed to be

$$u(x_1,x_2,...,x_n,t;p) = \sum_{n=0}^{\infty} p^n u_n(x_1,x_2,...,x_n,t). \tag{14}$$

From Equations (12) and (14), we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x_1,x_2,...,x_n,t) = &\tilde{u}_0(x_1,x_2,...,x_n,t) \\ &- p\big[\tilde{u}_0(x_1,x_2,...,x_n,t) - A(x_1,x_2,...,x_n,t) \\ &+ \mathcal{L}^{-1}\big[s^{-\alpha}\mathcal{L}\big[N\big(\sum_{n=0}^{\infty} p^n u_n(x_1,x_2,...,x_n,t)\big)\big]\big]\big]. \end{aligned} \tag{15}$$

By equating the coefficients of powers of p on both sides of Equation (15), we get

$$\begin{aligned} u_0(x_1,x_2,...,x_n,t) &= \tilde{u}_0(x_1,x_2,...,x_n,t), \\ u_1(x_1,x_2,...,x_n,t) &= A(x_1,x_2,...,x_n,t) - \tilde{u}_0(x_1,x_2,...,x_n,t) \\ &\quad - \mathcal{L}^{-1}\big[s^{-\alpha}\mathcal{L}\big[N(\tilde{u}_0(x_1,x_2,...,x_n,t))\big]\big] \\ u_m(x_1,x_2,...,x_n,t) &= -\mathcal{L}^{-1}\big[s^{-\alpha}\mathcal{L}\big[N(\tilde{u}_{m-1}(x_1,x_2,...,x_n,t))\big]\big], \\ &\qquad \forall\ m \geq 2. \end{aligned} \tag{16}$$

Thus, the approximate solution of Equation (7) will be

$$v(x_1,x_2,...,x_n,t) = \lim_{p\to 1} \sum_{m=0}^{\infty} u_m(x_1,x_2,...,x_n,t). \tag{17}$$

The convergence of the above series has been discussed in He [1999].

4 A Solution of Liouville-Caputo Time Fractional Black-Scholes Equation by LHPM

The use of LHPM to solve the Liouville-Caputo time fractional Black-Scholes equation for European call options with n variables is the topic of discussion in this section.

Theorem 4.1. *The solution of n-dimensional Black-Scholes equation discussed in Equation* (6)*, is given by*

$$\begin{aligned}
&\mathcal{C}(S_1, S_2, ..., S_n, \tau) = \\
&e^{-r(T-\tau)}\Bigg[\max\Big(\sum_{i=1}^{n} a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)} - K, 0\Big) + (T-\tau)^{\alpha} e^{-nr(T-\tau)}\Delta \\
&+\frac{1}{2}(T-\tau)^{\alpha}\sum_{i=1}^{n}\left[\sigma_i^2 \max\left(a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)}, 0\right) E_{\alpha,\alpha+1}\Big(\frac{1}{2}\sigma_i^2(T-\tau)^{\alpha}\Big)\right] \\
&+\frac{1}{2}(T-\tau)^{2\alpha}\,\Gamma(\alpha+1)\,e^{-nr(T-\tau)}\,\Delta\,E_{\alpha,2\alpha+1}(w)\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij} \\
&-(T-\tau)^{\alpha}\,\Gamma(\alpha+1)\,e^{-nr(T-\tau)}\Delta\,E_{\alpha,\alpha+1}(w)\Bigg],
\end{aligned}$$

where

$$\Delta = \prod_{i=1}^{n} S_i e^{\frac{1}{2}\sigma_i^2(T-\tau)} \quad \textit{and} \quad w = \frac{1}{2}(T-\tau)^{\alpha}\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij}. \tag{18}$$

Proof. We have Liouville-Caputo time fractional Black-Scholes equation for n variables discussed in Equation (6) is given by

$$D_t^{\alpha} v = \frac{1}{2}\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j}, \tag{19}$$

where initial and boundary conditions are given by Equations (3)-(5). Consider

$$N(v(x_1, x_2, ..., x_n, t)) = \frac{1}{2}\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j}.$$

Thus, Equation (19) will be

$$D_t^{\alpha} v(x_1, x_2, ..., x_n, t) = N(v(x_1, x_2, ..., x_n, t)). \tag{20}$$

Now, applying Laplace transform with respect to time variable t on both sides of Equation (20), we get

$$\begin{aligned}\mathcal{L}\big[v(x_1,x_2,...,x_n,t)\big] =& s^{-\alpha}\max\Big(\sum_{i=1}^{n}\tilde{a}_i e^{x_i}-K,0\Big)\\ &+\frac{1}{2}s^{-\alpha}\mathcal{L}\Big[\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij}\frac{\partial^2 v}{\partial x_i\partial x_j}\Big].\end{aligned} \tag{21}$$

The inverse Laplace transform of Equation (21) is obtained as

$$\begin{aligned}v(x_1,x_2,...,x_n,t) =& \max\Big(\sum_{i=1}^{n}\tilde{a}_i e^{x_i}-K,0\Big)\\ &+\frac{1}{2}\mathcal{L}^{-1}\Big[s^{-\alpha}\mathcal{L}\Big[\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij}\frac{\partial^2 v}{\partial x_i\partial x_j}\Big]\Big].\end{aligned} \tag{22}$$

Applying a homotopy discussed in Equation (12) to the Equation (22), we get

$$\begin{aligned}&(1-p)[v(x_1,x_2,...,x_n,t;p)-\tilde{v}_0(x_1,x_2,...,x_n,t)]\\ &\quad+p\Big[v(x_1,x_2,...,x_n,t;p)-\max\Big(\sum_{i=1}^{n}\tilde{a}_i e^{x_i}-K,0\Big)\\ &\quad-\frac{1}{2}\mathcal{L}^{-1}\Big[s^{-\alpha}\mathcal{L}\Big[\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij}\frac{\partial^2 v}{\partial x_i\partial x_j}\Big]\Big]\Big]=0,\end{aligned} \tag{23}$$

where $p\in[0,1]$ is an embedding parameter and $\tilde{v}_0(x_1,x_2,...,x_n,t)$ can be chosen independently. Here we choose

$$\tilde{v}_0(x_1,x_2,...,x_n,t)=\max\Big(\sum_{i=1}^{n}\tilde{a}_i e^{x_i}-K,0\Big)+t^{\alpha}\prod_{i=1}^{n}e^{x_i}. \tag{24}$$

Hence, Equation (23) reduces to

$$\begin{aligned}v(x_1,x_2,...,x_n,t;p) =& \max\Big(\sum_{i=1}^{n}\tilde{a}_i e^{x_i}-K,0\Big)+t^{\alpha}\prod_{i=1}^{n}e^{x_i}\\ &-p\Big[t^{\alpha}\prod_{i=1}^{n}e^{x_i}-\frac{1}{2}\mathcal{L}^{-1}\Big[s^{-\alpha}\mathcal{L}\Big[\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij}\frac{\partial^2 v}{\partial x_i\partial x_j}\Big]\Big]\Big].\end{aligned} \tag{25}$$

From HPM, the solution of Equation (19) can be assumed as

$$v(x_1,x_2,...,x_n,t;p)=\sum_{m=0}^{\infty}p^m u_m(x_1,x_2,...,x_n,t). \tag{26}$$

Thus, Equations (25) and (26), yields

$$\sum_{m=0}^{\infty} p^m u_m(x_1, x_2, ..., x_n, t) = \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) + t^\alpha \prod_{i=1}^{n} e^{x_i} - p\Big[t^\alpha \prod_{i=1}^{n} e^{x_i} - \frac{1}{2}\mathcal{L}^{-1}\Big[s^{-\alpha}\mathcal{L}\Big[\sum_{i,j=1}^{n} \sigma_i\sigma_j\rho_{ij} \sum_{m=0}^{\infty} p^m \frac{\partial^2 u_m}{\partial x_i \partial x_j}\Big]\Big]\Big]. \tag{27}$$

By equating the coefficients of powers of p on both sides of Equation (27), we get

$$\begin{aligned} u_0(x_1, x_2, ..., x_n, t) =& \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) + t^\alpha \prod_{i=1}^{n} e^{x_i} \\ u_m(x_1, x_2, ..., x_n, t) =& \frac{t^{m\alpha}}{2^m \Gamma(m\alpha + 1)} \Big(\sum_{i=1}^{n} \sigma_i^{2m} \max(\tilde{a}_i e^{x_i}, 0)\Big) \\ &+ \frac{t^{(m+1)\alpha}\Gamma(\alpha + 1)}{2^m \Gamma((m+1)\alpha + 1)} \prod_{i=1}^{n} e^{x_i} \Big(\sum_{i,j=1}^{n} \sigma_i\sigma_j\rho_{ij}\Big)^m \\ &- \frac{t^{m\alpha}\Gamma(\alpha + 1)}{2^{m-1} \Gamma(m\alpha + 1)} \prod_{i=1}^{n} e^{x_i} \Big(\sum_{i,j=1}^{n} \sigma_i\sigma_j\rho_{ij}\Big)^{m-1}, \ \forall\ m \geq 1. \end{aligned}$$

Substituting these values in Equation (26), we obtain

$$\begin{aligned} v(x_1, x_2, ..., x_n, t; p) =& \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0\Big) + t^\alpha \prod_{i=1}^{n} e^{x_i} \\ &+ \sum_{m=0}^{\infty} p^{m+1} \Big[\frac{t^{(m+1)\alpha}}{2^{m+1}\Gamma((m+1)\alpha + 1)} d \\ &+ \frac{t^{(m+2)\alpha}\Gamma(\alpha + 1)}{2^{m+1}\Gamma((m+2)\alpha + 1)} \prod_{i=1}^{n} e^{x_i} \Big(\sum_{i,j=1}^{n} \sigma_i\sigma_j\rho_{ij}\Big)^{m+1} \\ &- \frac{t^{(m+1)\alpha}\Gamma(\alpha + 1)}{2^m \Gamma((m+1)\alpha + 1)} \prod_{i=1}^{n} e^{x_i} \Big(\sum_{i,j=1}^{n} \sigma_i\sigma_j\rho_{ij}\Big)^{m}\Big]. \end{aligned}$$

where $d = \sum_{i=1}^{n} \sigma_i^{2(m+1)} \max(\tilde{a}_i e^{x_i}, 0)$

Taking $p \to 1$ in above Equation and simplifying, we get

$$\begin{aligned} v(x_1, x_2, , ..., x_n, t) =& \max\Big(\sum_{i=1}^{n} \tilde{a}_i e^{x_i} - K, 0 \Big) + t^{\alpha} \prod_{i=1}^{n} e^{x_i} \\ &+ \frac{t^{\alpha}}{2} \sum_{i=1}^{n} \left(\sigma_i^2 \ \max(\tilde{a}_i e^{x_i}, 0) \ E_{\alpha,\alpha+1}\Big(\frac{1}{2}\sigma_i^2 t^{\alpha}\Big) \right) \\ &+ \frac{t^{2\alpha}}{2} \Gamma(\alpha+1) \prod_{i=1}^{n} e^{x_i} E_{\alpha,2\alpha+1}\left(\frac{t^{\alpha}}{2}\delta\right)\delta \\ &- t^{\alpha}\Gamma(\alpha+1) \prod_{i=1}^{n} e^{x_i} E_{\alpha,\alpha+1}\left(\frac{t^{\alpha}}{2}\delta\right), \end{aligned} \tag{28}$$

where $E_{a,b}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(ak+b)}$ is the Mittag-Leffler function, with $a > 0$ and $b \in \mathbb{R}$ and $\delta = \sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij}$. But $x_i = \ln(S_i) - \big(r - \frac{1}{2}\sigma_i^2\big)\tau, \mathcal{C}(x_1, x_2, ..., x_n, \tau) = e^{-r(T-\tau)} v(x_1, x_2, ..., x_n, \tau)$ and $t = T - \tau$, Equation (28) can be written as

$$\begin{aligned} &\mathcal{C}(S_1, S_2, ..., S_n, \tau) = \\ &e^{-r(T-\tau)} \Bigg[\max\Big(\sum_{i=1}^{n} a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)} - K, 0 \Big) + (T-\tau)^{\alpha} e^{-nr(T-\tau)} \Delta \\ &+ \frac{1}{2}(T-\tau)^{\alpha} \sum_{i=1}^{n} \left[\sigma_i^2 \max\left(a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)}, 0\right) \ E_{\alpha,\alpha+1}\Big(\frac{1}{2}\sigma_i^2 (T-\tau)^{\alpha}\Big) \right] \\ &+ \frac{1}{2}(T-\tau)^{2\alpha} \ \Gamma(\alpha+1) \ e^{-nr(T-\tau)} \ \Delta \ E_{\alpha,2\alpha+1}(w) \sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \\ &- (T-\tau)^{\alpha} \ \Gamma(\alpha+1) \ e^{-nr(T-\tau)} \Delta \ E_{\alpha,\alpha+1}(w) \Bigg], \end{aligned}$$

where Δ and w are given in Equation (18). □

Similarly, one can derive solution of Liouville-Caputo time fractional Black-Scholes equation for n variables of put option using LHPM. This formula is stated without proof.

Theorem 4.2. *The solution of n-dimensional Black-Scholes equation for European put option discussed in Section 2, is given by*

$$\begin{aligned}
&\mathcal{P}(S_1, S_2, ..., S_n, \tau) = \\
&e^{-r(T-\tau)}\Bigg[\max\Big(K - \sum_{i=1}^{n} a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)}, 0\Big) + (T-\tau)^{\alpha} e^{-nr(T-\tau)}\Delta \\
&+\frac{1}{2}(T-\tau)^{\alpha}\sum_{i=1}^{n}\Big[\sigma_i^2 \max\big(-a_i S_i e^{(r-\frac{1}{2}\sigma_i^2)(T-\tau)}, 0\big)\; E_{\alpha,\alpha+1}\Big(\frac{1}{2}\sigma_i^2(T-\tau)^{\alpha}\Big)\Big] \\
&+\frac{1}{2}(T-\tau)^{2\alpha}\;\Gamma(\alpha+1)\; e^{-nr(T-\tau)}\;\Delta\; E_{\alpha,2\alpha+1}(w)\sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{ij} \\
&-(T-\tau)^{\alpha}\;\Gamma(\alpha+1)\; e^{-nr(T-\tau)}\Delta\; E_{\alpha,\alpha+1}(w)\Bigg],
\end{aligned}$$

where Δ and w are given in Equation (18)*.*

Theorem 4.3. *S. J. Ghevariya, C. N. Patel and Fadugba [2022] Using the Laplace transform homotopy perturbation method (LTHPM), the n-dimensional Black-Scholes model for European options is given by*

$$\begin{aligned}
\mathcal{C}_{\text{LTHPM}} &= \max\Big(\sum_{i=1}^{n} a_i S_i - K, 0\Big) \\
&+ \sum_{\substack{i,j=1\\ i<j}}^{n}\rho_{ij}S_i^2\Big((\sigma_i^2+r)E_{2,\sigma_i^2+r}(T-\tau) - E_{1,\sigma_i^2+r}(T-\tau) + (T-\tau)\Big) \\
&+ \sum_{\substack{i,j=1\\ i<j}}^{n}\rho_{ij}S_j^2\Big((\sigma_j^2+r)E_{2,\sigma_j^2+r}(T-\tau) - E_{1,\sigma_j^2+r}(T-\tau) + (T-\tau)\Big) \\
&- rE_{1,-r}(T-\tau)\Big(\sum_{i=1}^{n}\max(a_i,0)S_i + \max\Big(\sum_{i=1}^{n} a_i S_i - K, 0\Big)\Big),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{P}_{\text{LTHPM}} &= \max\Big(K - \sum_{i=1}^{n} a_i S_i, 0\Big) \\
&+ \sum_{\substack{i,j=1 \\ i<j}}^{n} \rho_{ij} S_i^2 \Big((\sigma_i^2 + r) E_{2,\sigma_i^2+r}(T-\tau) - E_{1,\sigma_i^2+r}(T-\tau) + (T-\tau)\Big) \\
&+ \sum_{\substack{i,j=1 \\ i<j}}^{n} \rho_{ij} S_j^2 \Big((\sigma_j^2 + r) E_{2,\sigma_j^2+r}(T-\tau) - E_{1,\sigma_j^2+r}(T-\tau) + (T-\tau)\Big) \\
&- rE_{1,-r}(T-\tau)\Big(\sum_{i=1}^{n} \max(-a_i, 0) S_i - \max\Big(K - \sum_{i=1}^{n} a_i S_i, 0\Big)\Big),
\end{aligned}$$

where $E_{a,b}(t) = t^a \sum_{k=0}^{\infty} \frac{(bt)^k}{\Gamma(a+k+1)}$ *is the Mellin-Ross function with* $a, b > 0$.

Theorem 4.4. *[Haug, 2007, P.2] The Black-Scholes model for plain vanilla payoffs are given by*

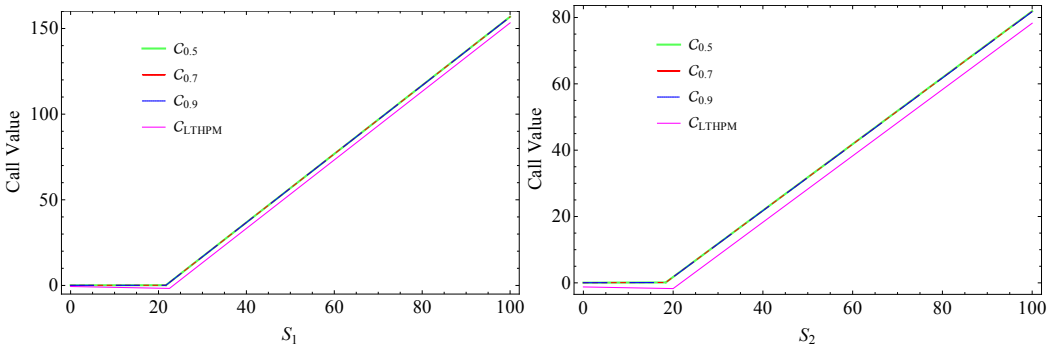
$$\begin{aligned}
\mathcal{C}_{\text{Vanilla}} &= SN(d_1) - Ke^{-r(T-t)} N(d_2) \\
\mathcal{P}_{\text{Vanilla}} &= Ke^{-r(T-t)} N(d_2) - SN(-d_1)
\end{aligned}$$

where $N(\cdot)$ *is cumulative distribution function of standard normal random variable and* $d_1 = \frac{\ln(S/K) - (r - \frac{1}{2}\sigma^2)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$.

5 An Illustrative Example

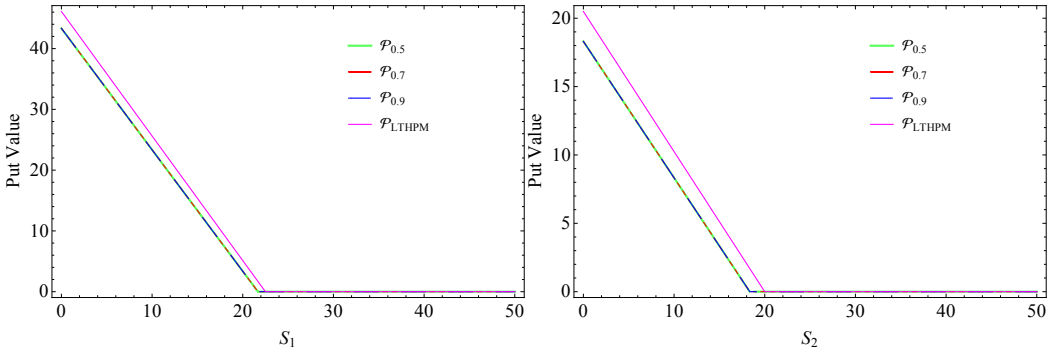
In this section, we consider Black-Scholes models using LTHPM and Liouville-Caputo time fractional derivative sense for $n = 2$. Further, the comparisons of these models for $n = 1$ with the Black-Scholes model for plain vanilla payoffs have been discussed. We examine call and put values against different values of α $(0.5, 0.7, 0.9)$ and asset prices. For that, take the striking price $K = 70$, the risk free interest rate, $r = 0.05$, $a_1 = 2, a_2 = 1, \sigma_1 = 0.05, \sigma_2 = 0.1, t = 0$ and time to expiration, $T = 0.5$. In Figures 1 & 2, we consider the correlation coefficient, $\rho = 0.25$. Figure 1(a) (Figure 2(a)) represents values of call (put) option against values of S_1 by taking $S_2 = 25$, while Figure 1(b) (Figure 2(b)) represents values of call (put) option against values of S_2 by taking $S_1 = 25$. Figures 3(a) & 3(b) represents values of call and put option against values of S_1 by taking $a_1 = 1$. Further, it can be seen from Figures 1 & 2 that the variations of call and put values for different values of α with the values from LTHPM are not far away. Also,

from Figure 3, we observe that the model using Liouville-Caputo time fractional derivative sense for $\alpha = 0.9$ coincide with the model for plain vanilla payoffs. This suggest that our model is more accurate than the model using LTHPM.



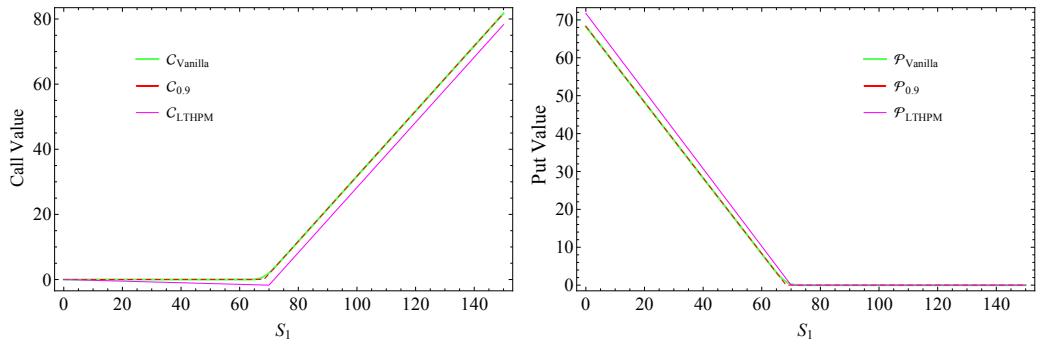
(a) $S_2 = 25$ (b) $S_1 = 25$

Figure 1: Call value with $n = 2$



(a) $S_2 = 25$ (b) $S_1 = 25$

Figure 2: Put value with $n = 2$



(a) Call (b) Put

Figure 3: $\mathcal{C}_{\text{Vanilla}}$, $\mathcal{C}_0.9$ and $\mathcal{C}_{\text{LTHPM}}$ with $n = 1$

Table 1 shows the relative errors of call option prices for $n = 2$ from Theorem 4.1 of different values of α with values from Theorem 4.3. For that take $K = 70, r = 0.05, \sigma_1 = 0.05, \sigma_2 = 0.1, \rho_{12} = 0.25, a_1 = 2, a_2 = 1, S_1 = 100, S_2 = 25, T = 0.5, t = 0$. From Theorem 4.3, we get $\mathcal{C}_{\text{LTHPM}} = 153.272$.

α	$\mathcal{C}_\alpha$	Error (%)
0.1	156.909	2.318
0.2	156.897	2.310
0.3	156.881	2.300
0.4	156.861	2.288
0.5	156.840	2.275
0.6	156.818	2.261
0.7	156.795	2.247
0.8	156.772	2.233
0.9	156.750	2.219
1	156.728	2.205

Table 1: Relative error $\mathcal{C}_\alpha$ and $\mathcal{C}_{\text{LTHPM}}$ with $n = 2$

Further, for put option, consider $S_1 = 10, S_2 = 12$. From Theorem 4.2 and Theorem 4.3, we get $\mathcal{P}_{\text{LTHPM}} = 38.938$ and $\mathcal{P}_\alpha = 36.314$. Hence the relative error for put option is 7.226%. Table 2 shows the the relative errors of call option prices for $n = 1$ from Theorems 4.1 & 4.4 of different values of α with values from Theorem 4.3. For that take $K = 70, r = 0.05, \sigma = 0.05, a_1 = 1, S = 150, T = 0.5, t = 0$. From Theorem 4.3 & 4.4, we get $\mathcal{C}_{\text{LTHPM}} = 153.272$ and $\mathcal{C}_{\text{Vanilla}} = 81.728$.

α	C_α	Error (%)	
		$\mathcal{C}_\alpha$ & $\mathcal{C}_{\text{Vanilla}}$	$\mathcal{C}_\alpha$ & $\mathcal{C}_{\text{LTHPM}}$
0.1	81.819	0.110	4.335
0.2	81.813	0.103	4.328
0.3	81.804	0.093	4.318
0.4	81.795	0.081	4.307
0.5	81.784	0.068	4.295
0.6	81.773	0.055	4.282
0.7	81.762	0.041	4.268
0.8	81.750	0.027	4.255
0.9	81.739	0.013	4.242
1	81.728	0.000	4.229

Table 2: Relative error $\mathcal{C}_\alpha$, $\mathcal{C}_{\text{LTHPM}}$ and $\mathcal{C}_{\text{Vanilla}}$ with $n = 1$

Further, for put option, consider $S = 10$. From Theorems 4.2, 4.3 and 4.4, we get $\mathcal{P}_{\text{LTHPM}} = 61.481, \mathcal{P}_\alpha = 58.278$ and $\mathcal{P}_{\text{Vanilla}} = 58.272$. Hence the relative error for $\mathcal{P}_{\text{Vanilla}}$ and $\mathcal{P}_\alpha$ is 0.011%, while relative error for $\mathcal{P}_\alpha$ and $\mathcal{P}_{\text{LTHPM}}$ is 5.497%.

6 Discussion

The famous Black-Scholes model is used for valuing option price with single asset as given in Theorem 4.4 derived by Fischer Black and Myron Scholes in 1973. Many authors have tried to improve it and in 2017, K. Trachoo,W. Sawangtong and P. Sawangtong presented a formula for valuing options based on two assets using LTHPM K. Trachoo, W. Sawangtong and Sawangtong [2017]. In 2022, S. J. Ghevariya, C. N. Patel and S. E. Fadugba had generalized it for n assets S. J. Ghevariya, C. N. Patel and Fadugba [2022]. The researchers have also presented a formula for valuing option price with two assets using Liouville-Caputo fractional derivative Prathumwan and Trachoo [2020], P. Sawangtong, K. Trachoo, W. Sawangtong and Wiwattanapataphee [2018]. In this paper we derived the formula for valuing option price based on n assets using Liouville-Caputo fractional derivative.

For $n = 2$, examples were presented to analyze the option prices derived from Liouville-Caputo time fractional Black-Scholes equation and LTHPM given in Theorem 4.3 by considering different values of α, while these option prices compared with plain vanilla options given in Theorem 4.4 by considering $n = 1$. Figures 1 and 2 shows values of call and put options with different values of α against the underlying asset prices, in the context of fixed values of the parameters: $K,\ r,\ a_1,\ a_2,\ \sigma_1,\ \sigma_2, T, \rho$. Figure 3 indicates the value of option prices

from Liouville-Caputo time fractional Black-Scholes equation, LTHPM and plain vanilla. It is observed from all Figures that the values of call option increase and the value of put option decrease linearly with values of asset prices.

7 Conclusion

Tables 1 & 2 show the performance, accuracy and validation of solution derived in the context of Liouville-Caputo fractional derivative sense with respect to LTHPM and plain vanilla options. Figures 1 & 2 indicate how close the call and put values derived by authors and LTHPM for different values of α. Also Figure 3 shows that our solution is quite close to the well known model for plain vanilla options for single asset.

The formulas derived in this paper are limited to assets paying no dividend. Some extensions of the methodology can be explored for further research instead of linear payoffs and assets paying no dividend.

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B-Gabor type frames in separable Hilbert spaces

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Abstract

It is guaranteed that Gabor like structured frames do exist in any finite dimensional Hilbert space via an invertible map from $l^2(\mathbb{Z}_N)$ J Thomas and Nambudiri [2022]. Hence the question: whether it is possible to obtain structured class of frames in separable Hilbert spaces? is relevant. In this article we obtain a structured class of frames for separable Hilbert spaces which are generalizations of Gabor frames for $L^2(\mathbb{R})$ in their construction aspects. We call them as B-Gabor type frames and present a characterization of the frame operators associated with these frames when B is a unitary map. Some significant properties of the associated frame operators are discussed.
Keywords: Gabor frame; frame operator; translation; modulation.
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1 Introduction

Elegant and fast growing theory of frames in Hilbert spaces clenches its own space in both pure and applied mathematics due to its wide applications. Time-frequency analysis of signals in $L^2(\mathbb{R})$, as suggested by D. Gabor [1946] in *Theory of Communication*, requires a special system of the form $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$, where $g \in L^2(\mathbb{R})$ and E_b, T_a $(a, b > 0)$ are the modulation and translation operators on $L^2(\mathbb{R})$ respectively. In their study of non harmonic Fourier series Duffin and Schaeffler [1952] were introduced frames in Hilbert spaces. M. Janssen [1981] designed it an independent topic of mathematical investigation by his outstanding work. Frames were brought to life by I. Daubechies and Meyer [1986] with the fundamental works and put forth the idea of combining Gabor analysis with frame theory.

Ample generalizations of the concept of frames have been proposed; frame of subspaces Cazassa and Kutyniok [2004], pseudo-frames Li and Ogawa [2004], oblique frames Christensen and Eldar [2004] and so on, where in all, the mathematical theory of Gabor frames (also known as Weyl-Heisenberg frames) plays the crucial role. These frames are very specific as they are being generated by translations and modulations of a single vector in the space concerned and hence they have a specific structure. Another class of structured frames in $L^2(\mathbb{R})$ are wavelet frames Nambudiri and Parthasarathy [2021]. Motivated from these class of structured frames, here we introduce a structured class of frames for separable Hilbert spaces $\mathcal{H}$ which are generalizations of Gabor frames for $L^2(\mathbb{R})$ in their construction aspects. The concept of Gabor type unitary systems in D. Han [2000] is generalized by considering a system of invertible operators in place of unitary systems hence we obtain a class of structured frames in $\mathcal{H}$, we call them as B-Gabor type frames.

A foremost object in frame theory is the frame operator associated with a given frame. In particular, Gabor frame operators, which are very special in their construction, are acquiring notable research attention. A complete characterization of Gabor frame operators on $L^2(\mathbb{R})$ is recently appeared in Nambudiri and Parthasarathy [2013]. In this paper we present a characterization of the frame operators associated with B-Gabor type frames when B is a unitary map.

The arrangement of this article is, Section 2 provides some basic definitions and results which are very essential for the study. B-Gabor type frames in separable Hilbert spaces and a complete characterization of frame operators associated with these frames is presented in Section 3. Our basic references for both abstract frame theory and the theory of Gabor frames are Christensen [2016] and Gröchenig [2001].

2 Preliminaries

A countable sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a Hilbert space $\mathcal{H}$ is said to be a *frame* in $\mathcal{H}$, if there are constants $\alpha, \beta > 0$, such that

$$\alpha\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq \beta\|f\|^2 \ \forall\, f \in \mathcal{H}.$$

These constants $\alpha,\ \beta$ are called *frame bounds*. $\{f_k\}_{k=1}^{\infty}$ is called a *tight frame*, if the frame bounds are equal and is called a *Parseval frame or normalized tight frame* when $\alpha = \beta = 1$. Whenever a sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ satisfies the upper frame inequality, then it is said to be a *Bessel sequence* or a *semi-frame sequence*.

If $\{f_k\}_{k=1}^{\infty}$ is a frame in a Hilbert space $\mathcal{H}$, then the map S defined using this collection by $Sf = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$ for all $f \in \mathcal{H}$ is a bounded linear operator on $\mathcal{H}$, called the *frame operator* associated with the frame $\{f_k\}_{k=1}^{\infty}$. It is easy to observe that the frame operator of a tight frame is a scalar multiple of the identity operator and that of a normalized tight frame is the identity operator, Christensen [2016]. For a Bessel sequence $\{f_k\}_{k=1}^{\infty}$, the operator defined above may not be invertible and it is called a *semi-frame operator*.

Remark 2.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S and frame bounds $\alpha,\ \beta$ in a Hilbert space $\mathcal{H}$. Then,*

(i) S is bounded, invertible, self-adjoint and positive.

(ii) $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a frame with frame bounds $\beta^{-1},\ \alpha^{-1}$ and $\{S^{-1/2}f_k\}_{k=1}^{\infty}$ is a normalized tight frame.

(iii) If α and β are the optimal frame bounds for $\{f_k\}_{k=1}^{\infty}$, then the bounds β^{-1}, α^{-1} are the optimal frame bounds for $\{S^{-1}f_k\}_{k=1}^{\infty}$. The frame operator for $\{S^{-1}f_k\}_{k=1}^{\infty}$ is S^{-1}. Further we have $\beta^{-1}I \leq S^{-1} \leq \alpha^{-1}I$.

Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S in $\mathcal{H}$, then the frame $\{S^{-1}f_k\}_{k=1}^{\infty}$ is called the *(canonical) dual frame* of the frame $\{f_k\}_{k=1}^{\infty}$. As follows, every frame in $\mathcal{H}$ admits the *frame decomposition* in two ways.

Theorem 2.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S in a Hilbert space $\mathcal{H}$. Then for all $f \in \mathcal{H}$, $f = \sum_{k=1}^{\infty}\langle f, S^{-1}f_k\rangle f_k$ and $f = \sum_{k=1}^{\infty}\langle f, f_k\rangle S^{-1}f_k$. Both the series converge unconditionally for all $f \in \mathcal{H}$.*

Gabor frames in $L^2(\mathbb{R})$ are generated by two classes of operators on $L^2(\mathbb{R})$, namely the translation and modulation operators. For $a, b \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$,

the translation operator T_a on $L^2(\mathbb{R})$ is defined by $(T_a f)(x) = f(x-a),\ x \in \mathbb{R}$ and the modulation operator E_b on $L^2(\mathbb{R})$ by $(E_b f)(x) = e^{2\pi i b x} f(x),\ x \in \mathbb{R}$.

A frame in $L^2(\mathbb{R})$ of the form $\{E_{mb} T_{na} g : m, n \in \mathbb{Z}\}$ for some non zero g in $L^2(\mathbb{R})$ (known as *window function or generating function*) and $a, b > 0$ is called a *Gabor frame or Weyl-Heisenberg frame*. Gabor analysis aims at representing functions $f \in L^2(\mathbb{R})$ as superposition of translated and modulated versions of a fixed function $g \in L^2(\mathbb{R})$.

3 Gabor type frames and operators in separable Hilbert spaces

We begin with a simple finding, analogous to Corollary 5.3.2 in Christensen [2016], on the interplay of bounded linear operators between separable Hilbert spaces and is given as a lemma which is very essential for our further discussion.

Lemma 3.1. *Every invertible bounded linear operator between two separable Hilbert spaces maps frames in one to frames in the other.*

Proof. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be an invertible bounded linear operator between separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and let $B = (A^*)^{-1}$. Then $B : \mathcal{H}_1 \to \mathcal{H}_2$ is also an invertible bounded linear operator and $A = (B^{-1})^*$. Let $\{g_k\}_{k\in\mathbb{N}}$ be a frame in $\mathcal{H}_1$ with frame bounds $0 < \alpha \leq \beta < \infty$. Then for all $f \in \mathcal{H}_1$,

$\alpha\|f\|^2 \leq \sum\limits_{k\in\mathbb{N}} |\langle f, g_k\rangle|^2 \leq \beta\|f\|^2$

$\Rightarrow \alpha\|B^{-1}x\|^2 \leq \sum\limits_{k\in\mathbb{N}} |\langle B^{-1}x, g_k\rangle|^2 \leq \beta\|B^{-1}x\|^2$ for all $x \in \mathcal{H}_2$

$\Rightarrow \|B\|^{-2}\alpha\|x\|^2 \leq \sum\limits_{k\in\mathbb{N}} |\langle x, B^{-1*}g_k\rangle|^2 \leq \beta\|B^{-1}\|^2\|x\|^2$ for all $x \in \mathcal{H}_2$

since $\alpha\|x\|^2 \leq \alpha\|B\|^2\|B^{-1}x\|^2 \Rightarrow \|B\|^{-2}\alpha\|x\|^2 \leq \alpha\|B^{-1}x\|^2$.

Thus $\{B^{-1*}g_k\}_{k\in\mathbb{N}} = B^{-1*}(\{g_k\}_{k\in\mathbb{N}}) = A(\{g_k\}_{k\in\mathbb{N}})$ is frame in $\mathcal{H}_2$ with frame bounds $0 < \|B\|^{-2}\alpha \leq \beta\|B^{-1}\|^2 < \infty$. □

Note that, if $\{g_k\}_{k\in\mathbb{N}}$ is a frame in the Hilbert space $\mathcal{H}_1$, with frame operator S and if $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded invertible map, then $\{Ag_k\}_{k\in\mathbb{N}}$ is a frame in $\mathcal{H}_2$ and has frame operator ASA^* Nambudiri and Parthasarathy [2012].

For a couple of invertible bounded linear operators B and C on a separable Hilbert space $\mathcal{H}$, the family $(B, C, x_0) = \{B^m C^n x_0 : m, n \in \mathbb{Z}\}$ generated by a fixed x_0 in $\mathcal{H}$ is of interest of discussion here. We look at different aspects of these systems when they yield Bessel sequences or frames in $\mathcal{H}$. The adjoint and the inverse of each one of the operator acting in this system admits an interesting relation as follows.

Lemma 3.2. *Let B and C be two invertible bounded linear operators on a separable Hilbert space $\mathcal{H}$. If, for $x_0 \in \mathcal{H}$, $\{B^m C^n x_0 : m, n \in \mathbb{Z}\}$ is a Bessel sequence in $\mathcal{H}$, then $B^m S_B (B^*)^m = S_B$ and $C^n S_C (C^*)^n = S_C$ for all $m, n \in \mathbb{Z}$ where S_B and S_C are semi-frame operators of the Bessel sequences $\{B^m x_0 : m \in \mathbb{Z}\}$ and $\{C^n x_0 : n \in \mathbb{Z}\}$ respectively.*

Proof. The proof follows by observing

$$\begin{aligned} S_B B^* x &= \sum_{m\in\mathbb{Z}} \langle B^* x, B^m x_0\rangle B^m x_0 \\ &= \sum_{m\in\mathbb{Z}} \langle x, B^{m+1} x_0\rangle B^m x_0 \\ &= \sum_{k\in\mathbb{Z}} \langle x, B^k x_0\rangle B^{k-1} x_0 \\ &= B^{-1}(\sum_{k\in\mathbb{Z}} \langle x, B^k x_0\rangle B^k x_0) \\ &= B^{-1} S_B x \text{ for all } x \in \mathcal{H}. \end{aligned}$$

Similarly it is true that, $S_C C^* x = C^{-1} S_C x$ for all $x \in \mathcal{H}$. Now the conclusion follows easily. □

Here is another interesting observation,

Lemma 3.3. *Let B and C be two invertible bounded linear operators on a separable Hilbert space $\mathcal{H}$. If the family $\{B^m C^n x_0 : m, n \in \mathbb{Z}\}$, $x_0 \in \mathcal{H}$, is a Bessel sequence in $\mathcal{H}$ with semi-frame operator S, then $S(B^*)^l = (B^{-1})^l S$ for all $l \in \mathbb{Z}$.*

Proof. For each $l \in \mathbb{Z}$, the image of the frame $\mathcal{B} = \{B^m C^n x_0 : m, n \in \mathbb{Z}\}$ under the operator B^l is $\mathcal{B}$ itself since $B^l(\mathcal{B}) = B^l(\{B^m C^n x_0 : m, n \in \mathbb{Z}\}) = \{B^{m+l} C^n x_0 : m, n \in \mathbb{Z}\} = \{B^k C^n x_0 : k, n \in \mathbb{Z}\} = \mathcal{B}$. Since the frame operator of this image frame is $B^l S (B^*)^l$ we obtain $B^l S (B^*)^l = S$ for all $l \in \mathbb{Z}$. □

The rest of our discussion focuses on specific operators B and C in the system $(B, C, x_0) = \{B^m C^n x_0 : m, n \in \mathbb{Z}\}$ which are generated by translation and modulation operators on $L^2(\mathbb{R})$ through invertible bounded linear operators $B : L^2(\mathbb{R}) \to \mathcal{H}$. We will use following definitions for upcoming discussions and results.

Definition 3.1. *For an invertible bounded linear operator $B : L^2(\mathbb{R}) \to \mathcal{H}$ and a parameter $\alpha \in \mathbb{R}$, we define B-translation T_α^B on $\mathcal{H}$ by $T_\alpha^B = B T_\alpha B^{-1}$ and B-modulation E_α^B on $\mathcal{H}$ by $E_\alpha^B = B E_\alpha B^{-1}$ where T_α and E_α are respectively the translation and modulation operators on $L^2(\mathbb{R})$.*

Following properties about B-translation T_α^B and B-modulation E_α^B can be easily acquired from the definition.

1. The operators T_α^B and E_α^B are invertible operators on $\mathcal{H}$. They are unitary if $B : L^2(\mathbb{R}) \to H$ is unitary.

2. For all $k \in \mathbb{Z}$, $(T_\alpha^B)^k = T_{k\alpha}^B$ and $(E_\alpha^B)^k = E_{k\alpha}^B$.
   Hence $(E_b^B)^m (T_a^B)^n = E_{mb}^B T_{na}^B$ for all $m, n \in \mathbb{Z}$.

3. Using the commutator relation of E_b and T_a, we see $E_b^B T_a^B = e^{2\pi iba} T_a^B E_b^B$.

4. Since $T_a^* = T_{-a} = (T_a)^{-1}$ we have, $(T_a^B)^* = T_{-a}^{(B^*)^{-1}}$
   if B is self adjoint operator $(T_a^B)^* = T_{-a}^{B^{-1}}$,
   if B is unitary operator $(T_a^B)^* = T_{-a}^B$.
   Similarly, since $E_b^* = E_{-b} = (E_b)^{-1}$, it follows that $(E_a^B)^* = E_{-a}^{(B^*)^{-1}}$,
   if B is self adjoint operator $(E_a^B)^* = E_{-a}^{B^{-1}}$,
   if B is unitary operator $(E_a^B)^* = E_{-a}^B$

Definition 3.2. *For $a > 0$ and $b > 0$, the family $\{E_{mb}^B T_{na}^B\ x_0 : m, n \in \mathbb{Z}\}$ generated by $x_0 \in \mathcal{H}$ is called a B-Gabor type family in $\mathcal{H}$. Such a family is called a B-Gabor type frame (B-Gabor type Bessel sequence) if it forms a frame (Bessel sequence) in $\mathcal{H}$, where $B : L^2(\mathbb{R}) \to \mathcal{H}$ is invertible.*

Proposition 3.1. *Suppose that $B : L^2(\mathbb{R}) \to \mathcal{H}$ is an invertible map. The family $\{E_{mb}^B T_{na}^B\ x_0 : m, n \in \mathbb{Z}\}$ forms a B-Gabor type frame in $\mathcal{H}$ if and only if $\{E_{mb} T_{na} B^{-1} x_0 : m, n \in \mathbb{Z}\}$ forms a Gabor frame in $L^2(\mathbb{R})$.*

Proof. Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ is invertible. Assume that the family $\{E_{mb}^B T_{na}^B\ x_0 : m, n \in \mathbb{Z}\}$ forms a B-Gabor type frame in $\mathcal{H}$. By lemma 3.1, $B^{-1}\{E_{mb}^B T_{na}^B\ x_0 : m, n \in \mathbb{Z}\}$ is a frame in $L^2(\mathbb{R})$.
But $B^{-1}\{E_{mb}^B T_{na}^B\ x_0 : m, n \in \mathbb{Z}\} = \{E_{mb} T_{na} B^{-1} x_0 : m, n \in \mathbb{Z}\}$, is a Gabor frame in $L^2(\mathbb{R})$.

Conversely assume that, $\{E_{mb} T_{na} g_0 : m, n \in \mathbb{Z}\}$ forms a Gabor frame in $L^2(\mathbb{R})$. Again by lemma 3.1, $B\{E_{mb} T_{na}\ g_0 : m, n \in \mathbb{Z}\} = \{B E_{mb} B^{-1} B T_{na} B^{-1} B g_0 : m, n \in \mathbb{Z}\} = \{E_{mb}^B T_{na}^B B g_0 : m, n \in \mathbb{Z}\}$ is a B-Gabor type frame in $\mathcal{H}$. □

Following is a useful observation about the existence of B-Gabor type Parseval frames in separable Hilbert space $\mathcal{H}$.

Proposition 3.2. *For each pair of B-translation and B-modulation parameters $a, b > 0$ satisfying $0 < ab \leq 1$, there exists B-Gabor type frame in $\mathcal{H}$ with BB^* as its frame operator.*

Proof. For each pair of translation and modulation parameters $a, b > 0$ satisfying $0 < ab \leq 1$, there exists a tight Gabor frame in $L^2(\mathbb{R})$ with identity operator as frame operator (see Nambudiri and Parthasarathy [2013]), say $\{E_{mb} T_{na} g_0 :$

$m, n \in \mathbb{Z}\}$. Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ is a bounded invertible linear map. Then image of this Gabor frame $\{E_{mb}T_{na}g_0 : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ under B is a Gabor type frame in $\mathcal{H}$ with frame operator as $BIB^* = BB^*$. □

Corolary 3.1. *In the above proposition, if B is a unitary operator then the identity operator I is the frame operator of the B-Gabor type frame in $\mathcal{H}$ and hence it is a Parseval frame in $\mathcal{H}$.*

If the family $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$, $x_0 \in \mathcal{H}$ is a B-Gabor type frame (Bessel sequence) in $\mathcal{H}$ then its frame operator S, defined by,
$S(x) = \sum_{m,n\in\mathbb{Z}} \langle x, E^B_{mb}T^B_{na}x_0\rangle E^B_{mb}T^B_{na}x_0$, for all $x \in \mathcal{H}$ is called *B-Gabor type frame operator (B-Gabor type semi-frame operator).*

We look into some properties of these operators and present characterization of such operators. In the above situation, if S is the frame operator of the Gabor frame $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ and if S' is the frame operator of the B-Gabor type frame $\{E^B_{mb}T^B_{na}Bg : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$, then

$$\begin{aligned} S'x &= \sum_{k\in\mathbb{N}} \langle x, E^B_{mb}T^B_{na}Bg\rangle E^B_{mb}T^B_{na}Bg \\ &= B(\sum_{k\in\mathbb{N}} \langle B^*x, E_{mb}T_{na}g\rangle E_{mb}T_{na}g) \\ &= BSB^*(x), \text{ for all } x \in \mathcal{H}. \end{aligned}$$

Now we give a result analogous to lemma 12.3.1 in Christensen [2016], which says that the semi-frame operator S of a Bessel sequence $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ commutes with the involved modulation and translation operators. We will prove that, if $B : L^2(\mathbb{R}) \to \mathcal{H}$ is a unitary operator then the semi-frame operator S for a B-Gabor type Bessel sequence $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$ commutes with involved B-modulation and B-translation operators.

Lemma 3.4. *Let $\mathcal{H}$ be a separable Hilbert space, $B : L^2(\mathbb{R}) \to \mathcal{H}$ is a unitary operator. For $x_0 \in \mathcal{H}$ and $a, b > 0$, if $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ is a B-Gabor type Bessel sequence in $\mathcal{H}$ with semi-frame operator S. Then the following holds:*

1. *$SE^B_{mb}T^B_{na} = E^B_{mb}T^B_{na}S$ for all $m, n \in \mathbb{Z}$.*

2. *If $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ is a B-Gabor type frame for $\mathcal{H}$, then $S^{-1}E^B_{mb}T^B_{na} = E^B_{mb}T^B_{na}S^{-1}$ for all $m, n \in \mathbb{Z}$.*

Proof. From the definition of a B-Gabor type semi-frame operator S on $\mathcal{H}$ and from the properties of B-modulation and B-translation operators,

we have for all $x \in \mathcal{H}$,

$$\begin{aligned} SE^B_{mb}T^B_{na}x &= \sum_{m',n' \in \mathbb{Z}} \langle E^B_{mb}T^B_{na}x, E^B_{m'b}T^B_{n'a}x_0\rangle E^B_{m'b}T^B_{n'a}x_0 \\ &= \sum_{m',n' \in \mathbb{Z}} \langle x, T^B_{-na}E^B_{-mb}E^B_{m'b}T^B_{n'a}x_0\rangle E^B_{m'b}T^B_{n'a}x_0 \\ &= \sum_{m',n' \in \mathbb{Z}} \langle x, T^B_{-na}E^B_{(m'-m)b}T^B_{n'a}x_0\rangle E^B_{m'b}T^B_{n'a}x_0 \end{aligned}$$

using commutator relation of E_{mb} and T_{na} properly and performing the suitable change of variables, a direct computation shows that,

$$SE^B_{mb}T^B_{na}x = E^B_{mb}T^B_{na}Sx$$

This proves (1). In order to prove (2) we recall the fact that, frame operator is invertible. Hence, whenever $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ is a B-Gabor type frame for $\mathcal{H}$; the result follows by applying S^{-1} to both sides of (1). □

In particular we can observe that, the frame operator S of the B-Gabor type frame $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$ commutes with all B-translations T^B_{na} and all B-modulations E^B_{mb}. Construction of frames with a given operator as their frame operator is an interesting theme having practical relevance in the context of frames. As shown in Nambudiri and Parthasarathy [2012], positive and invertible operators can become frame operators on separable Hilbert spaces. However, in view of lemma 3.4, those operators on $\mathcal{H}$ which are positive and invertible but fail to commute with certain B-translation and B-modulation operators cannot correspond to any B-Gabor type frame in $\mathcal{H}$. Here, we present a complete characterization of frame operators of B-Gabor type frame in $\mathcal{H}$.

Theorem 3.1. *A bounded linear operator on a separable Hilbert space $\mathcal{H}$ can be realized as a B-Gabor type frame operator if and only if it is positive, invertible and commutes with all the B-translations T^B_{na} and all B-modulations E^B_{mb}. Where $B : L^2(\mathbb{R}) \to \mathcal{H}$ is a unitary operator.*

Proof. Assume that $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ is a B-Gabor type frame in $\mathcal{H}$ with frame operator S, then it is clear that S is positive, invertible bounded linear operator on $\mathcal{H}$ which commutes with, from lemma 3.4, all the B-translations T^B_{na} and all B-modulations E^B_{mb}.

Conversely assume that S is positive, invertible bounded linear operator on $\mathcal{H}$ which commutes with all the B-translations T^B_{na} and all the B-modulations E^B_{mb}. Then its positive square root $S^{1/2}$ is also having these properties. Now by corollary 3.1, there is a B-Gabor type frame $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$ with

identity operator I as the frame operator. Invertibility and commutativity of $S^{1/2}$ ensures that $\{S^{1/2}E^B_{mb}T^B_{na}x_0 = E^B_{mb}T^B_{na}S^{1/2}x_0 : m,n \in \mathbb{Z}\}$ is also a B-Gabor type frame.

Moreover for all $x \in \mathcal{H}$,

$$\sum_{m,n\in\mathbb{Z}} \langle x, E^B_{mb}T^B_{na}S^{1/2}x_0\rangle E^B_{mb}T^B_{na}S^{1/2}x_0 = S^{1/2}\sum_{m,n\in\mathbb{Z}} \langle S^{1/2}x, E^B_{mb}T^B_{na}x_0\rangle E^B_{mb}T^B_{na}x_0$$
$$= S^{1/2}I(S^{1/2}x) = S(x).$$

Hence the frame operator of $\{E^B_{mb}T^B_{na}S^{1/2}x_0 : m,n \in \mathbb{Z}\}$ is nothing but S. □

Two more characterization of B-Gabor type frame operator in $\mathcal{H}$ with independent proofs using the idea in J. Laurence [2005] is given below. Here note that the operator B need not be unitary instead an invertible operator will work.

Theorem 3.2. *A bounded linear operator S on a separable Hilbert space $\mathcal{H}$ is a B-Gabor type frame operator if and only if it is of the form $S = TT^*$ where $T = BA$, A is a surjective operator on $L^2(\mathbb{R})$ which commutes with some translation T_a and some modulation E_b for some $a, b > 0$ with $ab \leq 1$ and $B : L^2(\mathbb{R}) \to \mathcal{H}$ is invertible.*

Proof. Assume that $S = TT^*$, where $T = BA$, $B : L^2(\mathbb{R}) \to \mathcal{H}$ is invertible and A is a surjective operator on $L^2(\mathbb{R})$ which commutes with some translation T_a and some modulation E_b for some $a, b > 0$ with $ab \leq 1$. There is always a Parseval Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$. Surjectivity and commutativity of A ensure that $A(\{E_{mb}T_{na}g\}) = \{AE_{mb}T_{na}g = E_{mb}T_{na}Ag : m,n \in \mathbb{Z}\}$ is also a Gabor frame in $L^2(\mathbb{R})$. Hence $B(\{E_{mb}T_{na}Ag : m,n \in \mathbb{Z}\})$ is a B-Gabor type frame in $\mathcal{H}$.

Let $BAg = x_0$. Then for all $x \in \mathcal{H}$,

$$\sum_{m,n\in\mathbb{Z}} \langle x, E^B_{mb}T^B_{na}x_0\rangle E^B_{mb}T^B_{na}x_0$$
$$= \sum_{m,n\in\mathbb{Z}} \langle x, BE_{mb}T_{na}B^{-1}BAg\rangle BE_{mb}T_{na}B^{-1}BAg$$
$$= \sum_{m,n\in\mathbb{Z}} \langle x, BE_{mb}T_{na}Ag\rangle BE_{mb}T_{na}Ag$$
$$= \sum_{m,n\in\mathbb{Z}} \langle x, BAE_{mb}T_{na}g\rangle BAE_{mb}T_{na}g$$
$$= BA\sum_{m,n\in\mathbb{Z}} \langle A^*B^*x, E_{mb}T_{na}g\rangle E_{mb}T_{na}g$$
$$= BAA^*B^*x = TT^*(x) = S(x).$$

Hence the frame operator of $\{E^B_{mb}T^B_{na}x_0 : m,n \in \mathbb{Z}\}$ is nothing but S.

Conversely, if S is the frame operator of a B-Gabor type frame $\{E^B_{mb}T^B_{na}x_0 : m,n \in \mathbb{Z}\}$ in $\mathcal{H}$, then $B^{-1}(\{E^B_{mb}T^B_{na}x_0 : m,n \in \mathbb{Z}\})$ is a Gabor frame in $L^2(\mathbb{R})$. By Theorem 7 in Nambudiri and Parthasarathy [2021], the frame operator S' of this Gabor frame takes the form $S' = AA^*$ where A is surjective and commutes with all T_{na} and E_{mb}. Since B maps this Gabor frame to the B-Gabor type frame

$\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$, the resulting frame operator is $BS'B^* = BAA^*B^*$. Thus $S = BA(BA)^*$, as desired. □

In the next characterization of B-Gabor type frame operator, note that the map A need not be commutative with modulations and translations and in the proof we use only operator algebra.

Theorem 3.3. *An operator $S : \mathcal{H} \to \mathcal{H}$ is a B-Gabor type frame operator if and only if there is a surjective map $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and an invertible operator $B : L^2(\mathbb{R}) \to \mathcal{H}$ such that B^*B and AA^* are Gabor frame operator on $L^2(\mathbb{R})$, with the property $S = \eta\eta^*$, where $\eta = BA$.*

Proof. Suppose that S is a B-Gabor type frame operator on $\mathcal{H}$. Then there is a Gabor frame operator S' on $L^2(\mathbb{R})$ and an invertible operator $B : L^2(\mathbb{R}) \to \mathcal{H}$ so that $BS'B^* = S$. Since S' is a Gabor frame operator on $L^2(\mathbb{R})$ by Theorem 7 of Nambudiri and Parthasarathy [2021], $S' = AA^*$ for some surjective map A on $L^2(\mathbb{R})$. Thus $S = BAA^*B^* = \eta\eta^*$, where $\eta = BA$.
Conversely assume that $S = \eta\eta^*$ with the properties in hypothesis. Since AA^* is a Gabor frame operator on $L^2(\mathbb{R})$, the image of this frame under B is a B-Gabor type frame and having the frame operator $S = B(AA^*)B^*$, hence S is a B-Gabor type frame operator on $\mathcal{H}$. □

Now we make an interesting specific observation on Gabor frames in $L^2(\mathbb{R})$. Let $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ be a Gabor frame in $L^2(\mathbb{R})$. Since $ab \leq 1$, (by interchanging the parameters a and b,) there is $\tilde{g} \in L^2(\mathbb{R})$ such that $\{E_{na}T_{mb}\tilde{g} : m, n \in \mathbb{Z}\}$ is also a Gabor frame in $L^2(\mathbb{R})$. Fourier transform is one of the very important operator in harmonic analysis (see Christensen [2016]). The commutator relations of the Fourier transform $\mathcal{F}$ on $L^2(\mathbb{R})$ with translations and modulations yield

$$\begin{aligned}\mathcal{F}(\{E_{na}T_{mb}\tilde{g} : m, n \in \mathbb{Z}\}) &= \{\mathcal{F}E_{na}T_{mb}\tilde{g} : m, n \in \mathbb{Z}\}\\ &= \{\mathcal{F}E_{na}\mathcal{F}^{-1}\mathcal{F}T_{mb}\mathcal{F}^{-1}(\mathcal{F}\tilde{g}) : m, n \in \mathbb{Z}\}\\ &= \{E^{\mathcal{F}}_{na}T^{\mathcal{F}}_{mb}(\mathcal{F}\tilde{g}) : m, n \in \mathbb{Z}\}, \mathcal{F}\text{-Gabor type frame}\end{aligned}$$

$$\begin{aligned}\text{Again, } \mathcal{F}(\{E_{na}T_{mb}\tilde{g} : m, n \in \mathbb{Z}\}) &= \{\mathcal{F}E_{na}T_{mb}\tilde{g} : m, n \in \mathbb{Z}\}\\ &= \{\mathcal{F}E_{na}T_{mb}\mathcal{F}^{-1}(\mathcal{F}\tilde{g}) :\in \mathbb{Z}\}\\ &= \{T_{na}\mathcal{F}\mathcal{F}^{-1}E_{-mb}(\mathcal{F}\tilde{g}) :\in \mathbb{Z}\}\\ &= \{T_{na}E_{-mb}(\mathcal{F}\tilde{g}) :\in \mathbb{Z}\}\\ &= \{T_{na}E_{kb}h : k, n \in \mathbb{Z}\}\text{ , where } h = \mathcal{F}\tilde{g}.\end{aligned}$$

Now we state another interesting observation as one theorem.

Theorem 3.4. *Associated with every Gabor frame $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$, there is a frame in $L^2(\mathbb{R})$ of the form $\{T_{na}E_{mb}h : m, n \in \mathbb{Z}\}$ for some $h \in L^2(\mathbb{R})$ and this frame is a $\mathcal{F}$-Gabor type frame in $L^2(\mathbb{R})$.*

Consider an invertible map $B : L^2(\mathbb{R}) \to \mathcal{H}$ such that B^*B is a Gabor frame operator on $L^2(\mathbb{R})$ and $\mathcal{F}$ is the Fourier transform on $L^2(\mathbb{R})$. If S is a $B\mathcal{F}$-Gabor type frame operator on $\mathcal{H}$. Then there is Gabor frame operator S' on $L^2(\mathbb{R})$ such that $(B\mathcal{F})S'(B\mathcal{F})^* = S$.

Now, $SE_{mb}^{B\mathcal{F}} = (B\mathcal{F})S'(B\mathcal{F})^*(B\mathcal{F})E_{mb}(B\mathcal{F})^{-1}$
$= (B\mathcal{F})S'\mathcal{F}^*(B^*B)\mathcal{F}E_{mb}\mathcal{F}^{-1}B^{-1}$
$= (B\mathcal{F})S'\mathcal{F}^*(B^*B)T_{mb}\mathcal{F}\mathcal{F}^{-1}B^{-1}$
$= (B\mathcal{F})S'\mathcal{F}^*(B^*B)T_{mb}B^{-1}$
$= (B\mathcal{F})S'\mathcal{F}^*T_{mb}(B^*B)B^{-1}$
$= (B\mathcal{F})S'\mathcal{F}^{-1}T_{mb}B^*$
$= (B\mathcal{F})S'E_{mb}\mathcal{F}^{-1}B^*$
$= (B\mathcal{F})E_{mb}S'\mathcal{F}^*B^*$
$= (B\mathcal{F})E_{mb}(\mathcal{F}^{-1}B^{-1})(B\mathcal{F})S'\mathcal{F}^*B^*$
$= E_{mb}^{B\mathcal{F}}S$

Thus, $SE_{mb}^{B\mathcal{F}} = E_{mb}^{B\mathcal{F}}S$, in a similar manner we can see that, $ST_{na}^{B\mathcal{F}} = T_{na}^{B\mathcal{F}}S$.

Hence, every $B\mathcal{F}$-Gabor type frame operator on Hilbert space $\mathcal{H}$ commutes with $B\mathcal{F}$-modulations and $B\mathcal{F}$-translations. Also it can be easily verify that $(B\mathcal{F})^*(B\mathcal{F})$ is invertible, positive and self adjoint. Hence $(B\mathcal{F})^*(B\mathcal{F})$ is a Gabor frame operator and hence commutes with modulation and translation by Nambudiri and Parthasarathy [2013]. This operator $B\mathcal{F}$ has the following important property:

(1). $(B\mathcal{F}B^{-1})E_{mb}^B = B\mathcal{F}(B^{-1}B)E_{mb}B^{-1} = B\mathcal{F}E_{mb}B^{-1}$
$= BT_{mb}\mathcal{F}B^{-1} = BT_{mb}B^{-1}B\mathcal{F}B^{-1} = T_{mb}^B(B\mathcal{F}B^{-1})$

(2). $(B\mathcal{F}B^{-1})T_{na}^B = B\mathcal{F}(B^{-1}B)T_{na}B^{-1} = B\mathcal{F}T_{na}B^{-1}$
$= BE_{-na}\mathcal{F}B^{-1} = BE_{-na}B^{-1}B\mathcal{F}B^{-1} = E_{-na}^B(B\mathcal{F}B^{-1})$

If $\mathcal{H}$ is a separable Hilbert space and $B : L^2(\mathbb{R}) \to \mathcal{H}$ is an invertible operator with B^*B Gabor frame operator on $L^2(\mathbb{R})$. Then $(B\mathcal{F}B^{-1})$ can be viewed as a generalized B-Fourier transform on $\mathcal{H}$.

Proposition 3.3. *For a given $A \in \mathcal{B}(\mathcal{H}_1)$ and invertible operators $B : \mathcal{H}_1 \to \mathcal{H}_2$ and $C : \mathcal{H}_2 \to \mathcal{H}_1$, the operator $BC \in \mathcal{B}(\mathcal{H}_2)$ commutes with $BAC \in \mathcal{B}(\mathcal{H}_2)$ if and only if $CB \in \mathcal{B}(\mathcal{H}_1)$ commutes with A.*

Proof. Since both B and C are invertible,

$(BC)(BAC) = (BAC)(BC)$
$\Leftrightarrow C(BAC)(C)^{-1} = B^{-1}(BAC)B$
$\Leftrightarrow (CB)A = A(CB).$ □

A useful consequence of Proposition 3.3 is the following.

Corolary 3.2. *Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ and $C : \mathcal{H} \to L^2(\mathbb{R})$ be invertible. Then $BC \in \mathcal{B}(\mathcal{H})$ commutes with the operators $BT_{na}C$ and $BE_{mb}C$, $m, n \in \mathbb{Z}$ if and only if CB commutes with T_{na} and E_{mb}, $m, n \in \mathbb{Z}$ on $L^2(\mathbb{R})$.*

For a given pair of invertible operators $B : L^2(\mathbb{R}) \to \mathcal{H}$ and $C : \mathcal{H} \to L^2(\mathbb{R})$, assume that CB commutes with T_{na} and E_{mb} for all $m, n \in \mathbb{Z}$. Then each Gabor frame $\mathcal{G} = \{E_{mb}T_{na}g :\in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ also yields a Gabor type frame of the form $\{(BE_{mb}C)(BT_{na}C)x_0 :\in \mathbb{Z}\}$ in $\mathcal{H}$. Indeed, this family is a frame, as is seen below.

Since CB is invertible, $(CB)^2(\mathcal{G})$ is a Gabor frame in $L^2(\mathbb{R})$. Hence its image under B is also a frame in $\mathcal{H}$. This frame is given by

$$\begin{aligned} B((CB)^2)(\mathcal{G}) &= B((CB)^2)(\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}) \\ &= B(\{E_{mb}(CB)T_{na}(CB)g : m, n \in \mathbb{Z}\}) \\ &= \{(BE_{mb}C)(BT_{na}C)Bg : m, n \in \mathbb{Z}\} \end{aligned}$$

and hence the claim.

In particular, let $C = B^*$. Assume that $CB = B^*B$ commutes with T_{na} and E_{mb} for all $m, n \in \mathbb{Z}$. As above, each Gabor frame $\mathcal{G} = \{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ yields a Gabor type frame of the form $\{(BE_{mb}B^*)(BT_{na}B^*)x_0 : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$.

Now, $(BT_aB^*)^2 = (BT_aB^*)(BT_aB^*) = BT_aT_aB^*BB^* = BT_{2a}B^*(BB^*)$. Hence, in general, $(BT_{na}B^*) \neq (BT_aB^*)^n$ and $(BE_{mb}B^*) \neq (BE_bB^*)^m$ even if B^*B commutes with T_{na} and E_{mb} for all $m, n \in \mathbb{Z}$. This makes B-Gabor type frames more significant in the aspects of structure of Gabor frames.

However, it is possible to get more types of Gabor frames in separable Hilbert spaces by relaxing the invertibility of the operator B by surjectivity. In addition, if $\mathcal{G}$ is a Parseval Gabor frame in $L^2(\mathbb{R})$, then the frame operator of this frame is $B(B^*B)^2B^* = B(B^*B)(B^*B)B^* = (BB^*)^3$ which commutes with the operators $BT_{na}B^*$ and $BE_{mb}B^*$, as is seen in Corollary 3.2.

More generally, there can be different Gabor type frames in $\mathcal{H}$ with same frame operator and generator. For instance, fix an invertible map $B : L^2(\mathbb{R}) \to \mathcal{H}$ and choose a Gabor frame $\mathcal{G} = \{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ with frame operator S. Then each invertible map $D : \mathcal{H} \to L^2(\mathbb{R})$ (preferably, $D \neq B^{-1}, B^*$) yields a frame in $\mathcal{H}$ of the form $B(\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}) = \{(BE_{mb}D)(D^{-1}T_{na}B^{-1})Bg : m, n \in \mathbb{Z}\}$. Obviously, Bg is the generator of this frame and BSB^* is its frame operator.

4 Conclusion

In this article we introduced a class of structured frames in separable Hilbert spaces $\mathcal{H}$ using an invertible map B from $L^2(\mathbb{R})$. Since this structure is similar to Gabor structure existing in literature, we name them as B-Gabor type frames. In light of the characterization of the Gabor frame operators on $L^2(\mathbb{R})$, recently appeared in Nambudiri and Parthasarathy [2013], we presented a characterization of frame operators associated with B-Gabor type frames in $\mathcal{H}$ when B is unitary.

Several significant properties of B-Gabor type frame operators are discussed. One can check whether such structured frames exists in separable Hilbert space $\mathcal{H}$ by replacing the invertible map with some other suitable maps.

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Common fixed points of self-maps over the generalized cone $\mathbb{b}$-metric spaces

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Abstract

The main aim of this research paper is to establish the most generalized common fixed-point theorem for two self-maps that are commuting to each other under $\mathfrak{T}$-Kannan type contractive condition over a generalized cone $\mathbb{b}$-metric spaces. The novelty of this research paper is to find a common fixed point of two weekly compatible self-maps over a generalized cone $\mathbb{b}$-metric spaces without assuming the normality condition of a cone. We illustrate our main result with a suitable example.

Keywords: weekly compatible self-maps; cone; common fixed points; normality condition of a cone; generalized $\mathbb{b}$-cone metric space.

2010 AMS subject classification: 54H25, 47H10.†

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1. Introduction

M. Frechet [6] introduced the metric spaces. It is well known that the Banach Contraction Principle [2] is the first fixed point theorem in metric spaces. A. Branciari [4] introduced the generalized (rectangular) metric spaces by replacing the triangular inequality in the definition of metric spaces with a similar one that involves four or more points instead of three points. I. A. Bakhtin [1] defined b-metric spaces by replacing triangular inequality in a metric space with a $\mathbb{b}$-triangular inequality as a generalization of metric spaces. L.G. Huang et al. [7] defined cone metric spaces which are obtained by replacing the set of real numbers in a metric mapping with a Real Banach space. A. Azam et al. [1] defined cone rectangular metric space by replacing triangular inequality in cone metric space with a rectangular inequality. N. Hussain et al. [9] defined cone $\mathbb{b}$-metric spaces by replacing triangular inequality in cone metric spaces with a $\mathbb{b}$-triangular inequality as a generalization of cone metric spaces. The authors improved the Banach Contraction Principle in such spaces. Recently, many authors ([5], [8], [14]) established some fixed-point theorems satisfying some generalized contractive conditions in cone $\mathbb{b}$-metric spaces over Banach algebras.

Reny George et al. [12] defined generalized cone $\mathbb{b}$-metric spaces by replacing $\mathbb{b}$-triangular inequality in a cone $\mathbb{b}$-metric space with a $\mathbb{b}$-rectangular inequality and proved basic Banach fixed point theorem on these spaces. Thereafter some fixed-point theorems ([10], [11], [13]) proved under various contractive type conditions.

The main objective of this article is to establish a generalized common fixed-point theorem for two self-maps that are commuting to each other under the $\mathfrak{T}$-Kannan type contractive condition over a generalized cone $\mathbb{b}$-metric space. We present an example to illustrate our result. Our main result generalizes many results in generalized cone $\mathbb{b}$-metric spaces without assuming the normality condition of a cone.

We recall some basic definitions and lemmas required in section 2 and establish main result in section 3 with suitable example.

2. Preliminaries

Definition 2.1 Let $\dot{X}$ be a non-empty set. An element $\dot{x} \in \dot{X}$ is said to be a fixed point of a self-map $\mathfrak{T}: \dot{X} \to \dot{X}$ if $\mathfrak{T}(\dot{x}) = \dot{x}$.

Definition 2.2 A result giving a set of conditions on $\mathfrak{T}$ and $\dot{X}$ under which $\mathfrak{T}$ has a fixed point is known as a fixed-point theorem.

Definition 2.3 (L.G. Huang and X. Zhang. [7]). If $\hat{\mathbf{E}}$ is a real Banach space and $\widehat{\mathrm{P}} \subseteq \hat{\mathrm{E}}$, then $\widehat{\mathrm{P}}$ is said to be a cone if it satisfies the following conditions:

(C1) $\widehat{\mathrm{P}}$ is a closed and nonempty subset of $\hat{\mathbf{E}}$ with $\widehat{\mathrm{P}} \neq \{\widehat{\mathbf{0}}\}$;

(C2) If $c_1, c_2 \geq 0$ and $\hat{x}, \hat{y} \in \widehat{\mathrm{P}}$, then $c_1\hat{x} + c_2\hat{y} \in \widehat{\mathrm{P}}$;

(C3) Intersection of $\widehat{\mathrm{P}}$ and $-\widehat{\mathrm{P}}$ is $\{\widehat{\mathbf{0}}\}$.

Definition 2.4 (L.G. Huang and X. Zhang. [7]). A partial-order relation on a cone $\widehat{\mathrm{P}} \subseteq \hat{\mathrm{E}}$ is denoted by $\lesssim$ and is defined as $\hat{x} \lesssim \hat{y} \Leftrightarrow \hat{y} - \hat{x} \in \widehat{\mathrm{P}}$.

Definition 2.5 (L.G. Huang and X. Zhang. [3]). The interior of a cone $\widehat{\mathrm{P}}$ is denoted by $\mathrm{int}(\widehat{\mathrm{P}})$and a partial-order relation $\prec\prec$ on $\mathrm{int}(\widehat{\mathrm{P}}) \subseteq \widehat{\mathrm{P}} \subseteq \hat{\mathrm{E}}$ defined as $\hat{u} \prec\prec \hat{v} \Leftrightarrow \hat{v} - \hat{u} \in \mathrm{int}(\widehat{\mathrm{P}})$.

Definition 2.6 (L.G. Huang and X. Zhang. [7]). If $\dot{\mathrm{X}}$ is a non-empty set and a mapping $\mathrm{d}_{\hat{\mathrm{E}}}: \dot{\mathrm{X}} \times \dot{\mathrm{X}} \to \hat{\mathbf{E}}$ satisfies:
(CMS 1) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) > \widehat{\mathbf{0}}, \forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$ and $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \widehat{\mathbf{0}} \Leftrightarrow \dot{x} = \dot{y}$;
(CMS 2) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \mathrm{d}_{\hat{\mathrm{E}}}(\dot{y}, \dot{x}), \forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$;
(CMS 3) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) \lesssim \mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{z}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{z}, \dot{y}), \forall\, \dot{x}, \dot{y}, \dot{z} \in \dot{\mathrm{X}}$,
then $\mathrm{d}_{\hat{\mathrm{E}}}$ is called a cone metric on $\dot{\mathrm{X}}$ and $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is called a cone metric space.
Definition 2.7 (A. Azam et al. [1]). If $\dot{\mathrm{X}}$ is a non-empty set and a mapping $\mathrm{d}_{\hat{\mathrm{E}}}: \dot{\mathrm{X}} \times \dot{\mathrm{X}} \to \hat{\mathbf{E}}$ satisfies:
(CRMS 1) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) > \widehat{\mathbf{0}}, \forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$ and $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \widehat{\mathbf{0}} \Leftrightarrow \dot{x} = \dot{y}$;
(CRMS 2) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \mathrm{d}_{\hat{\mathrm{E}}}(\dot{y}, \dot{x}), \forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$;
(CRMS 3) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) \lesssim \mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{u}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{u}, \dot{v}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{v}, \dot{y}), \forall\, \dot{x}, \dot{y}, \dot{u}, \dot{v} \in \dot{\mathrm{X}}$ with $\dot{u} \neq \dot{v}$,
then $\mathrm{d}_{\hat{\mathrm{E}}}$ is called a cone rectangular metric on $\dot{\mathrm{X}}$ and $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is called a cone rectangular metric space.
Definition 2.8 (N. Hussain et al. [9]). If $\dot{\mathrm{X}}$ is a non-empty set and a mapping $\mathrm{d}_{\hat{\mathrm{E}}}: \dot{\mathrm{X}} \times \dot{\mathrm{X}} \to \hat{\mathbf{E}}$ satisfies:
(CbMS 1) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) > \widehat{\mathbf{0}}, \forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$ and $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \widehat{\mathbf{0}} \Leftrightarrow \dot{x} = \dot{y}$;
(CbMS 2) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \mathrm{d}_{\hat{\mathrm{E}}}(\dot{y}, \dot{x}), \forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$;
(CbMS 3) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) \lesssim b[\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{z}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{z}, \dot{y})], \forall\, \dot{x}, \dot{y}, \dot{z} \in \dot{\mathrm{X}}$,
then $\mathrm{d}_{\hat{\mathrm{E}}}$ is called a cone 𝕓-metric on $\dot{\mathrm{X}}$ and $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is called a cone 𝕓-metric space.
Definition 2.9 (R. George et al. [12]). If $\dot{\mathrm{X}}$ is a non-empty set and a mapping $\mathrm{d}_{\hat{\mathrm{E}}}: \dot{\mathrm{X}} \times \dot{\mathrm{X}} \to \hat{\mathbf{E}}$ satisfies:
(GCbMS 1) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) > \widehat{\mathbf{0}}, \forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$ and $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \widehat{\mathbf{0}} \Leftrightarrow \dot{x} = \dot{y}$;
(GCbMS 2) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) = \mathrm{d}_{\hat{\mathrm{E}}}(\dot{y}, \dot{x}), \forall\, \dot{x}, \dot{y} \in \dot{\mathrm{X}}$;
(GCbMS 3) $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{y}) \lesssim b[\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \dot{u}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{u}, \dot{v}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{v}, \dot{y})], \forall\, \dot{x}, \dot{y}, \dot{u}, \dot{v} \in \dot{\mathrm{X}}$ with $\dot{u} \neq \dot{v}$,
then $\mathrm{d}_{\hat{\mathrm{E}}}$ is called a generalized cone 𝕓-metric on $\dot{\mathrm{X}}$ and $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is called a generalized cone 𝕓-metric space.
Definition 2.10 (R. George et al. [12]). Let $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ be a generalized cone 𝕓-metric space. Then
(1) If for every $\hat{t} \in \hat{\mathbf{E}}$ with $\hat{0} \prec\prec \hat{t}$, there exists a natural number N such that $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}_n, \dot{x}) \prec\prec \hat{t}$, for all n > N, then $\{\dot{x}_n\}$ converges to $\dot{x}$
(2) If for every $\hat{t} \in \hat{\mathbf{E}}$ with $\hat{0} \prec\prec \hat{t}$, there exists a natural number N such that $\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}_m, \dot{x}_n) \prec\prec \hat{t}$, for all m, n > N, then $\{\dot{x}_n\}$ converges to $\dot{x}$
(3) If every Cauchy sequence is convergence in $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$, then $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is a complete generalized cone 𝕓-metric space.

Lemma 2.11 (R. George et al. [12]). Let $\widehat{\mathrm{P}}$ be a cone and $\{\dot{x}_n\}$ be a sequence in $\hat{\mathbf{E}}$. If $\hat{t} \in int\,(\widehat{\mathrm{P}})$ and $\hat{0} \leq \dot{x}_n \to \hat{0}$ as there exists a natural number N such that for all n > N, we have $\dot{x}_n \prec\prec \hat{t}$.

3. Main Result

Theorem 3.1 Let $(\dot{X}, d_{\hat{E}})$ be a generalized cone $\mathbb{b}$-metric space. If the two self-maps $\mathfrak{f}$ and $\mathfrak{T}$ defined on $\dot{X}$ satisfy the following conditions:

(3.1) $d_{\hat{E}}\left(\mathfrak{T}\mathfrak{f}\dot{x}, \mathfrak{T}\mathfrak{f}\dot{y}\right) \lesssim \gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{x}, \mathfrak{T}\mathfrak{f}\dot{x}\right) + d_{\hat{E}}\left(\mathfrak{T}\dot{y}, \mathfrak{T}\mathfrak{f}\dot{y}\right)\right]$, $\forall \dot{x}, \dot{y} \in \dot{X}$, where, $0 \leq \gamma < \frac{1}{b+1}$,

(3.2) $\mathfrak{T}$ is one-to-one mapping and

(3.3) $\mathfrak{T}(\dot{X})$ is a complete subspace of $\dot{X}$, then the mapping $\mathfrak{f}$ has a unique fixed point in $\dot{X}$. Further,

(3.4) If $\mathfrak{f}$ and $\mathfrak{T}$ are commutes at the fixed point of $\mathfrak{f}$, then there exists a unique common fixed point of $\mathfrak{f}$ and $\mathfrak{T}$ in $\dot{X}$.

Pf. Let $\{\dot{y}_p\}$ be a sequence in $\dot{X}$ such that $\dot{y}_{p+1} = \mathfrak{f}\dot{y}_p$, $\forall\, p = 0,1,2, \ldots$ If $\dot{y}_m = \dot{y}_{m+1}$, for some natural number m, then $\dot{y}_m = \mathfrak{f}\dot{y}_m$, i.e., $\dot{y}_m$ is a fixed point of $\mathfrak{f}$ in $\dot{X}$. Suppose, $\dot{y}_p \neq \dot{y}_{p+1}$, $\forall\, p = 0,1,2, \ldots$, Then from (3.1) we obtain,

$$\begin{aligned} d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p, \mathfrak{T}\dot{y}_{p+1}\right) &= d_{\hat{E}}\left(\mathfrak{T}\mathfrak{f}\dot{y}_{p-1}, \mathfrak{T}\mathfrak{f}\dot{y}_p\right) \\ &\lesssim \gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1}, \mathfrak{T}\mathfrak{f}\dot{y}_{p-1}\right) + d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p, \mathfrak{T}\mathfrak{f}\dot{y}_p\right)\right] \\ &\lesssim \gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1}, \mathfrak{T}\dot{y}_p\right) + d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p, \mathfrak{T}\dot{y}_{p+1}\right)\right] \\ &\lesssim \frac{\gamma}{1-\gamma} d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1}, \mathfrak{T}\dot{y}_p\right), \end{aligned}$$

which implies that, $d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p, \mathfrak{T}\dot{y}_{p+1}\right) \lesssim \eta d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1}, \mathfrak{T}\dot{y}_p\right)$, $\forall\, p = 1,2, \ldots$, where $0 \leq \eta = \frac{\gamma}{1-\gamma} < \frac{1}{b}$. Therefore, by induction, we have,

$$\begin{aligned} d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p, \mathfrak{T}\dot{y}_{p+1}\right) &\lesssim \eta^2 d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-2}, \mathfrak{T}\dot{y}_{p-1}\right) \\ &\vdots \\ &\lesssim \eta^p d_{\hat{E}}\left(\mathfrak{T}\dot{y}_0, \mathfrak{T}\dot{y}_1\right), \ \forall\, p = 1,2, \ldots, \text{ where } 0 \leq \eta < \frac{1}{b} \end{aligned} \tag{3.5}$$

Since, $\gamma \leq \eta$ and therefore by using Kannan inequality (3.1) and $\mathbb{b}$-rectangular inequality, we get,

$$\begin{aligned} d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p, \mathfrak{T}\dot{y}_{p+2}\right) &= d_{\hat{E}}\left(\mathfrak{T}\mathfrak{f}\dot{y}_{p-1}, \mathfrak{T}\mathfrak{f}\dot{y}_{p+1}\right) \\ &\lesssim \gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1}, \mathfrak{T}\mathfrak{f}\dot{y}_{p-1}\right) + d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1}, \mathfrak{T}\mathfrak{f}\dot{y}_{p+1}\right)\right] \\ &\lesssim \gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p-1}, \mathfrak{T}\dot{y}_p\right) + d_{\hat{E}}\left(\mathfrak{T}\dot{y}_{p+1}, \mathfrak{T}\dot{y}_{p+2}\right)\right] \end{aligned}$$

$$\lesssim b\gamma\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p-1},\,\mathfrak{T}\dot{y}_{p}\right)+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p},\,\mathfrak{T}\dot{y}_{p+2}\right)+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+1}\right)\right]$$
$$+\gamma\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)$$
$$=b\gamma\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p-1},\,\mathfrak{T}\dot{y}_{p}\right)+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p},\,\mathfrak{T}\dot{y}_{p+2}\right)\right]+(1+b)\gamma\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)$$
$$\lesssim b\gamma\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p-1},\,\mathfrak{T}\dot{y}_{p}\right)+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p},\,\mathfrak{T}\dot{y}_{p+2}\right)\right]+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)$$

which implies that,

$$\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+2}\right)\lesssim\frac{1}{1-b\gamma}\left[b\gamma\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p-1},\,\mathfrak{T}\dot{y}_{p}\right)+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)\right]$$
$$\lesssim\frac{1}{1-b\gamma}\left[b\eta\eta^{p-1}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)+\eta^{p+1}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)\right]$$
$$\lesssim\left[\frac{b+\eta}{1-b\gamma}\right]\eta^{p}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{0},\,\mathfrak{T}\dot{y}_{1}\right)$$
$$=\alpha\eta^{p}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{0},\,\mathfrak{T}\dot{y}_{1}\right),\qquad(3.6)$$

where $\alpha=\dfrac{b+\eta}{1-b\gamma}\geq 0,\ \forall\, p\geq 1$. For the sequence $\{\mathfrak{T}\dot{y}_p\}$, we consider $\mathrm{d}_{\hat{\mathbf{E}}}(\mathfrak{T}\dot{y}_p, \mathfrak{T}_{p+n})$ in two different cases. If $n = 2r + 1$, where $r \geq 1$, then by applying 𝕓-rectangular inequality we get,

$$\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+2r+1}\right)\lesssim b\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+1}\right)+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+2r+1}\right)\right]$$
$$\lesssim b\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+1}\right)+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)\right]+b^{2}\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+3}\right)\right.$$
$$\left.+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+3},\mathfrak{T}\dot{y}_{p+4}\right)+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+4},\mathfrak{T}\dot{y}_{p+2r-1}\right)\right]$$
$$\lesssim b\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p},\mathfrak{T}\dot{y}_{p+1}\right)+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+1},\mathfrak{T}\dot{y}_{p+2}\right)\right]+b^{2}\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+2},\mathfrak{T}\dot{y}_{p+3}\right)\right.$$
$$\left.+\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+3},\mathfrak{T}\dot{y}_{p+4}\right)\right]+\cdots+b^{r}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{p+2r},\mathfrak{T}\dot{y}_{p+2r+1}\right)$$
$$\lesssim b\left[\eta^{p}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)+\eta^{p+1}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)\right]+b^{2}\left[\eta^{p+2}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)\right.$$
$$\left.+\eta^{p+3}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)\right]+\cdots+b^{r}\eta^{p+2r}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_{0},\mathfrak{T}\dot{y}_{1}\right)$$

$$\lesssim \eta^p \left[1 + b\eta^2 + \cdots\right] \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_0, \mathfrak{T}\, \dot{y}_1\right)$$

$$+ b\eta^{p+1}\left[1 + b\eta^2 + \cdots\right] \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_0, \mathfrak{T}\, \dot{y}_1\right)$$

$$= (1+\eta) b\eta^p \left[1 + b\eta^2 + \cdots\right] \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_0, \mathfrak{T}\, \dot{y}_1\right)$$

Hence, $\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_p, \mathfrak{T}\, \dot{y}_{p+2r+1}\right) \lesssim \left(\frac{1+\eta}{1-b\eta^2}\right) b\mu^p \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_0, \mathfrak{T}\, \dot{y}_1\right)$, for all natural numbers p and r.

Suppose, $\hat{0} \prec\prec \hat{k}$. Since, $b\eta^2 < 1$, we notice that $\left(\frac{1+\eta}{1-b\eta^2}\right) b\eta^p \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_0, \mathfrak{T}\, \dot{y}_1\right) \to \hat{0}$, as $n \to \infty$. Therefore for some $\hat{k} \in$ int ($\widehat{\mathrm{P}}$), we obtain a natural number $N_1 \in \mathbb{N} \ni \left(\frac{1+\eta}{1-b\eta^2}\right) b\eta^p \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_0, \mathfrak{T}\, \dot{y}_1\right) \prec\prec \hat{k}, \forall p > N_1$.

Thus, $\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_p, \mathfrak{T}\, \dot{y}_{p+2r+1}\right) \lesssim \left(\frac{1+\eta}{1-b\eta^2}\right) b\eta^p \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_0, \mathfrak{T}\, \dot{y}_1\right) \prec\prec \hat{k}, \forall p > N_1$ and $r \geq 1$.

Suppose, $p = 2r$, for $r \geq 1$. Since, $b\eta^2 < 1$ and by using inequalities (3.5) and (3.6) we get,

$$\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_p, \mathfrak{T}\, \dot{y}_{p+2r}\right) \lesssim b\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_p, \mathfrak{T}\, \dot{y}_{p+1}\right) + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+1}, \mathfrak{T}\, \dot{y}_{p+2}\right) + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+2}, \mathfrak{T}\, \dot{y}_{p+2r}\right)\right]$$

$$\lesssim b\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_p, \mathfrak{T}\, \dot{y}_{p+1}\right) + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+1}, \mathfrak{T}\, \dot{y}_{p+2}\right)\right] + b^2\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+2}, \mathfrak{T}\, \dot{y}_{p+3}\right)\right.$$

$$\left. + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+3}, \mathfrak{T}\, \dot{y}_{p+4}\right) + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+4}, \mathfrak{T}\, \dot{y}_{p+2r}\right)\right]$$

$$\lesssim b\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_p, \mathfrak{T}\, \dot{y}_{p+1}\right) + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+1}, \mathfrak{T}\, \dot{y}_{p+2}\right)\right] + b^2\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+2}, \mathfrak{T}\, \dot{y}_{p+3}\right)\right.$$

$$\left. + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+3}, \mathfrak{T}\, \dot{y}_{p+4}\right)\right] + \cdots + b^{r-1}\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+2r-4}, \mathfrak{T}\, \dot{y}_{p+2r-3}\right)\right.$$

$$\left. + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+2r-3}, T\, \dot{y}_{p+2r-2}\right) + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+2r-2}, \mathfrak{T}\, \dot{y}_{p+2r}\right)\right]$$

$$\lesssim b\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_p, \mathfrak{T}\, \dot{y}_{p+1}\right) + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+1}, \mathfrak{T}\, \dot{y}_{p+2}\right)\right] + b^2\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+2}, \mathfrak{T}\, \dot{y}_{p+3}\right)\right.$$

$$\left. + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+3}, \mathfrak{T}\, \dot{y}_{p+4}\right)\right] + \cdots + b^{r-1}\left[\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+2r-4}, \mathfrak{T}\, \dot{y}_{p+2r-3}\right)\right.$$

$$\left. + \mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+2r-3}, \mathfrak{T}\, \dot{y}_{p+2r-2}\right)\right] + b^{r-1}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\, \dot{y}_{p+2r-2}, \mathfrak{T}\, \dot{y}_{p+2r}\right)$$

$$\lesssim b\left[\eta^{p}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)+\eta^{p+1}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\right]+b^{2}\left[\eta^{p+2}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\right.$$

$$\left.+\eta^{p+3}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\right]+\cdots+\eta^{r-1}\left[\eta^{p+2r-4}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\right.$$

$$\left.+\eta^{p+2r-3}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\right]+b^{m-1}\alpha\eta^{p+2r-2}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$$

$$\lesssim b\eta^{p}\left[1+b\eta^{2}+\cdots\right]\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$$

$$+b\eta^{p+1}\left[1+b\eta^{2}+\cdots\right]\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)+b^{r-1}\alpha\eta^{p+2r-2}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$$

$$=(1+\eta)b\eta^{p}\left[1+b\eta^{2}+\cdots\right]\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)+b^{r-1}\alpha\eta^{p+2r-2}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$$

That is, $\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+2r}\right)\lesssim\left(\frac{1+\eta}{1-b\eta^{2}}\right)b\eta^{p}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)+b^{r-1}\alpha\eta^{p+2r-2}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$

i.e., $\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+2r}\right)\lesssim\left(\frac{1+\eta}{1-b\eta^{2}}+b^{r-2}\alpha\eta^{2r-2}\right)b\eta^{p}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)$, for all natural numbers p and r, where $\alpha\geq 0$.

Suppose, $\hat{0}\prec\prec\hat{t}$. Since, $b\eta^2 < 1$, we obtain that $\left(\frac{1+\eta}{1-b\eta^{2}}+b^{r-2}\alpha\eta^{2r-2}\right)b\eta^{p}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\to\hat{0}$, as $p\to\infty$. Therefore, for any $\hat{t}\in$ int $(\widehat{\mathrm{P}})$, we obtain $N_2\in\mathbb{N}\ni\left(\frac{1+\eta}{1-b\eta^{2}}+b^{r-2}\alpha\eta^{2r-3}\right)b\eta^{p}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\prec\prec\hat{t}$, $\forall\, p > N_2$, $r\geq 1$ and $\alpha=\frac{b+\eta}{1-b\eta}\geq 0$. Thus, $\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+2r}\right)\lesssim\left(\frac{1+\eta}{1-b\eta^{2}}+b^{r-2}\alpha\eta^{2r-2}\right)b\eta^{p}\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_0,\mathfrak{T}\dot{y}_1\right)\prec\prec\hat{z}$, $\forall p > N_2$ and $r\geq 1$. Let $N_0 = \max\{N_1, N_2\}$. Thus for each $\hat{t}\in$ Int $(\widehat{\mathrm{P}})$, we get, $\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+n}\right)\prec\prec\hat{k}$, $\forall\, p > N_0$ and $n\geq 1$.

Hence in both cases, $\{\mathfrak{T}\dot{y}_p\}$ is a Cauchy sequence in $\dot{\mathrm{X}}$. By the hypothesis we have, $\mathfrak{T}(\dot{\mathrm{X}})$ is a complete subspace of $\dot{\mathrm{X}}$, therefore $\exists\ \dot{z}\in\mathfrak{T}(\dot{\mathrm{X}})\ni\lim_{p\to\infty}\mathfrak{T}\dot{y}_{p+1}=\lim_{p\to\infty}\mathfrak{T}\mathfrak{f}\dot{y}_p=\dot{z}$. Here, we find $\dot{y}\in\dot{\mathrm{X}}\ni\dot{z}=\mathfrak{T}\dot{y}$. Suppose, $\hat{0}\prec\prec\hat{c}$. we find N_3, $N_4\in\mathbb{N}\ni\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_p,z\right)\prec\prec\frac{\hat{c}(1-b\gamma)}{2b}$, $\forall\, p > N_3$ and $\mathrm{d}_{\hat{\mathbf{E}}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)\prec\prec\frac{\hat{c}(1-b\gamma)}{2b(1+\gamma)}$, $\forall p > N_4$. Let $N = \max\{N_3, N_4\}$. Using 𝕓-rectangular inequality and (3.5) we obtain,

$$d_{\hat{E}}\left(\dot{z},\mathfrak{T}\mathfrak{f}\dot{y}\right)\lesssim b\left[d_{\hat{E}}\left(\dot{z},\mathfrak{T}\dot{y}_p\right)+d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\mathfrak{f}\dot{y}_p\right)+d_{\hat{E}}\left(\mathfrak{T}\mathfrak{f}\dot{y}_p,\mathfrak{T}\mathfrak{f}\dot{y}\right)\right]$$

$$\lesssim bd_{\hat{E}}\left(\dot{z},\mathfrak{T}\dot{y}_p\right)+bd_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)+b\gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\mathfrak{f}\dot{y}_p\right)+d_{\hat{E}}\left(\mathfrak{T}\dot{y},\mathfrak{T}\mathfrak{f}\dot{y}\right)\right]$$

$$=bd_{\hat{E}}\left(\dot{z},\mathfrak{T}\dot{y}_p\right)+bd_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)+b\gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)+d_{\hat{E}}\left(\dot{z},\mathfrak{T}\mathfrak{f}\dot{y}\right)\right]$$

Hence, $d_{\hat{E}}\left(\dot{z},\mathfrak{T}\mathfrak{f}\dot{y}\right)\lesssim\frac{1}{1-b\gamma}\left[bd_{\hat{E}}\left(\dot{z},\mathfrak{T}\dot{y}_n\right)+b(1+\gamma)d_{\hat{E}}\left(\mathfrak{T}\dot{y}_p,\mathfrak{T}\dot{y}_{p+1}\right)\right]\prec\prec\frac{\hat{c}}{2}+\frac{\hat{c}}{2}=\hat{c},\forall p\geq 1.$

Thus $d_{\hat{E}}\left(\dot{z},\mathfrak{T}\mathfrak{f}\dot{y}\right)\prec\prec\hat{k},\ \forall\,\hat{c}\in\text{Int}\,(\widehat{\mathcal{P}})$, since $\hat{c}$ is any arbitrary in Int $(\widehat{\mathcal{P}})$, we get $d_{\hat{E}}(z,\mathfrak{T}\mathfrak{f}\dot{y})=\hat{0}$. That is, $\mathfrak{T}\mathfrak{f}\dot{y}=\mathfrak{T}\dot{y}=\dot{z}$. Since $\mathfrak{T}$ is one-to-one it implies that $\dot{y}=\mathfrak{f}\dot{y}$, i.e., there exists a fixed point $\dot{y}\in\dot{X}$. Suppose $\dot{y}'\in\dot{X}$ is also a fixed point of $\mathfrak{f}$, i.e., $\dot{y}'=\mathfrak{f}\dot{y}'$. Therefore, by using Kannan Contraction (3.1), we get,

$d_{\hat{E}}\left(\mathfrak{T}\dot{y},\mathfrak{T}\dot{y}'\right)=d_{\hat{E}}\left(\mathfrak{T}\mathfrak{f}\dot{y},\mathfrak{T}\mathfrak{f}\dot{y}'\right)\lesssim\gamma\left[d_{\hat{E}}\left(\mathfrak{T}\dot{y},\mathfrak{T}\dot{y}\right)+d_{\hat{E}}\left(\mathfrak{T}\dot{y}',\mathfrak{T}\dot{y}'\right)\right]=\hat{0}$, and hence we get, $\mathfrak{T}\dot{y}=\mathfrak{T}\dot{y}'$. Since $\mathfrak{T}$ is one-to-one mapping, we get that $\dot{y}=\dot{y}'$. Since, the two self-maps $\mathfrak{f}$ and $\mathfrak{T}$ are commute at the fixed point $\dot{y}$ of $\mathfrak{f}$, we get that, $\mathfrak{f}\mathfrak{T}\dot{y}=\mathfrak{T}\mathfrak{f}\dot{y}$, i.e., $\mathfrak{f}\mathfrak{T}\dot{y}=\mathfrak{T}\dot{y}$, i.e., $\mathfrak{T}\dot{y}$ is also a fixed point of $\mathfrak{f}$. Since, we know that $\mathfrak{f}$ has a unique fixed point, therefore $\mathfrak{T}\dot{y}=\dot{y}$ and hence $\mathfrak{T}\dot{y}=\mathfrak{f}\dot{y}=\dot{y}$. Therefore, there exists a unique common fixed point $\dot{y}\in\dot{X}$ of $\mathfrak{f}$ and $\mathfrak{T}$. □

Put $\mathfrak{T}=\mathfrak{I}$, the identity mapping of $\dot{X}$ in Theorem 3.1, we get the following corollary.

Corollary 3.2 If a self-map $\mathfrak{f}:\dot{X}\to\dot{X}$ defined on a complete generalized cone $\mathbb{b}$-metric space $(\dot{X},d_{\hat{E}})$ satisfying an inequality:

$$d_{\hat{E}}(\mathfrak{f}\dot{x},\mathfrak{f}\dot{y})\lesssim\gamma\,[d_{\hat{E}}(\dot{x},\mathfrak{f}\dot{x})+d_{\hat{E}}(\dot{y},\mathfrak{f}\dot{y})],$$

$\forall\,\dot{x},\dot{y}\in\dot{X}$, $0\leq\gamma<\frac{1}{b+1}$ and $b>1$, then there exists a unique fixed point of $\mathfrak{f}$ in $\dot{X}$.

We illustrate our main theorem 3,1 with the following example.

Example 3.3 Suppose $\dot{X}=\dot{\mathcal{A}}\cup\dot{\mathcal{B}}$, if $\dot{\mathcal{A}}=\{0\}\cup\left\{\frac{1}{n}:n\in\{2,3,4\}\right\}$ and $\dot{\mathcal{B}}=[1,2]$, $\hat{\mathbf{E}}=C_{\mathbb{R}}(\dot{X})$, $\widehat{\mathcal{P}}=\{\hat{f}:f(\dot{x})\geq 0,\ \dot{x}\in\dot{X}\}\subset\hat{\mathbf{E}}$, It is known that $\widehat{\mathcal{P}}$ is a cone in $\hat{\mathbf{E}}$. Define $d_{\hat{E}}:\dot{X}\times\dot{X}\to\hat{\mathbf{E}}$ as $d_{\hat{E}}(\dot{t},\dot{s})=d_{\hat{E}}(\dot{s},\dot{t})$, $\forall\dot{t},\dot{s}\in\dot{X}$ and

$$\begin{cases} \mathrm{d}_{\hat{\mathrm{E}}}\left(0,\frac{1}{2}\right)=\mathrm{d}_{\hat{\mathrm{E}}}\left(\frac{1}{3},\frac{1}{4}\right)=0.03e^{\dot{x}}; \\ \mathrm{d}_{\hat{\mathrm{E}}}\left(0,\frac{1}{3}\right)=\mathrm{d}_{\hat{\mathrm{E}}}\left(\frac{1}{4},\frac{1}{2}\right)=0.06e^{\dot{x}}. \\ \mathrm{d}_{\hat{\mathrm{E}}}\left(0,\frac{1}{4}\right)=\mathrm{d}_{\hat{\mathrm{E}}}\left(\frac{1}{2},\frac{1}{3}\right)=0.02e^{\dot{x}}; \\ \mathrm{d}_{\hat{\mathrm{E}}}\left(\dot{t},\dot{s}\right)=|\dot{t}-\dot{s}|^2\ e^{\dot{x}}, \text{ otherwise,} \end{cases}$$

Here $e^{\dot{x}} \in \hat{\mathbf{E}}$. Then $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is neither a cone metric space, since, $\mathrm{d}_{\hat{\mathrm{E}}}\left(0, \frac{1}{3}\right) = 0.06\ e^{\dot{x}} > \mathrm{d}_{\hat{\mathrm{E}}}\left(0, \frac{1}{2}\right) + \mathrm{d}_{\hat{\mathrm{E}}}\left(\frac{1}{2}, \frac{1}{3}\right) = 0.03\ e^{\dot{x}} + 0.02\ e^{\dot{x}} = 0.05\ e^{\dot{x}}$ nor a cone rectangular metric space, since, $\mathrm{d}_{\hat{\mathrm{E}}}(0, 1) = e^{\dot{x}} > \mathrm{d}_{\hat{\mathrm{E}}}\left(1, \frac{1}{2}\right) + \mathrm{d}_{\hat{\mathrm{E}}}\left(\frac{1}{2}, \frac{1}{3}\right) + \mathrm{d}_{\hat{\mathrm{E}}}\left(\frac{1}{3}, 0\right) = 0.25\ e^{\dot{x}} + 0.02\ e^{\dot{x}} + 0.06e^{\dot{x}} = 0.33\ e^{\dot{x}}$. However, it is easy to see that $(\dot{\mathrm{X}}, \mathrm{d}_{\hat{\mathrm{E}}})$ is a complete generalized cone 𝕓-metric space with coefficient $b = 4 > 1$.

Further, let $\mathfrak{f}$ and $\mathfrak{T}: \dot{\mathrm{X}} \to \dot{\mathrm{X}}$ be the mappings defined by:

$$\mathfrak{f}\left(\dot{x}\right) = \begin{cases} \frac{1}{4}, & \dot{x} \in \dot{\mathcal{A}} \\ \frac{1}{2}, & \dot{x} \in \dot{\mathcal{B}} \end{cases}$$

$$\text{and } \mathfrak{T}(\dot{x}) = \begin{cases} \frac{1}{2} - \dot{x}, & \dot{x} \in \{0\} \cup \left\{\frac{1}{n} : n \in \{2,4\}\right\} \\ \dot{x}, & \dot{x} \in \dot{\mathcal{B}} \cup \left\{\frac{1}{3}\right\} \end{cases}$$

We observe that $\mathfrak{T}$ is one-to-one and self-maps $\mathfrak{f}$ and $\mathfrak{T}$ are satisfying Kannan type contraction (3.1) with $\gamma = \frac{1}{5}$. Suppose $\dot{x} \in \dot{\mathcal{A}}$ and $y \in \dot{\mathcal{B}}$. Therefore, $\mathrm{d}_{\hat{\mathrm{E}}}(\mathfrak{T}\mathfrak{f}\dot{x}, \mathfrak{T}\mathfrak{f}y) = \mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\left(\frac{1}{4}\right), \mathfrak{T}\left(\frac{1}{2}\right)\right) = \mathrm{d}_{\hat{\mathrm{E}}}\left(\frac{1}{4}, 0\right) = 0.02\ e^{\dot{x}}$. Suppose $\dot{x} \in \{0\} \cup \left\{\frac{1}{n} : n \in \{2,4\}\right\}$ and $\dot{y} \in \dot{\mathcal{B}}$. Therefore, $\mathrm{d}_{\hat{\mathrm{E}}}(\mathfrak{T}\dot{x}, \mathfrak{T}\mathfrak{f}\dot{x}) + \mathrm{d}_{\hat{\mathrm{E}}}(\mathfrak{T}\dot{y}, \mathfrak{T}\mathfrak{f}\dot{y}) = \mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{x}, \mathfrak{T}\left(\frac{1}{4}\right)\right) + \mathrm{d}_{\hat{\mathrm{E}}}\left(\mathfrak{T}\dot{y}, \mathfrak{T}\left(\frac{1}{2}\right)\right) = \mathrm{d}_{\hat{\mathrm{E}}}\left(\frac{1}{2} - \dot{x}, \frac{1}{4}\right) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{y}, 0)$. Then clearly, we can find $\gamma = \frac{1}{5} \in \left(0, \frac{1}{4}\right)$ satisfying $\mathfrak{T}$-Kannan type contractive condition (3.1).

Suppose $\dot{x} \in \{0\} \cup \left\{\frac{1}{n} : n \in \{2,4\}\right\}$ and $\dot{y} \in \left\{\frac{1}{3}\right\}$. Therefore we can find $\gamma = \frac{1}{5} \in \left(0, \frac{1}{4}\right)$ satisfying $\mathfrak{T}$-Kannan type contractive condition (3.1). Similarly in all other cases, $\mathfrak{f}$ satisfies $\mathfrak{T}$-Kannan type contractive condition (3.1). Thus $\mathfrak{f}$ satisfies all the conditions of Theorem 3.1 and there exists a unique fixed point $\frac{1}{4} \in \dot{\mathrm{X}}$ of $\mathfrak{f}$. Finally, $\mathfrak{f}$ and $\mathfrak{T}$ are commuting at $\frac{1}{4} \in \dot{\mathrm{X}}$, and hence $\dot{x} = \frac{1}{4} \in \dot{\mathrm{X}}$ is a unique common fixed point of the self-maps $\mathfrak{f}$ and $\mathfrak{T}$. However, $\mathfrak{f}$ does not satisfy $\mathfrak{T}$-contractive condition of Corollary 3.2 at $\dot{x} =$

0 and $\dot{y} = 1$ as $\mathrm{d}_{\hat{\mathrm{E}}}(\mathfrak{f}\dot{x}, \mathfrak{f}\dot{y}) = \mathrm{d}_{\hat{\mathrm{E}}}\big(\mathfrak{f}(0), \mathfrak{f}(1)\big) = \mathrm{d}_{\hat{\mathrm{E}}}\left(\frac{1}{4}, \frac{1}{2}\right) = 0.06\, e^{\dot{x}} > \frac{1}{5}[\mathrm{d}_{\hat{\mathrm{E}}}(\dot{x}, \mathfrak{f}\dot{x}) + \mathrm{d}_{\hat{\mathrm{E}}}(\dot{y}, \mathfrak{f}\dot{y})] = \frac{1}{5}\big[\mathrm{d}_{\hat{\mathrm{E}}}\big(0, \mathfrak{f}(0)\big) + \mathrm{d}_{\hat{\mathrm{E}}}\big(1, \mathfrak{f}(1)\big)\big] = \frac{1}{5}\left[\mathrm{d}_{\hat{\mathrm{E}}}\left(0, \frac{1}{4}\right) + \mathrm{d}_{\hat{\mathrm{E}}}\left(1, \frac{1}{2}\right)\right] = 0.054 e^{\dot{x}}$

4 Discussion and Conclusions

In Theorem 3.1 we have constructed a common fixed-point theorem for two self-maps that are commuting to each other under the $\mathfrak{T}$-Kannan-type contractive condition over a generalized cone $\mathbb{b}$-metric space without assuming the normality condition of a cone. If we put $\mathfrak{T} = \mathfrak{I}$, the identity mapping of $\dot{\mathrm{X}}$ in Theorem 3.1, we get the fixed-point theorem as corollary 3.2. We have also given an example 3.3 which satisfies all the conditions of our main result. Our result is generalized and extended to many recent results. Our result may be the vision for other authors to extend and improve many results in generalized cone $\mathbb{b}$-metric spaces.

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Antimagic Labeling of Some Degree Splitting Graphs

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Abstract

A graph with q edges is called *antimagic* if its edges can be labeled with $1, 2, 3, ..., q$ without repetition such that the sums of the labels of the edges incident to each vertex are distinct. As Wang et al. [2012], proved that not all graphs are antimagic, it is interesting to investigate antimagic labeling of graph families. In this paper we discussed antimagic labeling of the larger graphs obtained using degree splitting operation on some known antimagic graphs. As discussed in Krishnaa [2016], antimagic labeling has many applications, our results will be used to expand the network on larger graphs.

Keywords: Antimagic labeling, Antimagic graph, Graph operation, Degree splitting graph.
2020 AMS subject classifications: 05C78, 05C76. [1]

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1 Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. For all standard terminologies and notations we follow Clark and Holton [1969]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1. *A* graph labeling *is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or edge labeling).*

For an extensive survey on graph labeling and bibliographic references we refer to Gallian [2022].

Hartsfield and Ringel [1990] have introduced antimagic labeling as follows.

Definition 1.2. *A graph with q edges is called* antimagic *if its edges can be labeled with $1, 2, 3, ..., q$ without repetition such that the sums of the labels of the edges incident to each vertex are distinct.*

They have also discussed antimagic labeling for paths, cycles, wheels and complete graph. Wang [2005] has shown that the toroidal grids $C_{n_1} \times C_{n_2} \times \ldots \times C_{n_k}$ are antimagic. Cheng [2008] has proved that all Cartesian products of two or more regular graphs are antimagic.

Vaidya and Vyas [2012, 2013] have discuss antimagic labeling of some path and cycle related graphs as well as graph obtained using switching of a vertex. Alon et al. [2004] have obtained some conditions on degree of a vertices for graph to be antimagic.

Krishnaa [2016] has illustrated some applications of antimagic labeling.

Definition 1.3. *Let $G = (V(G), E(G))$ be a graph with $V = S_1 \cup S_2 \cup S_3 \cup ...S_t \cup T$, where each S_i is a set of vertices having same degree (at least two vertices) and $T = V \setminus \cup_{i=1}^{t} S_i$. The* degree splitting graph *of G denoted by $DS(G)$ is obtained from the graph G by adding vertices $w_1, w_2, w_3, ..., w_t$ and joining w_i to each vertex of S_i for $1 \leqslant i \leqslant t$.*

Definition 1.4. *The* Cartesian product of graphs *G_1 and G_2 denoted by $G_1 \square G_2$ is the graph with vertex set $V(G_1) \times V(G_2) = \{(u, v)/u \in V(G_1)$ and $v \in V(G_2)\}$ and (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(G_2)$ or $v = v'$ and $uu' \in E(G_1)$.*

Definition 1.5. *The* ladder graph *L_n is defined as $L_n = P_n \square K_2$.*

Definition 1.6. *Let G_1 and G_2 be two graphs with no vertex in common, We define the* join *of G_1 and G_2, denoted by G_1+G_2, to be the graph with vertex set $V(G_1+G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1+G_2) = E(G_1) \cup E(G_2) \cup J$, where $J = \{x_1x_2 : x_1 \in V(G_1), x_2 \in V(G_2)\}$.*

Definition 1.7. *The* fan graph *F_n is defined as $F_n = P_n + K_1$.*

Definition 1.8. *Let G and H be two graphs. The* corona product *of G and H, denoted by $G \odot H$, is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the i^{th} copies of H to the i^{th} vertex of G, for $1, 2, 3, \ldots, |V(G)|$.*

Definition 1.9. *Let C_n be a cycle of n vertices. The* crown graph *is defined as $C_n \odot K_1$. .*

Definition 1.10. *Let P_n be a path of n vertices. The* comb graph *is defined as $P_n \odot K_1$.*

Definition 1.11. *Let P_n be a path of n vertices. The* double comb graph *is defined as $P_n \odot 2K_1$.*

In this paper, we have discuss antimagic labeling of degree splitting graphs of path, ladder graph, fan graph, crown graph, comb graph and double comb graph.

2 Main Results

Theorem 2.1. *The graph $DS(P_n)$ is an antimagic graph.*

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of P_n with $d(v_1) = d(v_2) = 1$. To construct $DS(P_n)$, add vertices w_1 and w_2 such that w_1 is adjacent to each vertex of degree 1 and w_2 is adjacent to each vertex of degree 2 in P_n. Then $|V(DS(P_n))| = n + 2$ and $|E(DS(P_n))| = 2n - 1$ for $n \geqslant 4$.
We define $f : E(DS(P_n)) \to \{1, 2, \ldots, 2n - 1\}$, as per following three cases.
Case - 1 For $n = 2$ and 3.
$DS(P_2)$ and $DS(P_3)$ are also known as cycle C_3 and cycle C_4 respectively. The antimagic labeling of C_3 and C_4 have been investigated by Hartsfield and Ringel [1990].
Case - 2 For $n = 4$.

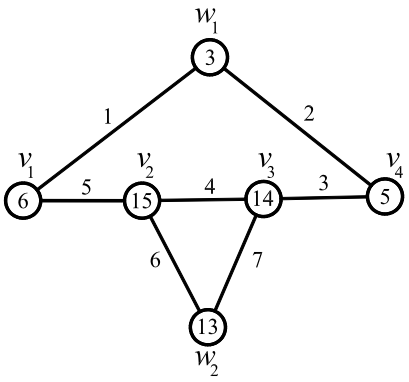

Figure 1: $DS(P_4)$ and its antimagic labeling.

Case - 3 For $n > 4$.
$f(v_1w_1) = 2(n-1)$,
$f(v_nw_1) = 2n-1$,
$f(v_iv_{i+1}) = i$; For $1 \leqslant i \leqslant n-1$,
$f(v_iw_2) = 2n-i-1$; For $2 \leqslant i \leqslant n-1$.
Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(P_n)$. Thus, f is an antimagic labeling.
Hence, the graph $DS(P_n)$ is an antimagic graph. □

Illustration 2.1. *The graph $DS(P_7)$ and its antimagic labeling is shown in Figure 2.*

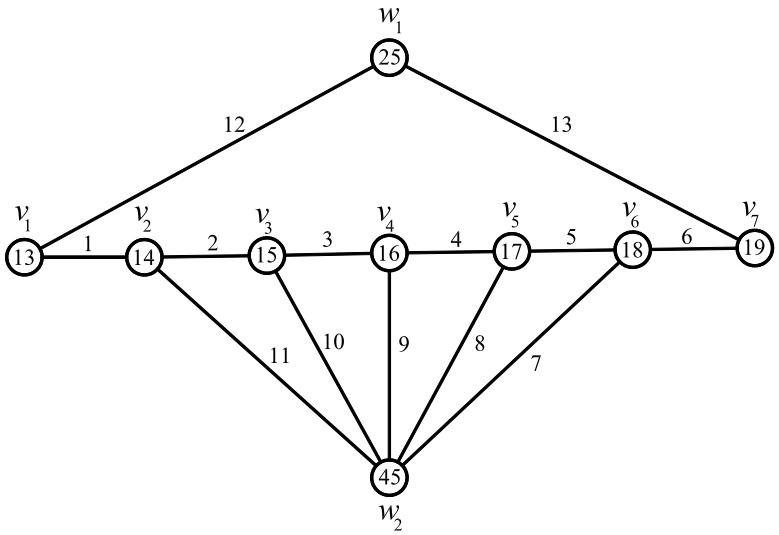

Figure 2: $DS(P_7)$ and its antimagic labeling.

Theorem 2.2. *The graph $DS(L_n)$ is an antimagic graph.*

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of first path and $v_{n+1}, v_{n+2}, v_{n+3}, \ldots, v_{2n}$ be the vertices of second path in ladder L_n. To construct $DS(L_n)$, add vertices w_1 and w_2 such that w_1 is adjacent to each vertex of degree 2 and w_2 is adjacent to each vertex of degree 3 in ladder L_n. Then $|V(DS(L_n))| = 2n + 2$ and $|E(DS(L_n))| = 5n - 2$ for $n \geqslant 3$.
We define $f : E(DS(L_n)) \to \{1, 2, \ldots, 5n - 2\}$ as per following two cases.
Case - 1 For $n = 2$.
$DS(L_2)$ is also known as wheel W_4. The antimagic labeling of W_4 has been investigated by Hartsfield and Ringel [1990].
Case - 2 For $n \geqslant 3$.

$f(v_i v_{i+1}) = i$; For $1 \leqslant i \leqslant n - 1$,
$f(v_i v_{n+i}) = 2n - i$; For $1 \leqslant i \leqslant n$,
$f(v_{n+i} v_{n+1+i}) = 2n + i - 1$; For $1 \leqslant i \leqslant n - 1$,
$f(v_i w_2) = 3(n - 1) + i$; For $2 \leqslant i \leqslant n - 1$,
$f(v_{n+i} w_2) = 4n - 5 + i$; For $2 \leqslant i \leqslant n - 1$,
$f(v_1 w_1) = 5(n - 1) + 2$,
$f(v_n w_1) = 5(n - 1) + 1$,
$f(v_{n+1} w_1) = 5(n - 1) + 3$,
$f(v_{2n} w_1) = 5(n - 1)$.

Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(L_n)$. Thus, f is an antimagic labeling.
Hence, the graph $DS(L_n)$ is an antimagic graph. □

Illustration 2.2. *The graph $DS(L_6)$ and its antimagic labeling is shown in Figure 3.*

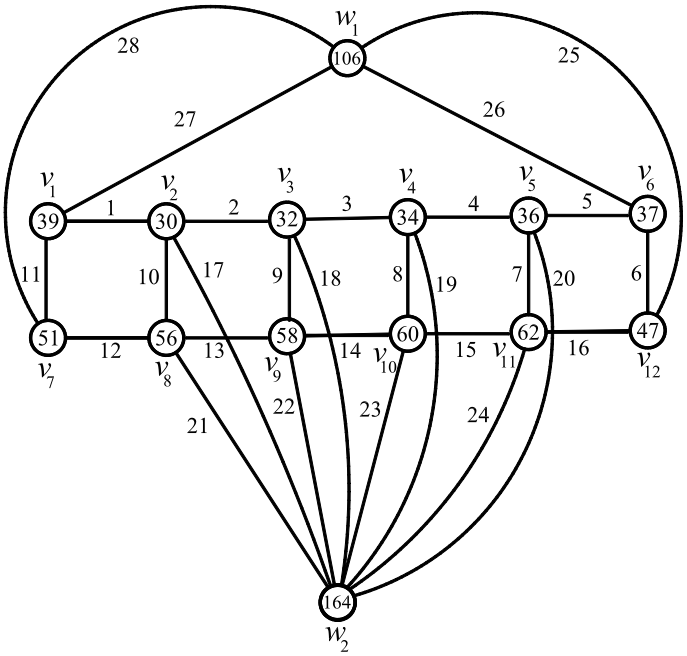

Figure 3: $DS(L_6)$ and its antimagic labeling.

Theorem 2.3. *The graph $DS(F_n)$ is an antimagic graph.*

Proof. Let v be a apex vertex and $v_1, v_2, v_3, ..., v_n$ be the vertices corresponding to path P_n in graph $F_n = P_n + K_1$. To construct $DS(F_n)$, add vertices w_1 and w_2 such that w_1 is adjacent to each vertex of degree 2 and w_2 is adjacent to each vertex of degree 3 in fan F_n. Then $|V(DS(F_n))| = n + 3$ and $|E(DS(F_n))| = 3n - 1$ for $n \geqslant 4$.

We define $f : E(DS(F_n)) \to \{1, 2, ..., 3n - 1\}$, as per following three cases.

Case - 1 For $n = 2$.

$DS(F_2)$ is also known as wheel W_3. The antimagic labeling of W_3 has been investigated by Hartsfield and Ringel [1990].

Case - 2 For $n = 3$.

The graph $DS(F_3)$ is an antimagic graph by Alon et al. [2004].

Case - 3 For $n \geqslant 4$.

$f(v_i v_{i+1}) = i$; For $1 \leqslant i \leqslant n - 1$,
$f(v_i w_2) = 2n - i - 1$; For $2 \leqslant i \leqslant n - 1$,
$f(vv_i) = 2n + i - 1$; For $1 \leqslant i \leqslant n$,
$f(v_1 w_1) = 2(n - 1)$,
$f(v_n w_1) = 2n - 1$.

Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(F_n)$. Thus, f is an antimagic labeling.

Hence, the graph $DS(F_n)$ is an antimagic graph. □

Illustration 2.3. *The graph $DS(F_7)$ and its antimagic labeling is shown in Figure 4.*

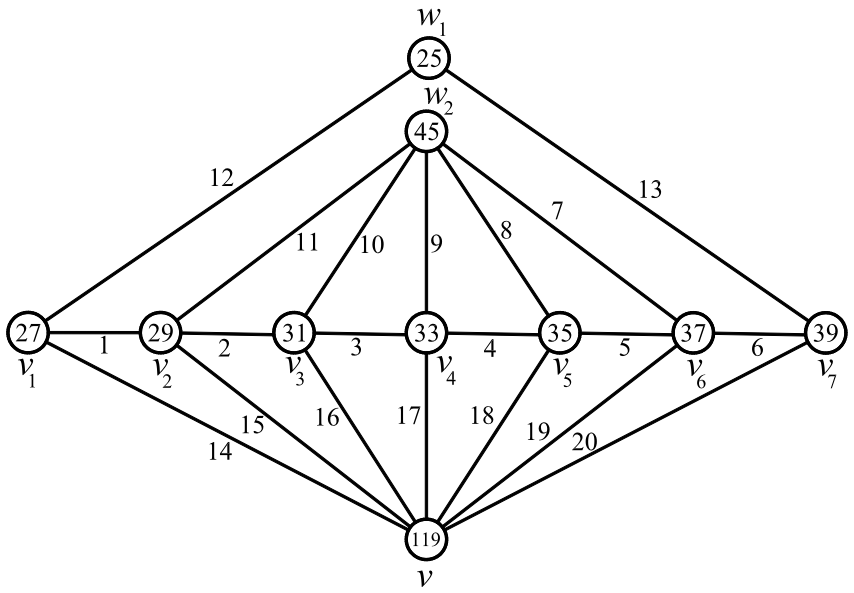

Figure 4: $DS(F_7)$ and its antimagic labeling

Theorem 2.4. *The graph $DS(C_n \odot K_1)$ is an antimagic graph.*

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices corresponding to cycle C_n and v_i' be the pendant vertices attached to v_i for each $i = 1, 2, \ldots, n$ respectively in graph $C_n \odot K_1$. To construct $DS(C_n \odot K_1)$, add vertices w_1 and w_2 such that w_1 is adjacent to each vertex of degree 3 and w_2 is adjacent to each vertex of degree 1 in $C_n \odot K_1$. Then $|V(DS(C_n \odot K_1))| = 2n + 2$ and $|E(DS(C_n \odot K_1))| = 4n$. We define $f : E(DS(C_n \odot K_1)) \to \{1, 2, \ldots, 4n\}$ as follows.

$f(v_i v_{i+1}) = 2n - i + 1$; For $1 \leqslant i \leqslant n-1$,
$f(v_1 v_n) = n + 1$,
$f(v_i v_i') = i$; For $1 \leqslant i \leqslant n$,
$f(v_1 w_1) = 2n + 1$,
$f(v_i w_1) = 3n - i + 2$; For $2 \leqslant i \leqslant n$,
$f(v_i' w_2) = 3n + i$; For $1 \leqslant i \leqslant n$.

Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(C_n \odot K_1)$. Thus, f is an antimagic labeling.
Hence, the graph $DS(C_n \odot K_1)$ is an antimagic graph. □

Illustration 2.4. *The graph $DS(C_6 \odot K_1)$ and its antimagic labeling is shown in Figure 5.*

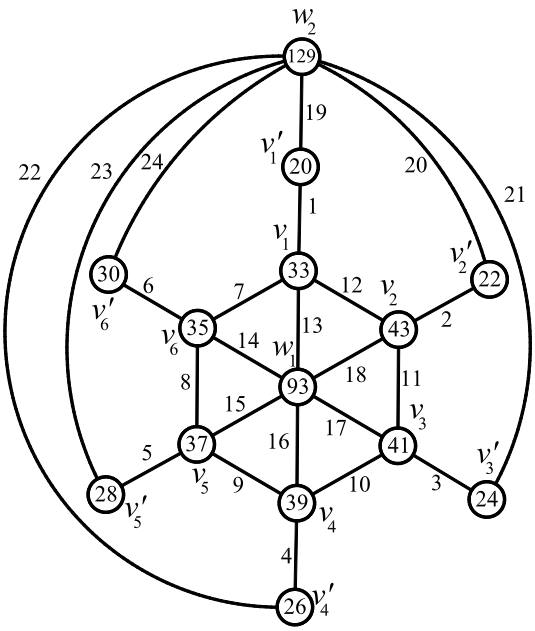

Figure 5: $DS(C_6 \odot K_1)$ and its antimagic labeling.

Theorem 2.5. *The graph $DS(P_n \odot K_1)$ is an antimagic graph.*

Proof. $v_1, v_2, v_3, \ldots, v_n$ be the vertices corresponding to path P_n and v_i' be the pendant vertices attached to v_i for each $i = 1, 2, \ldots, n$ respectively in graph $P_n \odot K_1$. To construct $DS(P_n \odot K_1)$, add vertices w_1, w_2 and w_3 such that w_1 is adjacent to each vertex of degree 1, w_2 is adjacent to each vertex of degree 2 and w_3 is adjacent to each vertex of degree 3 in $P_n \odot K_1$. Then $|V(DS(P_n \odot K_1))| = 2n + 3$ and $|E(DS(P_n \odot K_1))| = 4n - 1$ for $n \geqslant 4$.
We define $f : E(DS(P_n \odot K_1)) \to \{1, 2, \ldots, 4n-1\}$ as per following two cases.
Case - 1 For $n = 2$ and 3.

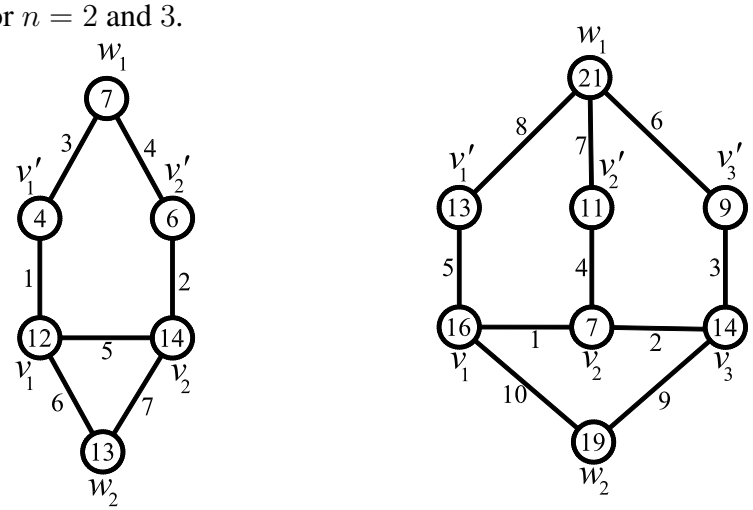

Figure 6: $DS(P_2 \odot K_1)$ and $DS(P_3 \odot K_1)$ and their antimagic labeling.

Case - 2 For $n \geqslant 4$.

$f(v_i v_i') = i$; For $1 \leqslant i \leqslant n$,
$f(v_i v_{i+1}) = 2n - i$; For $1 \leqslant i \leqslant n-1$,
$f(v_i w_3) = 4n - i - 1$; For $2 \leqslant i \leqslant n-1$,
$f(v_i' w_1) = 2n + i - 1$; For $1 \leqslant i \leqslant n$,
$f(v_1 w_2) = 2(2n-1)$,
$f(v_n w_2) = 4n - 1$.

Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(P_n \odot K_1)$. Thus, f is an antimagic labeling.
Hence, the graph $DS(P_n \odot K_1)$ is an antimagic graph. □

Illustration 2.5. *The graph $DS(P_8 \odot K_1)$ and its antimagic labeling is shown in Figure 7.*

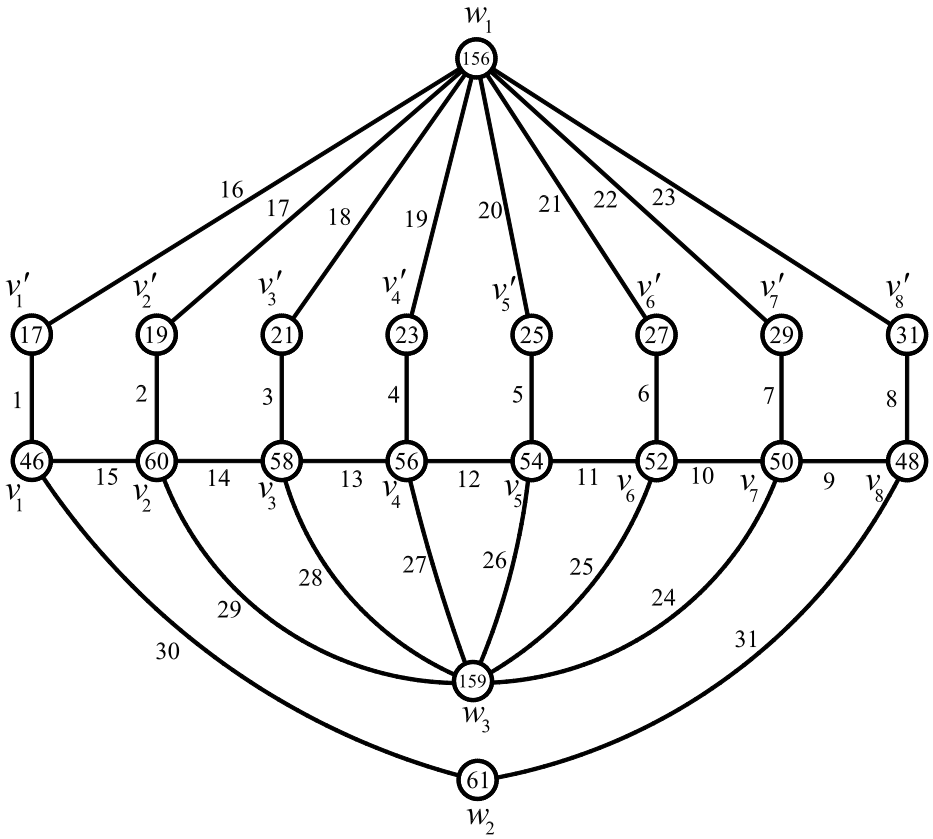

Figure 7: $DS(P_8 \odot K_1)$ and its antimagic labeling.

Theorem 2.6. *The graph $DS(P_n \odot 2K_1)$ is an antimagic graph.*

Proof. $v_1, v_2, v_3, \ldots, v_n$ be the vertices corresponding to path P_n and v_i' and v_i'' be the pendant vertices attached to v_i for each $i = 1, 2, \ldots, n$ respectively in graph

$P_n \odot 2K_1$. To construct $DS(P_n \odot 2K_1)$, add vertices w_1, w_2 and w_3 such that w_1 is adjacent to each vertex of degree 1, w_2 is adjacent to each vertex of degree 3 and w_3 is adjacent to each vertex of degree 4 in $P_n \odot 2K_1$. Then $|V(DS(P_n \odot 2K_1))| = 3n + 3$ and $|E(DS(P_n \odot 2K_1))| = 6n - 1$ for $n \geqslant 4$.
We define $f : E(DS(P_n \odot 2K_1)) \to \{1, 2, \ldots, 6n - 1\}$ as per following two cases.
Case - 1 For $n = 2$ and 3.

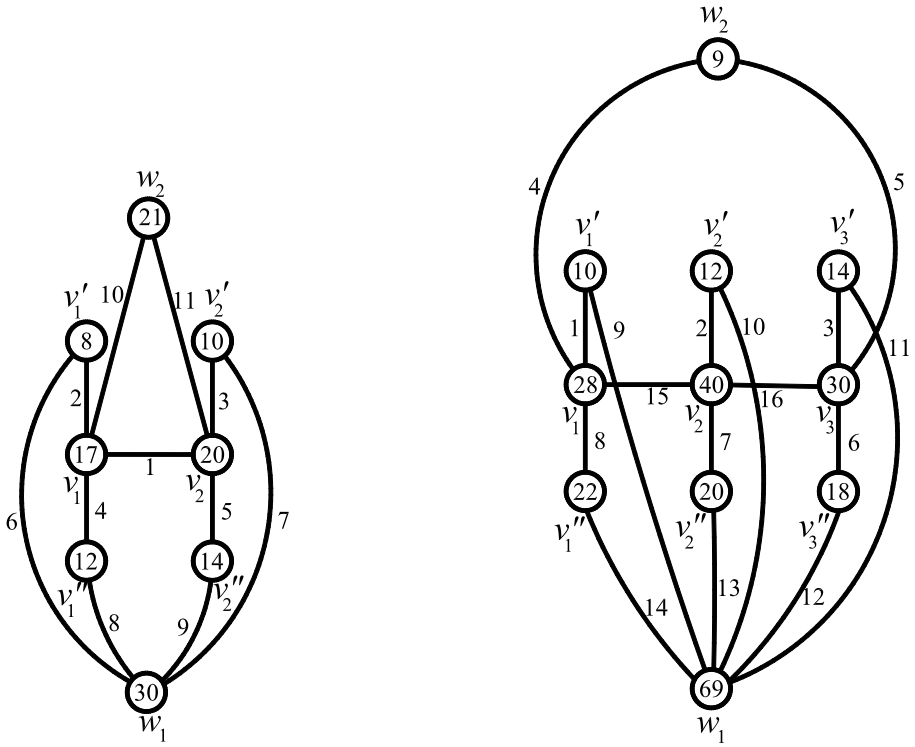

Figure 8: $DS(P_2 \odot 2K_1)$ and $DS(P_3 \odot 2K_1)$ and their antimagic labeling.

Case - 2 For $n \geqslant 4$.

$f(v_i v_{i+1}) = i$; For $1 \leqslant i \leqslant n - 1$,
$f(v_i v_i') = n - 1 + i$; For $1 \leqslant i \leqslant n$,
$f(v_i v_i'') = 2n - 1 + i$; For $1 \leqslant i \leqslant n$,
$f(v_i w_3) = 6n - 1 - i$; For $2 \leqslant i \leqslant n - 1$,
$f(v_i' w_1) = 3n - 1 + i$; For $1 \leqslant i \leqslant n$,
$f(v_i'' w_1) = 4n - 1 + i$; For $1 \leqslant i \leqslant n$,
$f(v_i w_2) = 2(3n - 1)$,
$f(v_n w_2) = 6n - 1$.

Above define edge labeling function will generate distinct vertex labels for all the vertices of $DS(P_n \odot 2K_1)$. Thus, f is an antimagic labeling.
Hence, the graph $DS(P_n \odot 2K_1)$ is an antimagic graph. □

Illustration 2.6. *The graph $DS(P_5 \odot 2K_1)$ and its antimagic labeling is shown in Figure 9.*

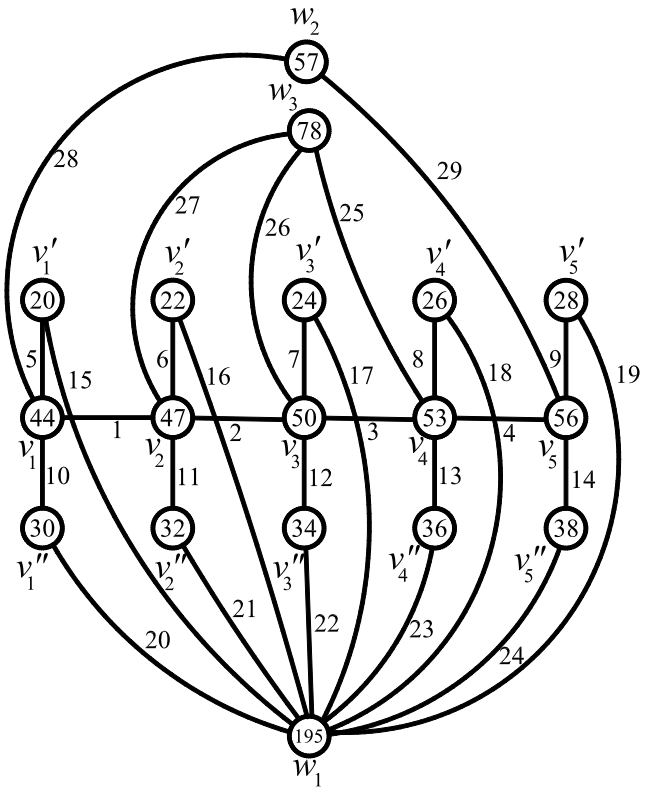

Figure 9: $DS(P_5 \odot 2K_1)$ and its antimagic labeling.

3 Conclusions

Hartsfield and Ringel [1990] has proved that path is an antimagic graph. Cheng [2007] has derived the antimagic labeling of ladder graph. While Sridharan and Umarani [2012] have discussed antimagic labeling of fan graph and crown graph. In this paper, we have discussed antimagic labeling of larger graphs obtained from Path, ladder graph, fan graph and crown graph using the degree splitting operation. Also antimagic labeling of degree splitting graphs of comb graph and double comb graph have been obtained. To investigated antimagic labeling for different graph families as well as in the context of various graph operations is an open area of research.

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Non-deterministic-M-fuzzy Lattices

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Abstract

Goran Trajakoviski and *Tepavčević* introduced the L-fuzzy lattice in which a bounded lattice is fuzzified using a complete lattice. Using a complete consistent distributive multilattice M, we broaden the concept of L-fuzzy lattice to Nd-M-fuzzy lattice in this study. We also propose the idea of Nd-M-order relation on a set X. Additionally, we demonstrated that the equivalent condition for a Nd-M-fuzzy subset to be a Nd-M-fuzzy sublattice of a bounded lattice L.

Keywords: Nd-M-fuzzy subset, Nd-M-fuzzy relation, Nd-M-fuzzy lattice.
2020 AMS subject classifications: 06D50, 06D72, 06F99 [1]

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1 Introduction

Giving suitable fuzzification to crisp concepts is a significant challenge that many academics find intriguing. The membership function, however, provides a unique value in L for each member of its domain in L-fuzzy sets [Birkhoff [1940], Goguen [1967], Klir and Yuan [1995]]. In this case, we employ a membership function to fuzzify a clear idea by providing each member in its domain with a set of values. Consequently, by employing non-deterministic membership functions [Gireesan and Thrivikraman [2015], K.K.Gireesan [2014]], a non-deterministic M-fuzzy set was created, where M is a multilattice [Cordero et al. [2008], Avallone [2002], Johnston [1990], Medina et al. [2006]]. A different concept for a multilattices that is connected to lattices was recently presented by Cordero [Cordero et al. [2008]]. We are looking on how to extend fuzzy-related concepts using these kinds of structures.

Goran Trajakoviski and Tepavc ˇevic ´ developed L-fuzzy lattice , who used a complete lattice to fuzzify a bounded lattice [Tepavčević and Trajkovski [2001]]. Two types of fuzzy lattice were proposed by them. The first form of fuzzy lattice is created by fuzzying up the membership of the elements in a crisp lattice, and the second type is created by fuzzying up the order relation in a crisp lattice. They concluded that these two varieties of fuzzy lattices are identical.

As shown below, the document is organised. Defining the words multilattice, Nd-M-fuzzy subsets, and "L-fuzzy Lattice" in Section 2 as well as outlining some preliminary (theoretical) findings. The concept of Nd-M-fuzzy order relation is explored in more detail in section 3. The concept of Nd-M-fuzzy lattice is introduced and studied in section 4.

2 Preliminary Results

Assume that $F \subseteq M$ and that (M,$\leq$) is a partially ordered set. The minimal element of the set of upper bounds of F is called the multisupremum of F, and is denoted by Multisup(F). Similarly the maximal element of the set of lower bounds of F is called the multiinfimum of F,and is denoted by multiinf(F).

Definition 2.1. *[Cordero et al. [2008]] A partially ordered set (M,$\leq$) is said to be ordered multilattice iff for all i, j, and k with $i \leq k$ and $j \leq k$, there exist $q \in Multisup\{i, j\}$ such that $q \leq k$.*

It should be emphasised that our sets will instead contain a set of multisuprema rather than a supremum. Therefore, here we present some ordering, namely the Hoare ordering, the Smyth ordering, and the Egli-Milner ordering, respectively among the subsets of posets.

Definition 2.2. *[Medina et al. [2006]] let $G, H \in 2^M$, then*
$G \sqsubseteq_H H$ iff $\forall\, g \in G\ \exists\, h \in H$ such that $g \leq h$;
$G \sqsubseteq_S H$ iff $\forall\, h \in H\ \exists\, g \in G$ such that $g \leq h$;
$G \sqsubseteq_{EM} H$ iff $G \sqsubseteq_H H$ and $G \sqsubseteq_S H$.

Definition 2.3. *[Johnston [1990]] Let G and H are two subsets of a algebraic multilattice$(M, \wedge, \vee)$ and $m \in M$, then*
$m \wedge G = \cup\{(m \wedge g)/g \in G\}$;
$m \vee G = \cup\{(m \vee g)/g \in G\}$;
$G \wedge H = \ \cup\{(g \wedge h)/g \in G, h \in H\}$;
$G \vee H = \cup\{(g \vee h)/g \in G, h \in H\}$.

Definition 2.4. *[Gireesan and Thrivikraman [2015], K.K.Gireesan [2014]] Let $A \neq \emptyset$ and M be a complete distributive multi lattice. Then a function from A to 2^M is called Non deterministic M-Fuzzy subset of A (Nd-M-fuzzy subset) and Non-deterministic-M-fuzzy space is the set of all Nd-M-fuzzy subsets of A and is denoted by $(2^M)^A$.*

Definition 2.5. *If a completely distributive multilattice M has an order reversing involution $' : M \to M$, then M is said to be F multilattice and for any $P \in (2^M)^A$, $P'(a) = [P(a)]' = \{p' \mid p \in P(a)\}$*

Definition 2.6. *[Gireesan and Thrivikraman [2015]] Set relations rules on $(2^M)^X$.*
Suppose H and K belongs to $(2^M)^A$. Then

1. *$H = K$, if $H(i) = K(i)$, for every $i \in A$;*
2. *$H \leq K$, if $H(i) \sqsubseteq_{EM} K(i)$;*
3. *$L = H \vee K$, if $L(i) =$ multisup $\{(H(i), K(i)) \mid \forall\, i \in A\} = \cup\{(h \vee k) \mid h \in H(i), k \in K(i)\}$, $\forall\, i \in A$;*
4. *$I = H \wedge K$, if $I(i) =$ multiinf $\{(H(i), K(i)) \mid \forall\, i \in A\} = \cup\{(h \wedge k) \mid h \in H(i), k \in K(i)\}$, $\forall\, i \in A$.*

Definition 2.7. *[Ajmal and Thomas [1994], Tepavčević and Trajkovski [2001]] Suppose $(A, \vee, \wedge)$ be a lattice and $(L, \vee_L, \wedge_L)$ is a complete lattice with 0_L and 1_L and $\Omega \in L^A$. Then the set $\Omega_\beta = \{t \in A \mid \Omega(t) \geq \beta\}$ is called the β cut of Ω . Also Ω is called a fuzzy sub lattice of L, if*

$$\Omega(s \wedge t) \wedge_L \Omega(s \vee t) \geq \min(\Omega(s), \Omega(t)) \ \ s, t \in A.$$

That is,

$$\Omega(s \wedge t) \wedge_L \Omega(s \vee t) \geq \Omega(s) \wedge_L \Omega(t).$$

Remark 2.1. *$\Omega \in L^A$ is an L-fuzzy sub lattice of A iff Ω_β is a sublattice of A for each $\beta \in L$.*

3 Non-deterministic-M-fuzzy relation

Definition 3.1. *Suppose $(M, \wedge_M, \vee_M)$) be a complete and consistent multilattice with greatest element 1_M and least element 0_M, where $\vee_M$ and $\wedge_M$ are multi-supremum and multiinfimum of elements of M and A be a set,$A \neq \phi$. Then a Non-deterministic M-fuzzy relation on A is a mapping $\Phi : A \times A \to A$.*

Definition 3.2. *Assume $\beta \in 2^M$, then the set $\Phi_\beta = \{(a, b) : \beta \sqsubseteq_{EM} \Phi(a, b)\}$ is called the β-level of Φ.*

Definition 3.3. *An Nd-M-fuzzy relation Φ is called*

1. *Nd-M- reflexive, if $\Phi(i, j) = \{1\}$, $\forall\, i, j \in A$;*
2. *Weakly Nd-M- reflexive, if*

$$\Phi(i, j) \sqsubseteq_{EM} \Phi(i, i) \text{ and } \Phi(j, i) \sqsubseteq_{EM} \Phi(i, i) \ \forall\ , i, j \in A;$$

3. *Nd-M- anti-symmetric, if*

$$\Phi(i, j) \wedge_M \Phi(j, i) = \{0\} \ \forall\ i, j \in A \ with \ i \neq j;$$

4. *Nd-M- transitive, if*

$$\Phi(i, j) \wedge \Phi(j, i) \sqsubseteq_{EM} \Phi(i, k) \ , \forall\ i, j, k \in A.$$

If Φ is Nd-M-reflexive, Nd-M- antisymmetric and Nd-M-transitive, then Φ is called Nd-M-fuzzy ordering relation on A.

Definition 3.4. *Suppose $(L, \vee_L, \wedge_L, 0, 1)$ is a bounded lattice and $(M, \vee_M, \wedge_M)$ be a non-trivial complete and consistent multilattice. Let $\Omega \in (2^M)^L$. Then the $\Omega_\beta = \{t \in L \mid \beta \sqsubseteq_{EM} \Omega(t)\}$ is the β -level of Ω, where $\beta \in 2^M$.*

4 Non-deterministic-M-fuzzy lattice

Definition 4.1. *If a Nd-M fuzzy subset Ω belongs to $(2^M)^L$ and if $\beta \sqsubseteq_{EM} \Omega(s) \wedge_M \mu(t)$, $\forall\, s, t \in \Omega_\beta$ and Ω_β is a sub lattice of L for each $\beta \in 2^M$, then Ω is called a non-deterministic-M-fuzzy sub lattice (Nd-M-fuzzy sub lattice) of L.*

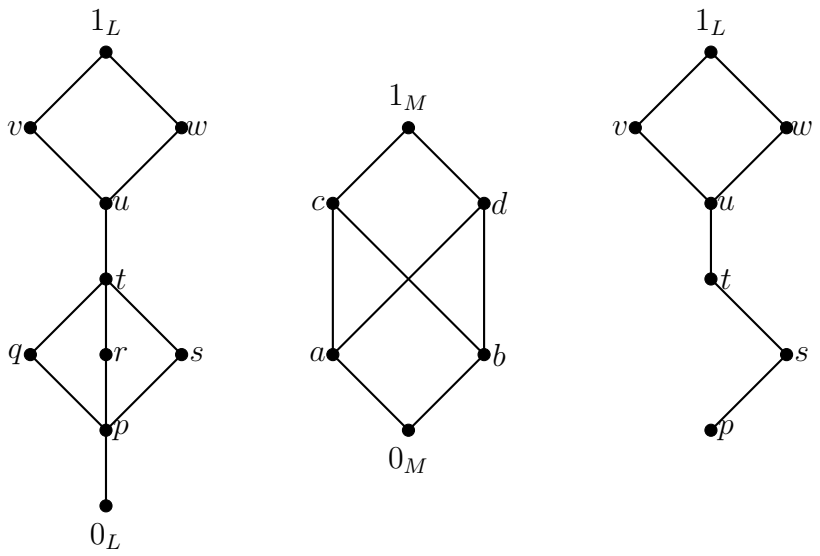

Figure 1: The lattice L, the multilattice M and the β-level Ω_β where $\beta = \{a\}$ in Example 4.1

Example 4.1. *1. Any L-fuzzy lattice is a Nd-M-fuzzy lattice.*

2. *Choose $\beta \in 2^M$ such that $\beta \wedge \beta = \beta$. Let $\Omega : L \to 2^M$ defined by $\Omega(t) = \beta$,$\forall$ $t \in L$, Ω is a Nd-M- fuzzy lattice.*

3. *If $(M, \wedge_M, \vee_M)$ be a multilattice with $0_M$and 1_M and $(L, \vee_L, \wedge_L)$ be a lattice where $M = \{0_M, a, b, c, d, 1_M\}$ and $L = \{0_L, p, q, r, s, t, u, v, w, 1_L\}$, then*

$$\Omega = \begin{pmatrix} 0_L & p & q & r & s & t & u & v & w & 1_L \\ \{b\} & \{a\} & \{b\} & \{a, b\} & \{a\} & \{d\} & \{d\} & \{c, d\} & \{d\} & 1_M \end{pmatrix}$$

is an $Nd - M-$ fuzzy lattice (see Fig.1).
If $\beta = \{a\}$, then $\Omega_\beta = \{p, s, t, u, v, w, 1_L\}$.

Theorem 4.1. *Assume Ω_β,where $\beta \in 2^M$ and $\Omega \in (2^M)^L$, satisfies $\beta \sqsubseteq_{EM} \Omega(s) \wedge_M \Omega(t), \forall s, t \in \Omega_\beta$, then Ω is a Nd-M-fuzzy lattice of L iff for every $s, t \in L$,*

$$multiinf\{\Omega(s), \Omega(t)\} \sqsubseteq_{EM} multiinf\{\Omega(s \wedge_L t), \Omega(s \vee_L t)\}.$$

That is,

$$\Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \wedge_L t) \wedge_M \Omega(s \vee_L t).$$

Proof. Assume $\Omega_\beta = \{t \in L : \beta \sqsubseteq_{EM} \Omega(t)\}$ satisfies $\beta \sqsubseteq_{EM} \Omega(s) \wedge_M \Omega(t)$,$\forall$ $s, t \in \Omega_\beta$. Consider $P = \Omega(s) \wedge_M \Omega(t)$, then $P \sqsubseteq_{EM} \Omega(s)$ and $P \sqsubseteq_{EM} \Omega(t)$. That is, $s, t \in \Omega_P$. But by our hypothesis, Ω_P is a sub lattice of L, then $s \wedge_L t$ and $s \vee_L t$ belongs to Ω_P and so $P \sqsubseteq_{EM} \Omega(s \vee_L t)$ and $P \sqsubseteq_{EM} \Omega(s \wedge_M t$. Since $s \wedge_L t$ and $s \vee_L t$ belongs to Ω_P, $P \sqsubseteq_{EM} \Omega(s \vee_L t) \wedge_M \Omega(s \wedge_L t)$. Therefore, $P = \Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \vee_L t) \wedge_M \Omega(s \wedge_L t)$.

Conversely, assume that Ω satisfies $\Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \vee_L t) \wedge_M \Omega(s \wedge_L t)$. Let P belongs to 2^M. Then For every $s, t \in \Omega_P$, Then $P \sqsubseteq_{EM} \Omega(s)$ and $P \sqsubseteq_{EM} \Omega(t)$. Hence $P \sqsubseteq_{EM} \Omega(s) \wedge_M \Omega(t)$. But our assumption, we have $P \sqsubseteq_{EM} \Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \wedge_L t) \wedge_M \Omega(s \vee_L t)$. Hence $P \sqsubseteq_{EM} \Omega(s \wedge_L t)$ and $P \sqsubseteq_{EM} \Omega(s \vee_L t)$. Since $s \vee_L t \in \Omega_P$ and $s \wedge_L t \in \Omega_P$, we obtain Ω_P is a sublattice of L, and thus Ω is an Nd-M fuzzy lattice.

□

Theorem 4.2. *Suppose $(L, \wedge_L, \vee_L)$ be a lattice $(M, \wedge_M, \vee_M)$ be a bounded multilattice. Then $\Omega : L \to 2^M$ is an Nd-M- fuzzy lattice, the following relations eqivalent.*

1. $\Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \wedge_L t)$;
2. $\Omega(s) \wedge_M \Omega(t) \sqsubseteq_{EM} \Omega(s \vee_L t)$.

The steps for creating a Nd-M- fuzzy lattice with a family of lattices as its family of level sets are shown below.

Consider two posets S_1 and S_2 that have disjoint underlying sets. The posets $(S_1 \cup S_2, \leq)$ is the disjoint union of posets S_1 and S_2, where $\leq$ is defined by $s \leq$ iff $s, t \in S_1$ and $s \leq t$ in S_1 or $s, t \in S_2$ and $s \leq$ in S_2 or $s \in S_1$ and $t \in S_2$. The sum of Posets S_1 and S_2 is called linear sum and is denoted by $(S_1 \oplus S_2, \leq)$.

Theorem 4.3. *Let κ be the disjoint family of lattices. Then there is an Nd-M-fuzzy lattice whose non-trivial β-levels are precisely the lattices from κ.*

Proof. Suppose κ i a collection of lattices $(L_i, \wedge_i, \vee_i)$ with smallest element 0_i and greatest element 1_i. lowest element 0_i and greatest element 1_i, $(i \in I)$.Then create a Nd-M- fuzzy lattice using the assortment of lattices in the manner shown below.

Assume $M = \{0_M, 1_M\} \cup \{m_i, n_i : i \in I\}$ be the multilattice, where the order M is defined by $\forall\ i\ m_i \geq n_i, \forall i$ and $m_1 \geq n_2$ and $m_2 \geq n_1$, where n_i,'s are atoms and m_i's are coatoms. Define the linear sum $\oplus$ of these lattices is the set $L = 0_L \oplus \bigcup_{i \in I} L_i \oplus 1_L$, where 0_L and 1_L are the single element lattices. Now L is a lattice. Then define $\Omega : L \to 2^M$ by

$$\Omega\{s\} = \{m_i, n_i\} \text{ iff } s \in L_i, i \in I \text{ and } \Omega\{0_L\} = \{0_M\}, \Omega\{1_L\} = \{1_M\}.$$

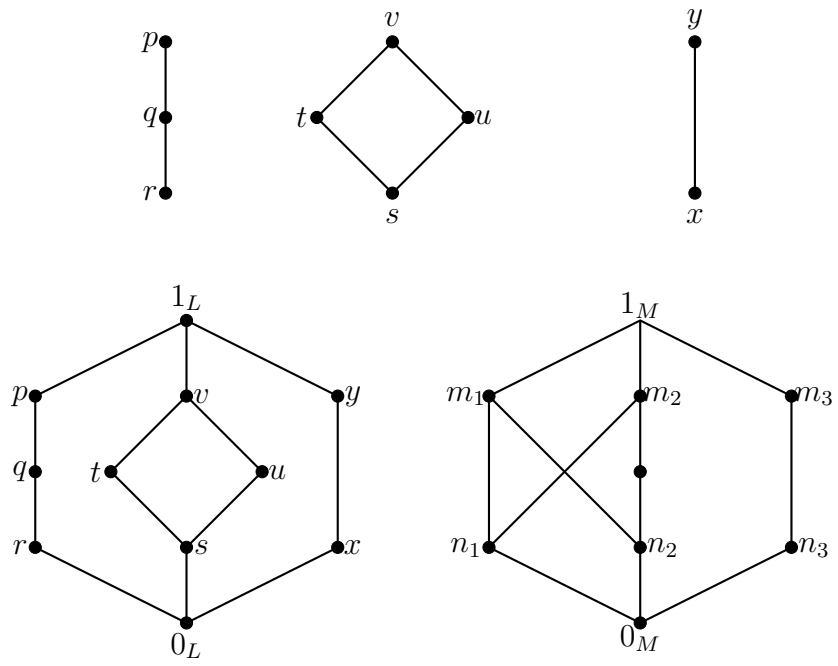

Figure 2: The lattices and the multilattice in Example 4.2

It is then evident that for every $\beta \in 2^M$, all of the non-trivial - β levels of Ω are precisely the lattices in κ. □

Example 4.2. *Let κ has three lattices L_1, L_2 and L_3 (see figure 2). Next, create L and M using the prior theorem.Following the mapping Ω provides the necessary Nd-M- fuzzy lattices.*

$$\bar{L} = \begin{pmatrix} p & q & r & s & t & u \\ \{m_1, n_1\} & \{m_1, n_1\} & \{m_1, n_3\} & \{m_2, n_2\} & \{m_2, n_3\} & \{m_2, n_3\} \\ v & x & y & 0_L & 1_L & \\ \{m_2, n_2\} & \{m_3, n_3\} & \{m_3, n_3\} & 0_M & 1_M & \end{pmatrix}.$$

5 Conclusion

The terms L-fuzzy relation and L-fuzzy lattice were generalised to Nd-M-fuzzy relation and Nd-M-fuzzy lattice, respectively. Furthermore, we proved the necessary and sufficient condition for a Nd-M-fuzzy subset to be a Nd-M-fuzzy sublattice of L. Finally, we noticed that when the membership value in an L-fuzzy lattice is replaced by a set of values in a multilattice in M, the expansion of L-fuzzy lattices to Nd-M-fuzzy lattices is true with regard to the Egli- Milner ordering.

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On Ideals in Partially Ordered Ternary Semigroups

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Abstract

The concept of ideals has been extensively studied through various algebraic structures such as near-rings, involution rings, regular rings, gamma-rings, semigroups, ordered semigroups, ternary semigroups and ordered ternary semigroups. In this article, we have investigated some intriguing properties of ideals, bi-ideals and pseudo symmetric ideals in partially ordered ternary semigroup T. We establish conditions for $(UVW]$ to be an ideal, bi-ideal and pseudo symmetric ideal of T, where U, V, W are non-empty subsets of T. We prove that the family $\mathcal{P}$ of all pseudo symmetric ideals of T forms a Moore family of subsets of T. We also prove that the collection $\mathcal{P}$ of all pseudo symmetric ideals of T is a Brouwerian lattice. Also we define a pseudo symmetric partially ordered ternary semigroup.

Keywords: partially ordered ternary semigroup; bi-ideal; pseudo symmetric ideal; brouwerian lattice.
2020 AMS subject classifications: 20M12; 20N99; 06F99.[1]

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1 Introduction

Algebraic structures play a significant role in development of abstract algebra in particular and mathematics in general. Algebraic structure has a wide variety of applications in various fields such as theoretical physics, computing science, information technology, control engineering, topological spaces and many more. The study of different algebraic structures has attracted the attention of many researchers.

The concept of a semigroup in particular is a very simple mathematical structure. It plays incredible role in the evolution of abstract algebra. Many applications of semigroup structure can be found in many disciplines such as physics, combinatorics, coding theory, automata theory and so on. In probability theory, the semigroups are associated with their relationship to the Markov process. A semigroup is an algebraic structure consisting of a non-empty set together with an associative binary operation. A variety of studies have been conducted on the generalizations to the concept of semigroups viz. ordered semigroups, ternary semigroups and ordered ternary semigroups. Following this, researchers have expanded the research in a myriad of ways. N. Kehayopulu developed the idea of ordered semigroups and invented the concept of ordered bi-ideals in an ordered semigroup. He primarily devoted his time to study ordered ideals, ordered quasi-ideals and ordered bi-ideals in an ordered semigroup. In addition, he proposed the idea of regular ordered semigroups.

There is an extensive research literature that deals with the ternary operation. Firstly, ternary algebraic operations were introduced during the 19^{th} century by Cayley. The idea of ternary algebraic structures was developed by D. H. Lehmer in 1932. In [3], D. H. Lehmer studied the certain algebraic systems called as triplexes, which were discovered to be commutative ternary groups. Ideals have been extensively studied in various algebraic structures such as near-rings, involution rings, regular rings, gamma-rings, semigroups, ordered semigroups, ternary semigroups and ordered ternary semigroups. The ideal theory in n-ary semigroups and ternary semigroup was developed by Sioson F. M. [4] in 1965. The properties of quasi-ideals and bi-ideals in ternary semigroups studied by authors V. N. Dixit and S. Dewan [7].

A. Iampan in [1, 2] invented the notion of ordered ternary semigroup which is a generalization of the notion of ordered semigroups together with the notion of ternary semigroups and characterized the minimality and maximality of ordered lateral ideals in the ordered ternary semigroups. He also developed the theory of ordered ternary semigroups along the line of the theory of ordered semigroups. In 2014, V. Siva Rami Reddy et al. [9, 10] have developed the ideal theory of a partially ordered ternary semigroup. They also defined and studied the notions of prime ideals in partially ordered ternary semigroups. In [6], V. Jyothi et al.

introduced and studied the notion of pseudo symmetric ideals in partially ordered ternary semigroups.

In this article, we investigate several intriguing properties of ideals in partially ordered ternary semigroup T. We define the conditions under which $(UVW]$ is an ideal, bi-ideal and pseudo symmetric ideal of T, where U, V, W are non-empty subsets of T. We prove that the family $\mathcal{P}$ of all pseudo symmetric ideals of T forms a Moore family of subsets of T. Also, we prove that the family $\mathcal{P}$ of all pseudo symmetric ideals of T forms a Brouwerian lattice. Also we define a pseudo symmetric partially ordered ternary semigroup.

2 Preliminaries

In this section, we collect some definitions and fundamental results from [1, 2, 3, 6, 8], which we need for the development of this article.

A non-empty set T is said to be a ternary semigroup [3] if there exists a ternary operation $[\ ]$, from $T \times T \times T$ to T written as $(u, v, w) \longrightarrow [uvw]$, satisfies the law, $[u\ v\ w\ s\ t] = [[u\ v\ w]\ s\ t] = [u\ [v\ w\ s]\ t] = [u\ v\ [w\ s\ t]]$, for all $u, v, w, s, t \in T$.

Let U, V and W be three non-empty subsets of a ternary semigroup T, $[UVW] = \{[uvw] \ : \ u \in U, v \in V \ \text{ and } \ w \in W\}$. For easiness, we write, $[UVW]$ as $UVW, [uvw] = uvw$ and $[UUU] = U^3$.

A ternary semigroup T is called a partially ordered ternary semigroup [1] if there exist a partial ordering relation $\leq$ on T such that, $s \leq t \Rightarrow uvs \leq uvt, usv \leq utv, suv \leq tuv\ \forall s, t, u, v \in T$. In this article, let T be a partially ordered ternary semigroup, unless otherwise specified. For $U \subseteq T$, let $(U] = \{s \in T \ : \ s \leq u, \text{ for some } \ u \in U\}$. A partially ordered ternary semigroup T is said to be commutative if $uvw = wuv = vwu = vuw = wvu = uwv\ \forall u, v, w \in T$.

A left (resp., a right, a lateral) ideal of T is a non-empty subset I of T such that $TTI \subseteq I$ (resp., $ITT \subseteq I$, $TIT \subseteq I$) and $(I] = I$. A non-empty subset I of T is called a ideal of T if I is a left ideal, a right ideal and a lateral ideal of T. A bi-ideal of T is a non-empty subset U of T such that $UTUTU \subseteq U$ and $(U] = U$. An ideal P of T is said to be a pseudo symmetric ideal [6] of T provided $u, v, w \in T,\ uvw \in P \Rightarrow usvtw \in P\ \forall s, t \in T$.

Result 2.1 ([8]). *Let U, V and W be three non-empty subsets of T then the following hold,*
(1) $U \subseteq (U]$.
(2) $((U]] = (U]$.
(3) $(U](V](W] \subseteq (UVW]$.
(4) If $U \subseteq V$ *then* $(U] \subseteq (V]$.

3 Some Properties of Ideals

In this section, we study the various properties of ideals in a partially ordered ternary semigroup.

Theorem 3.1. *If U is a non-empty subset of T. Then $(TUT]$ is a two-sided ideal of T.*

Proof. As U is non-empty subset of T then $(TUT]$ is a non-empty subsets of T. Consider, $TT(TUT] = ((TTT)UT] \subseteq (TUT]$. Similarly, we can show that, $(TUT]TT \subseteq (TUT]$. Now, let $v \in (TUT]$ and $u \in T$ such that $u \leq v$. $v \in (TUT] \Rightarrow v \leq w$ for some $w \in TUT$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in TUT \Rightarrow u \in (TUT]$. Hence $(TUT]$ is a two-sided ideal of T. □

Theorem 3.2. *If U and V are two non-empty subsets of T, then $(UVT]$ is a right ideal of T.*

Proof. Since U and V are two non-empty subsets of T then $(UVT]$ is a non-empty subsets of T. Consider, $(UVT]TT = (UV(TTT)] \subseteq (UVT]$. Now, let $v \in (UVT]$ and $u \in T$ such that $u \leq v$. $v \in (UVT] \Rightarrow v \leq w$ for some $w \in UVT$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UVT \Rightarrow u \in (UVT]$. Hence $(UVT]$ is a right ideal of T. □

Theorem 3.3. *If U is a left ideal and V is a non-empty subset of T, then $(UVT]$ is a two-sided ideal of T.*

Proof. As U, V are non-empty subset of T. So $(UVT]$ is a non-empty subset of T. Consider, $TT(UVT] = ((TTU)VT] \subseteq (UVT]$ (since U is a left ideal of T). Similarly, we can prove that, $(UVT]TT \subseteq (UVT]$. Now, let $v \in (UVT]$ and $u \in T$ such that $u \leq v$. $v \in (UVT] \Rightarrow v \leq w$ for some $w \in UVT$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UVT \Rightarrow u \in (UVT]$. Hence $(UVT]$ is two-sided ideal of T. □

Remark 3.1. *Taking $V = T$ in particular, in Theorem 3.2 we get $(UTT]$ is right ideal of T. But the set $(UTT]$ is not a left and a lateral ideal of T. To establish this, consider the non-commutative partially ordered ternary semigroup T. If U is a left ideal of T then $(UTT]$ is a two-sided ideal of T.*

Theorem 3.4. *If U and V are two non-empty subsets of T, then $(TUV]$ is a left ideal of T.*

Proof. Similar to the proof of Theorem 3.2 □

Theorem 3.5. *If V is a right ideal of T and U is a non-empty subset of T, then $(TUV]$ is a two-sided ideal of T.*

Proof. Similar to the proof of Theorem 3.3 □

Remark 3.2. *Taking $U = T, V = U$ in particular, in Theorem 3.4 we get $(TTU]$ is a left ideal of T. But the set $(TTU]$ is not a right and a lateral ideal of T. To establish this, consider the non-commutative partially ordered ternary semigroup T. If U is a right ideal of T then $(TTU]$ is a two-sided ideal of T.*

Theorem 3.6. *If U is a left ideal and V is a right ideal of T then $(UTV]$ is two-sided ideal of T.*

Proof. As U, V are non-empty subset of T. So $(UTV]$ is a non-empty subset of T. Consider, $TT(UTV] = ((TTU)TV] \subseteq (UTV]$ (since U is a left ideal of T). Similarly, we can prove that, $(UTV]TT \subseteq (UTV]$. Now, let $v \in (UTV]$ and $u \in T$ such that $u \leq v$. $v \in (UTV] \Rightarrow v \leq w$ for some $w \in UTV$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UTV \Rightarrow u \in (UTV]$. Hence $(UTV]$ is two-sided ideal of T. □

Corolary 3.1. *If U is a left ideal, W is a right ideal and V is a non-empty subset of T, then $(UVW]$ is two-sided ideal of T.*

Definition 3.1 ([5])**.** *A partially ordered ternary semigroup T is called regular partially ordered ternary semigroup if for each $u \in T$ there exists $v \in T$ such that $u \leq uvu$.*

Lemma 3.1 ([5])**.** *If T is a regular partially ordered ternary semigroup, then $U \subseteq (UTU]$ for any $U \subseteq T$.*

Theorem 3.7. *If U, V, W are three ideals of a regular partially ordered ternary semigroup T. Then $(UVW]$ is an ideal of T.*

Proof. Since U, V, W are three ideals of a regular partially ordered ternary semigroup T. By Lemma 3.1, $U \subseteq (UTU], V \subseteq (VTV]$ and $W \subseteq (WTW]$. Consider, $TT(UVW] = (T](T](UVW] = (TT(UVW)] = ((TTU)VW)] \subseteq (UVW]$. Similarly, we can show that, $(UVW]TT \subseteq (UVW]$. Now, consider $T(UVW]T = (T(UVW)T] \subseteq (T(UTU]V(WTW]T] \subseteq ((TUTUVWTWT]]$ $\subseteq (TUTUVWTWT] \subseteq ((TUT)UV(WTW)T] \subseteq (UUVW(TWT)] \subseteq$ $(UUVWW] \subseteq (UTVTW] \subseteq (U(TVT)W] \subseteq (UVW]$. Let $v \in (UVW]$ and $u \in T$ such that $u \leq v$. $v \in (UVW] \Rightarrow v \leq w$ for some $w \in UVW$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UVW \Rightarrow u \in (UVW]$. Hence $(UVW]$ is an ideal of T. □

Note 3.1. *In general, if V is an ideal of U and U is an ideal of T. Then V is not an ideal of T.*

Theorem 3.8. *If U is an ideal of T and V is an ideal of U such that $V = V^3$. Then V is an ideal of T.*

Proof. As U is an ideal of T and V is an ideal of U such that $V = V^3$. Consider, $TTV = TT(VVV) \subseteq (TTU)VV \subseteq UVV \subseteq UUV \subseteq V$. Similarly, we can prove that, $VTT \subseteq V$. Now, consider $TVT = T(VVV)T \subseteq (TTU)VT \subseteq UVT = U(VVV)T \subseteq (UUV)VT \subseteq VVT = V(VVV)T \subseteq VV(UTT) \subseteq VVU \subseteq VUU \subseteq V$. Let $v \in V$ and $u \in T$ such that $u \leq v$. $v \in V \subseteq U, u \in T, u \leq v$ and U is an ideal of $T \Rightarrow u \in U$. Now, we have V is an ideal of U, $v \in V$ and $u \in U \subseteq T$ such that $u \leq v \Rightarrow u \in V$. Therefore V is an ideal of T. □

4 Results on Bi-ideals

In this section, we study the various properties of bi-ideals in a partially ordered ternary semigroup.

Theorem 4.1. *Let U be a left ideal and V be a bi-ideal of T. Then $(UVV]$ is a left ideal and a bi-ideal of T.*

Proof. As U, V are non-empty subsets of T. So $(UVV]$ is a non-empty subset of T. Consider $TT(UVV] = (T](T](UVV] = (TT(UVV)] = ((TTU)VV)] \subseteq (UVV]$. By definition of bi-ideal, consider $(UVV]T(UVV]T(UVV]$
$= (UVVTUVVTUVV] = (U(V(VTU)V(VTU)V)V] \subseteq (U(VTVTV)V]$ (since $(VTU) \subseteq (TTT) \subseteq T) \subseteq (UVV]$ (since V is a bi-ideal of T, $(VTVTV) \subseteq V$). Now, let $v \in (UVV]$ and $u \in T$ such that $u \leq v$. $v \in (UVV] \Rightarrow v \leq w$ for some $w \in UVV$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UVV \Rightarrow u \in (UVV]$. Hence $(UVV]$ is a left ideal and a bi-ideal of T. □

Theorem 4.2. *Let U be a right ideal and V be a bi-ideal of T. Then $(VVU]$ is a right ideal and a bi-ideal of T.*

Proof. Similar to the proof of Theorem 4.1 □

Theorem 4.3. *Let U be a lateral ideal and V be a bi-ideal of T. Then $(VUV]$ is a bi-ideal of T.*

Proof. As U, V are non-empty subsets of T, $(VUV]$ is a non-empty subset of T. By definition of bi-ideal, consider $(VUV]T(VUV]T(VUV] =$
$(VUVTVUVTVUV] = (V(U(VTV)U(VTV)U)V] \subseteq (V(UTUTU)V]$ (since

$(VTV) \subseteq (TTT) \subseteq T) \subseteq (VU(TUT)UV] \subseteq (VU(U)UV]$ (since U is a lateral ideal of $T, (TUT) \subseteq U) \subseteq (V(UUU)V] \subseteq (VUV]$ (since $(UUU) \subseteq U$). Now, let $v \in (VUV]$ and $u \in T$ such that $u \leq v$. $v \in (VUV] \Rightarrow v \leq w$ for some $w \in VUV$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in VUV \Rightarrow u \in (VUV]$. Hence $(VUV]$ is a bi-ideal of T. □

Note 4.1. *In general, if V is a bi-ideal of T and U is a bi-ideal of V. Then U is not a bi-ideal of T.*

Theorem 4.4. *If V is a bi-ideal of T and U is a bi-ideal of V such that $U = U^3$, then U is a bi-ideal of T.*

Proof. As V is a bi-ideal of T, $VTVTV \subseteq V$ and U is a bi-ideal of V, $UVUVU \subseteq U$. Consider $UTUTU = (UUU)TUT(UUU) = UU(UTUTU)UU$ $\subseteq UU(VTVTV)UU \subseteq UUVUU = UUVU(UUU) \subseteq U(UVUVU)U \subseteq UUU = U$. Let $v \in U$ and $u \in T$ such that $u \leq v$. $v \in U \subseteq V, u \in T, u \leq v$ and V is a bi-ideal of $T \Rightarrow u \in V$. Now, we have U is a bi-ideal of V, $v \in U$ and $u \in V \subseteq T$ such that $u \leq v \Rightarrow u \in U$. Hence U is a bi-ideal of T. □

5 Properties of Pseudo Symmetric Ideals

In this section, we study the various properties of pseudo symmetric ideals in a partially ordered ternary semigroup.

Theorem 5.1. *If I is a pseudo symmetric ideal of T and U is a subsemigroup of T then $I \cap U$ is a pseudo symmetric ideal of U, provided $I \cap U \neq \emptyset$.*

Proof. Assume that $V = I \cap U \neq \emptyset$. Since $V \subseteq I$, it follows that $UUV \subseteq TTI \subseteq I$. Since $V \subseteq U$ and U is subsemigroup of T, we have $UUV \subseteq U$. Then $UUV \subseteq I \cap U \Rightarrow UUV \subseteq V$. Similarly, we can show that $VUU \subseteq V$ and $UVU \subseteq V$. Let $v \in V$ and $u \in U$ such that $u \leq v$. Since $v \in V \Rightarrow v \in I, v \in U$ and $u \leq v \Rightarrow u \in I$ and $u \in U \Rightarrow u \in V \Rightarrow (V] \subseteq V$. Obviously, $V \subseteq (V]$. This shows that $(V] = V$. Hence V is an ideal of U. Let $u, v, w \in U$ such that $uvw \in V \Rightarrow uvw \in I$ and $uvw \in U$. If $uvw \in I$ and I is pseudo symmetric ideal then for all $s, t \in U \subseteq T \Rightarrow usvtw \in I$. If $uvw \in U$ and U is subsemigroup of T then for all $s, t \in U \subseteq T$, consider $usvtw = (usv)tw \in UUU \subseteq U$. This shows that $usvtw \in V$. Hence $V = I \cap U$ is a pseudo symmetric ideal of U. □

Theorem 5.2. *If U is an ideal of T such that $T^3 \subseteq U$ and I is a pseudo symmetric ideal of T then $U \cap I$ is a pseudo symmetric ideal of T, provided $U \cap I \neq \emptyset$.*

Proof. Assume that, $C = U \cap I \neq \emptyset$. Therefore C is non-empty subset of T. Since, U is an ideal of T and I is a pseudo symmetric ideal of T then $U \cap I = C$

is an ideal of T. Now, let $u, v, w \in T, uvw \in C \Rightarrow uvw \in U$ and $uvw \in I$. If $uvw \in U$ and U is an ideal of T. For all $s, t \in T$, consider, $usvtw = (usv)tw \in UTT \subseteq U \Rightarrow usvtw \in U$. If $uvw \in I$ and I is an pseudo symmetric ideal of T then $usvtw \in I$, for all $s, t \in T$. Therefore $usvtw \in C$. Hence $C = U \cap I$ is a pseudo symmetric ideal of T. □

Remark 5.1. *Every ideal of commutative partially ordered ternary semigroup is a pseudo symmetric ideal.*

Remark 5.2. *Let I be a pseudo symmetric ideal of partial ordered ternary semigroup T, then $(I]$ is a pseudo symmetric ideal of T.*

Theorem 5.3. *Let U, V, W be three non-empty subsets of a commutative partially ordered ternary semigroup T then $(UVW]$ is a pseudo symmetric ideal of T, if one of the following condition holds in T.*
(1) U is a left or a right or a lateral ideal of T.
(2) V is a left or a right or a lateral ideal of T.
(3) W is a left or a right or a lateral ideal of T.

Proof. Since U, V, W are three non-empty subsets of a commutative partially ordered ternary semigroup T then $(UVW]$ is also non-empty subsets of T. Here we give the proof of statement (1) only. Similarly we can write for (2) and (3). If U is a left or a right or a lateral ideal of T then we prove that $(UVW]$ is a pseudo symmetric ideal of T. Firstly we prove $(UVW]$ is an ideal of T. Consider, $TT(UVW] = (T](T](UVW] = (TT(UVW)] = ((TTU)VW)] \subseteq (UVW]$. Similarly, we can show that, $(UVW]TT \subseteq (UVW]$ and $T(UVW]T \subseteq (UVW]$. Now, let $v \in (UVW]$ and $u \in T$ such that $u \leq v$. $v \in (UVW] \Rightarrow v \leq w$ for some $w \in UVW$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UVW \Rightarrow u \in (UVW]$. This shows that $(UVW]$ is an ideal of T. By Remark 5.1, $(UVW]$ is a pseudo symmetric ideal of T. □

Corolary 5.1. *Let U be any non-empty subset of a commutative partially ordered ternary semigroup T. Then $(TUT], (TTU]$ and $(UTT]$ are pseudo symmetric ideals of T.*

Corolary 5.2. *Let U and V be any two non-empty subsets of a commutative partially ordered ternary semigroup T then $(UVT], (UTV]$ and $(TUV]$ are pseudo symmetric ideals of T.*

Definition 5.1. *A partially ordered ternary semigroup T is said to be pseudo symmetric if every ideal of T is pseudo symmetric ideal of T.*

Remark 5.3. *Every commutative partially ordered ternary semigroup is a pseudo symmetric partially ordered ternary semigroup.*

Lemma 5.1. *Let T be a partially ordered ternary semigroup such that $T = T^3$, then T is a pseudo symmetric partially ordered ternary semigroup.*

Proof. Let I be any ideal in T and let $u, v, w \in T$ such that $uvw \in I$. Since $T = T^3$, consider $vwu = (vwu)(vwu)(vwu) = vw(uvw)(uvw)u \in I \Rightarrow vwu \in I$. Similarly, $wuv = (wuv)(wuv)(wuv) = w(uvw)(uvw)uv \in I \Rightarrow wuv \in I$. Now, if $s, t \in T$ then $usvtw = (usvtw)(usvtw)(usvtw) = usvt[wu(svt)(wus)v]tw \in I \Rightarrow usvtw \in I$. Therefore I is a pseudo symmetric ideal of T. Hence T is a pseudo symmetric partially ordered ternary semigroup. □

Proposition 5.1. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ forms a poset with respect to the inclusion of sets.*

Definition 5.2 ([8])**.** *A family Ω of subsets of a poset P is a Moore family if it satisfies the following conditions,*
(1) $P \in \Omega$.
(2) Ω closed with respect to arbitrary intersection.

Theorem 5.4. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ forms a Moore family of subsets of T.*

Proof. As T itself is a pseudo symmetric ideal, then $T \in \mathcal{P}$. We know that, intersection of an arbitrary collection of pseudo symmetric ideals of T is a pseudo symmetric ideal of T. Therefore $\mathcal{P}$ is closed with respect to arbitrary intersection. Thus $\mathcal{P}$ forms a Moore family of subsets of T. □

Lemma 5.2. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ is a lattice under the operations of intersection and union.*

Proof. Since, $\sup\{U_i, V_j\} = U_i \cup V_j$ and $\inf\{U_i, V_j\} = U_i \cap V_j$ exists for all $U_i, V_j \in \mathcal{P}$. Therefore $\mathcal{P}$ is a lattice. □

Lemma 5.3. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ is a complete lattice under the binary operations of intersection and union.*

Proof. Let H be the subset of $\mathcal{P}$. Since, $\bigwedge H = \bigcap_i I_i$ and $\bigvee H = \bigcup_i I_i$ exist for all $I_i \in H$. Therefore $\bigwedge H$ and $\bigvee H$ exists for any subset H of $\mathcal{P}$. Hence $\mathcal{P}$ is a complete lattice. □

Theorem 5.5. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ is a distributive lattice under the binary operations of intersection and union.*

Remark 5.4. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ is a modular lattice under the binary operations of intersection and union.*

Theorem 5.6. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ is a Brouwerian lattice.*

Proof. Let U and V be any two pseudo symmetric ideals of T such that $U \cap V \neq \emptyset$. Consider the family of ideals of T, $K = \{I \in \mathcal{P} : I \cap U \subseteq V\}$. Then by Zorn's lemma there exist a maximal element say M in K. Select $I \in \mathcal{P}$ such that $I \cap U \subseteq V$. To show that M is the greatest element of K. If $M = T$ then the result follows. Assume that, $M \neq T$. Then we have, $U \cap (I \cup M) = (U \cap I) \cup (U \cap M) \subseteq V \cup V \subseteq V$. Thus $U \cap (I \cup M) \subseteq V$. As M is maximal in $\mathcal{P}$, we have $I \cup M = M$. This gives $I \subseteq M$. Thus the set K has the greatest element M. Hence $\mathcal{P}$ is a Brouwerian lattice. □

6 Conclusions

This paper is a continuation of the study of ideals in a partially ordered ternary semigroup. We have studied the several intriguing properties of ideals in partially ordered ternary semigroup T. We have established conditions for $(UVW]$ to be an ideal, bi-ideal and pseudo symmetric ideal of T, where U, V, W are non-empty subsets of T. We have proved that the family $\mathcal{P}$ of all pseudo symmetric ideals of T forms a Moore family of subsets of T. Furthermore, we have proved that the family $\mathcal{P}$ of all pseudo symmetric ideals of T forms a Brouwerian lattice. We further extend this study to other ternary algebraic structures in the future.

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Relatively Prime Detour Domination Number of Switching Graphs

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Abstract

In this paper, we introduce the concept of relatively prime detour domination number for switching graph. If a set $S \subseteq V$ is a detour set, a dominating set with at least two elements, and has $(deg(u), deg(v)) = 1$ for each pair of vertices u and v, then it is said to be a relatively prime detour dominating set of a graph G. The relatively prime detour domination number of a graph G is known as $\gamma_{rpdn}(G)$ and represents the lowest cardinality of a relatively prime detour dominating set. First, we explain the concept of a switching graph before producing some conclusions based on the switching graphs of helm, fan, complete, spider, and sunlet graphs that have relatively prime detour domination numbers.

Keywords: Domination, Detour domination, Relatively prime domination, Relatively prime detour domination, Switching graphs.

2020 AMS subject classifications: 05C12, 05C69.[1]

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1 Introduction

Consider a finite undirected graph with no loops or multiple edges, adhering to graph theoretic terminology as outlined in the book by Chartrand and Lesniak[3]. The detour distance, denoted as $D(u, w)$, between vertices u and w is defined as the length of the longest $u - w$ path within a connected graph. A path with a length of $u - w$ is referred to as a detour of length $D(u, w)$, and an $u - w$ path with a length of $D(u, w)$ is termed an $u - w$ detour [1].

The closed interval $I_D[u, w]$ encompasses all vertices situated on some $u - w$ detour of the graph G. For a set $S \subseteq V$, $I_D[S]$ is represented as $I_D[u, w]$ for every $u, w \in S$. A set S of vertices is deemed a detour set if $I_D[S]$ covers the entire vertex set V, and the detour number $dn(G)$ is defined as the minimum cardinality of such a detour set. A detour set with cardinality $dn(G)$ is specifically referred to as a minimum detour set [2].

A set S, a subset of the vertex set $V(G)$, is recognized as a dominating set of the graph G if every vertex in $V(G)$ not in S is adjacent to at least one vertex in S. The domination number of G is denoted by $\gamma(G)$ and signifies the smallest size of a dominating set in G; such a set is referred to as a γ-set. The exploration of the domination number in graph G was conducted in [4]. The concept of detour domination number in a graph was initially introduced in [8]. In a graph $G = (V, E)$, the degree of a vertex u is defined as the count of edges incident with u, denoted by $deg(u)$.

A set S, a subset of V, is considered a relatively prime dominating set if a set S is dominant set and it contains at least two elements, and for every pair of vertices u and w in S, the greatest common divisor of their degrees is 1 (i.e., $(deg(u), deg(w)) = 1$). The smallest cardinality of such a relatively prime dominating set in a graph G is termed the relatively prime domination number and is denoted by $\gamma_{rpd}(G)$ [7]. The concept of switching in graphs, initially introduced by Lint and Seidel [9], is also discussed.

This paper establishes that a dn-set is a minimal detour dominating set, and it introduces the notion of the relatively prime detour domination number for a switching graph, determining the corresponding value denoted as $\gamma_{rpdn}(G^v)$.

Definition 1.1. *[6] A set $S \subseteq V$ is said be relatively prime detour dominating set of a graph G if it is a detour set and a dominating set with at least two elements and for every pair of vertices u and v such that $(deg(u), deg(v)) = 1$. The relatively prime detour domination number is represented by the symbol $\gamma_{rpdn}(G)$ and defines the lowest cardinality of a relatively prime detour dominating set of order $\gamma_{rpdn}(G)$ is said to as a γ_{rpdn}- set of G. If the relatively prime detour dominating set does not exist, then the relatively prime detour domination number is zero.*

Definition 1.2. *[5]For a finite undirected graph $G(V, E)$ and a vertex $v \in V$, the*

switching of G by v is defined as the graph G^v by removing all edges incident to v and adding edges which are not adjacent to v.

Example 1.1. *Consider the graph G and G^{v_1} are given in Figure 1. Here, $S = \{v_1, v_4, v_6\}$ is a γ_{rpdn}- set of G^{v_1}. Thus $\gamma_{rpdn}(G^{v_1}) = 3$.*

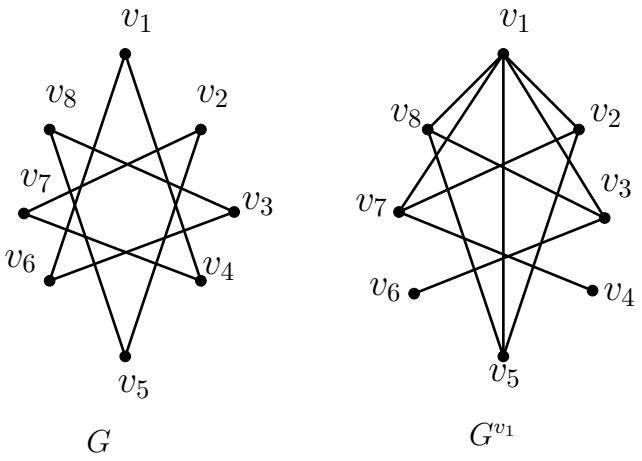


Figure 1: G and Switching of G by v_1 is G^{v_1}

2 Relatively Prime Detour Domination Number of Some Switching Graphs

Theorem 2.1. *Let v be a vertex of helm graph H_n. Then for $n \geq 4$,*

$$\gamma_{rpdn}\left(H_n^v\right) = \begin{cases} 2 & if \ \ deg_{H_n}(v) = 1 \ and \ 2n-3 \not\equiv 0 (mod\ 15) \\ 0 & otherwise \end{cases}$$

.

Proof. Let $v_1v_2...v_{n-1}v_1$ be the cycle C_{n-1}. Add a vertex u which is adjacent to $v_r, 1 \leq r \leq n$. The resultant graph is the wheel W_n. Add u_r, which is adjacent to v_r, for $1 \leq r \leq n-1$. The resultant graph is the helm graph H_n. Then $deg_{H_n}(u) = n-1$, $deg_{H_n}(v_r) = 4$ and $deg_{H_n}(u_r) = 1$ for each $r \in \{1, 2, ..., n\}$.
Case 1. $deg_{H_n}(v) = n-1$

Clearly, v is u and the graph H_n and H_n^u are given in Figure 2. Clearly, $E(H_n^u) = \{uu_r, u_rv_r, v_1v_{n-1}, v_sv_{s+1} : 1 \leq r \leq n-1, 1 \leq s \leq n-2\}$. In H_n^u, $deg(u) = n-1$, $deg(u_r) = 2$ and $deg(v_r) = 3$, $1 \leq r \leq n-1$

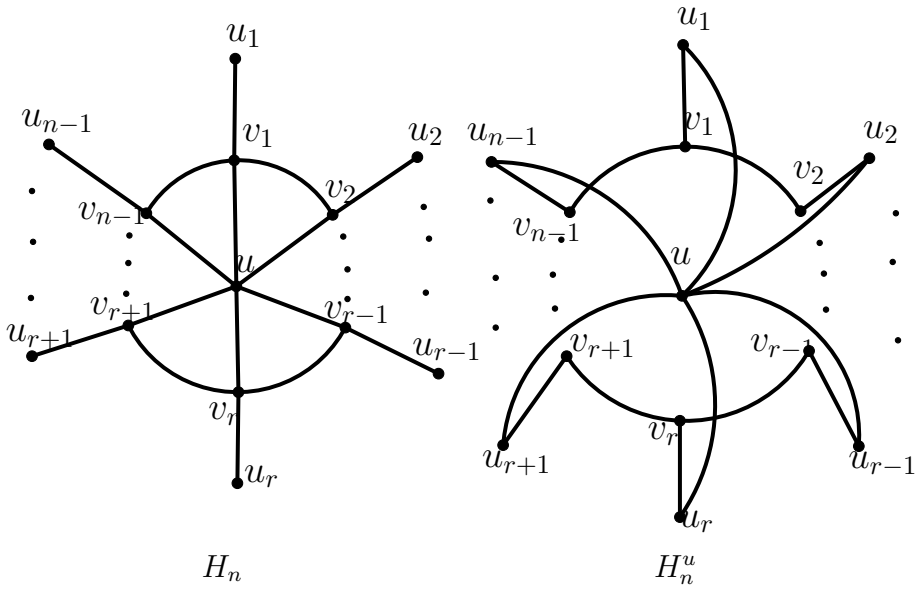


Figure 2: Helm graph H_n and switching of the helm graph by u

and also u is adjacent to all u_r and non-adjacent to all v_r, $1 \leq r \leq n-1$. Since $H_n^u[\{v_1, v_2, ..., v_{n-1}\}] = C_{n-1}$, $S = \{u, v_1, v_3, ..., v_{n-2}\}$ is a γ- set and also $I_D[S] = V(H_n^u)$. Therefore, it follows that S is a γ_{dn}- set of H_n^u. Since S cannot be a γ_{rpdn}- set because the degree sequence of its elements is $(n-1, 3, 3, ..., 3)$, $\gamma_{rpdn}(H_n^u) = 0$.

Case 2. $deg_{H_n}(v) = 4$

Here v is v_r, for some r, $1 \leq r \leq n$ the graph $H_n^{v_r}$ is given in Figure 3. Clearly, $H_n^{v_r}$ is a disconnected graph with the isolated vertex u_r and so $\gamma_{rpdn}(H_n^{v_r}) = 0$.

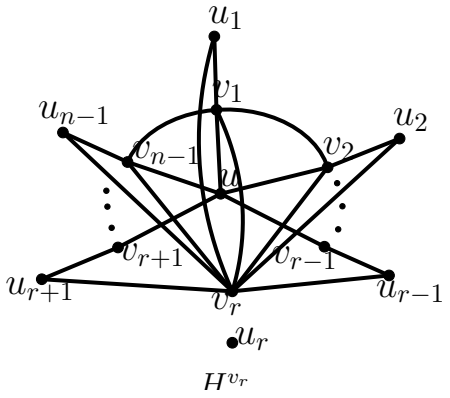


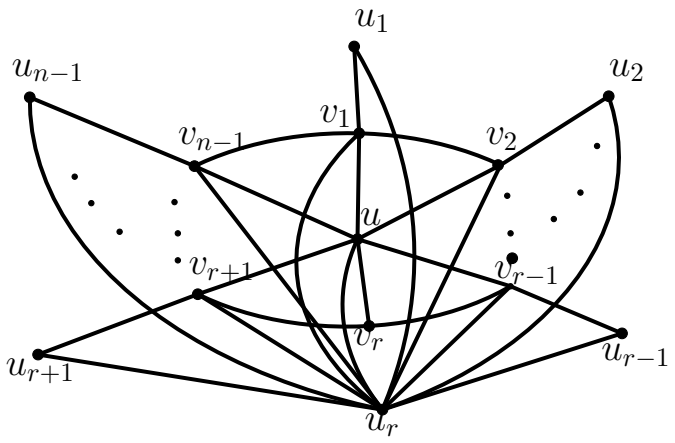
Figure 3: Switching of the helm graph H_n by v_r

Case 3. $deg_{H_n}(v) = 1$ and $2n - 3 \not\equiv 0(mod\ 15)$

In this case v is u_r for some r, for $1 \leq r \leq n-1$ the graph $H_n^{u_r}$ is given in Figure 4. Clearly, $E(H_n^{u_r}) = \{uv_l, v_tv_{t+1}, v_{n-1}v_1, v_ku_k,\ u_ru, u_rv_k, u_ru_k\ :$ $1 \leq l \leq n-1, 1 \leq k \leq n-1,\ 1 \leq t \leq n-2$ and $k \neq r\}$ and in $H_n^{u_r}$, $deg(u_r) = 2n-3, deg(v_r) = 3, deg(u) = n$, $deg(u_k) = 2$ and $deg(v_k) = 5$. In $H_n^{u_i}$, u_i is non-adjacent to v_i and v_{i-1} is adjacent to v_i and thereby $S = \{u_i, v_{i-1}\}$ is a γ- set of $H_n^{u_i}$ and also $I_D[S] = V({H_n}^v)$. Hence, S is a γ_{dn}- set of $H_n^{u_i}$. Now $(deg(u_i), deg(v_{i-1})) = (2n-3, 5) = 1$ if and only if $2n - 3 \not\equiv 0(mod\ 5)$. Therefore, S is a γ_{rpdn}- set if and only if $2n-3 \not\equiv 0(mod\ 5)$ and hence $\gamma_{rpdn}\ (H_n^{u_i}) = 2$ if and only if $2n - 3 \not\equiv 0(mod\ 5)$. . . . (1)

Also in $H_n^{u_r}$, u_r is non-adjacent to v_r and thereby $S^* = \{u_r, v_r\}$ is a γ-set of $H_n^{u_r}$ and also $I_D[S^*] = V({H_n}^v)$. Hence, S^* is a γ_{dn}- set of $H_n^{u_i}$. Now $(deg(u_r), deg(v_r)) = (2n-3, 3) = 1$ if and only if $2n - 3 \not\equiv 0(mod\ 3)$. Therefore, S^* is a γ_{rpdn}- set if and only if $2n-3 \not\equiv 0(mod\ 3)$ and hence $\gamma_{rpdn}\ (H_n^{u_r}) = 2$ if and only if $2n - 3 \not\equiv 0(mod\ 3)$. . . . (2)

Since $2n-3$ is odd, from (1) and (2), we have $\gamma_{rpdn}\ (H_n^{u_r}) = 2$ if and only if $2n - 3 \not\equiv 0(mod\ 15)$. □



$H_n^{u_r}$

Figure 4: Switching of the helm graph H_n by u_r

Theorem 2.2. *Let v be any vertex of the fan graph $F_{1,n}$. Then*

$$\gamma_{rpdn}\left(F_{1,n}^{v}\right) = \begin{cases} 2 & if\ \ n > 2 \text{ and either } deg_{F_{1,n}}(v) = 2 \text{ or } deg_{F_{1,n}}(v) = 3,\ v \\ & \quad \text{is non adjacent to a vertex of degree 2 in } F_{1,n} \text{ and} \\ & \quad n \text{ is even} \\ 3 & if\ \ v \text{ is adjacent to a vertex of degree 2 in } F_{1,n} \\ & \quad \text{and } n > 2 \text{ is even} \\ 0 & otherwise \end{cases}$$

Proof. Let u be a vertex of K_1 and $v_1v_2...v_n$ be a path P_n. The join $P_n + K_1 = F_{1,n}$ is a fan graph with $deg_{F_{1,n}}(u) = n, deg_{F_{1,n}}(v_1) = deg_{F_{1,n}}(v_n) = 2$ and $deg_{F_{1,n}}(v_r) = 3, 2 \leq r \leq n-1$. Let v be any vertex of $F_{1,n}$ and let $F_{1,n}^{v}$ be the vertex switching of $F_{1,n}$ with respect to v.

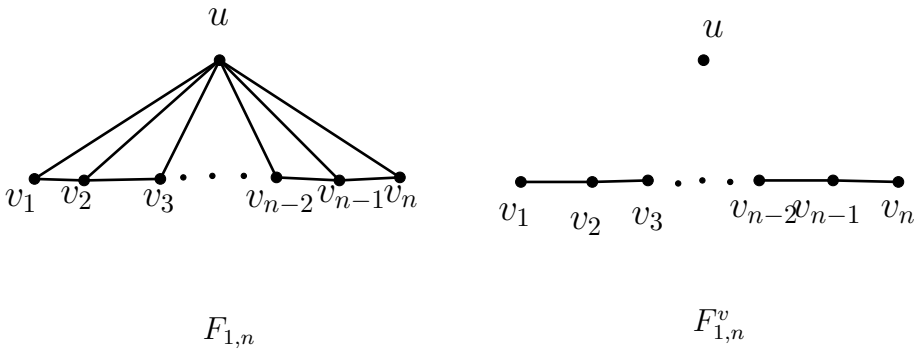


Figure 5: Fan graph $F_{1,n}$ and switching of the fan graph by v

Case 1. $n = 2$

Then $F_{1,2} = C_3$ and for any vertex v in $F_{1,2}$, $F_{1,2}^{v} = K_2 \cup K_1$, is a disconnected graph and so $\gamma_{rpdn}\left(F_{1,n}^{v}\right) = 0$.

Case 2. $n > 2$

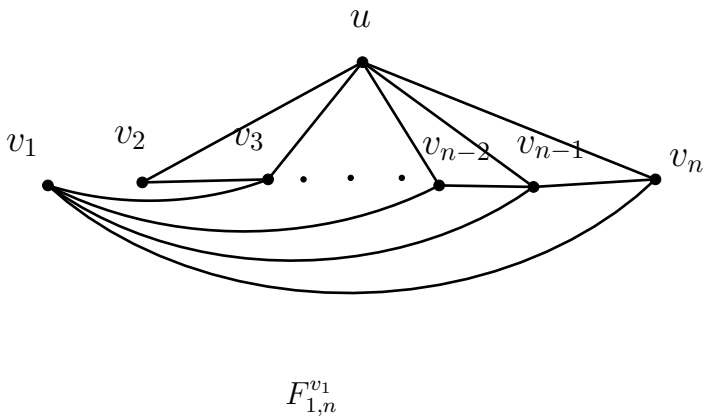
Subcase 2.1. $deg_{F_{1,n}}(v) = n$

Here v is u. The graph $F_{1,n}$ and $F_{1,n}^{v} = P_n \cup K_1$ are given in Figure 5 which is a disconnected graph and so $\gamma_{rpdn}\left(F_{1,n}^{v}\right) = 0$.

Subcase 2.2. $deg_{F_{1,n}}(v) = 2$

Clearly, v is either v_1 or v_n and $F_{1,n}^{v_1} \cong F_{1,n}^{v_n}$. Let v be v_1. Now $E(F_{1,n}^{v}) = \{uv_r, vv_k, v_sv_{s+1} : 2 \leq r \leq n, 2 \leq s \leq n-1, 3 \leq k \leq n\}$ and in $F_{1,n}^{v}$, $deg(v) = n-2, deg(u) = n-1, deg(v_2) = 2, deg(v_n) = 3$ and $deg(v_l) = 4, 3 \leq l \leq n-1$. The graph $F_{1,n}^{v_1}$ is given in Figure 6. In $F_{1,n}^{v_1}$, v is non-adjacent to only v_2 and u and

the vertices u and v_2 are adjacent and so $S = \{u, v\}$ is a γ- set. Also $I_D[u, v] = V[F_{1,n}^{v_1}]$. Hence, S is a γ_{rpdn}- set since $(deg(u), deg(v)) = (n-1, n-2) = 1$ and thereby $\gamma_{rpdn}\left(F_{1,n}^{v_1}\right) = 2$.

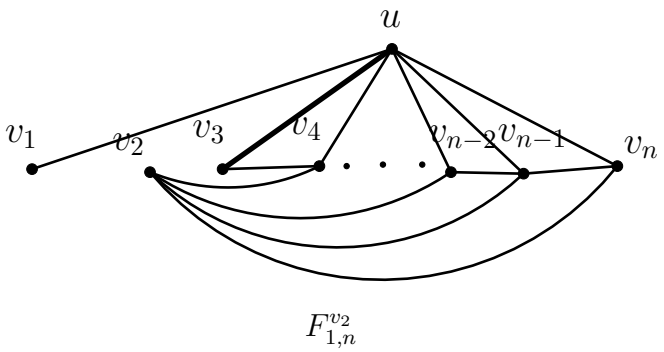


$F_{1,n}^{v_1}$

Figure 6: Switching of the fan graph $F_{1,n}$ by v_1

Subcase 2.3. $deg_{F_{1,n}}(v) = 3$
Subcase 2.3.1. v is adjacent to a vertex of degree 2 in $F_{1,n}$

Clearly, v is either v_2 or v_{n-1} and $F_{1,n}^{v_2} \cong F_{1,n}^{v_{n-1}}$. Let v be v_2. The graph $F_{1,n}^{v_2}$ is given in Figure 7. Now $E(F_{1,n}^{v_2}) = \{uv_r, vv_s, v_kv_{k+1} : 1 \leq r \leq n, r \neq 2, 4 \leq s \leq n, 3 \leq k \leq n-1\}$ and in G^v, $deg(v) = n-3, deg(u) = n-1, deg(v_1) = 1, deg(v_3) = 2\ deg(v_n) = 3$ and $deg(u_k) = 4, 4 \leq k \leq n-1$.



$F_{1,n}^{v_2}$

Figure 7: Switching of the fan graph $F_{1,n}$ by v_2

In $F_{1,n}^{v_2}$, v is non-adjacent to v_1, v_3 and u and u is non-adjacent to only vertex v_2 and so $S = \{u, v\}$ is a γ- set for $F_{1,n}^{v_2}$. But $I_D[u, v] \neq V[F_{1,n}^{v_2}]$. For the set $S_1 =$

$S \cup \{v_1\}$, $I_D[S_1] = V[F_{1,n}^{v_2}]$ and so S_1 is a γ_{dn}- set of G^v. Now $(deg(u), deg(v)) = (n-1, n-3) = 1$ if and only if n is even. Thus, S is a γ_{rpdn}- set if and only if n is even and hence $\gamma_{rpdn}\left(F_{1,n}^{v_2}\right) = 3$ if and only if n is even.
Subcase 2.3.2. v is non-adjacent to a vertex of degree 2 in $F_{1,n}$

Here v is v_r, for some r, $3 \leq r \leq n-2$. The graph $F_{1,n}^{v_r}$ is given in Figure 8. Now $E(F_{1,n}^{v_r}) = \{v_r v_s, v_k v_{k+1}, uv_l : 1 \leq r, s \leq n, s \neq r \pm 1, 1 \leq l, k \leq n-1, l \neq r, k \neq r-1, r\}$ and $deg(v_r) = n-3, deg(u) = n-1$ and $deg(v_{r-1}) = 2 = deg(v_{r+1})$. In $F_{1,n}^{v_r}$, v_r is non-adjacent to v_{r-1}, v_{r+1} and u and u is non-adjacent to v_r and so $S = \{u, v_r\}$ is aγ- set for $F_{1,n}^{v_r}$. Also $I_D[u, v_r] = V[F_{1,n}^{v_r}]$. Hence, S is a γ_{dn}- set and also $(deg(u), deg(v)) = (n-1, n-3) = 1$ if and only if n is even. Thus, S is a γ_{rpdn}- set if and only if n is even and hence $\gamma_{rpdn}\left(F_{1,n}^{v_r}\right) = 2$ if and only if n is even. □

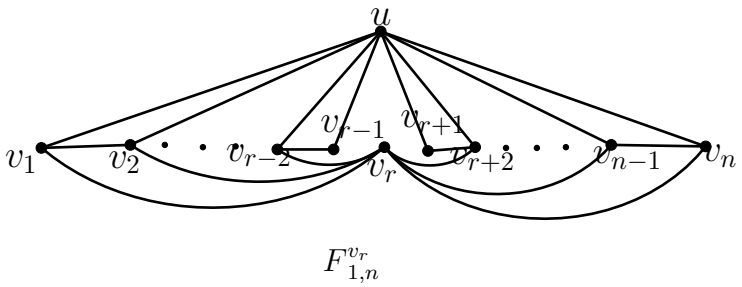


Figure 8: Switching of the fan graph $F_{1,n}$ by v_r

Theorem 2.3. *If v is a vertex of the complete graph K_n, then $\gamma_{rpdn}(K_n^v) = 0$.*

Proof. Since each vertex in K_n is a full vertex, K_n^v is disconnected graph and so $\gamma_{rpdn}(K_n^v) = 0$. □

Theorem 2.4. *Let G be a spider graph $K_{1,n,n}$ with a centre u and let v be any vertex of G. Then*

$$\gamma_{rpdn}(G^v) = \begin{cases} n+1 & if \ \ v \ is \ the \ centre \ vertex \\ 0 & if \ \ v \ is \ a \ to \ support \ vertex \\ 2 & if \ \ v \ is \ an \ end \ vertex \end{cases}$$

.

Proof. Let u be the central vertex and $v_1, v_2, ..., v_n$ be the end vertices of $K_{1,n}$, and let u_r, $1 \leq r \leq n$ be the respective vertices connected with v_r, $1 \leq r \leq n$. The

spider graph $K_{1,n,n}$ is the resulting graph, and its vertex set $V(G) = \{u, u_r, v_r : 1 \leq r \leq n\}$ and $E(G) = \{uu_r, u_rv_r : 1 \leq r \leq n\}$. Let v be any vertex of G. Then v is either the centre vertex or an end vertex or a support vertex.

Case 1. v is the centre vertex

Here v is u. The graph $K_{1,n,n}$ and $K^v_{1,n,n}$ are given in Figure 9. Clearly, G^v is a spider graph with end vertices u_r, $1 \leq r \leq n$ and support vertices v_r, $1 \leq r \leq n$. Now $S = \{v, u_1, u_2, ..., u_n\}$ is a γ- set and $I_D[S] = V[G^v]$. In G^v, $deg(v) = n$ and $deg(u_r) = 1, 1 \leq r \leq n$. This gives that S is a γ_{rpdn}- set of G^v and hence $\gamma_{rpdn}(G^v) = n + 1$.

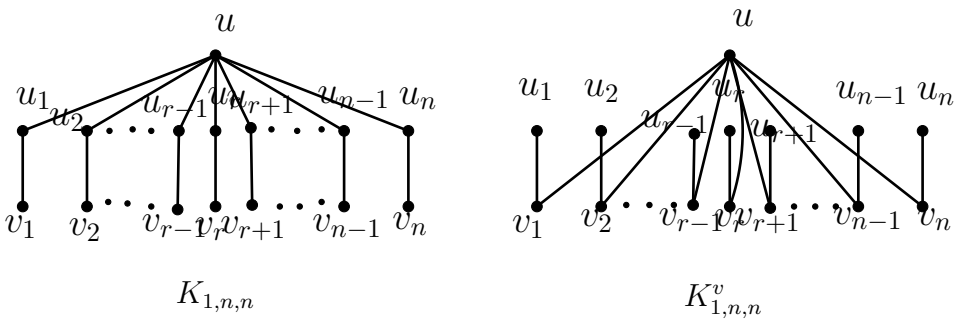


Figure 9: Spider graph $K_{1,n,n}$, Switching of the spider graph $K_{1,n,n}$ by v

Case 2. v is an end vertex

Here v is v_r, for some $r, 1 \leq r \leq n$. Clearly, $G^{v_1} \cong G^{v_2} \cong ... \cong G^{v_n}$. Let v be v_1. The graph G^{v_1} is given in Figure 10.

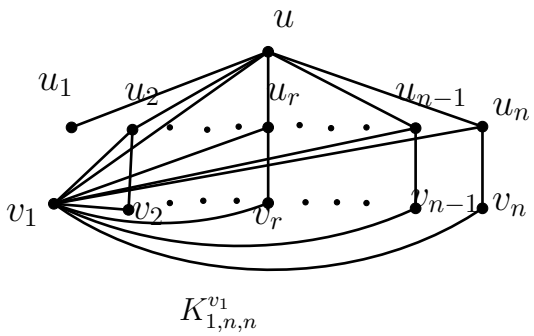


Figure 10: Switching of the spider graph $K_{1,n,n}$ by v_1

Now $E(G^v) = \{vu, vu_r, vv_k, uv_r, u_l v_l : 1 \leq r \leq n, 2 \leq k, l \leq n\}$ and in G^v, $deg(v) = 2n-1$, $deg(u_1) = 1$ $deg(v_r) = 2, deg(u_r) = 3$ and $deg(u) = n+1$, $2 \leq r \leq n$. In G^v, v is adjacent all the vertices other than u_1 and so $S = \{v, u_1\}$ is a γ- set and also $I_D[v, u_1] = V[G^v]$. Hence, S is a γ_{rpdn}- set of G^v since $(deg(v), deg(u_1)) = 1$ and thereby $\gamma_{rpdn}(G^v) = 2$.

Case 3. v is a support vertex

Then v is u_r, for some $r, 1 \leq r \leq n$. Clearly, $G^{u_1} \cong G^{u_2} \cong ... \cong G^{u_n}$ and G^v is a disconnected graph and so $\gamma_{rpdn}(G^v) = 0$. □

Theorem 2.5. *Let v be any vertex of sunlet graph S_n. Then $\gamma_{rpdn}(S_n^v) = 0$.*

Proof. The sunlet graph S_n is obtained from a cycle $C_n : u_1u_2...u_nu_1$ by attaching a pendent vertex v_r at each vertex u_r of the cycle, $1 \leq r \leq n$. Clearly, $V(G)$ = $\{u_r, v_r : 1 \leq r \leq n\}$ and $E(G)$ = $\{u_r v_r, u_r u_{r\pm1} : 1 \leq r \leq n\}$.

Case 1. v is an end vertex

Then v is v_r, for some r, $1 \leq r \leq n$. The graphs S_n and S_n^v are given in Figure 11. Clearly, $G^{v_1} \cong G^{v_2} \cong ... \cong G^{v_n}$. So let v be v_r. In G^{v_r}, the distinct degrees of the vertices are $2(n-1)$, 4 and 2 and hence $\gamma_{rpdn}(G^v) = 0$.

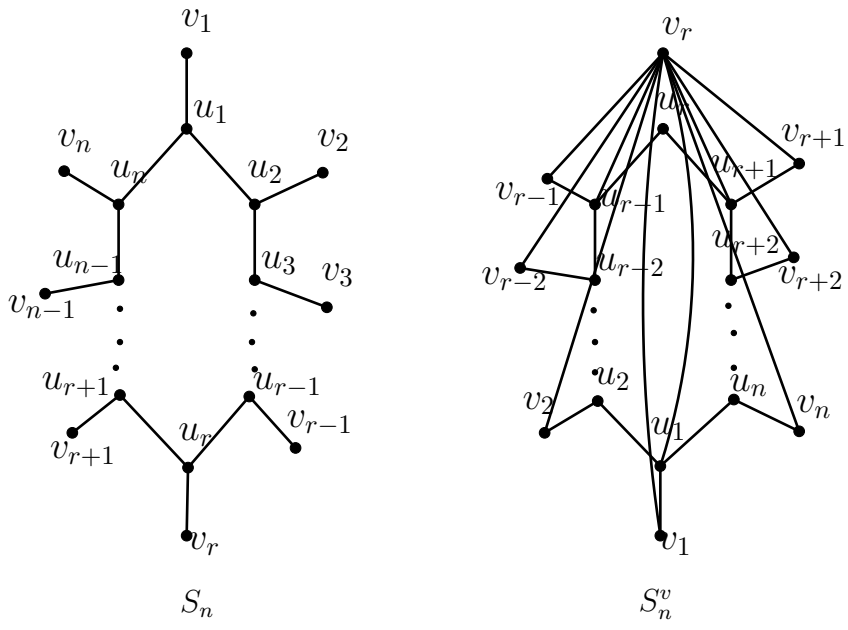


Figure 11: Sunlet graph S_n and switching of the sunlet graph S_n by v

Case 2. v is not an end vertex

Here $v = u_r$, for some r, $1 \leq r \leq n$. Clearly, G^v is disconnected graph with the isolated vertex v_r. Hence, $\gamma_{rpdn}(G^v) = 0$. □

3 Conclusion

Motivated by the concepts of detour sets and relatively prime dominating sets, we present the relatively prime detour dominating set specifically tailored for switching graphs. Our investigation extends to determining the relatively prime detour domination numbers for switching graphs derived from helm graphs, fan graphs, complete graphs, spider graphs, and sunlet graphs. The validity of our findings is bolstered by illustrative examples. Notably, our investigation extends beyond these specific graphs, as we aasert that relatively prime detour domination number is applicable to a broader range of graphs. Looking ahead, we envisage the development of an algorithm capable of determining whether a given graphs have been relatively prime detour domination number.

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Strongly*-2 Divisor Cordial Labeling of Cycle Related Graphs

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Abstract

A strongly*-2 divisor cordial labeling of a graph G with the vertex set $V(G)$ is a bijection $a : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ such that each edge cd assigned the label 1 if the lower integal value of sum of $a(c)$, $a(d)$ and $a(c)a(d)$ divided by 2 is odd and 0 if lower integal value of sum of $a(c)$, $a(d)$ and $a(c)a(d)$ divided by 2 is even, then the number of edges labeled with 0 and the number of edges labeled with 1 differs by atmost 1 or $|e_a(0) - e_a(1)| \leq 1$ where $e_a(0)$ denotes the number of edges labeled with 0 and $e_a(1)$ denotes the number of edges labeled with 1. A graph which admits a strongly*-2 divisor cordial labeling is called a srongly*-2 divisor cordial graph. In this paper, we prove that the wheel graph W_g and sunflower graph SF_g are strongly*-2 divisor cordial graphs.

Keywords: Bijection, Cordial labeling, Strongly*-graph, Strongly*-2 divisor cordial graph.
2020 AMS subject classifications: 05C78 [1]

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1 Introduction

Graph theory, a mathematical discipline exploring relationships between objects, utilizes structures called graphs. A graph is represented as $G = (V, E)$, where V signifies the vertex set (a finite assortment of elements), and E stands for the edge set (consisting of unordered pairs of distinct elements from V).
In the 1960s, the concept of graph labeling emerged, involving the assignment of integers to vertices, edges, or both, subject to specific conditions. We pursue Harary [1988] for looking over the fundamental terms and notations and Gallian [2018] for a comprehensive understanding of graph labeling.

In 1987, Cahit [1987] hailed cordial labeling. A graph is deemed cordial if it can be labeled with 0s and 1s on its vertices in a way that, when the edges are labeled with the difference of the labels at their endpoints, the difference in the counts of vertices (or edges) labeled with ones and zeros is, at most, one. R. Varatharajan [2011] pioneered the concept of divisor cordial labeling.

A graph is pertained to as a strongly*-graph of order n if it can be labeled with values from 1 to n assigned to its vertices in such a manner that, for any edge labeled with the value $m + z + mz$, where m and z are the labels of its connecting vertices, every edge label is unique. C. Adiga [1999] demonstrated a significant result by proving that all trees, cycles, and grids inherently possess the characteristics of strongly*-graphs.

S.T.A. Mary [2018] establised the strongly*-graph labeling of some more graphs. J. Baskar [2009] have demonstrated that wheels, paths, fans, crowns, and umbrellas can be characterized as vertex strongly*-graphs. M.A. Seoud [2012] offered some technical necessary conditions for a graph to be strongly*-graph.
Stimulated by these, C. Jayasekaran [2020] introduced strongly*-2 divisor cordial labeling. In this paper, we prove that the wheel graph W_g and sunflower graph SF_g are strongly*-2 divisor cordial graphs. Also we delivered the vital definitions that are obligatory for our eternal work.

Definition 1.1. *A wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle.*

Definition 1.2. *The sunflower graph is obtained from a wheel graph with the central vertex b, the cycle $C_g : f_1 f_2 ... f_g f_1$ and additional vertices $j_1, j_2, ..., j_g$ where j_s is joined by edges to f_s, f_{s+1} where f_{s+1} is taken modulo g.*

2 Main Results

Theorem 2.1. *The wheel graph W_g is a strongly*-2 divisor cordial graph.*

Proof. Let $V(W_g) = \{b, f_h : 1 \leq h \leq g\}$ and $E(W_g) = \{bf_h, f_sf_{s+1}, f_gf_1 : 1 \leq h \leq g, 1 \leq s \leq g-1\}$. Then W_g has $g+1$ vertices and $2g$ edges.

Case 1. $g \equiv 0(mod\ 4)$

Define $a : V(W_g) \to \{1, 2, ..., g+1\}$ by $a(b) = 1$ and $a(f_h) = h+1$, $1 \leq h \leq g$. Let a^* be the induced edge label of a and let $g = 4d$, $d \in N$.

Consider the edges $bf_h, 1 \leq h \leq g$. Then $a^*(bf_h)$ = $\lfloor \frac{a(b)+a(f_h)+a(b)a(f_h)}{2} \rfloor$ = $\lfloor \frac{1+h+1+h+1}{2} \rfloor$ = $\lfloor \frac{2h+3}{2} \rfloor$ = $h+1$ which is even if h is odd and odd if h is even. Therefore, the edges $bf_h, 1 \leq h \leq g$ get label 0 if h is odd and 1 if h is even. (1)

For the edges $f_hf_{h+1}, 1 \leq h \leq g-1$, $a^*(f_hf_{h+1})$ = $\lfloor \frac{h+1+h+2+(h+1)(h+2)}{2} \rfloor$ = $\lfloor \frac{h^2+5h+5}{2} \rfloor$ which is even if $h \equiv 0, 3(mod\ 4)$ and odd if $h \equiv 1, 2(mod\ 4)$. Therefore, the edges f_hf_{h+1} get label 0 if $h \equiv 0, 3(mod\ 4)$ and 1 if $h \equiv 1, 2(mod\ 4), 1 \leq h \leq g-1$. (2)

For the edge f_gf_1, $a^*(f_gf_1)$ = $\lfloor \frac{g+1+1+g+1}{2} \rfloor$ = $\lfloor \frac{2g+3}{2} \rfloor$ = $g+1$ which is even since g is even. Therefore, the edge f_gf_1 gets label 0.

Let us find the number of edges with label 0 and 1. Now $g = 4d$ implies that there are $2g = 8d$ edges. Among these, there are $g = 4d$ edges of the form $bf_h, 1 \leq h \leq g$ and so there are $2d$ even numbers and odd numbers. By (1), there are $2d$ edges get label 0 and $2d$ edges get label 1. Also there are $g-1 = 4d-1$ edges of the form $f_hf_{h+1}, 1 \leq h \leq g-1$ and so there are d numbers which are congruent to h modulo 4, $(h = 1, 2, 3)$ and $d-1$ numbers congruent to 0 modulo 4. By (2), there are $2d-1$ edges get label 0 and $2d$ edges get label 1 and the edge f_gf_1 gets label 0. Hence there are $4d$ edges with label 0 and $4d$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

Case 2. $g \equiv 1(mod\ 4)$

Define $a : V(W_g) \to \{1, 2, ..., g+1\}$ by $a(b) = g+1$ and $a(f_h) = h$, $1 \leq h \leq g$. Let a^* be the induced edge label of a and let $g = 4d+1$, $d \in N$.

For the edges $bf_h, 1 \leq h \leq g$, $a^*(bf_h)$ = $\lfloor \frac{g+1+h+(g+1)h}{2} \rfloor$ = $\lfloor \frac{g+1+gh+2h}{2} \rfloor$ = $\lfloor \frac{4d+1+1+(4d+1)h+2h}{2} \rfloor$ = $2d(1+h)$+$\lfloor \frac{3h+2}{2} \rfloor$ which is even if $h \equiv 1, 2(mod\ 4)$ and odd if $h \equiv 0, 3(mod\ 4)$. Therefore, the edges $bf_h, 1 \leq h \leq g$ get label 0 if $h \equiv 1, 2(mod\ 4)$ and 1 if $h \equiv 0, 3(mod\ 4)$. (1)

Consider the edges $f_hf_{h+1}, 1 \leq h \leq g-1$. Then $a^*(f_hf_{h+1})$ = $\lfloor \frac{h+h+1+h(h+1)}{2} \rfloor$ = $\lfloor \frac{h^2+3h+1}{2} \rfloor$ which is even if $h \equiv 0, 1(mod\ 4)$ and odd if $h \equiv 2, 3(mod\ 4)$. Therefore, the edges $f_hf_{h+1}, 1 \leq h \leq g-1$ get label 0 if $h \equiv 0, 1(mod\ 4)$ and 1 if $h \equiv 2, 3(mod\ 4)$. (2)

For the edge f_gf_1, $a^*(f_gf_1)$ = $\lfloor \frac{1+g+g}{2} \rfloor$ = $\lfloor \frac{2g+1}{2} \rfloor$ = g which is odd since g is odd and so the edge f_gf_1 gets label 1.

Let us find the number of edges with label 0 and 1. Now $g = 4d+1$ implies that there are $2g = 8d+2$ edges. Among these, there are $4d+1$ edges of the form $bf_h, 1 \leq h \leq g$ such that there are d numbers which are congruent to h

modulo 4, $(h = 0, 2, 3)$ and $d + 1$ numbers which are congruent to 1 modulo 4. By (1), there are $2d + 1$ edges get label 0 and $2d$ edges get label 1. Also there are $g - 1 = 4d$ edges of the form $f_h f_{h+1}, 1 \leq h \leq g - 1$ and so there are d numbers which are congruent to h modulo 4, $(h = 0, 1, 2, 3)$. By (2), there are $2d$ edges get label 0 and $2d$ edges get label 1 and the edge $f_g f_1$ gets label 1. Hence, there are $4d+1$ edges with label 0 and $4d+1$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

Case 3. $g \equiv o(mod\ 4)$, $o \in \{2, 3\}$

Define $a : V(W_g) \to \{1, 2, ..., g+1\}$ by $a(b) = 2$, $a(f_1) = 1$, $a(f_{g-1}) = 3$, $a(f_g) = g + 1$ and $a(f_h) = h + 2$, $2 \leq h \leq g - 2$. Let a^* be the induced edge label of a.

Consider the edges $bf_h, 1 \leq h \leq g$. For the edge bf_1, $f^*(bf_1) = \lfloor \frac{2+1+2}{2} \rfloor = \lfloor \frac{5}{2} \rfloor = 2$ which is even and thereby the edge bf_1 gets label 0.

For the edges $bf_h, 2 \leq h \leq g - 2$. $a^*(bf_h) = \lfloor \frac{2+h+2+2(h+2)}{2} \rfloor = 4 + \lfloor \frac{3h}{2} \rfloor$ which is even if $h \equiv 0, 3(mod\ 4)$ and odd if $h \equiv 1, 2(mod\ 4)$. Therefore, the edges $bf_h, 2 \leq h \leq g - 2$ get label 0 if $h \equiv 0, 3(mod\ 4)$ and 1 if $h \equiv 1, 2(mod\ 4)$. (1)

For the edge bf_{g-1}, $a^*(bf_{g-1}) = \lfloor \frac{2+3+6}{2} \rfloor = \lfloor \frac{11}{2} \rfloor = 5$ which is odd and thereby the edge bf_{g-1} gets label 1.

Consider the edges $f_h f_{h+1}, 1 \leq h \leq g - 1$. For the edge $f_1 f_2$, $a^*(f_1 f_2) = \lfloor \frac{1+4+4}{2} \rfloor = \lfloor \frac{9}{2} \rfloor = 4$ which is even and thereby the edge $f_1 f_2$ gets label 0.

For the edges $f_h f_{h+1}, 2 \leq h \leq g - 2$, $a^*(f_h f_{h+1}) = \lfloor \frac{h+2+h+3+(h+2)(h+3)}{2} \rfloor = \lfloor \frac{h^2+7h+11}{2} \rfloor$ which is even if $h \equiv 2, 3(mod\ 4)$ and odd if $h \equiv 0, 1(mod\ 4)$. Therefore, the edges $f_h f_{h+1}$ get label 0 if $h \equiv 2, 3(mod\ 4)$ and 1 if $h \equiv 0, 1(mod\ 4), 2 \leq h \leq g - 2$. (2)

For the edge $f_{g-1} f_g$, $a^*(f_{g-1} f_g) = \lfloor \frac{3+g+1+3(g+1)}{2} \rfloor = \lfloor \frac{4g+7}{2} \rfloor = 2g + 3$ which is odd and thereby the edge $f_{g-1} f_g$ gets label 1.

Subcase 3.1. $o = 2$

Here $g = 4d + 2$. For the edge bf_g, $a^*(bf_g) = \lfloor \frac{2+g+1+2(g+1)}{2} \rfloor = \lfloor \frac{3g+5}{2} \rfloor = \lfloor \frac{12d+11}{2} \rfloor = 6d + 5$ which is odd and thereby the edge bf_g gets label 1.

For the edge $f_g f_1$, $a^*(f_g f_1) = \lfloor \frac{g+1+1+g+1}{2} \rfloor = \lfloor \frac{2g+3}{2} \rfloor = g + 1$ which is odd since g is even and thereby the edge $f_n f_1$ gets label 1.

Let us find the number of edges with label 0 and 1. Now $g = 4d+2$ implies that there are $2g = 8d + 4$ edges. Among these, there are $g - 3 = 4d - 1$ edges of the form $bf_h, 2 \leq h \leq g - 2$ and so there are d numbers which are congruent to h modulo 4, $(h = 0, 2, 3)$ and $d-1$ numbers congruent to 1 modulo 4 . By (1), there are $2d$ edges get label 0 and $2d-1$ edges get label 1. Also there are $g-3 = 4d-1$ edges of the form $f_h f_{h+1}, 2 \leq h \leq g - 2$ and so there are d numbers which are congruent to h modulo 4, $(h = 0, 2, 3)$ and $d - 1$ numbers congruent to 1 modulo 4. By (2), there are $2d$ edges get label 0 and $2d - 1$ edges get label 1. Also the edges bf_{g-1}, bf_g, $f_{g-1} f_g$ and $f_g f_1$ get label 1 and the edges bf_1 and $f_1 f_2$ get label

0. Hence, there are $4d+2$ edges with label 0 and $4d+2$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

Subcase 3.2. $o = 3$

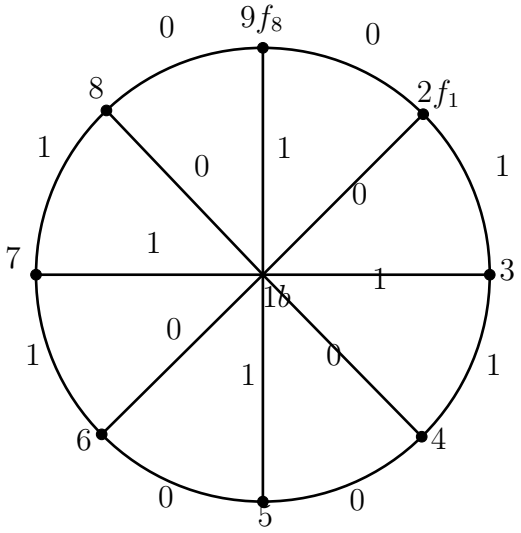
Here $g = 4d+3$. For the edge bf_g, as in subcase 2.1., $a^*(bf_g) = \lfloor \frac{3g+5}{2} \rfloor = \lfloor \frac{12d+14}{2} \rfloor = 6d+7$ which is odd and thereby the edge bf_g gets label 1.

For the edge $f_g f_1$, as in subcase 2.1., $a^*(f_g f_1) = \lfloor \frac{2g+3}{2} \rfloor = g+1$ which is even since g is odd and thereby the edge $f_g f_1$ gets label 0.

Let us find the number of edges with label 0 and 1. Now $g = 4d+3$ implies that there are $2g = 8d+6$ edges. Among these, there are $g-3 = 4d$ edges of the form $bf_h, 2 \leq h \leq g-2$ and so there are d numbers which are congruent to h modulo 4, $(0 \leq h \leq 3)$. By (1), there are $2d$ edges get label 0 and $2d$ edges get label 1. Also there are $g-3 = 4d$ edges of the form $f_h f_{h+1}, 2 \leq h \leq g-2$ and so there are d numbers which are congruent to h modulo 4, $(0 \leq h \leq 3)$. By (2), there are $2d$ edges get label 0 and $2d$ edges get label 1. Also the edges bf_{g-1}, bf_g and $f_{g-1}f_g$ get label 1 and the edges bf_1, $f_g f_1$ and $f_1 f_2$ get label 0. Hence, there are $4d+3$ edges with label 0 and $4d+3$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

Thus, $|e_a(0) - e_a(1)| \leq 1$ and thereby the wheel graph W_g is a strongly*-2 divisor cordial graph. □

Example 2.1. *A strongly*-2 divisor cordial labeling of the graph W_8 is shown in Figure 1. Here $e_a(0) = 8$, $e_a(1) = 8$ and hence $|e_a(0) - e_a(1)| = 0$.*



$Figure\ 1$: Strongly*-2 divisor cordial labeling of W_8

Theorem 2.2. *The sunflower graph SF_g is a strongly*-2 divisor cordial graph.*

Proof. Let $V(SF_g) = \{b, f_h, j_h : 1 \leq h \leq g\}$ and $E(SF_g) = \{bf_h, f_hj_h, f_sf_{s+1}, f_{s+1}j_s, f_gf_1, j_gj_1 : 1 \leq h \leq g, 1 \leq s \leq g-1\}$. Then SF_g has $2g+1$ vertices and $4g$ edges.

Case 1. $g \equiv o(mod\ 4)$, $o \in \{0, 1\}$

Define $a : V(SF_g) \to \{0, 1\}$ by $a(b) = 1$, $a(f_h) = h+1$, $1 \leq h \leq g$ and $a(j_h) = g+1+h$, $1 \leq h \leq g$. Let a^* be the induced edge label of a.

For the edges $bf_h, 1 \leq h \leq g$, $f^*(bf_h) = \lfloor \frac{1+h+1+1(h+1)}{2} \rfloor = \lfloor \frac{2h+3}{2} \rfloor = h+1$ is even if h is odd and odd if h is even. Therefore, the edges $bf_h, 1 \leq h \leq g$ get label 0 if h is odd and 1 if h is even. (1)

For the edges $f_hf_{h+1}, 1 \leq h \leq g-1$, $f^*(f_hf_{h+1}) = \lfloor \frac{h+1+h+2+(h+1)(h+2)}{2} \rfloor = \lfloor \frac{h^2+5h+5}{2} \rfloor$ which is even if $h \equiv 0, 3(mod\ 4)$ and odd if $h \equiv 1, 2(mod\ 4)$. Therefore, the edges f_hf_{h+1} get label 0 if $h \equiv 0, 3(mod\ 4)$ and 1 if $h \equiv 1, 2(mod\ 4)$. (2)

Subcase 1.1. $o = 0$

Here $g = 4d$, $d \in N$. For the edge f_gf_1, $a^*(f_gf_1) = \lfloor \frac{g+1+2+2(g+1)}{2} \rfloor = \lfloor \frac{3g+5}{2} \rfloor = \lfloor \frac{12d+5}{2} \rfloor = 6d+2$ which is even and so the edge f_gf_1 gets label 0.

For the edges $f_hj_h, 1 \leq h \leq g$, $a^*(f_hj_h) = \lfloor \frac{h+1+g+1+h+(h+1)(g+1+h)}{2} \rfloor = \lfloor \frac{2g+gh+h^2+4h+3}{2} \rfloor = \lfloor \frac{2(4d)+(4d)h+h^2+4h+3}{2} \rfloor = 2(4d+dh+h) + \lfloor \frac{h^2+3}{2} \rfloor$ which is even if h is odd and odd if h is even. Therefore, the edges $f_hj_h, 1 \leq h \leq g$, get label 0 if h is odd and 1 if h is even. (3)

For the edges $f_{h+1}j_h, 1 \leq h \leq g-1$, $a^*(f_{h+1}j_h) = \lfloor \frac{h+2+g+1+h+(h+2)(g+1+h)}{2} \rfloor = \lfloor \frac{3g+gh+h^2+5h+5}{2} \rfloor = \lfloor \frac{3(4d)+(4d)h+h^2+5h+5}{2} \rfloor = 2d(3+h) + \lfloor \frac{h^2+5h+5}{2} \rfloor$ which is even if $h \equiv 0, 3(mod\ 4)$ and odd if $h \equiv 1, 2(mod\ 4)$. Therefore, the edges $f_{h+1}j_h, 1 \leq h \leq g-1$ get label 0 if $h \equiv 0, 3(mod\ 4)$ and 1 if $h \equiv 1, 2(mod\ 4)$. (4)

For the edge j_gf_1, $a^*(j_gf_1) = \lfloor \frac{2g+1+2+2(2g+1)2}{2} \rfloor = \lfloor \frac{6g+5}{2} \rfloor = \lfloor \frac{24d+5}{2} \rfloor = 12d+2$ which is even and so the edge j_gf_1 gets label 0.

Let us find the number of edges with label 0 and 1. Now $g = 4d$ implies that there are $4g = 16d$ edges. Among these, there are $g = 4d$ edges of the form bf_h and $f_hj_h, 1 \leq h \leq g$ and so there are $2d$ odd and $2d$ even numbers. By (1) and (3), there are $4d$ edges get label 0 and $4d$ edges get label 1. Also there are $g-1 = 4d-1$ edges of the form f_hf_{h+1} and $f_{h+1}j_h$, $1 \leq h \leq g-1$ and so there are d numbers which are congruent to h modulo 4, $(h = 1, 2, 3)$ and $d-1$ numbers which are congruent to 0 modulo 4. By (2) and (4), there are $4d-2$ edges get label 0 and $4d$ edges get label 1. And the edgse f_gf_1 and j_gj_1 get label 0. Hence, there are $8d$ edges with label 0 and $8d$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

Subcase 1.2. $o = 1$

Here $g = 4d+1$, $d \in N$. For the edge f_gf_1, $a^*(f_gf_1) = \lfloor \frac{g+1+2+2(g+1)}{2} \rfloor = \lfloor \frac{3g+5}{2} \rfloor = \lfloor \frac{3(4d+1)+5}{2} \rfloor = 6d+4$ which is even and so the edge f_gf_1 gets label 0.

For the edges $f_hj_h, 1 \leq h \leq g$, $a^*(f_hj_h) = \lfloor\frac{h+1+g+1+h+(h+1)(g+1+h)}{2}\rfloor$ $= \lfloor\frac{2g+gh+h^2+4h+3}{2}\rfloor = \lfloor\frac{2(4d+1)+(4d+1)h+h^2+4h+3}{2}\rfloor = 2d(2+h) + \lfloor\frac{h^2+5h+5}{2}\rfloor$ which is even if $h \equiv 0, 3(mod\ 4)$ and odd if $h \equiv 1, 2(mod\ 4)$. Therefore, the edges $f_hj_h, 1 \leq h \leq g$, get label 0 if $h \equiv 0, 3(mod\,4)$ and 1 if $h \equiv 1, 2(mod\,4)$. (5)

For the edges $f_{h+1}j_h, 1 \leq h \leq g-1$, $a^*(f_{h+1}j_h) = \lfloor\frac{h+2+g+1+h+(h+2)(g+1+h)}{2}\rfloor$ $= \lfloor\frac{3g+gh+h^2+5h+5}{2}\rfloor = \lfloor\frac{3(4d+1)+(4d+1)h+h^2+5h+5}{2}\rfloor = 2d(3+h) + 4 + \lfloor\frac{h^2+6h}{2}\rfloor$ which is even if $h \equiv 0, 2(mod\ 4)$ and odd if $h \equiv 1, 3(mod\ 4)$. Hence, the edges $f_{h+1}j_h, 1 \leq h \leq g-1$, get label 0 if $h \equiv 0, 2(mod\,4)$ and 1 if $h \equiv 1, 3(mod\,4)$.(6)

For the edge j_gf_1, $a^*(j_gf_1) = \lfloor\frac{2g+1+2+(2g+1)2}{2}\rfloor = \lfloor\frac{6g+5}{2}\rfloor = \lfloor\frac{6(4d+1)+5}{2}\rfloor = 12d+5$ which is odd and so the edge j_gf_1 gets label 1.

Let us find the number of edges with label 0 and 1. Now $g = 4d+1$ implies that there are $4g = 16d+4$ edges. Among these, there are $g = 4d+1$ edges of the form $bf_h, 1 \leq h \leq g$ and so there are $2d+1$ odd and $2d$ even numbers. By (1), there are $2d+1$ edges get label 0 and $2d$ edges get label 1. Also there are $g-1 = 4d$ edges of the form f_hf_{h+1} and $f_{h+1}j_h, 1 \leq h \leq g-1$ and so there are d numbers which are congruent to h modulo 4, $(0 \leq h \leq g)$. By (2) and (6), there are $4d$ edges get label 0 and $4d$ edges get label 1. There are $g = 4d+1$ edges of the form $f_hj_h, 1 \leq h \leq g$ and so there are d numbers which are congruent to h modulo 4, $(h = 0, 2, 3)$ and there are $d+1$ numbers which are congruent to 1 modulo 4. Hence by (5), $2d+1$ edges get label 1 and $2d$ edges get label 0. And the edge f_gf_1 gets label 0 and the edge j_gf_1 gets label 1. Hence, there are $8d+2$ edges with label 0 and $8d+2$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

Case 2. $g \equiv l(mod\,4), l \in \{2, 3\}$

Define $a : V(SF_g) \to \{1, 2, ..., 2g+1\}$ by $a(b) = 1$, $a(f_h) = 2h$, $1 \leq h \leq g$ and $a(j_h) = 2h+1$, $1 \leq h \leq g$. Let a^* be the induced edge label of a.

For the edges $bf_h, 1 \leq h \leq g$, $a^*(bf_h) = \lfloor\frac{1+2h+2h}{2}\rfloor = \lfloor\frac{4h+1}{2}\rfloor = 2h$ which is even and thereby the edges bf_h get label 0 for $1 \leq h \leq g$. (1)

For the edges $f_hf_{h+1}, 1 \leq h \leq g-1$, $a^*(f_hf_{h+1}) = \lfloor\frac{2h+2h+2+2h(2h+2)}{2}\rfloor$ $= \lfloor\frac{4h^2+8h+2}{2}\rfloor = 2h^2+4h+1$ which is odd. Therefore, the edges f_hf_{h+1} get label 1 for $1 \leq h \leq g-1$. (2)

For the edges $f_hj_h, 1 \leq h \leq g$, $a^*(f_hj_h) = \lfloor\frac{2h+2h+1+2h(2h+1)}{2}\rfloor = \lfloor\frac{4h^2+6h+1}{2}\rfloor$ $= 2h^2+3h$ which is even if h is even and odd if h is odd. Therefore, the edges $f_hj_h, 1 \leq h \leq g$, get label 0 if h is even and 1 if h is odd. (3)

For the edges $f_{h+1}j_h, 1 \leq h \leq g-1$, $a^*(f_{h+1}j_h) = \lfloor\frac{2h+2+2h+1+(2h+2)(2h+1)}{2}\rfloor$ $= \lfloor\frac{4h^2+10h+5}{2}\rfloor = 2h^2+5h+2$ which is even if h is even and odd if h is odd. Therefore, the edges $f_{h+1}j_h, 1 \leq h \leq g-1$, get label 0 if h is even and 1 if h is odd. (4)

Subcase 2.1. $l = 2$

Here $g = 4d + 2$, $d \in N$. For the edge $f_g f_1$, $a^*(f_g f_1) = \lfloor \frac{2g+2+2g(2)}{2} \rfloor = \lfloor \frac{6g+2}{2} \rfloor = \lfloor \frac{6(4d+2)+2}{2} \rfloor = 12d + 7$ which is odd and so the edge $f_g f_1$ gets label 1.

For the edge $j_g f_1$, $a^*(j_g f_1) = \lfloor \frac{2g+1+2+(2g+1)2}{2} \rfloor = \lfloor \frac{6g+5}{2} \rfloor = \lfloor \frac{6(4d+2)+5}{2} \rfloor = 12d + 8$ which is even and so the edge $j_g f_1$ gets label 0.

Let us find the number of edges with label 0 and 1. Now $g = 4d+2$ implies that there are $4g = 16d+8$ edges. Among these, there are $g = 4d+2$ edges are of the form $bf_h, 1 \leq h \leq g$. By (1), $4d+2$ edges get label 0. There are $g-1 = 4d+1$ edges of the form $f_h f_{h+1}, 1 \leq h \leq g-1$. By (2), $4d+1$ edges get label 1. Also there are $g = 4d+2$ edges of the form $f_h j_h, 1 \leq h \leq g$ and so there are equal odd and even nubers. By (3), there are $2d+1$ edges get label 0 and $2d+1$ edges get label 1. There are $g-1 = 4d+1$ edges of the form $f_{h+1} j_h, 1 \leq h \leq g-1$ and so there are $2d$ even and $2d+1$ odd numbers. By (4), there are $2d$ edges get label 0 and $2d+1$ edges get label 1. And the edge $f_g f_1$ gets label 1 and the edge $j_g f_1$ gets label 0. Hence, there are $8d+4$ edges with label 0 and $8d+4$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

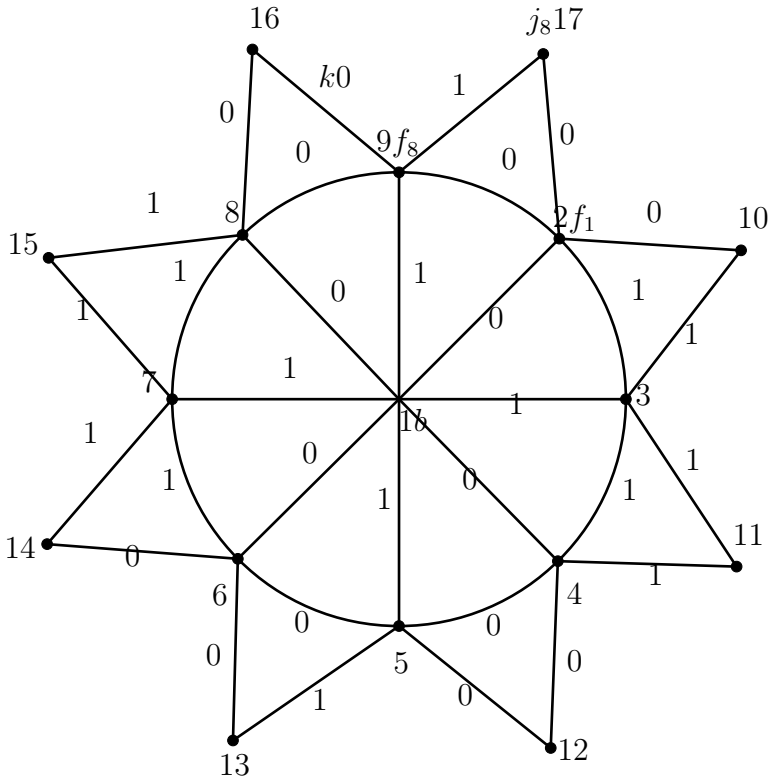
Subcase 2.2. $l = 3$

Here $g = 4d + 3$, $d \in N$. For the edge $f_g f_1$, as in subcase 2.1., $a^*(f_g f_1) = \lfloor \frac{6g+2}{2} \rfloor = \lfloor \frac{6(4d+3)+2}{2} \rfloor = 12d + 10$ which is even and so the edge $f_g f_1$ gets label 0.

For the edge $j_g f_1$, as in subcase 2.1., $a^*(j_g f_1) = \lfloor \frac{6g+5}{2} \rfloor = \lfloor \frac{6(4d+3)+5}{2} \rfloor = 12d + 11$ which is odd and so the edge $j_g f_1$ gets label 1.

Let us find the number of edges with label 0 and 1. Now $g = 4d + 3$ implies that there are $4g = 16d + 12$ edges. Among these, there are $g = 4d + 3$ edges of the form $bf_h, 1 \leq h \leq g$. By (1), $4d + 3$ edges get label 0. There are $g - 1 = 4d + 2$ edges of the form $f_h f_{h+1}, 1 \leq h \leq g-1$. By (2), $4d+2$ edges get label 1. Also there are $g = 4d + 3$ edges of the form $f_h j_h, 1 \leq h \leq g$ and so there are $2d + 2$ odd and $2d + 1$ even nubers. By (3), there are $2d + 1$ edges get label 0 and $2d + 2$ edges get label 1. There are $g - 1 = 4d + 2$ edges are of the form $f_{h+1} j_h, 1 \leq h \leq g-1$ and so there are equal even and odd numbers. By (4), there are $2d + 1$ edges get label 0 and $2d + 1$ edges get label 1. And the edge $f_g f_1$ gets label 0 and the edge $j_g f_1$ gets label 1. Hence, there are $8d + 6$ edges with label 0 and $8d + 6$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$. □

Example 2.2. *A strongly*-2 divisor cordial labeling of the graph SF_8 is shown in Figure 2. Here $e_a(0) = 16$, $e_a(1) = 16$ and hence $|e_a(0) - e_a(1)| = 0$.*



$Figure\ 2$: Strongly*-2 divisor cordial labeling of SF_8

3 Conclusion

Motivated by the concepts of cordial labeling, divisor cordial labeling and strongly*- graphs in graph theory, we introduce a novel notion termed ”strongly*-2 divisor cordial labeling.” Our research includes a proof demonstrating that both the wheel graph W_g and the sunflower graph SF_g exhibit the property of being strongly*-2 divisor cordial graphs. Notably, our investigation extends beyond these specific graphs, as we assert that strongly*-2 divisor cordial labeling is applicable to a broader range of graphs. Looking ahead, we envision the development of an algorithm capable of determining whether a given graph admits a strongly*-2 divisor cordial labeling.

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Inventory Model for Quadratic Demand and Deteriorating Items Following Weibull Distribution with Trade Credit Policy

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Abstract

In this paper, an inventory model for deteriorating items following two parameter Weibull distribution with trade credit policy is developed, while demand is viewed as quadratic function of time. The supplier gives the retailer a trade credit period. Trade credit is a frequently used method of payment implemented by suppliers, and it generally leads to greater revenue and ultimately, higher income. The suggested inventory model seeks to calculate the ideal replenishment cycle duration in order to maximize the overall profit per unit of time. Shortages are permitted and partially backlogged. Two categories are applied to the mathematical model. Case I: When the payment to settle the account is made on or before the positive inventory. Case II: When the payment to settle the account is made after the inventory reaches to zero. The model is illustrated through numerical experiments, sensitivity analysis, and graphical depiction.

Keywords: Inventory; Quadratic Demand; Deterioration; Weibull Distribution; Trade Credit.‡

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1. Introduction

Inventory system management is a critical component in controlling corporate costs. Extensive research in this topic has looked at several models that take into account issues including item scarcity, deterioration, demand trends, order cycles, and their interactions. Notably, efficiently monitoring and regulating ageing products is a big difficulty in any inventory management system. Fruits, vegetables, electronics, and chemicals deteriorate with time, necessitating a comprehensive inventory plan that accounts for potential losses due to deterioration. One of the primary goals of inventory models for depreciating commodities is to determine appropriate inventory levels and procurement techniques that maximise overall profitability while accommodating consumer demand while accounting for the costs associated with product depreciation. Inventory demand, which denotes the number of items consumers intend to purchase within a particular timeframe, plays an important role in inventory management, assisting organisations in keeping acceptable stock levels while avoiding surplus inventory.

The study of inventory systems has seen significant development, as evidenced by a series of models proposed by various researchers in recent years. R. Roy Chowdhury (2015) focused on devising an inventory model catering to seasonal items, considering the degradation rate to determine optimal ordering quantities and pricing strategies for maximizing predicted profits. Similarly, K. Karthikeyan (2015) delved into the implications of the degradation factor, accounting for a cubic function of time in demand and time-dependent holding costs. Umakanta Mishra (2015) concentrated on Weibull degrading items, emphasizing the determination of the most suitable procurement quantity. Sumit Saha (2017) developed an inventory model for goods with negative exponential demand and time-proportional deterioration, allowing partial backlogging. Sandeep Kumar (2017) studied perishable goods with exponentially declining demand, incorporating partial backordering and considering the impact of replenishment timing on backlog rate. Furthermore, the incorporation of credit risk in inventory management was explored by Gour Chandra Mahata (2017), which provided insightful guidelines for retailers selling depreciating goods. Bidyadhara Bishi (2018) investigated a deterministic inventory model for decaying goods, allowing shortages and partial backlogging, with a quadratic demand rate over time. Dr. D. Chitra (2019) designed an inventory model for non-instantaneous decaying items, considering a parabolic backlog pattern in a dynamic demand scenario. Biswaranjan Mandal (2020) examined a time-varying degrading item EOQ inventory model with cubic demand. P. D. Khatri (2020) proposed an EPQ model for deteriorating goods with a ramp-type demand rate and time-varying holding costs, accommodating shortages and backlogs. Cardenas-Barron (2020) introduced a trade credit EOQ inventory model, highlighting the influence of nonlinear stock-dependent holding costs and demand on inventory management. Additionally, Ashish Sharma

(2021) and Maryam Esmaeili (2021) explored partially backlogged inventory models with ramp-type demand, taking into account time, price, and availability constraints, and inflation for decaying items, respectively. To handle the complexities of inventory systems with degrading products, recent studies by Khyati and A. K. Saxena (2022), A. K. Saxena and R. K. Yadav (2011), Shaikh, A. A., Khakzad, A. (2020), and Duary, A. (2022) proposed various models considering different aspects of inventory management, such as trade credit, stock-dependent demand, inspection times, and capacity constraints, each contributing valuable insights into the field. Kumari, M. (2023) introduced a comprehensive mathematical model considering the influence of both selling price and inventory levels on consumer demand, emphasizing the benefits of a complete trade credit period for both vendors and buyers.

This research provides a quadratic demand inventory model for degradable products that takes into account trade credit rules. Employing a quadratic function of time demand, this model simplifies the estimation of optimal ordering quantities and pricing methods over defined periods. This approach is especially beneficial for businesses dealing with products with fluctuating demand levels, such as seasonal goods or items subject to changing consumer preferences, as it allows for more informed inventory and pricing decisions, boosting profitability and market competitiveness. Furthermore, incorporating a trade credit policy into the inventory model digs into a thorough review of credit costs, which includes monitoring expenses and revenue creation from credit sales. This strategy tries to reduce overall inventory costs, match customer demand, and maximise credit sales revenues by optimising order quantities and reorder points. Taking into account numerous characteristics such as loan period length, interest rates on outstanding accounts, and default risk, the model tries to find an optimal balance between the costs and benefits of providing credit, assuring successful credit management practises.

2. Notations and Assumptions

There are some notations and assumptions which are necessary for mathematical formulation of model. These are as follows:

Notations

$D(t) = a + bt + ct^2$: quadratic demand.

$\theta(t) = \alpha\beta t^{\beta-1}$: The Weibull distribution deterioration rate where $0 < \alpha < 1$the scale parameter and $\beta > 0$ is the shape parameter.

t_0 = The period of time after which interest is payable.

t_1 = Length of time with a positive item stock.

T= Cycle Length.
Q= Ordered Quantity per Cycle.
t_0= Earlier Period of Deterioration.
M = Period of trade credit provided by supplier.
Z_e= Earned interest per dollar per unit of time.
Z_{t_0}= Interest charges per dollar per unit item by the supplier for the period (M, t_0).
Z_{t1}= Interest charges per dollar per unit item by the supplier for the period (t_0, T).
C_1 = Ordering Price per unit.
C_2 = Cost of inventory deterioration per item.
C_3 = Holding cost per item
C_4 = Backordering cost per item
C_5 = Sales revenue cost per item
C_p= The constant purchase cost.

Assumptions

- The demand rate is linearly time dependent.
- A single warehouse and a single item are considered.
- Shortages are allowed and partially backlogged.
- Lead time is assumed to be zero.
- The deterioration rate follows the two parameter Weibull distribution.
- There is no repair or replenishment of the deteriorated items during the inventory cycle.
- The inventory is replenished only one in each cycle.
- The retailer can increase returns and produce interest, which will be followed by customer payments in purchasing costs from the retailer in anticipation of the dealer's permissible payment time being completed.

3. Mathematical Formulation of Model

Let Q represent the greatest amount of inventory, including backorders, that can be filled. During time period $(0, t_1)$, the inventory depletion occurs due to deterioration and demand. The inventory level becomes zero at $t_1$and shortages are allowed in interval (t_1, T). And finally, inventory replenished at time T. The quadratic function of time demand is typically of the form:

$D(t) = a + bt + ct^2$ where D(t) represents the demand for the item at time t, a representing the baseline demand, b reflecting the rate of change in demand over time, and c expressing the acceleration in the rate of change in demand over time.

The following differential equations describes the states of inventory level $Z(t)$ at time t:

$$\frac{dZ(t)}{dt} + \theta(t)Z(t) = -D(t) \qquad 0 \le t \le T \tag{1}$$

Where $\theta(t) = \alpha\beta e^{\beta-1}$ and $\mathrm{D(t)} = \mathrm{a + bt} + ct^2$

$$\frac{dZ(t)}{dt} + \alpha\beta e^{\beta-1}Z(t) = -(\mathrm{a + bt} + ct^2) \qquad 0 \le t \le T \tag{2}$$

From equation (2), we get.

$$\mathrm{Z(t)} = -e^{-\alpha t^{\beta}}\left[at + \frac{bt^2}{2} + \frac{ct^3}{3} + \alpha\left(\frac{at^{\beta+1}}{\beta+1} + \frac{bt^{\beta+2}}{\beta+2} + \frac{ct^{\beta+2}}{\beta+3}\right) + \mathrm{c_1}\right] \tag{3}$$

Showing the boundary conditions $Z(t) = Q$ at $t = 0$, we get

$$\mathrm{c_1} = Q$$

Putting the value of $\mathrm{c_1}$ in (3), we get,

$$\mathrm{Z(t)} = -e^{-\alpha t^{\beta}}\left[Q - \left[at + \frac{bt^2}{2} + \frac{ct^3}{3} + \alpha\left(\frac{at^{\beta+1}}{\beta+1} + \frac{bt^{\beta+2}}{\beta+2} + \frac{ct^{\beta+3}}{\beta+3}\right)\right]\right] 0 \le t \le T \tag{4}$$

Since $\mathrm{Z(t)} = 0$ at $t = t_1$

We get

$$Q = at_1 + \frac{bt_1{}^2}{2} + \frac{ct_1{}^3}{3} + \alpha\left(\frac{at_1{}^{\beta+1}}{\beta+1} + \frac{bt_1{}^{\beta+2}}{\beta+2} + \frac{ct_1{}^{\beta+3}}{\beta+3}\right) \tag{5}$$

Putting the value of Qin (4)

$$\mathrm{Z(t)} = -e^{-\alpha t^{\beta}}\left[a(t_1 - t) + \frac{b(t_1-t)^2}{2} + \frac{c(t_1-t)^3}{3} + \alpha\left(\frac{a(t_1-t)^{\beta+1}}{\beta+1} + \frac{b(t_1-t)^{\beta+2}}{\beta+2} + \frac{c(t_1-t)^{\beta+3}}{\beta+3}\right)\right]$$

(6)

Also $\mathrm{Z(t)} = 0$ in $t_1 \le t \le T$

1. Ordering Cost

$$\mathrm{O_c} = \frac{C_1}{T}, \tag{7}$$

2. Deterioration Cost

$$D_c = \frac{C_2}{T}\left[Q - \int_0^{t_1} Z(t)dt\right]$$

$$D_c = \frac{C_2}{T}\left[\alpha\left(\frac{at_1{}^{\beta+1}}{\beta+1} + \frac{bt_1{}^{\beta+2}}{\beta+2} + \frac{ct_1{}^{\beta+3}}{\beta+3}\right)\right] \tag{8}$$

3. Holding cost:

$$H_c = \frac{C_3}{T}\left[\int_0^{t_1} Z(t)dt\right]$$

$$H_c = \frac{C_3}{T}\left[\frac{at_1{}^2}{2} + \frac{bt_1{}^3}{3} + \frac{ct_1{}^4}{4} + \alpha\left(1 - \frac{1}{\beta+1}\right)\left(\frac{at_1{}^{\beta+2}}{\beta+2} + \frac{bt_1{}^{\beta+3}}{\beta+3} + \frac{ct_1{}^{\beta+4}}{\beta+4}\right)\right] \tag{9}$$

4. Backorder cost:

$$B_c = \frac{C_4}{T}\left[\int_0^{T-t_1} D(t)tdt\right]$$

$$B_c = \frac{C_4}{T}\left[\frac{a(T-t_1)^2}{2} + \frac{b(T-t_1)^3}{3} + \frac{c(T-t_1)^4}{4}\right] \tag{10}$$

5. Sales Revenue

$$SR = \frac{C_p}{T}\left[\int_0^{t_1} D(t)dt\right]$$

$$SR = \frac{C_p}{T}\left[\frac{at_1{}^2}{2} + \frac{bt_1{}^3}{3} + \frac{ct_1{}^4}{4}\right] \tag{11}$$

In this paper, we examined two periods of permissible payment delay: (on the basis of the length of T and M):

Case-1: When the payment to settle the account is made on or before the time when all of the inventory has been utilized up i.e., when $M \le t_0 \le t_1 \le T$

$$ZC_1 = \frac{\mathrm{C_p}}{T}\left[Z_{t_0}\int_M^{t_0} Z(t)dt + Z_{t_1}\int_{t_0}^{t_1} Z(t)dt + Z_{t_1}\int_{t_1}^{T} Z(t)dt\right]$$

$$ZC_1 = \frac{\mathrm{C_p}}{T}\Bigg[(Z_{t_0} - Z_{t_1})\Bigg\{a\left(t_1 t_0 - \frac{{t_0}^2}{2}\right) + \frac{b}{2}\left({t_1}^2 t_0 - \frac{{t_0}^3}{3}\right) + \frac{c}{3}\left({t_1}^3 t_0 - \frac{{t_0}^4}{4}\right) + \alpha\Bigg\{\frac{a}{\beta+1}\left({t_1}^{\beta+1} t_0 - \frac{{t_0}^{\beta+2}}{\beta+2}\right) + \frac{b}{\beta+2}\left({t_1}^{\beta+2} t_0 - \frac{{t_0}^{\beta+3}}{\beta+3}\right) + \frac{c}{\beta+3}\left({t_1}^{\beta+3} t_0 - \frac{{t_0}^{\beta+4}}{\beta+4}\right)\Bigg\}\Bigg\} - Z_{t_0}\Bigg[a\left(t_1 M - \frac{M^2}{2}\right) + \frac{b}{2}\left({t_1}^2 M - \frac{M^3}{3}\right) + \frac{c}{3}\left({t_1}^3 M - \frac{M^4}{4}\right) + \alpha\Bigg\{\frac{a}{\beta+1}\left({t_1}^{\beta+1} M - \frac{M^{\beta+2}}{\beta+2}\right) + \frac{b}{\beta+2}\left({t_1}^{\beta+2} M - \frac{M^{\beta+3}}{\beta+3}\right) + \frac{c}{\beta+3}\left({t_1}^{\beta+1} M - \frac{M^{\beta+4}}{\beta+4}\right)\Bigg\}\Bigg] + Z_{t_1}\left[\frac{a{t_1}^2}{2} + \frac{b{t_1}^3}{3} + \frac{c{t_1}^4}{4} + \alpha\left(1 - \frac{1}{\beta+1}\right)\left(\frac{a{t_1}^{\beta+2}}{\beta+2} + \frac{b{t_1}^{\beta+3}}{\beta+3} + \frac{c{t_1}^{\beta+4}}{\beta+4}\right)\right]\Bigg] \tag{12}$$

The interest earned is:

$$ZE_1 = \frac{\mathrm{C_p}}{T} Z_e \int_0^{t_1} D(t)dt$$

$$ZE_1 = \frac{\mathrm{C_p}}{T} Z_e \left[a t_1 + \frac{b{t_1}^2}{2} + \frac{c{t_1}^3}{3}\right] \tag{13}$$

The total cost will be the sum of ordering cost, deteriorating cost, holding cost, backorder cost, the interest charged on a cycle-by-cycle basis for unsold inventory after the due date M (i.e., Z_{c1}) and the interest earned per cycle during the positive inventory level ZE_1.

$$TP_1 = \mathrm{SR} - \mathrm{O_c} - D_c - H_c - B_c - Z_{c1} + ZE_1$$

$$TP_1 = \frac{1}{T}\Bigg\{C_p\left[\frac{a{t_1}^2}{2} + \frac{b{t_1}^3}{3} + \frac{c{t_1}^4}{4}\right] + C_p Z_e\left[a t_1 + \frac{b{t_1}^2}{2} + \frac{c{t_1}^3}{3}\right] - C_1 - C_2\left[\alpha\left(\frac{a{t_1}^{\beta+1}}{\beta+1} + \frac{b{t_1}^{\beta+2}}{\beta+2} + \frac{c{t_1}^{\beta+3}}{\beta+3}\right)\right] - C_3\left[\frac{a{t_1}^2}{2} + \frac{b{t_1}^3}{3} + \frac{c{t_1}^4}{4} + \alpha\left(1 - \frac{1}{\beta+1}\right)\left(\frac{a{t_1}^{\beta+2}}{\beta+2} + \frac{b{t_1}^{\beta+3}}{\beta+3} + \frac{c{t_1}^{\beta+4}}{\beta+4}\right)\right] - C_4\left[\frac{a(T-t_1)^2}{2} + \frac{b(T-t_1)^3}{3} + \frac{c(T-t_1)^4}{4}\right] - \mathrm{C_p}\Bigg[(Z_{t_0} - Z_{t_1})\Bigg\{a\left(t_1 t_0 - \frac{{t_0}^2}{2}\right) + \frac{b}{2}\left({t_1}^2 t_0 - \frac{{t_0}^3}{3}\right) + \frac{c}{3}\left({t_1}^3 t_0 - \frac{{t_0}^4}{4}\right) + \alpha\Bigg\{\frac{a}{\beta+1}\left({t_1}^{\beta+1} t_0 - \frac{{t_0}^{\beta+2}}{\beta+2}\right) + \frac{b}{\beta+2}\left({t_1}^{\beta+2} t_0 - \frac{{t_0}^{\beta+3}}{\beta+3}\right) + \frac{c}{\beta+3}\left({t_1}^{\beta+3} t_0 - \frac{{t_0}^{\beta+4}}{\beta+4}\right)\Bigg\}\Bigg\} - Z_{t_0}\Bigg[a\left(t_1 M - \frac{M^2}{2}\right) + \frac{b}{2}\left({t_1}^2 M - \frac{M^3}{3}\right) + \frac{c}{3}\left({t_1}^3 M - \frac{M^4}{4}\right) + \alpha\Bigg\{\frac{a}{\beta+1}\left({t_1}^{\beta+1} M - \frac{M^{\beta+2}}{\beta+2}\right) + \frac{b}{\beta+2}\left({t_1}^{\beta+2} M - \frac{M^{\beta+3}}{\beta+3}\right) + \frac{c}{\beta+3}\left({t_1}^{\beta+1} M - \frac{M^{\beta+4}}{\beta+4}\right)\Bigg\}\Bigg] + Z_{t_1}\left[\frac{a{t_1}^2}{2} + \frac{b{t_1}^3}{3} + \frac{c{t_1}^4}{4} + \alpha\left(1 - \frac{1}{\beta+1}\right)\left(\frac{a{t_1}^{\beta+2}}{\beta+2} + \frac{b{t_1}^{\beta+3}}{\beta+3} + \frac{c{t_1}^{\beta+4}}{\beta+4}\right)\right]\Bigg]\Bigg\} \tag{14}$$

Case-2: The permissible trade credit period greater than the period with positive inventory of the item i.e., when $t_1 \leq M$:-

$$Z_{c2} = 0 \tag{15}$$

Since Interest earned = Interest earned during $t_1 \leq M$+ Interest earned during time (per cycle)

Hence the total interest earned per unit time (ZE_2) is given by.

$$ZE_2 = \frac{C_p}{T} Z_e \left[\int_0^{t_1} D(t)tdt + D(t_1)t_1(M - t_1) \right]$$

$$ZE_2 = \frac{C_p}{T} Z_e \left[\frac{at_1^2}{2} + \frac{bt_1^3}{3} + + \frac{ct_1^4}{4} + (at_1 + bt_1^2 + ct_1^3)(M - t_1) \right]$$

$$TP_2 = \mathrm{SR} - O_c - D_c - H_c - B_c - Z_{c2} + ZE_2 \tag{16}$$

$$\mathbf{TP_2} = \frac{1}{T}\left\{ \mathbf{C_p}\left[\frac{\mathbf{at_1}^2}{2} + \frac{\mathbf{bt_1}^3}{3} + \frac{\mathbf{ct_1}^4}{4}\right] + \mathbf{C_p Z_e}\left[\frac{\mathbf{at_1}^2}{2} + \frac{\mathbf{bt_1}^3}{3} + + \frac{\mathbf{ct_1}^4}{4} + \left(\mathbf{at_1} + \mathbf{bt_1}^2 + \mathbf{ct_1}^3\right)(\mathbf{M} - \mathbf{t_1})\right] - \mathbf{C_1} - \mathbf{C_2}\left[\boldsymbol{\alpha}\left(\frac{\mathbf{at_1}^{\beta+1}}{\beta+1} + \frac{\mathbf{bt_1}^{\beta+2}}{\beta+2} + \frac{\mathbf{ct_1}^{\beta+3}}{\beta+3}\right)\right] - \mathbf{C_3}\left[\frac{\mathbf{at_1}^2}{2} + \frac{\mathbf{bt_1}^3}{3} + \frac{\mathbf{ct_1}^4}{4} + \boldsymbol{\alpha}\left(\mathbf{1} - \frac{1}{\beta+1}\right)\left(\frac{\mathbf{at_1}^{\beta+2}}{\beta+2} + \frac{\mathbf{bt_1}^{\beta+3}}{\beta+3} + \frac{\mathbf{ct_1}^{\beta+4}}{\beta+4}\right)\right] - \mathbf{C_4}\left[\frac{\mathbf{a(T-t_1)}^2}{2} + \frac{\mathbf{b(T-t_1)}^3}{3} + \frac{\mathbf{c(T-t_1)}^4}{4}\right] - \mathbf{C_p}\left[\left(\mathbf{Z_{t_0}} - \mathbf{Z_{t_1}}\right)\left\{\mathbf{a}\left(\mathbf{t_1 t_0} - \frac{\mathbf{t_0}^2}{2}\right) + \frac{\mathbf{b}}{2}\left(\mathbf{t_1}^2\mathbf{t_0} - \frac{\mathbf{t_0}^3}{3}\right) + \frac{\mathbf{c}}{3}\left(\mathbf{t_1}^3\mathbf{t_0} - \frac{\mathbf{t_0}^4}{4}\right) + \boldsymbol{\alpha}\left\{\frac{\mathbf{a}}{\beta+1}\left(\mathbf{t_1}^{\beta+1}\mathbf{t_0} - \frac{\mathbf{t_0}^{\beta+2}}{\beta+2}\right) + \frac{\mathbf{b}}{\beta+2}\left(\mathbf{t_1}^{\beta+2}\mathbf{t_0} - \frac{\mathbf{t_0}^{\beta+3}}{\beta+3}\right) + \frac{\mathbf{c}}{\beta+3}\left(\mathbf{t_1}^{\beta+3}\mathbf{t_0} - \frac{\mathbf{t_0}^{\beta+4}}{\beta+4}\right)\right\}\right\}\right]\right\} \tag{17}$$

4. Theoretical Results and Optimality of Mathematical Model

Our aim is to find the optimal values of t_1 and T to obtain maximize TP_1. The necessary condition for maximization of total cost is as following:

$$\text{If } \frac{\partial TP_1}{\partial t_1} = 0 \text{ and } \frac{\partial TP_1}{\partial T} = 0,$$

Providing that $\frac{\partial^2 TP_1}{\partial^2 t_1} < 0$ and $\frac{\partial^2 TP_1}{\partial^2 T} < 0$. Then the objective function will be maximum for some values of t_1 and T.

Now we will use the theorems to prove the optimality of objective function in mathematically.

Theorem-1: Objective function TP_1 achieves its global maximum with respect to t_1 if $C_p[at_1 + bt_1^2 + ct_1^3 + Z_e(a + bt_1 + ct_1^2)] < 0$, providing decision variable T is fixed.

Proof: See the Appendix A. We can easily see that $\frac{\partial^2 TP_1}{\partial^2 t_1} < 0$. So, the objective function TC_1attains its global maximisation with respect to $t_1$while decision variable T is fixed.

Theorem -2: Objective function TP_1 achieves its global maximum with respect to T if $\frac{2(X-Y(T))-T^2\frac{\partial^2 Y(T)}{\partial T}+2T\frac{\partial Y(T)}{\partial T}}{T^3} < 0$, providing decision variable t_1 is fixed.

Proof: See the Appendix B. We can easily see that $\frac{\partial^2 TP_1}{\partial^2 T} < 0$. So, the objective function TP_1attains its global maximization with respect to Twhile decision variable t_1 is fixed.

Similarly, we can prove the optimality of second objective function TP_2. find the optimal values of t_1 and T to obtain maximize TP_2. The necessary condition for maximization of total cost is as following:

If $\frac{\partial TP_2}{\partial t_1} = 0$ and $\frac{\partial TP_2}{\partial T} = 0$,

Providing that $\frac{\partial^2 TP_2}{\partial^2 t_1} < 0$ and $\frac{\partial^2 TP_2}{\partial^2 T} < 0$. Then the objective function will be maximum for some values of t_1 and T.

Now we will use the theorems to prove the optimality of objective function in mathematically.

Theorem-3: Objective function TP_2 achieves its global maximum with respect to t_1 if $\mathrm{C_p} Z_e\big((2b + 6ct_1)(M - t_1)\big) < 0$, providing decision variable T is fixed.

Proof: See the Appendix A. We can easily see that $\frac{\partial^2 TP_2}{\partial^2 t_1} < 0$. So, the objective function TP_2attains its global maximisation with respect to $t_1$while decision variable T is fixed.

Theorem -4: Objective function TP_1 achieves its global maximum with respect to T if $\frac{2(U-V(T))-T^2\frac{\partial^2 V(T)}{\partial T}+T\frac{\partial V(T)}{\partial T}}{T^3} < 0$, providing decision variable t_1 is fixed.

Proof: See the Appendix B. We can easily see that $\frac{\partial^2 TP_2}{\partial^2 T} < 0$. So, the objective function TP_2attains its global maximization with respect to Twhile decision variable t_1 is fixed.

5. Numerical Example and Sensitivity Analysis

Let

$$a = 100, b = 50\ , c = 20, \alpha = 0.5, \beta = 1\ , \mathrm{C_1} = 200\ per\ order,\ \mathrm{C_2} = 0.6\ per\ unit\ ,\ \mathrm{C_3} = 2\ per\ unit\ per\ year, \mathrm{C_4} = 3\ per\ unit, Z_{t_0} = 0.12\ per\ year, Z_{t_1} = 0.15\ per\ year, \mathrm{C_p} = 10\ per\ unit, Z_e = 0.07\ per\ year, t_0 = 0.1$$

For case-I, $, M = 0.25\ year$ and for case-II, $M = 0.5\ year$ By putting all these values of parameter in total cost equations and solving in MATLAB software, we find the optimum values $t_1 = 1.036541, T = 1.628414\ and\ TC = 2596$

In order to explore the impact of under or overestimating the inventory system characteristics on the ideal values of the starting time period, cycle duration, as well as the maximum profit of the system, we did a sensitivity analysis on the numerical data mentioned above. The percentage variations from the previously mentioned ideal values are used as sensitivity indicators. Changes (increases and decreases) of -20% to +20% are made to the parameters to do the analysis. One parameter is changed at a time while the other parameters are left unchanged to produce the results. For both models, the findings of these analysis are presented in Tables-1 and Table-2.

5.1 Sensitivity analysis for Case-I: $M \le t_0 \le t_1 \le T$

Table-1. Variation in Total Profit, t_1, T, D and Q on changing the value of one parameter while other parameters are constant:

Parameter	% Change in Parameter	% Change in			
		TP^*	${t_1}^*$	T^*	Q^*
C_p	-20	3560.4507	1.2091	1.6228	678.1169
	-10	2890.5695	1.0795	1.5901	495.2035
	10	1976.7853	0.8976	1.4896	381.1964
	20	1031.9881	0.6973	1.4001	114.9168
C_1	-20	2819.9600	1.0859	1.5078	468.1512
	-10	2801.0034	1.0985	1.5090	470.3921
	10	2790.1133	1.1009	1.5102	472.7648
	20	2784.7814	1.1178	1.5125	474.4517
C_2	-20	1750.5130	0.7819	1.5351	169.1951
	-10	2510.9015	0.8617	1.5789	193.7615
	10	3018.8106	0.9901	1.5336	217.0015
	20	3981.7098	0.8181	1.4903	206.1109
C_3	-20	2567.1176	1.1581	1.5950	421.5214
	-10	2501.3940	1.0770	1.5894	375.8917
	10	2480.7890	1.0195	1.5476	301.9150
	20	2336.7861	1.0081	1.5001	285.7981

C_4	-20	1145.8761	1.0410	1.5601	380.9510
	-10	1131.7321	1.0434	1.5581	382.8150
	10	1113.8911	1.0479	1.5553	387.1506
	20	1107.5214	1.0509	1.5521	390.0015
M	-20	2771.0159	1.0685	1.5104	220.1513
	-10	2849.0078	1.0631	1.4975	218.9504
	10	2990.0195	1.0567	1.4515	210.1511
	20	3013.5330	1.0511	1.4017	206.8769
a	-20	2634.8526	1.0655	1.4967	390.2816
	-10	2965.3281	1.0601	1.4863	439.0867
	10	3156.2880	1.0593	1.4800	483.5671
	20	3320.6501	1.0586	1.4773	519.2041
b	-20	2529.2541	1.0642	1.5069	387.2657
	-10	2603.5963	1.0629	1.5132	389.2811
	10	2697.3214	1.0600	1.5201	390.0840
	20	2703.2824	1.0593	1.5293	391.2900
c	-20	2492.1280	1.0613	1.5179	385.2014
	-10	2505.1413	1.0609	1.5038	384.0371
	10	2581.6941	1.0600	1.4900	383.9869
	20	2597.0054	1.0596	1.4890	383.1084

5.2 Sensitivity analysis for Case-II: $t_1 \leq M$

Table-2. Variation in Total Profit, t_1, T, D and Q on changing the value of one parameter while other parameters are constant:

Parameter	% Change in Parameter	% Change in			
		TP^*	$t_1{}^*$	T^*	Q^*
C_p	-20	4015.6954	1.3206	1.7036	683.2654
	-10	3380.2148	1.1984	1.6925	502.3624
	10	2465.3584	0.9652	1.6825	396.3541
	20	1403.4367	0.7956	1.6726	204.9587
C_1	-20	2984.3645	1.0765	1.6354	490.6853
	-10	2978.2565	1.0864	1.6379	491.3687

	10	2953.6842	1.0965	1.6396	493.5298
	20	2921.3657	1.0836	1.6403	495.2150
C_2	-20	2065.8495	0.8652	1.5532	180.6965
	-10	2632.2451	0.9654	1.5436	201.3670
	10	3294.3256	1.0023	1.5395	223.6041
	20	4065.9451	0.9984	1.5203	228.9520
C_3	-20	2631.5468	1.2036	1.6132	425.5214
	-10	2536.2546	1.1362	1.6069	379.5241
	10	2498.3654	1.0365	1.5932	305.2146
	20	2331.3254	1.0036	1.5520	290.3894
C_4	-20	1250.5423	1.0568	1.5732	381.0261
	-10	1239.3256	1.0573	1.5701	383.2549
	10	1214.3256	1.0596	1.5635	389.5249
	20	1101.9556	1.0615	1.5596	392.6940
M	-20	2832.5436	1.0765	1.5536	223.6370
	-10	2910.3566	1.0712	1.5336	220.1856
	10	3050.6565	1.0651	1.5031	215.1025
	20	3089.2586	1.0592	1.4930	209.2367
a	-20	2736.5012	1.0764	1.5638	289.6216
	-10	2928.2691	1.0738	1.5328	297.3548
	10	3091.2800	1.0703	1.5036	301.9434
	20	3215.3652	1.0684	1.4979	305.5622
b	-20	2631.6902	1.0641	1.5152	295.6817
	-10	2770.0521	1.0621	1.5035	298.3640
	10	2836.2894	1.0603	1.4863	300.5122
	20	2964.3950	1.0583	1.4794	301.2819
c	-20	2694.3601	1.0651	1.5700	299.3617
	-10	2706.2891	1.0642	1.5682	300.2811
	10	2797.6318	1.0640	1.5602	301.5717
	20	2836.9525	1.0632	1.5574	302.6945

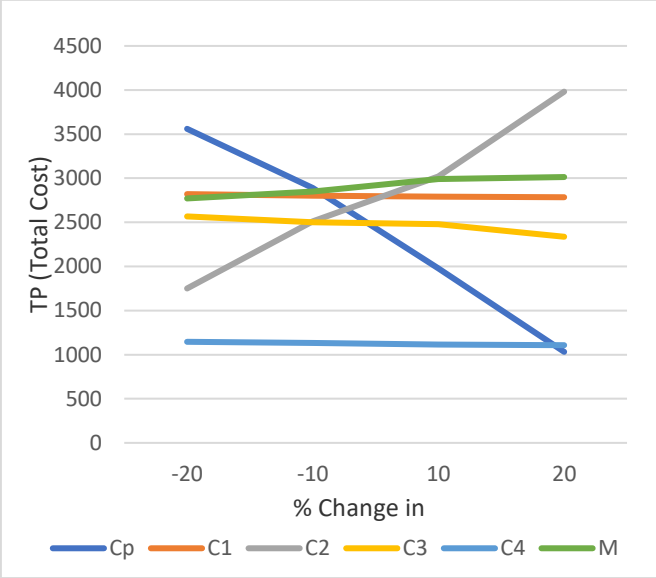

Figure:1 Variation in TP (Total Profit) with respect to $\mathbf{C_p}$, $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{C_4}$ and M

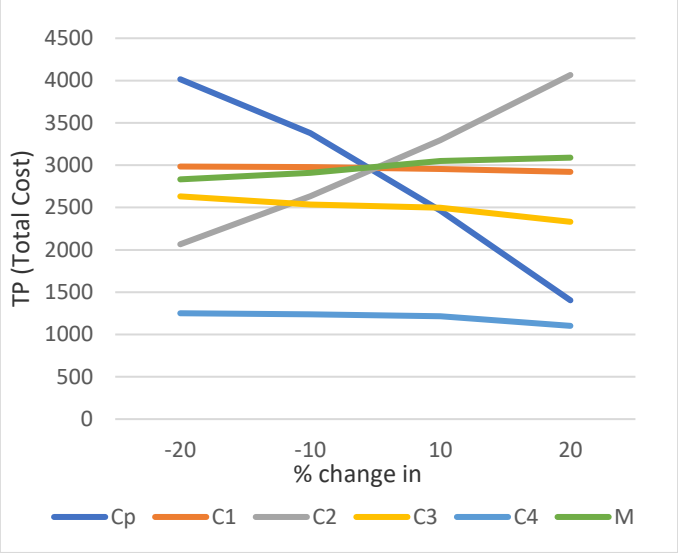

Figure:2 Variation in TP (Total Profit) with respect to $\mathbf{C_p}$, $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{C_4}$ and M

From Table-1 and Table-2 we can observe that:

1. If purchase cost C_p increases, the profit TP^* decreases and Q^*, $t_1{}^*$and T^* also decrease or if purchase cost C_p decreases, the profit TP^* increases and Q^*, $t_1{}^*$and T^* also increase.

2. If ordering cost C_1 increases, the profit TP^* decreases and Q^*, $t_1{}^*$and T^* increase or if ordering cost C_1 decreases, the profit TP^* increases and Q^*, $t_1{}^*$and T^* decrease.
3. If deterioration cost C_2 increases, the profit TP^*, Q^*increase and $t_1{}^*$, T^* decrease or if deterioration cost C_2 decreases, the profit TP^*, Q^*decrease and $t_1{}^*$, T^* increase.
4. If holding cost C_3 increases, the profit TP^* decreases and Q^*, $t_1{}^*$and T^* also decrease or if holding cost C_3 decreases, the profit TP^* increases and Q^*, $t_1{}^*$and T^* also increase.
5. If backlogging cost C_4 increases, the profit TP^*, T^* decreases and Q^*, $t_1{}^*$ increase or if backlogging cost C_4 decreases, the profit TP^*, T^* increases and Q^*, $t_1{}^*$ decrease.
6. If trade credit period M increases, the profit TP^* increases and Q^*, $t_1{}^*$and T^* decrease or if trade credit period M decreases, the profit TP^* decreases and Q^*, $t_1{}^*$and T^* increase.
7. If demand parameter a, b and c increases, the profit $TP^*$and Q^*, increases and $t_1{}^*$and T^*decreases or if demand parameter a, b and c decreases, the profit $TP^*$and Q^*, decreases and $t_1{}^*$and T^*increases.
8. Purchase cost C_p and deterioration cost C_2 are more sensitive to total profit TP^*, $T^*$and Q^*than other parameters.

6. Discussion

The decision-maker or management can be advised of the following findings based on the results of the sensitivity studies that were conducted for model:

From here, we derive our findings that higher purchasing costs result in reduced profit margins, a longer cycle time between inventory replenishments, and a drop in replenishment quantity. Higher ordering costs diminish profitability, but they also imply a shorter cycle length between inventory replenishments and an increase in replenishment quantities. Higher product deterioration costs result in higher profits and a greater quantity of replenishment. However, in order to reduce the negative consequences of deterioration, the cycle length must be reduced. Higher holding costs result in lower profitability, as well as a longer cycle length between inventory replenishments and a drop in replenishment quantity. Higher backlog expenses reduce profitability while also implying an increase in the quantity of replenishment. One significant finding is that profit will drop when the credit period supplied by the supplier to the retailer is shorter since the retailer is unable to receive interest on his profit during that time. Increased demand parameters result in higher profitability and a greater amount of replenishment. This results in a reduced cycle duration and lower holding costs. Retailers must boost inventory levels because as purchasing costs rise,

overall profit also rises. Additionally, this yields a longer replenishing cycle. Additionally, this benefits shops in avoiding the issue of a shortage.

7. Conclusion

We have developed an inventory model for quadratic demand and deteriorating items that follows the Weibull distribution with a trade credit policy involves incorporating various factors such as the rate of demand, rate of deterioration, credit terms, and holding costs into the decision-making process for stocking and selling goods. The Weibull distribution is used to model the probability of demand and deterioration over time, while the trade credit policy can be used to determine the optimal time to sell goods and receive payment. This can help to balance the trade-off between the benefits of holding inventory and the costs of carrying it, and can lead to more effective inventory management. This model can further be extended for the quadratic demand function of price.

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Appendix A

On differentiating the objective function with respect to t_1, we get

$$\frac{\partial TP_1}{\partial t_1} = \frac{1}{T}\Big[C_p(at_1 + bt_1{}^2 + ct_1{}^3) + C_pZ_e(a + bt_1 + ct_1{}^2) - C_1 - C_2\left(\alpha\left(at_1{}^\beta + bt_1{}^{\beta+1} + ct_1{}^{\beta+2}\right)\right) - (C_3 - Z_{t_1})\left[at_1 + bt_1{}^2 + ct_1{}^3 + \alpha\left(1 - \frac{1}{\beta+1}\right)\left(at_1{}^{\beta+1} + bt_1{}^{\beta+2} + ct_1{}^{\beta+3}\right)\right] + C_4[a(T - t_1) + b(T - t_1)^2 + c(T - t_1)^3] - \mathrm{C_p}\Big[\left(Z_{t_0} - Z_{t_1}\right)\Big\{t_0(a + bt_1 + ct_1{}^2) + \alpha\Big\{t_0\left(at_1{}^\beta + bt_1{}^{\beta+1} + ct_1{}^{\beta+2}\right)\Big\}\Big\} - Z_{t_0}M\left[a + bt_1 + ct_1{}^2 + \alpha\left\{at_1{}^\beta + bt_1{}^{\beta+1} + ct_1{}^{\beta+2}\right\}\right]\Big]\Big] \quad \text{(A.1)}$$

Differentiating the objective function again with respect to t_1, we have

$$\frac{\partial^2 TP_1}{\partial^2 t_1} = \frac{1}{T}\Big[C_p\left(a + 2bt_1 + 3ct_1{}^2\right) + C_pZ_e[b + 2ct_1] - C_2\Big[\alpha\left(a\beta t_1{}^{\beta-1} + b(\beta+1)t_1{}^\beta + c(\beta+2)t_1{}^{\beta+1}\right)\Big] - (C_3 - Z_{t_1})\left[a + 2bt_1 + 3ct_1{}^2 + \alpha\left(1 - \frac{1}{\beta+1}\right)\left(a(\beta+1)t_1{}^\beta + b(\beta+2)t_1{}^{\beta+1} + c(\beta+\right.\right.$$

$3){t_1}^{\beta+2})\Big] - C_4[a + 2b(T - t_1) + 3c(T - t_1)^2] - \mathrm{C_p}\Big[\big(Z_{t_0} - Z_{t_1}\big)\big\{t_0(b + 2ct_1) + \alpha\{t_0(a\beta {t_1}^{\beta-1} + b(\beta + 1){t_1}^{\beta} + c(\beta + 2){t_1}^{\beta+1})\}\big\} - Z_{t_0} M\big[b + 2ct_1 + \alpha(a\beta {t_1}^{\beta-1} + b(\beta + 1){t_1}^{\beta} + c(\beta + 2){t_1}^{\beta+1})\big]\Big]$

(A.2)

From equation (A.2) we can observe that

$\frac{\partial^2 TP_1}{\partial^2 t_1} < 0$

if

$C_p[at_1 + b{t_1}^2 + c{t_1}^3 + Z_e(a + bt_1 + c{t_1}^2)] < 0$

This demonstrates *Theorem-1*.

Appendix B

Differentiate the objective function TP_1 with respect to T, we get

$$\frac{\partial TP_1}{\partial T} = -\frac{X}{T^2} - \frac{\frac{\partial Y(T)}{\partial T}}{T} + \frac{Y(T)}{T^2} \quad \text{(B.1)}$$

Again, differentiate the objective function TP_1 with respect to T, we have

$$\frac{\partial^2 TP_1}{\partial^2 T} = \frac{2\big(X - Y(T)\big) - T^2\frac{\partial^2 Y(T)}{\partial T} + 2T\frac{\partial Y(T)}{\partial T}}{T^3} \quad \text{(B.2)}$$

From equation (B.2) we can observe that

$\frac{\partial^2 TC_1}{\partial^2 T} < 0$

If

$\frac{2\big(X - Y(T)\big) - T^2\frac{\partial^2 Y(T)}{\partial T} + T\frac{\partial Y(T)}{\partial T}}{T^3} < 0$, provided that $T > 0$

This demonstrates *Theorem- 2*.

Appendix C

On differentiating the objective function TP_2 with respect to t_1, we get

$\frac{\partial TP_1}{\partial t_1} = \frac{1}{T}\Big[C_p[at_1 + b{t_1}^2 + c{t_1}^3] + \mathrm{C_p} Z_e\big[at_1 + b{t_1}^2 + c{t_1}^3 + \big(a + 2bt_1 + 3c{t_1}^2\big)(M - t_1) - (at_1 + b{t_1}^2 + c{t_1}^3)\big] - C_2\big[\alpha\big(a{t_1}^{\beta} + b{t_1}^{\beta+1} + c{t_1}^{\beta+2}\big)\big] - C_3\Big[at_1 + b{t_1}^2 + c{t_1}^3 + \alpha\Big(1 - \frac{1}{\beta+1}\Big)\big(a{t_1}^{\beta+1} + b{t_1}^{\beta+2} + c{t_1}^{\beta+3}\big)\Big] + C_4[a(T - t_1) + b(T - t_1)^2 + c(T - t_1)^3] - \mathrm{C_p}\Big[\big(Z_{t_0} - Z_{t_1}\big)\big\{t_0(a + bt_1 + c{t_1}^2) + \alpha\big\{t_0\big(a{t_1}^{\beta} + b{t_1}^{\beta+1} + c{t_1}^{\beta+2}\big)\big\}\big\}\Big]\Big]$ (C.1)

Differentiating the objective function TP_2 again with respect to t_1, we have,

$\frac{\partial^2 TP_1}{\partial^2 t_1} = \frac{1}{T}\Big[\big[C_p(1 - Z_e)\big(a + 2bt_1 + 3c{t_1}^2\big) + \mathrm{C_p} Z_e\big((2b + 6ct_1)(M - t_1)\big)\big] - C_2\big[\alpha\big(a\beta {t_1}^{\beta-1} + b(\beta + 1){t_1}^{\beta} + c(\beta + 2){t_1}^{\beta+1}\big)\big] - C_3\Big[a + 2bt_1 + 3c{t_1}^2 + \alpha\Big(1 - \frac{1}{\beta+1}\Big)\big(a(\beta + 1){t_1}^{\beta} + b(\beta +$

$2){t_1}^{\beta+1} + c(\beta+3){t_1}^{\beta+2})\Big] - C_4[a + 2b(T-t_1) + 3c(T-t_1)^2] - \mathrm{C_p}\Big[(Z_{t_0} - Z_{t_1})\big\{t_0(b+2ct_1) + \alpha\big\{t_0\big(a\beta{t_1}^{\beta-1} + b(\beta+1){t_1}^{\beta} + c(\beta+2){t_1}^{\beta+1}\big)\big\}\big\}\Big]$ (C.2)

From equation (C.2) we can observe that

$\frac{\partial^2 TP_1}{\partial^2 t_1} < 0$ If

$\mathrm{C_p}Z_e\big((2b+6ct_1)(M-t_1)\big) < 0$ (C.3)

This demonstrates *Theorem-3*

Appendix D

Differentiate the objective function TP_2 with respect to T, we get

$\frac{\partial TP_2}{\partial T} = -\frac{U}{T^2} - \frac{\frac{\partial V(T)}{\partial T}}{T} + \frac{V(T)}{T^2}$ (D.1)

Again, differentiate the objective function TP_1 with respect to T, we have

$\frac{\partial^2 TP_2}{\partial^2 T} = \frac{2\big(U-V(T)\big) - T^2\frac{\partial^2 V(T)}{\partial T} + 2T\frac{\partial V(T)}{\partial T}}{T^3}$ (D.2)

From equation (D.2) we can observe that

$\frac{\partial^2 TC_2}{\partial^2 T} < 0$

If

$\frac{2\big(U-V(T)\big) - T^2\frac{\partial^2 V(T)}{\partial T} + T\frac{\partial V(T)}{\partial T}}{T^3} < 0$, provided that $T > 0$ (D.3)

This demonstrates *Theorem- 4.*

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On Prime Index of a Graph

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Abstract

In prime labeling, vertices are labeled from 1 to n, with the condition that any two adjacent vertices have relatively prime labels. Coprime labeling maintains the same criterion as prime labeling with adjacent vertices using any set of distinct positive integers. Minimum coprime number $\mathfrak{pr}G$, is the minimum value k for which G has coprime labeling. There are many graphs that do not possess prime labeling, and hence have coprime labeling. The primary purpose of this work is to change a coprime labeled graph into a prime graph by removing the minimum number of edges. Thus, the prime index $\varepsilon(G)$ is the least number of edges to be removed from a coprime graph G to form a prime graph. In this study, the prime index of various graphs is determined. Also, an algorithmic way to determine the prime index of the complete graph is found.

Keywords: Prime labeling, Coprime labeling, Minimum Coprime Number, Prime index

2020 AMS subject classifications: 05C76, 05C78 [1]

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1 Introduction

The fundamental concepts of combinatorics and number theory are used in the majority of graph labeling techniques. In number theory, one of the greatest achievements of mathematicians is the prime number theorem. It provides a way to find the number of prime numbers less than or equal to n and is denoted by $\pi(n)$ [1].

In recent years, graph labeling acts as a significant topic in the field of research. To face the wide range of application challenges, various labeling techniques have been developed. Tout et al. [2] investigated Entringer's concept of prime labeling. Many researchers have studied and introduced various definitions in the context of prime graphs[3]. As a continuation of the above work for edges, a new labeling technique known as relatively prime edge labeling for a simple graph is introduced [4]. A graph G with relatively prime edge labeling is said to be a relatively prime edge labeled graph.

Coprime labeling came from prime labeling where prime labeling uses labels from 1 to n, but in coprime labeling, the labels have no bound [5]. In coprime labeling, if the vertices are labeled from 1 to k, then the least k is the minimum coprime number of G. From the above work, to make a coprime graph into a prime graph the definition of prime index arises. The prime index $\varepsilon(G)$ is the least number of edges removed from the coprime graph G to make it a prime graph.

The next section is discussing on preliminaries of the current work. Section 3 is on defining the parameter prime index of a graph with some results followed by the prime index of the complete graph in section 4. Section 5 is discussed the results of relatively prime index of a graph and lastly, section 6 is the conclusion.

2 Preliminaries

This section briefs some definitions needed for further discussion.

A graph $G = (V, E)$ is a prime graph,
if there exists a bijection $f : V \to \{1, 2, 3, ..., | V |\}$ such that, for every edge $e(= uv) \in E$, $\gcd(f(u), f(v)) = 1$ [6].

Asplund et al. emphasize defining minimal coprime number for the coprime graphs [7]. In particular, the vertices of coprime labeling are labeled with relatively prime labels for the adjacent vertices from the set $\{1, 2, ..., k\}$, for some $k \geq n$, . Also, the minimum coprime number $\mathfrak{pr}G$ is the minimum number k for which G has coprime labeling [8]. The corresponding labeling of G is called minimal coprime labeling of G. If $\mathfrak{pr}G = n$, then a corresponding minimal coprime labeling of G is called prime labeling of G and we call G prime.

The coprime number of complete graph and wheel graph are found to be p_{n-1} and

$n+2$, respectively where p_{n-1} denotes the $n-1^{th}$ prime number [7]. Also, the minimum coprime number of the sum of path and cycles are found in [8].
Coxeter et al. introduced the concept of the generalized Petersen graph $P(n,k)$ [9] and it was named by Mark Watkins [10].
The generalized Petersen graphs $P(n,k)$ with $n \geq 3$ and $1 \leq k \leq \frac{n}{2}$ are defined to be a graph with $V(P(n,k)) = \{u_i, v_i \colon 1 \leq i \leq n\}$ and $E(P(n,k)) = \{v_1v_{i+1}, v_iu_i, u_iu_{i+k} \colon 1 \leq i \leq n$, subscript mod $n\}$. For the graph G and H, the sum $G+H$, is the graph obtained by taking disjoint copies of G and H and then adding every edge xy, where $x \in V(G)$ and $y \in V(H)$.
The Wheel graph W_n, $(n \geq 3)$ of $n+1$ vertices is a graph that contains a cycle with vertex set $\{v_1, v_2, ..., v_n\}$ and connects all the vertices of the cycle to the central vertex v. Here v denotes the central vertex and the other vertices $v_1, v_2, ..., v_n$ be on the rim of the wheel graph [12].

3 Prime Index of a Graph

Definition 3.1. *For a coprime graph G, the prime index is the least number of edges removed from G to form a prime graph G^*. And is denoted by, $\varepsilon(G)$. In other words, $\varepsilon(G) = min\{|\ E(H)\ | \colon H \subseteq G\ and\ G - E(H)\ is\ prime\}$*

Every prime graph is a coprime graph, but the converse is not true. The next theorem finds the prime index of the wheel graph W_n, for odd n.

Theorem 3.1. *For odd n, we have $\varepsilon(W_n) = 1$.*

Proof. The vertices of wheel graphs are $\{v_1, v_2, \cdots, v_{n+1}\}$ having v_1 as the central vertex, and the remaining n vertices form a cycle. Now as in Fig 1, label

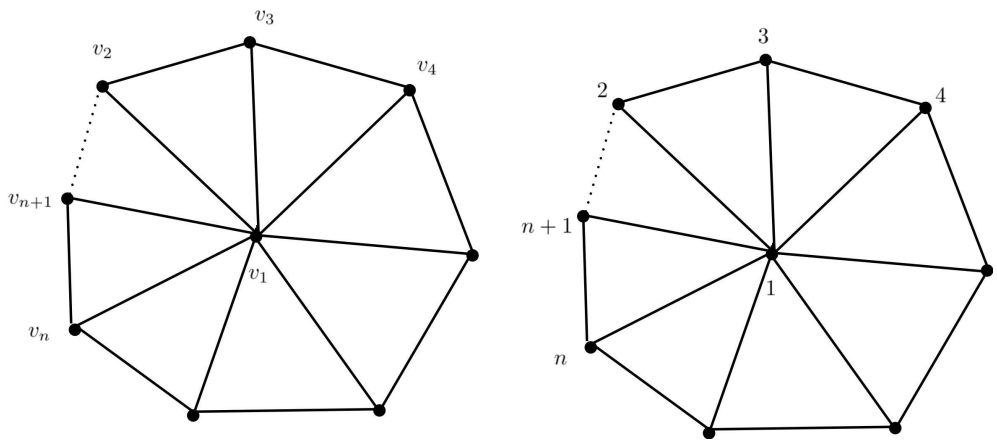

Figure 1: Wheel Graph, $\varepsilon(W_n) = 1$

the vertices v_i as i , for all $i \in \{1, 2, \cdots, n+1\}$, which fails to form prime graph. Since v_2 receives the label 2 and v_{n+1} receives the label $n+1$, which is even. Therefore, for odd n to make w_n a prime graph it is necessary to remove an edge v_2v_{n+1}. Hence $\varepsilon(W_n) = 1$, for odd n.

$P(n, k)$ is a coprime graph , for n odd is found by U. M. Prajapati et.al.[13]. The next theorem finds the prime index of $P(n, 2)$, for even n.

Theorem 3.2. *$P(n, k)$ contains d disjoint inner cycles of length $\frac{n}{d}$, only if $gcd(n, k) = d$. [13]*

Theorem 3.3. *For even n , the prime index of $P(n, 2)$ is $\varepsilon(P(n, 2)) \leq n/2$.*

Proof. We know that for n even, the Generalised Petersen graph $P(n, 2)$ is not prime. Hence it is enough to prove that the removal of $n/2$ edges from $P(n, 2)$ forms a prime graph. In $P(n, 2)$, $v_1, v_2, \cdots, v_n$ represents the outer vertices and $u_1, u_2, \cdots, u_n$ represents the inner vertices as in Fig 2.

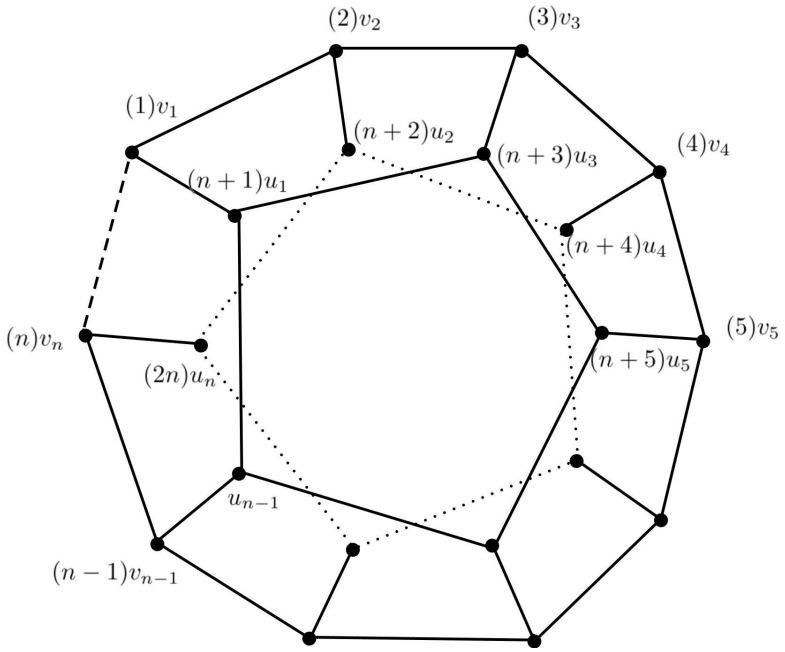

Figure 2: $P(n, 2)$

Edges of $P(n, 2)$ are $\{v_1v_{i+1}, v_iu_i, u_iu_{i+2} \colon 1 \leq i \leq n, subscript\ mod\ n\}$. From corollary 2, there exist 2 interior cycles of length $n/2$. Now, vertices of $P(n, 2)$ are labeled in such a way that $L(v_i) = i\ \&L(u_i) = n + i, 1 \leq i \leq n$. From the definition of the generalized Petersen graph, each u_i is adjacent to u_{i+2}. Thus, there are $n/2$ vertices having the label $n + 2, n + 4, \cdots, 2n$. Therefore,

by removing atleast $\{u_2u_4, u_4u_6, \ldots, u_{n-2}u_n\}$ edges from $P(n, 2)$ results a prime graph. Hence the proof.

It is obvious that, for even m, $P_m + P_2$ is not a prime graph.The removal of at least an edge from $P_m + P_2$ forms a prime graph. Thus, all possible spanning subgraph of $P_m + P_2$ is again a prime graph, for even m.

Theorem 3.4. *For even m, we have $\varepsilon(P_m + P_2) = 1$.*

Proof. The proof is obvious.

4 Prime Index of complete graph

Euler's function, $\phi(n)$ in number theory finds the number of integer k in the range 1 to n for which $gcd(n, k) = 1$. In this section, the prime index of a complete graph with an algorithm is derived.

Theorem 4.1. *For a complete graph K_n the prime index $\varepsilon(G)$ of a complete graph is $\varepsilon(K_n) = \binom{n}{2} - \sum_{k=2}^{n} \phi(k)$*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the n vertices of the complete graph K_n having nc_2 edges. Maximal subgraph in K_n containing the closed path is the cycle C_n. Now, label the vertices of K_n as $L(v_i) = i$. As every vertex is adjacent to every other vertices, it is enough to find the pairs for which $gcd(n, k) \neq 1$. Hence, the number of pairs in the range 1 to n for which $gcd(n, k) \neq 1$ equals the difference between the total number of pairs in the range 1 to n and the numbers of pairs in the range 1 to n for which $gcd(n, k) = 1$. That is, $\varepsilon(k_n) = nc_2 - \sum_{k=2}^{n} \phi(k)$.

Super prime graph SP_n is a graph with vertex set $V(SP_n) = \{1, 2, ..., n\}$ and two vertices $u, v \in V(SP_n)$ are adjacent iff $(u, v) = 1$.Also, the super prime graph SP_n is maximal and unique[15]. The next theorem shows the method for a complete graph to be a super-prime graph.

Theorem 4.2. *The removal of $\varepsilon(K_n)$ edges from a complete graph K_n results a super prime graph.*

Proof. The proof is obvious.

4.1 Algorithm to find prime Index

GCD of any two numbers can be found by using the Euclidean algorithm. Here, an algorithmic way to find the prime index of the complete graph is defined.

INPUT: No. of vertices $= n$.
OUTPUT: Prime Index of K_n.

1. Calculate, the number of edges $E = nC_2$, for the given input n.
2. Calculate $\gcd(r, s)$ as, if $r < s$, exchange r and s. Then, divide r by s, with remainder R.
   (a) If $R = 0$, report $s = \gcd(r, s)$
   (b) Else if $R \neq 0$, replace r by s and replace s by R. Return to the previous step a.
3. Run a loop for $i = 2$ to n,
   $\phi(n)$ counts the number of pairs having $\gcd(i, n) = 1$.
4. For $n = 2$ to n, $sum(n)$ counts the value $\phi(n)$.
5. Finally, $Prime\ index = E - sum(n)$

4.2 Program to find the prime index of complete graph

C program to find the prime index of complete graph is given below.

```
#include <stdio.h>
int phi(int n);
int gcd(int a, int b);
int sum(int n);
int main( )
{
     int i, n, E, PI;
     printf(" Number of vertices:  ");
     scanf("%d",&n);
     E = (n*(n-1))/2;
     printf("Number of Edges = %d \n",E);
for (i = 1; i <= n; i++)
     printf("phi(%d) = %d \n", i, phi(i));
     PI = E -  sum(n);
     printf("\n Prime index = %d",PI);
     return 0;
  }

// Function finds GCD of r and s
int gcd(int r, int s)
{
  if (r == 0)
       return s;
```

```
  return gcd(s % r, r);
}
// Function to return phi(n)
int phi(int n)
{
      int result = 1;
      for (int i = 2; i < n; i++)
  if (gcd(i, n) == 1)
    result++;
       return result;
}

// Function counts phi(n) from 2 to n
int sum(int n)
{
    int sum;
   for(int k=2; k<=n; k++)
           {
            if (k==2)
            sum =1;
            else
            sum = sum + phi(k);
             }
    return sum;
  }
}
```

```
Enter the number:  7
Edges = 21
phi(1) = 1
phi(2) = 1
phi(3) = 2
phi(4) = 2
phi(5) = 4
phi(6) = 2
phi(7) = 6

 Prime index = 4

...Program finished with exit code 0
Press ENTER to exit console.
```

Figure 3: Output

The necessary and sufficient condition for $K_{m,n}$ to be prime was put forward by Fu and Huang[14]. Also, in [14] by taking, $q(t, v)$ to be the set of all primes x such that $t < x \leq v$, the following proposition exists.

Theorem 4.3. *For the positive integers a, b with $a < b$, $K_{a,b}$ is prime iff $a \leq \mid q(\frac{a+b}{2}, a+b) \mid +1$.*

The next theorem finds the prime index of a complete bipartite graph $K_{m,m}$. For that, let V_1 with a set of odd numbers and V_2 with a set of even numbers. Define $\mu(V_1, V_2) = \{u_i v_i \colon u_i \in V_1 \ \& \ v_i \in V_2 \mid (u_i, v_i) \neq 1\}$.

Theorem 4.4. *The prime index of the complete bipartite graph $K_{m,m}$ with $m > \mid P(m, 2m) \mid +1$ is $\varepsilon(K_{m,m}) = \mid \mu(V_1, V_2) \mid$.*

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ be the two partite of $K_{m,m}$. Label the vertices of V_1 and V_2 with odd and even numbers respectively. Then, the resultant graph is not a prime. Thus to make $K_{m,m}$ to be a prime graph, its necessary to remove $\mu(V_1, V_2)$ edges from $K_{m,m}$. Hence the prime index $\varepsilon(K_{m,m}) = \mid \mu(V_1, V_2) \mid$.

As an application part of the prime index of a graph, consider there are certain workers having common skill sets. Represent them using a graph, if the vertex represents the workers and if they have a common skill set an edge can be drawn between them. The label represents the shift hours of each worker. Prime labeling can be used to find the optimal solution so that it covers the maximum skill set in the particular shift. Suppose every worker has some common skill set with every other worker, then the prime index helps to find the common skill that cannot be carried out to get the optimal solution.

5 On Relatively Prime Index of a Graph

The concept of relatively prime edge labeling is just an extension of prime labeling by considering edges into account. In prime labeling, adjacent vertices have relatively prime labels, whereas, in relatively prime edge labeling, adjacent edges have relatively prime labels[4]. The idea to make a coprime edge labeled graph into a relatively prime edge labeled graph results in a definition of a relatively prime index[4].

Definition 5.1. *[4] Let G be a coprime edge labeled graph. A relatively prime index $\varepsilon_r(G)$ is defined to be the minimum number of edges removed from G to form a relatively prime edge labeled graph G^*. In other words, $\varepsilon_r(G) = min\{\mid E(H) \mid : H \subseteq G \ \& \ G - E(H)$ is relatively prime edge labeled graph$\}$*

The next theorem finds the relatively prime index of the corona product of K_n and K_1.

Theorem 5.1. *For a graph $K_n \odot K_1$,*

$$\varepsilon_r(K_n \odot K_1) = \begin{cases} \frac{n(n-3)}{2} & \text{if } 2n+1 \equiv 0(mod\ 3) \\ \frac{n(n-3)}{2} - 1 & \text{if } 2n+1 \not\equiv 0(mod\ 3) \end{cases} \quad (1)$$

Proof. Let $G = K_n \odot K_1$ be the graph with $2n$ vertices and $\frac{n(n+1)}{2}$ edges and let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of $K_n \cdots K_1$.
Case 1: For $2n + 1 \equiv 0(mod\ 3)$.
We now prove that, the removal of $\frac{n(n-3)}{2}$ edges results in a relatively prime edge-labeled graph. Suppose the removal of $\frac{n(n-3)}{2} - 1$ edges in$K_n \odot K_1$ results in a relatively prime edge labeled graph. That is, remaining $\frac{n(n+1)}{2} - \frac{n(n-3)}{2} + 1 = 2n+1$ edges of K_n can be labeled from 1 to $2n+1$. Hence by removing $\frac{n(n-3)}{2} - 1$ interior edges of $K_n \odot K_1$, the resultant graph will be of the form C_n with n edges, n pendent vertices $(u_1, u_2, \ldots, u_n)$ connecting to the each vertex of C_n and an edge connecting any two non-adjacent vertices of C_n. Each vertex $v_1, v_2, \ldots, v_n$ of C_n is of degree 3. By labeling the edges of C_n with $1, 3, 5, \ldots, 2n-1$, each edge incident on the pendant vertices is labeled with $2, 4, 6, \ldots, 2n$ and an edge connecting any two non-adjacent vertices of C_n is labeled with $2n+1$. As $2n+1 \equiv 0(mod\ 3)$, that is $2n + 1 = 3m$, the label incident on the vertices of the edge with label $2n + 1$ fails to be relatively prime.
Case 2: For $2n + 1 \not\equiv 0(mod\ 3)$.
We now prove that the removal of $\frac{n(n-3)}{2} - 1$ edges results in a relatively prime edge labeled graph. Suppose the removal of $\frac{n(n-3)}{2} - 2$ edges in $K_n \odot K_1$ results in a relatively prime edge labeled graph. That is, remaining $\frac{n(n+1)}{2} - \frac{n(n-3)}{2} + 2 = 2n+2$ edges of K_n can be labeled from 1 to $2n+2$. Hence by removing $\frac{n(n-3)}{2} - 2$ interior edges of $K_n \odot K_1$, the resultant graph will be of the form C_n with n edges, n pendent vertices $(u_1, u_2, \ldots, u_n)$ connecting to the each vertex of C_n and two edges connecting any two non-adjacent vertices of C_n. Each vertex $v_1, v_2, \ldots, v_n$ of C_n is of degree 3. By labeling the edges of C_n with $1, 3, 5, \ldots 2n-1$, each edge incident on the pendant vertices is labeled with $2, 4, 6, \ldots, 2n$ and the edge connecting any two non-adjacent vertices of C_n is labeled with $2n+1, 2n+2$. As $2n + 1 \not\equiv 0(mod\ 3)$, the label incident on the vertices of the edge with label $2n + 2$ fails to be relatively prime.

A set is an edge-independent set of G, if any two edges in the set are not incident one another. The maximum cardinality of all edge-independent sets is the edge independence number of G, $\alpha'(G)$[16].

Theorem 5.2. *For a relatively prime edge labeled graph G having p vertices and q edges, then edge independence number $\alpha'(G) \geq \lfloor \frac{q}{2} \rfloor$*

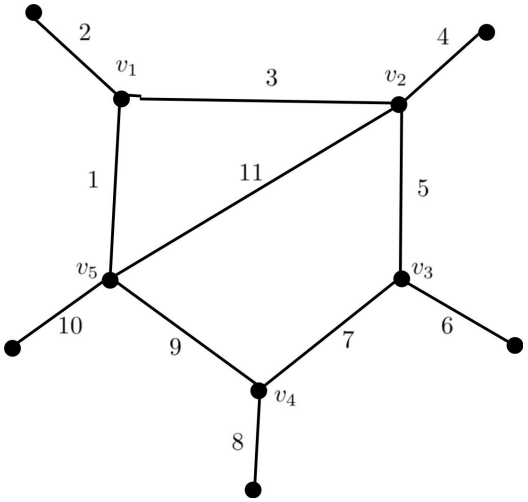

Figure 4: Relatively prime index of $K_5 \odot K_1$ = 4

Proof. Let G be a relatively prime edge labeled graph. The set of edges that are labeled with even numbers is an edge-independent set and its cardinality will be less than or equal to $\lfloor \frac{q}{2} \rfloor$. Hence the proof.

Theorem 5.3. *Let $K(G)$ be the clique graph of a graph G with q edges and if $\mid E(K(G)) \mid < \lfloor \frac{q}{2} \rfloor$, then G is not a relatively prime edge labeled graph.*

Proof. From the definition of clique graph, no two edges in the clique graph $K(G)$ can be independent. That is, $\alpha'(G) < \mid E(K(G) \mid < \lfloor \frac{q}{2} \rfloor$. Also by Theorem - 8, for a relatively prime edge labeled graph $\alpha'(G) \geq \lfloor \frac{q}{2} \rfloor$, which is a contradiction. Hence G is not a relatively prime edge labeled graph.

6 Conclusions

Prime labeling is a versatile technique that finds applications in various fields such as graph theory and optimization. One of its primary applications is in the domain of scheduling and resource allocation. For example, it can be used to determine the optimal way to assign workers to shifts to maximize the utilization of their skills. In network flow problems, prime labeling can help identify critical edges that should be included in any feasible flow solution to optimize the flow of a commodity through the network. In bioinformatics, prime labeling can be used to identify conserved regions in DNA sequences. By representing DNA sequences as graphs and labeling the vertices based on prime numbers, it can help identify regions of high conservation between different sequences.

In summary, prime labeling is a powerful technique that has wide-ranging applications in areas such as graph theory, optimization, and several other fields in science and engineering.
As a next, to employ the prime index of a graph in a practical way, we can think of a scenario where there are several workers who share certain skills. We can create a graph to represent this scenario, with each worker being a vertex and an edge connecting two vertices if the corresponding workers share a common skill. The label on each vertex indicates the shift hours for that worker. By using prime labeling, we can determine the best solution that covers the maximum number of skills required during a particular shift. If every worker shares at least one skill with every other worker, the prime index can help identify the shared skills that must be present to achieve the optimal solution.
The work presented here finds some new results in the idea of prime labeling of graphs. The main goal of this work is to convert a coprime-labeled graph into a prime graph by removing the fewest edges possible. Finally, a method for computing the prime index of the complete graph using an algorithm has been found. In the future, the prime index of a certain class of graphs can be found. Also, the importance of prime index will be discussed briefly.

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G-Supra and G-Infra space

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Abstract

The main idea of this paper is to generate supra and infra topologies from simple undirected graphs. For this, we have introduced two new operators namely supra and infra operators which are defined on the power set of the vertex set of a graph. Moreover, we have also proved that the supra operator satisfying Kuratowski's closure axiom will yield a topology. Further it was extended to develop the concept of connectedness and separation axioms on G-supra and G-infra spaces.

Keywords: supra operator, infra operator, G-supra topology, G-infra topology.
2020 AMS subject classifications: 05C40, 54C50, 54D10 [1]

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1 Introduction

Topology is a relatively new branch of mathematics; most of the research in topology has been done since 1900. Graph theory is another application oriented area of mathematics originated from konigsberg bridge problem. These two theories were connectively studied by many researchers. In 1983, Diesto and Gervacio [S.Diesto and S.Gervacio, 1983] proved that, a simple undirected graph generates a topology on its vertex set. In 1986, Guerrero and Gervacio [R.Guerrero and S.Gervacio, 1986] investigated further the graphs which generate the discrete and indiscrete topology.

In 1983, Mashhour et al.[A. S.Mashhour and F.H.Khedr, 1983] emerged a new idea of supra topological spaces by relaxing the finite intersection axiom of topological space.In 2015, Adel. M. Al-Odhari.[M.Adel and Al-Ddhari, 2015] introduced infra topological spaces by dropping arbitrary union axiom of topological space. Many researchers have induced topology from undirected graphs by generating base or subbase form the vertex set of graphs.

This made an intention to generate supra and infra topologies from graphs. In order to induce these types of topology, we have defined the supra and infra operators on the power set of the vertex set of a simple undirected graph. The properties of these operators were studied. Certain types of graphs namely null graph, complete graph and disconnected graph with complete components satisfies $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$ and $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$. So these graphs also generates topological space. Moreover, we have also proved that the disconnected graphs generates the disconnected G-supra and G-infra topologies. Using this operator we have made an attempt to define and charactereize separation axioms both in G-supra and G-infra spaces.

2 Preliminaries

In this section, we recalled some basic definitions related to graph theory and topological spaces.

Definition 2.1. *[Deo, 2016] A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \cdots\}$ called vertices, and another set $E = \{e_1, e_2, \cdots\}$, whose elements are called edges, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k.*

Definition 2.2. *[Deo, 2016] A graph with a finite number of vertices as well as a finite number of edges is called a finite graph; otherwise, it is an infinite graph.*

Definition 2.3. *[Deo, 2016] In a graph, an edge having the same vertex as both its end vertices is called a self-loop. Also more than one edge associated with a given pair of vertices are referred to as parallel edges. A graph that has neither self-loops nor parallel edges is called a simple graph.*

Definition 2.4. *[Deo, 2016] A vertex having no incident edge is called an isolated vertex. In a graph $G = (V, E)$, it is possible for the edge set E to be empty. Such a graph, without any edges, is called a null graph or totally disconnected graph.*

Definition 2.5. *[Deo, 2016] A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a circuit. A circuit is also called a cycle.*

Definition 2.6. *Munkres [2022] A topology on a set X is a collection τ of subsets of X having the following properties.(i) ϕ and X are in τ. (ii) The union of the elements of any subcollection of τ is in τ. (iii) The intersection of the elements of any finite subcollection of τ is in τ. A set X for which a topology τ has been specified is called a topological space. A subset U of X is an open set of X if U belongs to the collection τ.*

Definition 2.7. *[Munkres, 2022] The collection of all subsets of X is a topology on X, it is called the discrete topology. The collection consisting of X and ϕ only is also a topology on X, it is called the indiscrete topology.*

Definition 2.8. *[Munkres, 2022] A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space is said to be connected if there does not exist a separation of X. A space is totally disconnected if its only connected subspaces are one-point sets. If X has the discrete topology, then X is totally disconnected.*

Definition 2.9. *[A. S.Mashhour and F.H.Khedr, 1983] A subclass $\tau^* \subset \mathcal{P}(X)$ is called a supra topology on X if $X \in \tau^*$ and τ^* is closed under arbitrary union. (X, τ^*) is called a supra topological space. The members of τ^* are called the supra open sets.*

Definition 2.10. *[M.Adel and Al-Ddhari, 2015] An Infra-topological space on a non empty set X is a collection τ_{iX} of subsets of X such that (i) $\phi, X \in \tau_{iX}$. (ii) The intersection of the elements of any sub collection of τ_{iX} is in X.*

Throughout this paper a graph G means a finite simple undirected graph.

3 Supra operator on graphs

In this section, we have defined supra operator and G-supra topology on graphs.

Definition 3.1. *Let $G = (V, E)$ be a graph and let $\mathcal{P}(V)$ is the power set of V. The function $\mathcal{S} : \mathcal{P}(V) \to \mathcal{P}(V)$ defined by $\mathcal{S}(A) = A \cup \{v \in A^c / uv \in E, \forall u \in A\}$ is called the supra operator on $\mathcal{P}(V)$.*

Definition 3.2. *The G-supra topology μ_G on the vertex set V of a graph G is defined as the collection $\{\mathcal{S}(A) : A \subseteq V\}$. The elements of μ_G are called μ_G-open sets and its complements are called μ_G-closed sets.*

Theorem 3.1. ***(Properties of supra operator)*** *Let $G = (V, E)$ be a graph. Let A and B are subsets of V. Then,*

(i) $\mathcal{S}(\phi) = \phi$

(ii) $\mathcal{S}(V) = V$

(iii) $\mathcal{S}(A) \subseteq \mathcal{S}(B)$ *; when* $A \subseteq B$

(iv) $A \subseteq \mathcal{S}(A), \forall A \subseteq V$

(v) $\mathcal{S}(A \cup B) = \mathcal{S}(A) \cup \mathcal{S}(B)$

(v) $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A) \cap \mathcal{S}(B)$

Proof. (i) and (ii) it is obvious.
(iii) Let $A, B \subseteq V$ such that $A \subseteq B$.
Let $x \in \mathcal{S}(A)$. Then $x \in A \cup \{v \in A^c / uv \in E, \forall u \in A\}$
$\implies$ $x \in A$ or $x \in \{v \in A^c / uv \in E, \forall u \in A\}$
$\implies$ $x \in A$ or there exist $u \in A$ such that $uv \in E$
$\implies$ $x \in B$ or there exist $u \in B$ such that $uv \in E$, since $A \subseteq B$
$\implies$ $x \in B$ or $x \in \{v \in B^c / uv \in E, \forall u \in B\}$
$\implies$ $x \in B \cup \{v \in B^c / uv \in E, \forall u \in B\} \implies x \in \mathcal{S}(B)$
Hence $\mathcal{S}(A) \subseteq \mathcal{S}(B)$
(iv) It is clear from the definition.
(v) Let $x \in \mathcal{S}(A \cup B)$
$\iff$ $x \in (A \cup B) \cup \{v \in (A \cup B)^c / uv \in E, \forall u \in (A \cup B)\}$
$\iff$ $x \in (A \cup B)$ or $x \in \{v \in (A \cup B)^c / uv \in E, \forall u \in (A \cup B)\}$
$\iff$ $x \in A$ or $x \in B$ or there exist $u \in (A \cup B)$ such that $ux \in E$
$\iff$ $x \in A$ or $x \in B$ or there exist $u \in A$ such that $ux \in E$ or
there exist $u \in B$ such that $ux \in E$
$\iff$ $x \in A$ or $x \in B$ or $x \in \{v \in A^c / uv \in E, \forall u \in A\}$ or

$x \in \{v \in B^c / uv \in E, \forall u \in B\}$
$\iff x \in A$ or $x \in \{v \in A^c / uv \in E, \forall u \in A\}$
or $x \in B$ or $x \in \{v \in B^c / uv \in E, \forall u \in B\}$
$\iff x \in A \cup \{v \in A^c / uv \in E, \forall u \in A\}$ or $x \in B \cup \{v \in B^c / uv \in E, \forall u \in B\}$
$\iff x \in \mathcal{S}(A)$ or $x \in \mathcal{S}(B)$
$\iff x \in \mathcal{S}(A) \cup \mathcal{S}(B)$
Hence $\mathcal{S}(A \cup B) = \mathcal{S}(A) \cup \mathcal{S}(B)$.
(vi) $A \cap B \subseteq A$ and $A \cap B \subseteq B$
$\implies \mathcal{S}(A \cap B) \subseteq \mathcal{S}(A)$ and $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(B)$
$\implies \mathcal{S}(A \cap B) \subseteq \mathcal{S}(A) \cap \mathcal{S}(B)$ □

Theorem 3.2. *The collection $\mu_G = \{\mathcal{S}(A) : A \subseteq V\}$ forms a G-supra topology on the vertex set V of G.*

Proof. (i) Since $\mathcal{S}(\phi) = \phi$ and $\mathcal{S}(V) = V$. Hence $\phi, V \in \mu_G$.
(ii) Let $\{G_i\}$ be any collection of open sets in μ_G. By definition, each $G_i = \mathcal{S}(A_i)$, where $A_i \subseteq V$.
$\therefore \bigcup_{i \in \Lambda} G_i = \bigcup_{i \in \Lambda} [\mathcal{S}(A_i)] = \mathcal{S}(\bigcup_{i \in \Lambda} A_i) = \mathcal{S}(A_k)$ [Since $\bigcup_{i \in \Lambda} A_i = A_k$; $A_k \subseteq V$]
$= G_k \in \mu_G$. Hence arbitrary union of μ_G-open sets is μ_G-open. □

Example 3.1. *Consider the graph $G = (V, E)$ in figure 1 with vertex set $V = \{a, b, c, d\}$. $\mu_G = \{\phi, \{a, b\}, \{a, c, d\}, V\}$ is a G-supra topology on the vertex set V of G.*

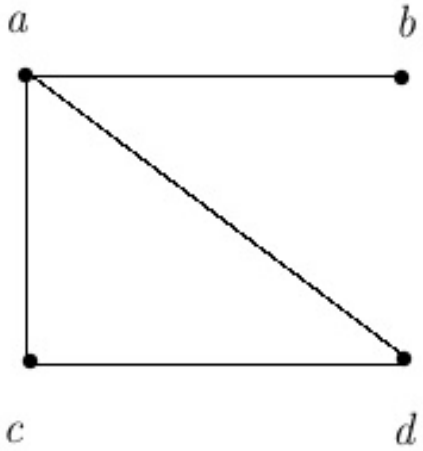

Figure 1: Graph for G-supra space

Proposition 3.1. *For a complete graph $G = (V, E)$, $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$, $\forall A \subseteq V$.*

Proof. Let $G = (V, E)$ be a complete graph. Since G is complete, each vertex is adjacent to every other vertices. Let $v \in V$. Then $\mathcal{S}(\{v\}) = V, \forall v \in V$. Then $\mathcal{S}(A) = V$, for every non empty subset A of V, which implies $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(V) = V = \mathcal{S}(A)$. Also $\mathcal{S}(\phi) = \phi \implies \mathcal{S}(\mathcal{S}(\phi)) = \mathcal{S}(\phi)$. Hence $\mathcal{S}(\mathcal{S}(A))$ = $\mathcal{S}(A), \forall A \subseteq V$. □

Proposition 3.2. *For a null graph $G = (V, E)$, $\mathcal{S}(A) = A, \forall A \subseteq V$.*

Proof. Let $G = (V, E)$ be a null graph with 'n' vertices. Since G is a null graph, each vertex is not adjacent to any other vertices. Let $v \in V$. Then $\mathcal{S}(\{v\}) = \{v\}, \forall v \in V$. Let $u, v \in V$. Then $\mathcal{S}(\{u, v\}) = \mathcal{S}(\{u\}) \cup \mathcal{S}(\{v\}) = \{u, v\}$. Similarly $\mathcal{S}(\{v_1, v_2, \cdots, v_n\}) = \mathcal{S}(\{v_1\}) \cup \mathcal{S}(\{v_2\}) \cup \cdots \cup \mathcal{S}(\{v_n\}) = \{v_1, v_2, \cdots, v_n\}$. Also $\mathcal{S}(\phi) = \phi$. Hence $\mathcal{S}(A) = A, \forall A \subseteq V$. □

Remark 3.1. *From the above proposition, for a null graph $G = (V, E)$, $\mathcal{S}(\mathcal{S}(A))$ = $\mathcal{S}(A), \forall A \subseteq V$.*

Remark 3.2. *The following example shows that the property $\mathcal{S}(\mathcal{S}(A))$ = $\mathcal{S}(A)$, $\forall A \subseteq V$ does not hold true for all graphs G with vertex set V.*

Example 3.2. *Consider the graph G in figure 1, with vertex set $V = \{a, b, c, d\}$. Here $\mathcal{S}(\{b\}) = \{a, b\}$ and $\mathcal{S}(\{a, b\}) = V$. Hence $\mathcal{S}(\mathcal{S}(A)) \neq \mathcal{S}(A), \forall A \subseteq V$.*

Theorem 3.3. *A graph $G = (V, E)$ is complete if and only if $\mu_G = \{\phi, V\}$.*

Proof. Let $G = (V, E)$ be a complete graph. Then each vertex is adjacent to every other vertices. Also $\mathcal{S}(\phi) = \phi$ and $\mathcal{S}(A) = V$, for every non empty subset A of V. Hence ϕ and V are the only μ_G-open sets. Therefore a complete graph G generates an indiscrete topology. Conversely, let $\mu_G = \{\phi, V\}$. Then for each $A \subseteq V, \mathcal{S}(A) = \phi$ or $\mathcal{S}(A) = V$. When $A = \phi$, $\mathcal{S}(\phi) = \phi$. When $A = \{u\}, \mathcal{S}(\{u\}) = \{u\} \cup \{v \in A^c / uv \in E\} \neq \phi$. Therefore the only possibility is $\mathcal{S}(\{u\}) = V$, which means that u is adjacent to every other vertices in G. Since $u \in V$ is arbitrary, $\mathcal{S}(\{x\}) = V, \forall x \in V$. Hence each vertex is adjacent to every other vertices in G. Hence G is complete. □

Theorem 3.4. *$G = (V, E)$ is a null graph if and only if μ_G is discrete topology.*

Proof. Let $G = (V, E)$ be a null graph. By proposition 3.2, $\mathcal{S}(A) = A$, for every subset A of V. Hence every subset of V is a μ_G-open set. Therefore a null graph G generates discrete topology. Conversely, let μ_G be discrete topology. Then for each $x \in V, \{x\} \in \mu_G$. From the definition of $\mathcal{S}(A)$, x is not adjacent to any other vertex in G. Hence G is a null graph. □

Theorem 3.5. *Let $G = (V, E)$ be a disconnected graph with all its components are complete, then $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$, where A is a subset of V.*

Proof. Let $G_1, G_2, \cdots, G_n$ be the 'n' components of a disconnected graph G. Let V_i be the vertex set of G_i. Since each G_i is complete, $\mathcal{S}(\mathcal{S}(A_i)) = \mathcal{S}(A_i)$, for every subset A_i of V_i. Let A be a subset of V such that its elements belong to the components $G_1, G_2, \cdots, G_m$ where $m \leq n$. Now $\mathcal{S}(A) = A \cup \{v \in A^c / uv \in E, \forall u \in A\} = \mathcal{S}(V_1) \cup \mathcal{S}(V_2) \cup \cdots \cup \mathcal{S}(V_m)$(Since $u \in A \implies u \in V_i$, for anyone of i = 1, 2, $\cdots$,n) = $\mathcal{S}(\mathcal{S}(V_1)) \cup \mathcal{S}(\mathcal{S}(V_2)) \cup \cdots \cup \mathcal{S}(\mathcal{S}(V_m))$ = $\mathcal{S}(\mathcal{S}(V_1) \cup \mathcal{S}(V_2) \cup \cdots \cup \mathcal{S}(V_m))$ = $\mathcal{S}(\mathcal{S}(A))$. Hence for every subset A of V, $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$. □

Theorem 3.6. *Let $G = (V, E)$ be a disconnected graph with all its components are complete then the collection $\mu_G = \{\mathcal{S}(A) : A \subseteq V\}$ forms a topology τ_G on the vertex set V of G.(i.e) $\mu_G = \tau_G$.*

Proof. Let $G = (V, E)$ be a disconnected graph 'n' complete components. Let V_i be the vertex set of the component G_i, $i = 1, 2, \cdots, n$. By theorem 3.2, $\phi, V \in \tau_G$ and arbitrary union of μ_G-open sets is μ_G-open. Let $x \in V$, then $x \in V_i$, for some $i = 1, 2, \cdots, n$. Now $\mathcal{S}(\{x\}) = V_i, i = 1, 2, \cdots, n$. For any subsets A and B of V, we have $\mathcal{S}(A) \cup \mathcal{S}(B) = \mathcal{S}(A \cup B)$. Therefore for any non empty subset A of V, $\mathcal{S}(A) = V_i$ or $\mathcal{S}(A) = \bigcup_{i \in \lambda} V_i$, $\lambda = \{1, 2, \cdots, n\}$. Suppose $\mathcal{S}(A)$ and $\mathcal{S}(B)$ are disjoint, then $\mathcal{S}(A) \cap \mathcal{S}(B) = \phi \in \mu_G$. Also when $\mathcal{S}(A) \subset \mathcal{S}(B)$, then $\mathcal{S}(A) \cap \mathcal{S}(B) = \mathcal{S}(A) \in \mu_G$ and when $\mathcal{S}(A) \cap \mathcal{S}(B) \neq \phi$ then $\mathcal{S}(A) \cap \mathcal{S}(B) = (\bigcup_{i \in \lambda} V_i) \cap (\bigcup_{j \in \lambda} V_j) = \bigcup_{k \in \lambda} V_j, \in \mu_G$. Therefore the finite intersection of elements of μ_G is in μ_G. Hence μ_G is a topology on the vertex set V of G. Hence $\mu_G = \tau_G$. □

Remark 3.3. *For null graph, complete graph and disconnected graph with complete components, the collection $\{\mathcal{S}(A) : A \subseteq V\}$ forms a topology. Also, for these type of graphs the supra operator obeys Kuratowski's closure axioms.*

4 Connectedness on G-supra topology

In this section, the concept of connectedness on graphs and G-supra space were compared.

Theorem 4.1. *If the graph G is disconnected then the G-supra topology μ_G generated by G is also disconnected.*

Proof. Let $G = (V, E)$ be a disconnected graph with 'n' components namely $G_1, G_2, \cdots, G_n$. Let $V_1, V_2, \cdots, V_n$ be the vertex set of $G_1, G_2, \cdots, G_n$ respectively. Clearly $V_1, V_2, \cdots, V_n$ are subsets of V and $V_1 \cup V_2 \cup \cdots \cup V_n = V$ and $V_1 \cap V_2 \cap \cdots \cap V_n = \phi$. Since each components G_i are connected, $\mathcal{S}(V_i) = V_i$, for $i = 1, 2, \cdots, n$. Also each V_i is a μ_G-open set, there exist separation of V. Hence (μ_G, V) is disconnected. □

Remark 4.1. *If the graph G is connected then the G-supra topology μ_G generated by G need not be connected, which can be illustrated by the following example.*

Example 4.1. *Consider the connected graph $G = (V, E)$ in figure.2 with vertex set $V = \{a, b, c, d\}$. The G-supra topology generated by the above graph G is*

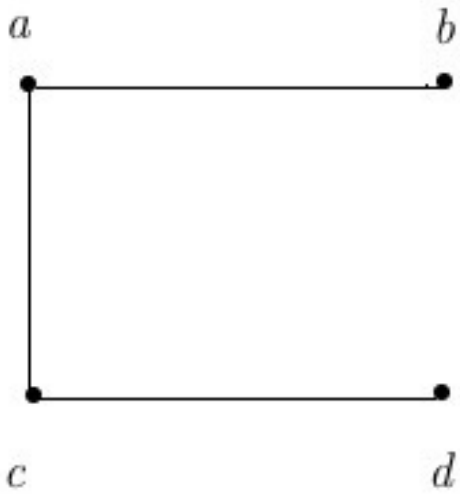

Figure 2: Graph for disconnected G-supra space

$\mu_G = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, V\}$*, which is disconnected.*

Proposition 4.1. *The G-supra topology μ_G generated by a totally disconnected graph is also totally disconnected.*

Proof. Let $G = (V, E)$ be a totally disconnected graph. Then $\mathcal{S}(A) = A, \forall A \subseteq V$. Hence the supra topology μ_G generated by G is discrete. Since every discrete topology is totally disconnected, μ_G is totally disconnected.□

5 Separation axioms in G-supra space

In this section, G-supra T_0-space, T_1-space and T_2-space were defined and illustrated with examples.

Definition 5.1. *Let μ_G be a G-supra topology generated by the vertex set V of a graph G. (V, μ_G) is called G-supra T_0-space if for every pair of distinct vertices there exist a μ_G-open set containing one but not other.*

Definition 5.2. *Let μ_G be a G-supra topology generated by the vertex set V of a graph G. (V, μ_G) is called G-supra T_1-space if for every pair of distinct vertices $u, v \in V$ there exist two μ_G-open sets X and Y such that $u \in X$ but $v \notin X$ and $v \in Y$ but $u \notin Y$.*

Example 5.1. *Let G be a graph in figure 3 with vertex set $V = \{a, b, c\}$. The G-*

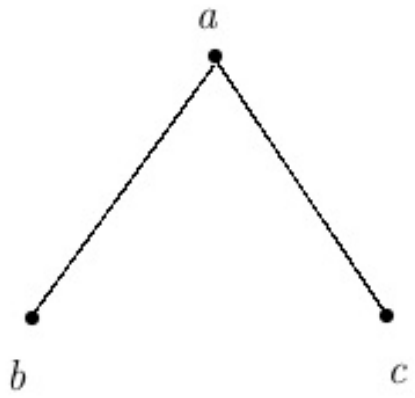

Figure 3: Graph for G-supra T_0-space

supra topology μ_G on the vertex set V of G is $\mu_G = \{\phi, \{a, b\}, \{a, c\}, V\}$. Since for each pair of distinct vertices, there exist a μ_G-open set containing one but not other. Hence (V, μ_G) is a G-supra T_0-space.

Example 5.2. *Let G be a graph in figure 4 with vertex set $V = \{a, b, c, d\}$. The*

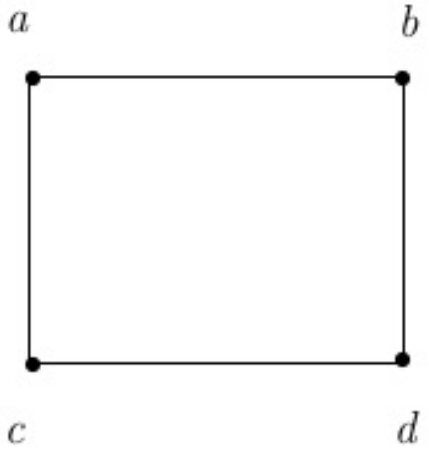

Figure 4: Graph for G-supra T_1-space

G-supra topology μ_G on the vertex set V of G is $\mu_G = \{\phi, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, V\}$. Since for each pair of distinct vertices $u, v \in V$, there exist two μ_G-open sets X and Y such that $u \in X$ but $v \notin X$ and $v \in Y$ but $u \notin Y$. Hence (V, μ_G) is a G-supra T_1-space.

Definition 5.3. *Let μ_G be a G-supra topology generated by the vertex set V of a graph G. (V, μ_G) is called G-supra T_2-space if for every pair of distinct vertices $u, v \in V$ there exist disjoint μ_G-open sets X and Y such that $u \in X$ and $v \in Y$.*

Example 5.3. *Let G be a graph in figure 5, with vertex set $V = \{a, b, c, d, e, f\}$.*

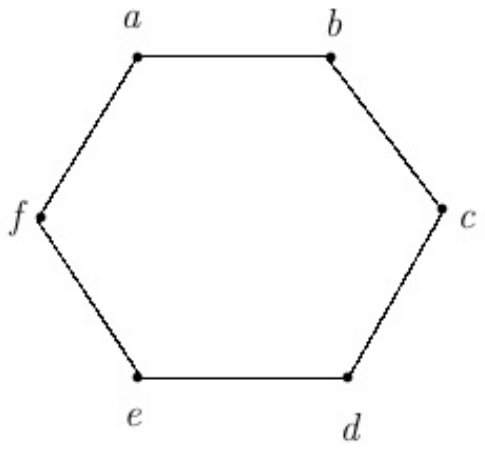

Figure 5: Graph for G-supra T_2-space

The G-supra topology μ_G on the vertex set V of G is $\mu_G = \{\phi, \{a, b, c\}, \{a, b, f\}, \{a, e, f\}, \{b, c, d\}, \{c, d, e\}, \{d, e, f\}, \{a, b, c, d\}, \{a, b, c, f\}, \{a, b, e, f\}, \{a, d, e, f\}, \{b, c, d, e\}, \{c, d, e, f\}, \{a, b, c, d, e\}, \{a, b, c, d, f\}, \{a, b, c, e, f\}, \{a, b, d, e, f\}, \{a, c, d, e, f\}, \{b, c, d, e, f\}, V\}$. Since for each pair of distinct vertices $u, v \in V$ there exist disjoint μ_G-open sets X and Y such that $u \in X$ and $v \in Y$. Hence (V, μ_G) is a G-supra T_2-space.

Theorem 5.1. *Let $G = C_n$, $n \geq 6$ be a cycle graph, then (V, μ_G) is a G-supra T_2-space.*

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$, $n \geq 6$ be the vertex set of G. Then $\mathcal{S}(\{v_i\}) = \{v_{i-1}, v_i, v_{i+1}\}$, for i = 1, 2, $\cdots$, n. Since G is a cycle, for $i = 1, v_{i-1} = v_n$ and for $i = n, v_{i+1} = v_1, v_{i+2} = v_2$ and so on. Now $\mathcal{S}(\{v_i\}) = \{v_{i-1}, v_i, v_{i+1}\}$ and $\mathcal{S}(\{v_{i+3}\}) = \{v_{i+2}, v_{i+3}, v_{i+4}\}$ are disjoint μ_G-open sets seperates 9 pairs of dictinct vertices of G. Hence there exist disjoint μ_G-open sets X and Y for every pair of distinct vertices v_i and v_j such that $v_i \in X$ and $v_j \in Y$. Hence (V, μ_G) is a G-supra T_2-space. □

Remark 5.1. *For C_3, C_4 and C_5, there does not exist disjoint μ_G-open sets for any pair of distinct vertices.*

6 Infra operator on graphs

In this section, G-infra topology was induced by the infra operator and its properties were studied.

Definition 6.1. *Let $G = (V, E)$ be a graph and let $\mathcal{P}(V)$ is the power set of V. The function $\mathcal{I} : \mathcal{P}(V) \to \mathcal{P}(V)$ defined by $\mathcal{I}(A) = [\mathcal{S}(A^c)]^c$is called the infra operator on $\mathcal{P}(V)$.*

Definition 6.2. *The G-infra topology δ_G on the vertex set V of a graph G is defined as the collection $\{\mathcal{I}(A) : A \subseteq V\}$. The elements of δ_G are called δ_G-open sets and its complemets are called δ_G-closed sets.*

Theorem 6.1. ***(Properties of infra operator)*** *Let $G = (V, E)$ be a graph. Let A and B are subsets of V. Then,*

(i) $\mathcal{I}(\phi) = \phi$

(ii) $\mathcal{I}(V) = V$

(iii) $\mathcal{I}(A) \subseteq \mathcal{I}(B)$ *; when* $A \subseteq B$

(iv) $\mathcal{I}(A) \subseteq A, \forall A \subseteq V$

(v) $\mathcal{I}(A \cap B) = \mathcal{I}(A) \cap \mathcal{I}(B)$

(vi) $\mathcal{I}(A) \cup \mathcal{I}(B) \subseteq \mathcal{I}(A \cup B)$

Proof. (i) $\mathcal{I}(\phi) = [\mathcal{S}(\phi^c)]^c = [\mathcal{S}(V)]^c = V^c = \phi$.
(ii) $\mathcal{I}(V) = [\mathcal{S}(V^c)]^c = [\mathcal{S}(\phi)]^c = \phi^c = V$.
(iii) Let $A, B \subseteq V$ such that $A \subseteq B$.
Let $x \in \mathcal{I}(A)$. Then $x \in [\mathcal{S}(A^c)]^c$. $\implies x \notin \mathcal{S}(A^c)$.
Since $A \subseteq B, B^c \subseteq A^c \implies \mathcal{S}(B^c) \subseteq \mathcal{S}(A^c)$.
$\therefore x \notin \mathcal{S}(B^c) \implies x \in [\mathcal{S}(B^c)]^c \implies x \in \mathcal{I}(B)$. Hence $\mathcal{S}(A) \subseteq \mathcal{I}(B)$.
(iv) Let $x \in \mathcal{I}(A)$. Then $x \in [\mathcal{S}(A^c)]^c \implies x \notin \mathcal{S}(A^c) \implies x \notin A^c$
$\implies x \in A$. Hence $\mathcal{I}(A) \subseteq A$.
(v) $\mathcal{I}(A \cap B) = [\mathcal{S}(A \cap B)^c]^c = [\mathcal{S}(A^c \cup B^c)]^c = [\mathcal{S}(A^c) \cup \mathcal{S}(B^c)]^c = [\mathcal{S}(A^c)]^c \cap [\mathcal{S}(B^c)]^c = \mathcal{I}(A) \cap \mathcal{I}(B)$.
(vi) $A \subseteq A \cup B$ and $B \subseteq A \cup B$
$\implies \mathcal{I}(A) \subseteq \mathcal{I}(A \cup B)$ and $\mathcal{I}(B) \subseteq \mathcal{I}(A \cup B)$
$\implies \mathcal{I}(A) \cup \mathcal{I}(B) \subseteq \mathcal{I}(A \cup B)$. □

Theorem 6.2. *The collection $\delta_G = \{\mathcal{I}(A) : A \subseteq V\}$ forms a G-infra topology on the vertex set V of G.*

Proof. (i) Since $\mathcal{I}(\phi) = \phi$ and $\mathcal{I}(V) = V$. Hence $\phi, V \in \delta_G$.
(ii) Let $\{G_i\}$ be a collection of open sets in δ_G. By definition each G_i = $\mathcal{I}(A_i)$, where $A_i \subseteq V$.
$\therefore \bigcap_{i\in\Lambda} G_i = \bigcap_{i\in\Lambda} [\mathcal{I}(A_i)] = \mathcal{I}(\bigcap_{i\in\Lambda} A_i) = \mathcal{I}(A_k)$ [Since $\bigcap_{i\in\Lambda} A_i = A_k$; $A_k \subseteq V$]
$= G_k \in \mu_G$. Hence intersection of δ_G-open sets is δ_G-open. □

Example 6.1. *Consider the graph $G = (V, E)$ in figure 1.*
$\delta_G = \{\phi, \{b\}, \{c, d\}, V\}$ is a G-infra topology on the vertex set V of G.

Proposition 6.1. *For a complete graph $G = (V, E)$, $\mathcal{I}(\mathcal{I}(A))$ = $\mathcal{I}(A)$, $\forall A \subseteq V$.*

Proof. $\mathcal{I}(\phi) = \phi \implies \mathcal{I}(\mathcal{I}(\phi)) = \mathcal{I}(\phi)$. Since G is a complete graph, $\mathcal{S}(A) = V$, $\forall A \subset V$. Let A be a proper subset of V. Then $\mathcal{I}(A) = [\mathcal{S}(A^c)]^c = V^c = \phi \implies \mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(\phi) = \phi = \mathcal{I}(A)$. Also $\mathcal{I}(V) = V \implies \mathcal{I}(\mathcal{I}(V)) = \mathcal{I}(V)$. Hence $\mathcal{I}(\mathcal{I}(A))$ = $\mathcal{I}(A)$, $\forall A \subseteq V$. □

Proposition 6.2. *For a null graph $G = (V, E)$, $\mathcal{I}(A) = A$, $\forall A \subseteq V$.*

Proof. Let $G = (V, E)$ be a null graph with 'n' vertices. Since $\mathcal{S}(A) = A, \forall A \subseteq V$. Now $\mathcal{I}(A) = [\mathcal{S}(A^c)]^c = [A^c]^c$ = A, $\forall A \subseteq V$. □

Remark 6.1. *From the above proposition, for a null graph $G = (V, E)$, $\mathcal{I}(\mathcal{I}(A))$ = $\mathcal{I}(A)$, $\forall A \subseteq V$.*

Remark 6.2. *The following example shows that the property $\mathcal{I}(\mathcal{I}(A))$ = $\mathcal{I}(A)$, $\forall A \subseteq V$ does not hold for all graphs G with vertex set V.*

Example 6.2. *Consider the graph G in figure 1, with vertex set $V = \{a, b, c, d\}$. Here $\mathcal{I}(\{a, c, d\}) = \{c, d\}$ and $\mathcal{I}(\{c, d\}) = \phi$. Hence $\mathcal{I}(\mathcal{I}(A) \neq \mathcal{I}(A), \forall A \subseteq V$.*

Theorem 6.3. *Let $G = (V, E)$ be a disconnected graph with all its components are complete, then $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$, where A is a subset of V.*

Proof. Let $G_1, G_2, \cdots, G_n$ be the 'n' components of a disconnected graph G. Let V_i be the vertex set of G_i. Since each G_i is complete, $\mathcal{I}(\mathcal{I}(A_i)) = \mathcal{I}(A_i)$, for every subset A_i of V_i. Let A be a sub set of V.
Case(i). Suppose $A = V_1 \cup V_2 \cup \cdots \cup V_m, m \leq n$. Then $\mathcal{I}(A) = [\mathcal{S}(V_1 \cup V_2 \cup \cdots \cup V_m)^c]^c = [\mathcal{S}(V_{m+1} \cup V_{m+2} \cup \cdots \cup V_n)]^c = [\mathcal{S}(V_{m+1}) \cup \mathcal{S}(V_{m+1}) \cup \cdots \cup \mathcal{S}(V_n)]^c = [V_{m+1} \cup V_{m+2} \cup \cdots \cup V_n]^c = V_1 \cup V_2 \cup \cdots \cup V_m = A$. $\therefore \mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$.
Case(ii). Suppose $A = U_1 \cup U_2 \cup \cdots \cup U_m$, where $U_j \subseteq V_j$, $j = 1, 2, \cdots, m$ and

$m \leq n$. Then $\mathcal{I}(A) = \mathcal{I}(U_1 \cup U_2 \cup \cdots \cup U_m) = [\mathcal{S}(U_1 \cup U_2 \cup \cdots \cup U_m)^c]^c$
= $[\mathcal{S}\{(V_1 - U_1) \cup (V_2 - U_2) \cup \cdots (V_m - U_m) \cup V_{m+1} \cup V_{m+2} \cup \cdots \cup V_n\}]^c$
=$[\mathcal{S}(V_1 - U_1) \cup \mathcal{S}(V_2 - U_2) \cup \cdots \mathcal{S}(V_m - U_m) \cup \mathcal{S}(V_{m+1}) \cup \mathcal{S}(V_{m+2}) \cup \cdots \cup \mathcal{S}(V_n)]^c$
=$[\mathcal{S}(W_1) \cup \mathcal{S}(W_2) \cup \cdots \mathcal{S}(W_m) \cup \mathcal{S}(V_{m+1}) \cup \mathcal{S}(V_{m+2}) \cup \cdots \cup \mathcal{S}(V_n)]^c$
$[where W_j = V_j - U_j; j = 1, 2, \cdots, m$. Also $W_j \subset V_j]$
= $[V_1 \cup V_2 \cup \cdots \cup V_m \cup V_{m+1} \cup \cdots \cup V_n]^c = V^c = \phi$.
Hence $\mathcal{I}(A) = \phi \implies \mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(\phi) = \phi = \mathcal{I}(A)$. $\therefore \mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$.
Case(iii). Suppose $A = V_1 \cup V_2 \cup \cdots \cup U_{k+1} \cup U_{k+2} \cup \cdots \cup U_m$,
where $k < m \leq n$ and $U_j \subset V_j, j = k+1, k+2, \cdots, m$. Then
$\mathcal{I}(A) = \mathcal{I}(V_1 \cup V_2 \cup \cdots \cup U_{k+1} \cup U_{k+2} \cup \cdots \cup U_m) = [\mathcal{S}(V_1 \cup V_2 \cup \cdots \cup U_{k+1} \cup U_{k+2} \cup \cdots \cup U_m)^c]^c = [\mathcal{S}\{(V_{k+1} - U_{k+1}) \cup \mathcal{S}\{(V_{k+2} - U_{k+2}) \cdots \cup (V_m - U_m) \cup V_{m+1} \cup V_{m+2} \cup \cdots \cup V_n\}]^c = [\mathcal{S}\{W_{k+1} \cup W_{k+2} \cup \cdots \cup W_m \cup V_{m+1} \cup V_{m+2} \cup \cdots \cup V_n\}]^c$, $[where\ W_j = V_j - U_j, j = k+1, k+2, \cdots, m$. Since $W_j \subset V_j \implies \mathcal{S}(W_j) = V_j]$.Now
$\mathcal{I}(A) = [V_{k+1} \cup V_{k+2} \cup \cdots \cup V_m \cup V_{m+1} \cup \cdots \cup V_n]^c = V_1 \cup V_2 \cup \cdots \cup V_k$.
Hence $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(V_1 \cup V_2 \cup \cdots \cup V_k) = [\mathcal{S}(V_1 \cup V_2 \cup \cdots \cup V_k)^c]^c = [\mathcal{S}(V_{k+1} \cup V_{k+2} \cup \cdots \cup V_m \cup V_{m+1} \cup \cdots \cup V_n)]^c = [\mathcal{S}(V_{k+1}) \cup \mathcal{S}(V_{k+1}) \cup \cdots \cup \mathcal{S}(V_m) \cup \mathcal{S}(V_{m+1}) \cup \cdots \mathcal{S}(V_n)]^c = [V_{k+1} \cup V_{k+2} \cup \cdots \cup V_m \cup V_{m+1} \cup \cdots V_n]^c = V_1 \cup V_2 \cup \cdots \cup V_k = \mathcal{I}(A)$.
$\therefore \mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$. □

Theorem 6.4. *Let $G = (V, E)$ be a a graph and let $A \subseteq V$, then the collections $\mu_G = \{\mathcal{S}(A)\}$ and $\delta_G = \{\mathcal{I}(A)\}$ are complement to each other.*

Proof. Let A be the subset of the vertex set V of G. Put $A^c = B$. Since $A, B \subseteq V$, $\mathcal{S}(A), \mathcal{S}(B) \in \mu_G$. Now, for $\mathcal{S}(B) \in \mu_G$, we have $[\mathcal{S}(B)]^c = [\mathcal{S}(A^c)]^c = \mathcal{I}(A) \in \delta_G$. Also for $\mathcal{I}(B) \in \delta_G$, we have $[\mathcal{I}(B)]^c$ = $[[\mathcal{S}(B^c)]^c]^c = [[\mathcal{S}(A)]^c]^c = \mathcal{S}(A) \in \mu_G$. Hence every μ_G-open set is the complement of δ_G-open set and vice-versa. □

Theorem 6.5. *Let $G = (V, E)$ be a disconnected graph with all its components are complete then the collection $\delta_G = \{\mathcal{I}(A) : A \subseteq V\}$ forms a topology τ_G on the vertex set V of G. (i.e) $\delta_G = \tau_G$.*

Proof. By theorem 6.2, $\phi, V \in \delta_G$ and intersection of finite number of δ_G-open sets is δ_G-open. Now by theorem 6.4, every δ_G-open set is μ_G-closed set. Also by theorem 3.6, finite intersection of μ_G-open sets is μ_G-open. This gives arbitrary union of δ_G-open sets is δ_G-open. □

Theorem 6.6. *A graph $G = (V, E)$ is complete if and only if $\delta_G = \{\phi, V\}$.*

Proof. Clearly, by theorem 3.3 and theorem 6.4. □

Theorem 6.7. *$G = (V, E)$ is a null graph if and only if δ_G is discrete topology.*

Proof. Clearly, by theorem 3.4 and theorem 6.4. □

Remark 6.3. *For null graph, complete graph and disconnected graph with complete components, the collection* $\{\mathcal{I}(A) : A \subseteq V\}$ *forms a topology. Also, for these type of graphs the infra operator obeys the axioms of interior operator.*

7 Connectedness on G-infra space

In this section, we have compared the concept of connectedness on graphs and G-supra space.

Theorem 7.1. *If the graph* G *is disconnected then the* G*-infra topology* δ_G *generated by* G *is also disconnected.*

Proof. Let $G = (V, E)$ be a disconnected graph with 'n' components namely $G_1, G_2, \cdots, G_n$. Let $V_1, V_2, \cdots, V_n$ be the vertex set of $G_1, G_2, \cdots, G_n$ respectively. Clearly $V_1, V_2, \cdots, V_n$ are subsets of V and $V_1 \cup V_2 \cup \cdots \cup V_n = V$ and $V_1 \cap V_2 \cap \cdots \cap V_n = \phi$.–(*). Now $\mathcal{I}(V_i) = [\mathcal{S}(V_i)^c]^c = [\mathcal{S}(V_1 \cup V_2 \cup \cdots \cup V_{i-1} \cup V_{i+1} \cup \cdots \cup V_n)]^c$ = $[\mathcal{S}(V_1) \cup \mathcal{S}(V_2) \cup \cdots \cup \mathcal{S}(V_{i-1}) \cup \mathcal{S}(V_{i+1}) \cdots \cup \mathcal{S}(V_n)]^c = [V_1 \cup V_2 \cup \cdots \cup V_{i-1} \cup V_{i+1} \cup \cdots \cup V_n]^c = V_i$. Since each $\mathcal{I}(V_i) = V_i$, for $i = 1, 2, \cdots, n$ is a $\delta_G - openset$, (δ_G, V) is disconnected (by*). □

Remark 7.1. *If the graph* G *is connected then the* G*-infra topology* δ_G *generated by* G *need not be connected, which can be illustrated by the following example.*

Example 7.1. *Consider the connected graph* $G = (V, E)$ *in figure.2 with vertex set* $V = \{a, b, c, d\}$. *The* G*-infra topology generated* G *is* $\delta_G = \{\phi, \{b\}, \{d\}, \{a, b\}, \{c, d\}, V\}$, *which is disconnected.*

Proposition 7.1. *The* G*-infra topology* δ_G *generated by a totally didconnected graph is also totally disconnected.*

Proof. Let $G = (V, E)$ be a totally disconnected graph. Then $\mathcal{I}(A) = A, \forall A \subseteq V$. Hence the infra topology δ_G generated by G is a discrete. Since every discrete topology is totally disconnected, δ_G is totally disconnected. □

8 Separation axioms in G-infra space

In this section, we have defined and discussed the separation axioms in G-infra space.

Definition 8.1. *Let* δ_G *be a* G*-infra topology generated by the vertex set* V *of a graph* G. (V, δ_G) *is called* G*-infra* T_0*-space if for every pair of distinct vertices there exist a* δ_G*-open set containing one but not other.*

Example 8.1. *Consider the graph G in figure 2. The G-infra topology δ_G on the vertex set V of G is $\delta_G = \{\phi, \{b\}, \{d\}, \{a, b\}, \{c, d\}, V\}$. Since for each pair of distinct vertices, there exist a δ_G-open set containing one but not other. Hence (V, δ_G) is a G-infra T_0-space.*

Definition 8.2. *Let δ_G be a G-infra topology generated by the vertex set V of a graph G. (V, δ_G) is called G-infra T_1-space if for every pair of distinct vertices $u, v \in V$ there exist two δ_G-open sets X and Y such that $u \in X$ but $v \notin X$ and $v \in Y$ but $u \notin Y$.*

Example 8.2. *Consider the graph G in figure 6, with vertex set $V = \{a, b, c, d\}$. The G-infra topology δ_G on the vertex set V of G is $\delta_G\{\phi, \{b\}, \{c\}, \{d\}, \{b, d\},$*

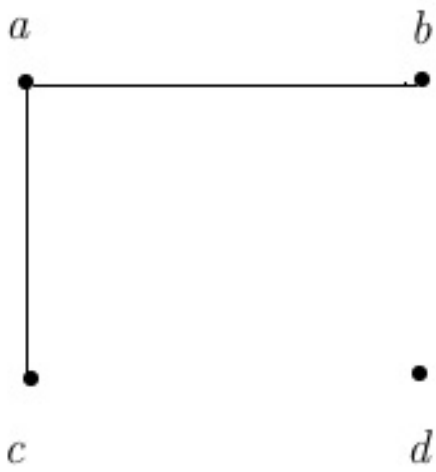

Figure 6: Graph generates G-infra T_1-space

$\{c, d\}, \{a, b, c\}, V\}$. Since for each pair of distinct vertices $u, v \in V$, there exist two δ_G-open sets X and Y such that $u \in X$ but $v \notin X$ and $v \in Y$ but $u \notin Y$. Hence (V, δ_G) is a G-infra T_1-space.

Definition 8.3. *Let δ_G be a G-infra topology generated by the vertex set V of a graph G. (V, δ_G) is called G-infra T_2-space if for every pair of distinct vertices $u, v \in V$ there exist disjoint δ_G-open sets X and Y such that $u \in X$ and $v \in Y$.*

Example 8.3. *Consider the graph G in figure 4. The G-infra topology δ_G on the vertex set V of G is $\delta_G = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, V\}$. Since for each pair of distinct vertices $u, v \in V$ there exist disjoint δ_G-open sets X and Y such that $u \in X$ and $v \in Y$. Hence (V, δ_G) is a G-infra T_2-space.*

Theorem 8.1. *Let $G = C_n$, $n \geq 4$ be a cycle graph, then (V, μ_G) is a G-infra T_2-space.*

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$, $n \geq 4$ be the vertex set of G. Let $A = \{v_n, v_1, v_2\}$. Then $\mathcal{I}(A) = [\mathcal{S}(A^c)]^c = [\mathcal{S}(\{v_3, v_4, \cdots, v_{n-1}\})]^c = [\{v_2, v_3, \cdots, v_n\}]^c = \{v_1\}$. For $i = 2, 3, \cdots, (n-1)$, $\mathcal{I}(\{v_{i-1}, v_i, v_{1+1}\}) = [\mathcal{S}(\{v_{i-1}, v_i, v_{1+1}\}^c)]^c = [\mathcal{S}(\{v_1, v_2, \cdots, v_{i-2}, v_{i+2}, \cdots, v_n\})]^c = [\{v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n\}]^c = \{v_i\}$. When $A = \{v_{n-1}, v_n, v_{n+1}\}$, $\mathcal{I}(A) = [\mathcal{S}(A^c)]^c = [\mathcal{S}(\{v_2, v_3, \cdots, v_{n-2}\})]^c = [\{v_1, v_2, \cdots, v_{n-1}\}]^c = \{v_n\}$. Hence each $\{v_i\}$, for $i = 1, 2, \cdots, n$ is a δ_G-open set. Therefore for every pair of distinct vertice $v_i, v_j \in V$ there exist disjoint δ_G-open sets $\{v_i\}$ and $\{v_j\}$ containing them respectively. Hence (V, μ_G) is a G-infra T_2-space. □

9 Conclusion

By employing supra and infra operator we have generated G-supra and G-infra topologies. Some properties of this operators were studied. Also we have proved that supra and infra operators obeys the criterion Kuratowski's closure axioms and interior operator respectively in some special types of graph namely null graph, complete graph and disconnected graph with all its components are complete. Also we have proved that the G-supra and G-infra topologies are complement to each other. In addition we have discussed the connectedness and separation axioms in G-supra and G-infra spaces. In future, this study can be extended on some other weaker forms of topology and also on directed graphs.

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Weaker forms of Contra Continuous Multifunctions

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Abstract

The purpose of this paper is to study the properties of αgs-continuous multifunctions.
Some basic characterizations, preservation theorems and several properties concerning upper and lower contra αgs-continuous multifunctions are investigated. Further, the basic properties of upper and lower almost contra αgs-continuous multifunctions are studied.
Keywords: αgs-closed set, αgs-open sets, αgs-continuous functions, αgs-contra continuous multifunctions.
2010 AMS subject classifications: 54A05, 54C08.[1]

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1 Introduction

The research in topology over last two decades has reached a high level in many directions. Topological methods are widely used in many other branches of modern mathematics such as differential equation, functional analysis, classical mechanics, general theory of relativity, mathematical economics, quantum theory, biology etc. One of the important and basic topics in the theory of classical point set topology and in several branches of mathematics, which has been investigated by many authors is continuity of functions. This concept has been extended to the setting of multifunctions.

The concept of upper and lower multifunctions were introduced and studied by Berge [2]. After this work several authors have given the several weaker and strong forms of continuity of multifunctions [1], [7], [8], [9], [10], [11], [15]). In this direction, we will continue the study of new type of multifunction called αgs-continuous multifunctions and study some of basic properties.

In this paper, (P, τ) and (Q, σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. A multifunction $f : (P, \tau) \to (Q, \sigma)$ is a point to set correspondence, and we always assume that $f(p) \neq \phi$, for every point $p \in P$. For each $A \subset P$ and each $B \subset Q$, $f(A) = \cup\{f(p) : p \in A\}$, $f^+(B) = \{p \in P : f(p) \subset B\}$ and $f^-(B) = \{p \in P : f(p) \cap B \neq \phi\}$.

A multifunction $f : P \to Q$ is said to be upper semi continuous [13] (resp. lower semi continuous) at a point $p \in P$, and each open set $V \subseteq Q$ with $f(p) \subset V$ ($f(p) \cap V \neq \phi$) there is an open neighborhood U of p such that $f(U) \subseteq V$ ($f(a) \cap V \neq \phi$ for each $a \in U$). Then f is semi continuous if and only if f is upper and lower semi continuous.

In this paper, the first section deals with the concept of Upper and Lower contra αgs-continuous multifunctions in topological spaces. Several characterizations of these functions have been introduced. The second section contains the properties of Upper and Lower Almost contra αgs-continuous functions in topological spaces. Many properties concerning to these functions have been introduced in this paper.

2 Preliminaries

Definition 2.1. *[12] A subset A of a topological space (P, τ) is called αgs-closed if αcl(A)$\subseteq U$ whenever $A \subseteq U$ and U is gs-open in P.*

The family of all αgs-open (resp. αgs-closed) sets of P is denoted by αgs-O(P) (resp. αgs-C(P)).
If $B \in \alpha$gs-O(P) then $P - B \in \alpha$gs-C(P).

Definition 2.2. *[12] Let $A \subset P$. Then*
(i) αgs-interior of A is denoted by αgs-int(A) and is defined as αgs-int(A) = $\cup\{G : G \subset A$, where $G \in \alpha$gs-O(P)$\}$.
(ii) αgs-closure of A is denoted by αgs-cl(A) and is defined as αgs-cl(A) = $\cap\{G : A \subset G$, where $G \in \alpha$gs-C(P)$\}$.

Definition 2.3. *[12] Let $f : P \to Q$ is a function. Then f is said to be*
(i) αgs-continuous, if $f^{-1}(U) \in \alpha$gs-C(P) for every $U \in C(Q)$.
(ii) αgs-irresolute, if $f^{-1}(U) \in \alpha$gs-C(P) for every $U \in \alpha$gs-C(Q).

Lemma 2.1. *[10] For a multifunction $f : (P, \tau) \to (Q, \sigma)$ we have the following:*
(i) $G_f^+(A \times B) = A \cap f^+(B)$
(ii) $G_f^-(A \times B) = A \cap f^-(B)$.
For any subsets A of P and B of Q, we have $G_f : P \to P \times Q$ is defined as $G_f(p) = \{p\} \times f(p)$ for every $p \in P$.

3 Upper and Lower Contra αgs-continuous Multifunctions

Definition 3.1. *A multifunction $f : (P, \tau) \to (Q, \sigma)$ is called a*
(i) Upper αgs-continuous if for each $p \in P$, $V \in (Q, f(p))$ there exists $U \in \alpha$gs-$O(P, p)$ such that $f(U) \subseteq V$.
(ii) Lower αgs-continuous if for each $p \in P$, $V \in O(Q, f(p))$ with $f(p) \cap V \neq \phi$, then there exists $U \in \alpha$gs-$O(P, p)$ such that $U \in f^{-1}(V)$.

Definition 3.2. *A function $f : P \to Q$ is said to be Contra αgs-continuous if for each $p \in P$ and $V \in O(Q, f(p))$ there exists $V \in \alpha$gs-$O(Q, f(p)$ such that $f(U) \subseteq V$.*

Definition 3.3. *A multifunction $f : (P, \tau) \to (Q, \sigma)$ is called a*
(i) Upper contra αgs-continuous if for each $p \in P$, $V \in C(Q, f(p))$ there exists $U \in \alpha$gs-$O(P, p)$ such that $f(U) \subseteq V$.
(ii) Lower contra αgs-continuous if for each $p \in P$, $V \in C(Q)$ with $f(p) \cap V \neq \phi$, then there exists $U \in \alpha$gs-$O(P, p)$ such that $U \in f(V)$.

Theorem 3.1. *The following statements are equivalent for a multifunction in $f : (P, \tau) \to (Q, \sigma)$*
(i) f is upper contra αgs-continuous

(ii) $f^+(V) \in \alpha gp\text{-}O(P)$*, for every* $V \in C(Q)$
(iii) for each $V \in O(Q)$*,* $\Rightarrow f^-(V) \in \alpha gs\text{-}C(Q)$
(iv) for each $p_1 \in P$ *and each* $K \in C(Q, f(p_1))$*, there exists* $U \in \alpha gs\text{-}O(P, p_1)$ *with* $q_1 \in U$*, then* $p_1(q_1) \subseteq K$.
Proof: $(i) \Leftrightarrow (ii)$*: Let* $V \in C(Q)$ *and* $p_1 \in f^+(V)$ *as* p *is upper contra* αgs*-continuous, there exists* $U \in \alpha gs\text{-}O(P, p_1)$ *with* $f(U) \subseteq V$*. Thus* $f^+(V) \in O(P)$*. We can prove the other part similarly.*
$(ii) \Rightarrow (iii)$*: We have* $f^+(Q \setminus V) = P \setminus f^-(V)$ *for every* $V \subseteq Q$.
$(iii) \Rightarrow (iV)$*: Obvious from the definition.*

Theorem 3.2. *The following statements are equivalent for a multifunction in* $f : (P, \tau) \to (Q, \sigma)$
(i) f *is lower almost contra* αgs*-continuous.*
(ii) $f^+(V) \in \alpha gs\text{-}O(P)$*, for each* $V \in C(Q)$.
(iii) $f^-(K) \in \alpha gs\text{-}C(P)$*, for each* $K \in O(Q)$
(iv) for each $p \in Q$ *and each* $K \in C(P)$ *with* $f(p_1) \cap K \neq \phi$ *there exists* $U \in \alpha gs\text{-}O(P, p_1)$ *such that* $q_1 \in U$ *then* $p_1(q_1) \subseteq K \neq \phi$.
Proof: *The proof is similar to the Theorem 3.1.*

Definition 3.4. *Let* $K \in P$*. Then* K *is said to be strongly S-closed(*αgs*-compact) relative to* P *if every cover of* K *by closed (resp.* αgs*-open) sets of* P *has a finite subcover.*
A space P *is said to be strongly S-closed(*αgs*-compact) if* P *is strongly S-closed relative to* P.

Theorem 3.3. *Let* $f : (P, \tau) \to (Q, \sigma)$ *be an upper contra* αgs*-continuous surjective multifunction and* $f(p)$ *is strongly S-closed relative to* Q *holds for each* $p \in P$*. Then* $f(A)$ *is strongly S-closed if* A *is* αgs*-compact relative to* P.
Proof: *Let* $\{A_i : i \in \alpha\}$ *be any cover of* $f(A)$ *by closed sets in* P*. Then for each* $p \in A$ *there exists a finite subset* $\alpha(p)$ *of* α *such that* $f(p) \subseteq \cup\{A_i : i \in \alpha(p)\}$.
Let $V(p) = \cup\{A_i : i \in \alpha(p)\}$*, then* $f(p) \subseteq V(P)$ *and so there exist* $U(P) \in \alpha gs\text{-}O(P, p)$ *with* $p(U(p)) \subseteq V(P)$ *as* $\{U(P) : p \in A\}$ *is cover of* A *by* αgs*-open sets in* P*. Then, there exist finite number of points say* $p_1, p_2, ...p_m$ *such that* $A \subseteq \cup\{U(p_i) : i = 1, 2, ...n\}$*. Hence* $f(A) \subseteq f(\bigcup_{i=1}^{n} U(p_i)) \subseteq \bigcup_{i=1}^{n} f(U(p_n)) \subseteq \bigcup_{i=1}^{n} V(p_i) \subseteq \bigcup_{i=1}^{n} \bigcup_{i=\alpha(p_i)} V_i$*. Thus* $f(A)$ *is strongly S-closed relative to* Q.

Lemma 3.1. *[12] Let* $A, B \subseteq P$*. Then*

(i) If $A \in \alpha gs\text{-}O(P)$ *and* $B \in O(P)$ *then* $A \cap B \in \alpha gs\text{-}O(P)$.

(ii) If $A \in \alpha gs\text{-}O(B)$ *and* $B \in \alpha gs\text{-}O(P)$ *then* $A \in \alpha gs$ *-*$O(P)$.

Theorem 3.4. *Let $f : (P, \tau) \to (Q, \sigma)$ be a multifunction with $V \subseteq O(P)$. If f is upper contra αgs-continuous (lower contra αgs-continuous) then $f_{|v} : U \to Q$ is upper contra αgs-continuous(lower contra αgs-continuous).*
Proof: *Let $V \subseteq C(Q)$ with $p \in U$ with $p \in P_{|v^{-1}}(V)$ by lower contra αgs-continuous of f. Then there exist $G \in \alpha$gs-$O(P, p)$ such that $G \subseteq P^{-1}(V)$ and so $p \in G \cap U \in \alpha$gs $O(A)$ with $G \cap U \subseteq F_{|U^{-1}}(V)$ which satisfies the conditions of lower contra αgs-continuous multifunction.*
Similarly, we can show for upper contra αgs-continuous multifunction.

Theorem 3.5. *Let $\{U_i : i \in \alpha\}$ be an open cover of a space P. A function $p : (P, \tau) \to (Q, \sigma)$ is upper contra αgs-continuous if and only if the restriction $P_{|U_i} : U_i \to Q$ is upper contra αgs-continuous for each $i \in \alpha$.*
Proof: *Let p is upper contra αgs-continuous and $i \in \alpha$. Let $P \in U_i$ and $V \subseteq C(N, P_{|U_i}(p_i)))$ as p is upper contra αgs-continuous and $P(m) = P_{|U_i}(p_1)$. Then there exists $G \in \alpha$gs-O(P, p_1) with $P(G) \subseteq V$. Put $U = G \cap U_i$, then $p_1 \in U \in \alpha$gs-$O(U, p_1)$ and $P_{|U_i}(U) = P(U) \subseteq V$. Hence $P_{|U_i}$ is upper contra αgs-continuous.*
Conversely, let $m \in P$, $V \in \alpha O(Q)$. Then there exist $i < \alpha$ with $p_1 \in U_i$ by upper contra αgs continuity of p and $P(p_1) = P_{|U_i}(p_1)$. Then there exists $U \in \alpha$gs-$O(U_i, p_1)$ with $P_{|U_i}(p_1) \subseteq V$. Thus $V \in \alpha$gs-O(P, p_1) and $P(U) \subseteq V$. Hence p is upper contra αgs-continuous.

Theorem 3.6. *The set of points p of P at which a multifunction $f : (P, \tau) \to (Q, \sigma)$ is not upper contra αgs-continuous (lower contra αgs-continuous) is identical with the union of the αgs-frontiers of the upper(lower) inverse images of a closed sets containing (meeting) $f(p)$.*
Proof: *Let $p \in P$ at which f is not upper contra αgs-continuous. Then there exist $V \in C(Q)$ with $U \cap (P \setminus f^+(V)) \neq \phi$ holds for each $U \in \alpha$gs-O(P, p) and so $p \in \alpha$gs-cl$(P \setminus f^+(V))$. As $p \in f^+(V)$, so $p \in \alpha$gs-cl$(f^+(Q))$ and so $p \in \alpha$gs-Fr$(f^+(V))$. On the other hand, let $V \in C(Q, f(p))$ with $p \in \alpha$gs-Fr$(f^+(V))$. Assume that f is upper contra αgs-continuous at point p, then there exists $U \in \alpha$gs-O(P, p) with $f(U) \subset V$. Thus, $p \in U \subset \alpha$gs-int$(f^+(V))$, which contradicts the fact that $p \in \alpha$gs-Fr$(f^+(V))$. Thus, f is not upper contra αgs-continuous.*
Similarly, we can prove for lower contra αgs-continuous.

4 Upper and Lower Almost Contra αgs-Continuous Multifunctions

Definition 4.1. *A multifunction $f : (P, \tau) \to (Q, \sigma)$ is said to be*
(i) upper almost contra αgs-continuous if for each $p \in P$ and $V \in RC(P)$ with

$p \in f^+(V)$, there exists $U \in \alpha gs$-O(P, p) with $U \subset f^+(V)$.
(ii) lower almost contra αgs-continuous if for each $p \in P$ and $V \in RC(P)$ with $p \in f^-(V)$, there exists $U \in \alpha gs$-O(P, p) with $U \subset f^-(V)$.

Theorem 4.1. *Every upper almost contra αgs-continuous (lower almost contra αgs-continuous) multifunction is upper (lower) weakly αgs-continuous.*
Proof: *Let $p \in P$ and $V \in O(Q)$ with $f(p) \subset V$, that is $cl(V)$ is regular closed with $f(p) \subseteq cl(V)$. By upper almost contra αgs-continuity, there exists $U \in \alpha gs$-O(P, p) with $U \subset f^+(cl(V))$ and so f is upper weakly αgs-continuous.*

Example 4.1. *Let $P = Q = \{p_1, p_2, p_3\}$. Let $\tau = \{P, \phi, \{p_1\}, \{p_1, p_3\}\}$ and $\sigma = \{Q, \phi, \{p_1\}, \{p_2\}, \{p_1, p_2\}, \{p_1, p_3\}\}$. Let $A = \{p_1, p_2\}$ and $U = \{p_1, p_2\}$. Then $p(U) = \{p_1, p_2\} \subset cl(A) = P$. Then f is weakly αgs-continuous but not upper almost contra αgs-continuous. For $V = \{p_2\}$ and $U = \{p_1, p_2\}$, we have $U \notin f^+(int(cl(V))$.*

Theorem 4.2. *Every upper contra αgs-continuous (lower contra αgs-continuous) multifunction is upper almost contra αgs-continuous (lower almost contraαgs-continuous).*
Proof: *Follows from the definition 4.1.*

Theorem 4.3. *The following are equivalent for a multifunction $f : (P, \tau) \to (Q, \sigma)$:*
(i) f is upper almost contra αgs-continuous
(ii) for every $A \subset RC(Q)$, $f^+(A) \subset \alpha gs$-O(Q)
(iii) for every $U \subset RO(Q)$, $f^-(U) \in \alpha gs$-C(P)
(iv) for every $A \subset O(Q)$, $f^-(int(cl(A))) \in \alpha gs$-C(P)
(v) for every $A \subset O(Q)$, $f^+(cl(int(A))) \in \alpha gs$-O(P).
(vi) for all $p \in P$ and $V \in SO(Q)$ with $f(p) \subset V$, there exists $U \in \alpha gs$-O(P, p) such that $f(U) \subset cl(V)$.
(vii) $f^+(V) \subset \alpha gs$-int$(f^+(cl(V)))$ holds for all $V \in SO(Q)$.
Proof: *$(i) \to (ii)$: Let $A \subset RC(Q)$ with $p \in f^+(A))$. As f is upper almost contra αgs-continuous, there exists $U \in \alpha gs$-O(P, p) such that $U \subset f^+(A)$. Thus, $f^+(A) \in \alpha gs$-O(P).*
$(ii) \to (i)$: Follows from the definition.
$(ii) \Leftrightarrow (iii)$ and $(iv) \Leftrightarrow (v)$: We have $f^+(Q \setminus A) = P \setminus f^-(A)$ holds for all $A \subset Q$.
$(iii) \Leftrightarrow (iv)$: Let $A \in O(Q)$. As $int(cl(A)) \in RO(Q)$ and so $f^-(int(cl(A)) \in \alpha gs$-C(P).
Similarly, we can show $(v) \Leftrightarrow (ii)$.
$(vi) \to (vii)$: Let $V \in SO(Q)$ and $p \in f^+(V)$. Then, $f(p) \subset V$. By definition, there exists $U \in \alpha gs$-O(P, p) such that $f(U) \subset cl(V)$, that is $p \in U \subset f^+(cl(V))$.

Thus, $p \in \alpha gs\text{-}int(f^+(cl(V)))$ and $f^+(V) \subset \alpha gs\text{-}int(f^+(cl(V)))$.
$(vii) \to (ii)$: Let $A \in RC(Q)$. As $A \in SO(Q)$ and from (vii), $f^+(A) \subset \alpha gs\text{-}int(f^+(cl(A)))$. Thus, $f^+(A) \in \alpha gs\text{-}O(P)$.
$(ii) \to (vi)$: Let $p \in P$ and $V \in SO(Q)$ with $f(p) \subset V$. As $cl(V) \in RC(Q)$, then there exists $A \in \alpha gs\text{-}O(P, p)$ with $A \subset f^+(cl(A))$. Thus, $f(A) \subset cl(V)$.

Lemma 4.1. *[14] If A and B are disjoint compact subsets of an Urysohn space P, then there exist $U \in O(P)$ and $V \in O(Q)$ with $A \subset U$, $B \subset V$ with $cl(U) \cap cl(V) = \phi$.*

Theorem 4.4. *Let $f : (P, \tau) \to (Q, \sigma)$ be upper almost contra αgs-continuous injective multifunction into an Urysohn space Q with $f(p)$ is compact for each point $p \in P$. Then P is $\alpha gs\text{-}T_2$ space.*
Proof: *Let $p, q \in P$ with $f(p) \cap f(q) = \phi$ by injectiveness of f, where $f(p)$ and $f(q)$ are disjoint compact sets in P. Then there exist $V_1, V_2 \in O(P)$ with $f(p) \subset V_1$ and $f(q) \subset V_2$ such that $cl(V_1) \cap cl(V_2) = \phi$. Since $cl(V_1)$ and $cl(V_2)$ are regular closed sets, then by upper almost contra αgs-continuity of f, there exist $U_1 \in \alpha gs\text{-}O(P, p)$ and $U_2 \in \alpha gs\text{-}O(P, q)$ with $f(U_1) \subset cl(V_1)$, $f(U_2) \subset cl(V_2)$. Thus, $U_1 \cap U_2 = \phi$. Thus P is $\alpha gs\text{-}T_2$ space.*

Discussion and Conclusion

Topology developed as a field of study of geometry and set theory, through analysis of such concepts as space, dimensions and transformation. The topological structures are modelled suitably in the field of Computer graphics, Pattern recognition, Artificial intelligence, Data mining, Rough set theory, Information systems, Quantum physics etc.

Continuous functions stands among the most fundamental point in the whole of the Mathematical Science. Many different forms of stronger and weaker forms of functions have been introduced over the years. As a generalization of closed sets, Levine introduced the concept of generalized closed sets which are weaker than closed sets in topological spaces.

In this way, this paper deals with the properties of weaker forms of continuous functions contra and almost contra continuous multifunctions in topological spaces. We have defined the new weaker froms of continuous multifunctions called contra and almost contra αgs-continuous multifunctions in topological spaces. The properties and preservation theorems on upper and lower contra αgs-continuous multifunctions are introduced. Further, upper and lower almost

contra αgs-continuous multifunctions are also introduced and studied basic characterizations of these multifunctions in topological spaces.

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Monophonic Distance Laplacian Energy of Transformation Graphs S_n^{++-}, S_n^{+-+}, S_n^{+++}

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Abstract

Let G be a simple connected graph of order n, v_i its vertex. Let $\delta_1^L, \delta_2^L, \ldots, \delta_n^L$ be the eigenvalues of the distance Laplacian matrix D^L of G. The distance Laplacian energy is denoted by $LE_D(G)$. This motivated us to defined the new graph energy monophonic distance Laplacian energy of graphs. The eigenvalues of monophonic distance Laplacian matrix $M^L(G)$ are denoted by $\mu_1^L, \mu_2^L, \ldots, \mu_n^L$ and are said to be M^L- eigenvalues of G and to form the M^L-spectrum of G, denoted by $Spec_{M^L}(G)$. Here $MT_G(v_j)$ is the j^{th} row sum of monophonic distance matrix of $M(G)$ and $\mu_1^L \leq \mu_2^L \leq, \ldots, \leq \mu_n^L$ be the eigenvalues of the monophonic distance Laplacian matrix is $M^L(G)$. The monophonic distance Laplacian energy is defined as $LE_M(G)$. In this paper we computed the monophonic distance Laplacian energy of S_n^{++-}, S_n^{+-+}, S_n^{+++} graphs based on its spectrum values.

Keywords: distance Laplacian energy, monophonic distance matrix, monophonic distance Laplacian energy, transformation graphs.

2020 AMS subject classifications: 05C12,05C50 [1]

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1 Introduction

The concept of energy stems from chemistry to approximate the total π electron energy of a molecule of a conjugated hydrocarbon. Conjugated hydrocarbon can be represented by a graph called molecular graph according to the rule: every carbon atom is represented by a vertex and every carbon-carbon bond by an edge, hydrogen atoms are ignored. The eigenvalues of the molecular graph represent the energy level of the electron in a molecule. An interesting quantity in H¨uckel theory is the sum of the energies of all the electrons in a molecule, the so called total π electron energy $E\pi$. The first results on energy of a graph was obtained as early as 1940's. The same concept of energy is extended to simple graphs.

Spectral graph theory studies the relation between graph properties and the spectra of certain matrices associated to it. It finds applications in other areas of graph theory such as Chemistry, Biology, Physics, Computer Science, Statistics, etc. Here we defind the monophonic distance Laplacian energy of graphs. The concept of energy of a graph was introduced by I.Gutman in 1978[4]. The energy of a graph G, indicated by $E(G)$, is defined as the absolute sum of the eigenvalues μ_i, $(1 \leq i \leq n)$ of the adjacency matrix of the graph $E(G) = \sum_{i=1}^{n} |\mu_i|$.

In 2006, Ivan Gutman and Bo Zhou[7] found the Laplacian energy of graphs. Let $\mu_1, \mu_2 \ldots \mu_n$ be the Laplacian eigenvalues of G. The Laplacian energy of G has defined as $LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$. Also, In 2008, I.Gutman et.al.,[3] have introduced the distance energy of graphs. The distance matrix or D- matrix of G is defined as $D = [d_{ij}]$, where d_{ij} is the distance between the vertices v_i and v_j in G. The eigenvalues $\mu_1, \mu_2, \ldots, \mu_p$ of the D- matrix of G are said to be the D-eigenvalues of G and to form the D-spectrum of G, denoted by $Spec_D(G)$.

The distance Laplacian energy of graphs was introduced by Jieshan yang et.al., in 2013[8]. The distance degree of the vertex v_i, denoted by D_i is given by $D_i = \sum_{j=1}^{n} d_{ij}$. The distance Laplacian matrix of a connected graph G is $D^L = D^L(G) = diag(D_i) - D(G)$, where $diag(D_i)$ denotes the diagonal matrix of the distance degrees. The distance Laplacian energy of G is denoted by $LE_D(G)$ is defined as $LE_D(G) = \sum_{i=1}^{n} \left| \delta_i^L - \frac{1}{n} \sum_{j=1}^{n} D_j \right|$. In 2015, V.S Shigehalli and Kenchappa S Betageri[11] determined the color Laplacian energy. The seidal Laplacian energy of graphs was initiated by H.S. Ramane and others in 2017[9]. The seidal signless Laplacian energy of graphs was derived by S. Harishchandra

and Ivan Gutman in 2017[6]. Path Laplacian energy was introduced by Shridhar Chandrakant Patekar and Maruti Mukinda Shikare in 2018[12]. The monophonic distance in graphs was introduced by A.P. Santhakumaran and P.Titus in 2011[10]. Based on these we introduce a new concept monophonic distance Laplacian energy of graphs.

2 Definitions

Definition 2.1. Let G be a connected graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The monophonic distance matrix G is defined as

$$M = M(G) = (d_{m_{ij}})_{n\times n}, where\ d_{m_{ij}} = \begin{cases} d_m(v_i, v_j) & \text{if } \ i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

Here $d_m(v_i, v_j)$ is the monophonic distance of v_i to v_j.

Definition 2.2. The monophonic transmission $MT_G(v)$ of a vertex v as $\sum_{u\in V(G)} d_m(u, v)$ and monophonic transmission matrix $MT(G)$ is the diagonal matrix $diag[MT_G(v_1), MT_G(v_2), \ldots, MT_G(v_n)]$. For $1 \leq i \leq n$, $MT_G(v_i)$ is the i^{th} row sum of $M(G)$. The monophonic distance Laplacian matrix of a connected graph G is defined as $M^L(G) = MT(G) - M(G)$. The eigen values of monophonic distance Laplacian matrix $M^L(G)$ are denoted by $\mu_1^L, \mu_2^L, \ldots, \mu_n^L$ and are said to be M^L- eigen values of G and to form the M^L-spectrum of G, denoted by $Spec_{M^L}(G)$. Since the monophonic distance Laplacian matrix is symmetric and its eigen values are real,it can be ordered as $\mu_1^L \leq \mu_2^L \leq, \ldots, \leq \mu_n^L$.

Definition 2.3. [2] The monophonic distance Laplacian energy of a graph is defined as $LE_M(G) = \sum_{i=1}^{n} \left| \mu_i^L - \frac{1}{n} \sum_{j=1}^{n} MT_G(v_j) \right|$.

3 Main Results

Definition 3.1. [1]Let $G = (V(G), E(G))$ be a graph and x, y, z be three variables taking values $+$ or $-$. The transformation graph G^{xyz} is the graph having $V(G) \cup E(G)$ as the vertex set and for $\alpha, \beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xyz} if and only if one of the following holds:
(i) $\alpha, \beta \in V(G)$, α and β are adjacent in G if $x = +$; α and β are not adjacent in

G if $x = -$.
(ii) $\alpha, \beta \in E(G)$, α and β are adjacent in G if $y = +$; α and β are not adjacent in G if $y = -$.
(iii) $\alpha \in V(G)$, $\beta \in E(G)$, α and β are incident in G if $z = +$; α and β are not incident in G if $z = -$.
Thus,we may obtain eight kinds of transformation graphs, in which G^{+++} is the total graph of G and G^{---} is its complement. Also G^{--+}, G^{-+-} and G^{-++} are the complements of G^{++-}, G^{+-+} and G^{+--} respectively.

Theorem 3.1. Let S_n be a star graph with $n \geq 3$ vertices, S_n^{++-} be the transformation graph and $M(S_n^{++-})$ be the monophonic distance matrix and its dimension is $(2n-1) \times (2n-1)$. Then
(a) the spectrum of S_n^{++-} is given by

$$Spec_{M^L}(S_n^{++-}) = \begin{pmatrix} 0 & 3n + \sqrt{n^2 - 4n + 8} & 3n - \sqrt{n^2 - 4n + 8} \\ 1 & n-1 & n-1 \end{pmatrix}$$

(b)the monophonic distance Laplacian energy of S_n^{++-} is

$$LE_M(S_n^{++-}) = \frac{2}{2n-1}\left[(3n^2 - 2n - 1) + (2n^2 - 3n + 1)\sqrt{n^2 - 4n + 8}\right], \text{ for } n \geq 3.$$

Proof. The graph S_n^{++-} as shown in Figure: 1

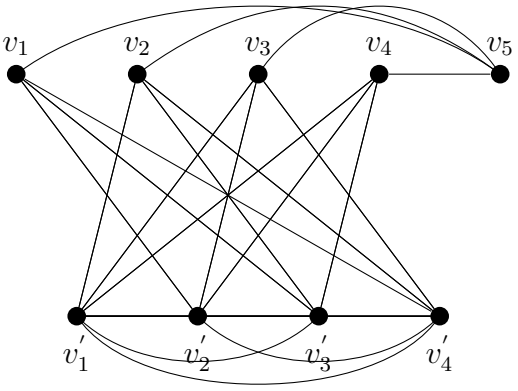

Figure 1: The graph S_n^{++-}, $n = 5$

We have $M(S_n^{++-})$=$(d_{m_{ij}})_{(2n-1)\times(2n-1)}$

$$where\ d_{m_{ij}} = \begin{cases} 0 & \text{if } v_i = v_j \\ 1 & \text{if } v_i\ and\ v_j \text{ are adjacent} \\ 3 & \text{if } v_i\ and\ v_j \text{ are non adjacent} \end{cases}$$

The monophonic distance Laplacian matrix of S_n^{++-} is
$M^L\left(S_n^{++-}\right) = MT\left(S_n^{++-}\right) - M\left(S_n^{++-}\right)$

$$= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \\ v_1^{'} \\ v_2^{'} \\ v_3^{'} \\ \vdots \\ v_n^{'} \end{array} \begin{array}{c} \begin{array}{cccccccccc} v_1 & v_2 & v_3 & \ldots & v_n & v_1^{'} & v_2{}^{'} & v_3^{'} & \ldots & v_n{}^{'} \end{array} \\ \begin{pmatrix} 4n-4 & -3 & -3 & \ldots & -1 & -3 & -1 & -1 & \ldots & -1 \\ -3 & 4n-4 & -3 & \ldots & -1 & -1 & -3 & -1 & \ldots & -1 \\ -3 & -3 & 4n-4 & \ldots & -1 & -1 & -1 & -3 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \ldots & 4n-4 & -3 & -3 & -3 & \ldots & -3 \\ -3 & -1 & -1 & \ldots & -3 & 2n+2 & -1 & -1 & \ldots & -1 \\ -1 & -3 & -1 & \ldots & -3 & -1 & 2n+2 & -1 & \ldots & -1 \\ -1 & -1 & -3 & \ldots & -3 & -1 & 1 & 2n+2 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \ldots & -3 & -1 & -1 & -1 & \ldots & 2n+2 \end{pmatrix} \end{array}$$

The M^L-eigenvalues of S_n^{++-} are $\begin{pmatrix} \mu_1^L \\ \mu_2^L \\ \vdots \\ \mu_n^L \\ \mu_{n+1}^L \\ \vdots \\ \mu_{2n-1}^L \end{pmatrix} = \begin{pmatrix} 0 \\ 3n - \sqrt{n^2-4n+8} \\ \vdots \\ 3n - \sqrt{n^2-4n+8} \\ 3n + \sqrt{n^2-4n+8} \\ \vdots \\ 3n + \sqrt{n^2-4n+8} \end{pmatrix}$.

The M^L- spectrum of S_n^{++-} is
$Spec_{M^L}(S_n^{++-}) = \begin{pmatrix} 0 & 3n-\sqrt{n^2-4n+8} & 3n+\sqrt{n^2-4n+8} \\ 1 & n-1 & n-1 \end{pmatrix}$.

Let $n \geq 4$

$$LE_M\left(S_n^{++-}\right) = \sum_{i=1}^{2n-1} \left|\mu_i^L - \frac{6n^2-4n-2}{2n-1}\right|$$

$$= \left|0 - \frac{6n^2-4n-2}{2n-1}\right| +$$

$$(n-1)\left|3n+1+\sqrt{n^2-4n+8} - \frac{6n^2-4n-2}{2n-1}\right| +$$

$$(n-1)\left|(3n+1)-\sqrt{n^2-4n+8} - \frac{6n^2-4n-2}{2n-1}\right|$$

Since $\mu_i^L - \frac{1}{2n-1}\sum_{j=1}^{2n-1} MT_G(v_j) < 0$, where $i = 2, 3, \ldots, n$, we have

$$LE_M\left(S_n^{++-}\right) = \left[\frac{6n^2-4n-2}{2n-1}\right] +$$

$$(n-1)\left[\frac{(6n^2-4n-2)-(2n-1)\left[(3n+1)+\sqrt{n^2-4n+8}\right]}{2n-1}\right]+$$
$$(n-1)\left[\frac{(3n+1)(2n-1)+(2n-1)\sqrt{n^2-4n+8}-6n^2+4n+2}{2n-1}\right]$$
$$=\frac{1}{2n-1}\left[(6n^2-4n-2)+2(n-1)(2n-1)\sqrt{n^2-4n+8}\right]$$
$$=\frac{2}{2n-1}\left[(3n^2-2n-1)+(2n^2-3n+1)\sqrt{n^2-4n+8}\right].$$

Theorem 3.2. The monophonic distance Laplacian energy of S_n^{+-+} is $LE_M\left(S_n^{+-+}\right)=\dfrac{2}{2n-1}[12n^2-32n+19]$.

Proof. The graph S_n^{+-+} as shown in Figure: 2

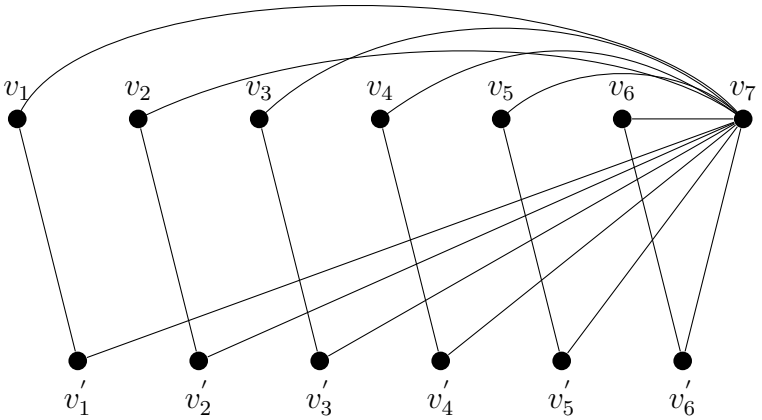

Figure 2: The graph S_n^{+-+}, $n=7$

We have $M(S_n^{+-+})$=$(d_{m_{ij}})_{(2n-1)\times(2n-1)}$

$$where\ d_{m_{ij}}=\begin{cases}0 & \text{if } v_i=v_j\\ 1 & \text{if } v_i\ and\ v_j \text{ are adjacent}\\ 2 & \text{if } v_i\ and\ v_j \text{ are non adjacent}\end{cases}$$

The monophonic distance Laplacian matrix of S_n^{+-+} is $M^L\left(S_n^{+-+}\right)=MT\left(S_n^{+-+}\right)-M\left(S_n^{+-+}\right)$

$$M^L\left(S_n^{+-+}\right) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \\ v_1^{'} \\ v_2^{'} \\ \vdots \\ v_n^{'} \end{array} \begin{array}{c} \begin{array}{ccccccccc} v_1 & v_2 & v_3 & \dots & v_n & v_1^{'} & v_2{}' & \dots & v_n{}' \end{array} \\ \begin{pmatrix} 4n-6 & -2 & -2 & \dots & -1 & -1 & -2 & \dots & -2 \\ -2 & 4n-6 & -2 & \dots & -1 & -2 & -1 & \dots & -2 \\ -2 & -2 & 4n-6 & \dots & -1 & -2 & -2 & \dots & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 2n-2 & -1 & -1 & \dots & -1 \\ -1 & -2 & -2 & \dots & -1 & 4n-6 & -2 & \dots & -2 \\ -2 & -1 & -2 & \dots & -1 & -2 & 4n-6 & \dots & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & -2 & \dots & -1 & -2 & -2 & \dots & 4n-6 \end{pmatrix} \end{array}$$

The M^L-eigenvalues of S_n^{+-+} are $\begin{pmatrix} \mu_1^L \\ \mu_2^L \\ \mu_3^L \\ \vdots \\ \mu_{n+1}^L \\ \mu_{n+2}^L \\ \vdots \\ \mu_{2n-1}^L \end{pmatrix} = \begin{pmatrix} 0 \\ 2n-1 \\ 4n-5 \\ \vdots \\ 4n-5 \\ 4n-3 \\ \vdots \\ 4n-3 \end{pmatrix}$.

The M^L- spectrum of S_n^{+-+} is

$Spec_{M^L}(S_n^{+-+}) = \begin{pmatrix} 0 & 2n-1 & 4n-5 & 4n-3 \\ 1 & 1 & n-1 & n-2 \end{pmatrix}$.

Then, $LE_M\left(S_n^{+-+}\right) = \sum_{i=1}^{2n-1} \left|\mu_i^L - \frac{8n^2-18n+10}{2n-1}\right|$

$$= \left|0 - \frac{8n^2-18n+10}{2n-1}\right| + \left|(2n-1) - \frac{8n^2-18n+10}{2n-1}\right| + (n-1)\left|(4n-5) - \frac{8n^2-18n+10}{2n-1}\right| + (n-2)\left|(4n-3) - \frac{8n^2-18n+10}{2n-1}\right|$$

Since $\mu_i^L - \frac{1}{2n-1}\sum_{j=1}^{2n-1} MT_G(v_j) < 0$, where $i = 2$, we have

$$LE_M\left(S_n^{+-+}\right) = \left[\frac{8n^2-18n+10}{2n-1}\right] + \left[\frac{(8n^2-18n+10)-(2n-1)^2}{2n-1}\right] + (n-1)\left[\frac{(4n-5)(2n-1)-(8n^2-18n+10)}{2n-1}\right] + (n-2)\left[\frac{(2n-1)(4n-3)-(8n^2-18n+10)}{2n-1}\right]$$

$$= \frac{1}{2n-1}\left[12n^2 - 32n + 19 + (n-1)(4n-5) + (n-2)(8n-7)\right]$$
$$= \frac{2}{2n-1}\left[12n^2 - 32n + 19\right].$$

Theorem 3.3. Let S_n^{+++} be the transformation graph of order 2n-1. Then the monophonic distance Laplacian energy of S_n^{+++} is
$LE_M\left(S_n^{+++}\right) = \frac{2}{2n-1}[(6n^2 - 14n + 7) + (2n^2 - 5n + 2)\sqrt{n^2 - 2n + 2}$, for $n \geq 4$.

Proof. The graph S_n^{+++} as shown in Figure: 3

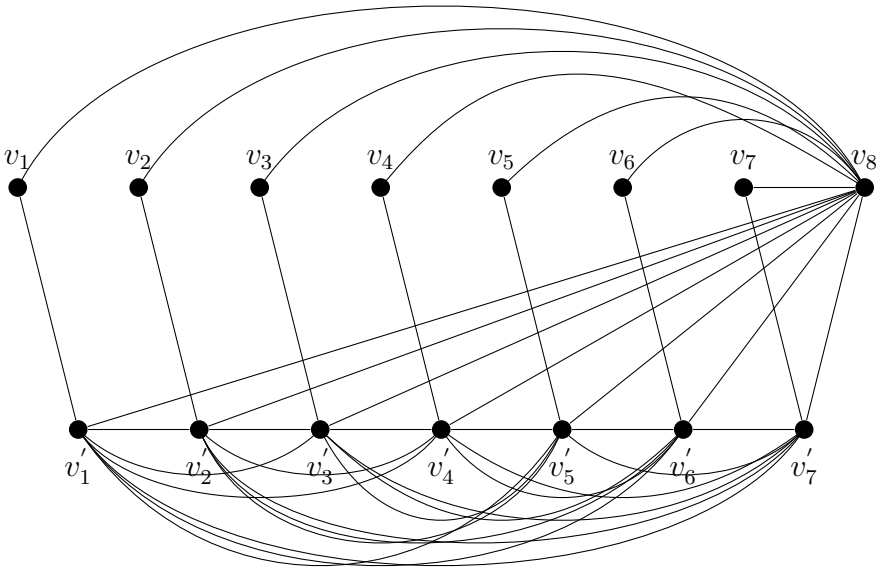

Figure 3: The graph S_n^{+++}, $n = 8$

The monophonic distance matrix of S_n^{+++} is

$$M\left(S_n^{+++}\right) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \\ v_1^{'} \\ v_2^{'} \\ v_3^{'} \\ \vdots \\ v_n^{'} \end{array} \begin{array}{c} \begin{array}{cccccccccc} v_1 & v_2 & v_3 & \dots & v_n & v_1^{'} & v_2{}' & v_3^{'} & \dots & v_n{}' \end{array} \\ \begin{pmatrix} 0 & 3 & 3 & \dots & 1 & 1 & 2 & 2 & \dots & 2 \\ 3 & 0 & 3 & \dots & 1 & 2 & 1 & 2 & \dots & 2 \\ 3 & 3 & 0 & \dots & 1 & 2 & 2 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 1 & 0 & 1 & 1 & \dots & 1 \\ 2 & 1 & 2 & \dots & 1 & 1 & 0 & 1 & \dots & 1 \\ 2 & 2 & 1 & \dots & 1 & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & 1 & 1 & 1 & 1 & \dots & 0 \end{pmatrix} \end{array}$$

The monophonic distance Laplacian matrix of S_n^{+++} is
$M^L\left(S_n^{+++}\right) = MT\left(S_n^{+++}\right) - M\left(S_n^{+++}\right)$

$$M^L\left(S_n^{+++}\right) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \\ v_1^{'} \\ v_2^{'} \\ \vdots \\ v_n^{'} \end{array} \begin{array}{c} \begin{array}{ccccccccc} v_1 & v_2 & v_3 & \dots & v_n & v_1^{'} & v_2{}' & \dots & v_n{}' \end{array} \\ \begin{pmatrix} 5n-8 & -3 & -3 & \dots & -1 & -1 & -2 & \dots & -2 \\ -3 & 5n-8 & -3 & \dots & -1 & -2 & -1 & \dots & -2 \\ -3 & -3 & 5n-8 & \dots & -1 & -2 & -2 & \dots & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 2n-2 & -1 & -1 & \dots & -1 \\ -1 & -2 & -2 & \dots & -1 & 3n-4 & -1 & \dots & -1 \\ -2 & -1 & -2 & \dots & -1 & -1 & 3n-4 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & -2 & \dots & -1 & -1 & -1 & \dots & 3n-4 \end{pmatrix} \end{array}$$

The M^L-eigenvalues of S_n^{+++} are $\begin{pmatrix} \mu_1^L \\ \mu_2^L \\ \mu_3^L \\ \vdots \\ \mu_n^L \\ \mu_{n+1}^L \\ \mu_{n+2}^L \\ \vdots \\ \mu_{2n-1}^L \end{pmatrix} = \begin{pmatrix} 0 \\ 2n-1 \\ (4n-4)-\sqrt{n^2-2n+2} \\ \vdots \\ (4n-4)-\sqrt{n^2-2n+2} \\ 4n-5 \\ (4n-4)+\sqrt{n^2-2n+2} \\ \vdots \\ (4n-4)+\sqrt{n^2-2n+2} \end{pmatrix}$

The M^L-spectrum of S_n^{+++} is

$$Spec_M^L(S_n^{+++}) = \begin{pmatrix} 0 & 2n-1 & (4n-4)-\sqrt{n^2-2n+2} & 4n-5 & (4n-4)+\sqrt{n^2-2n+2} \\ 1 & 1 & n-2 & 1 & n-2 \end{pmatrix}$$

Let $n \geq 4$

$$LE_M\left(S_n^{+++}\right) = \sum_{i=1}^{2n-1}\left|\mu_i^L - \frac{8n^2-18n+10}{2n-1}\right|$$
$$= \left|0 - \frac{8n^2-18n+10}{2n-1}\right| + (n-1)\left|(2n-1) - \frac{8n^2-18n+10}{2n-1}\right| +$$
$$(n-1)\left|(4n-5) - \frac{8n^2-18n+10}{2n-1}\right| +$$
$$(n-2)\left|(4n-4) + \sqrt{n^2-2n+2} - \frac{8n^2-18n+10}{2n-1}\right| +$$
$$(n-2)\left|(4n-4) - \sqrt{n^2-2n+2} - \frac{8n^2-18n+10}{2n-1}\right|$$

Since $\mu_i^L - \frac{1}{2n-1}\sum_{j=1}^{2n-1} MT_G(v_j) < 0$, where $i = 2, 3, \ldots, n$, we have

$$LE_M\left(S_n^{+++}\right) = \left[\frac{8n^2-18n+10}{2n-1}\right] + (n-1)\left[\frac{(8n^2-18n+10)-(2n-1)^2}{2n-1}\right] +$$
$$(n-1)\left[\frac{(4n-5)(2n-1)-(8n^2-18n+10)}{2n-1}\right] +$$
$$(n-2)\left[\frac{((4n-4)+\sqrt{n^2-2n+2})(2n-1)-(8n^2-18n+10)}{2n-1}\right] +$$
$$(n-2)\left[\frac{((8n^2-18n+10)-(4n-4)-\sqrt{n^2-2n+2})}{2n-1}\right]$$
$$= \frac{1}{2n-1}\left[12n^2-28n+14+2(n-2)(2n-1)\sqrt{n^2-2n+2}\right]$$
$$= \frac{2}{2n-1}[(6n^2-14n+7)+(2n^2-5n+2)\sqrt{n^2-2n+2}].$$

4 Conclusion

In this paper we derived a new graph energy called monophonic distance Laplacian energy. We obtained monophonic distance Laplacian energy of transformation graphs S_n^{++-}, S_n^{+-+}, S_n^{+++}. In this study we computed monophonic distance Laplacian spectrum based on these eigenvalues. Future scope of this study is comparitive study of monophonic distance Laplacian energy of graphs with other graphs. We can also extend this concept monophonic distance Laplacian energy of graph to various graph distance parameters.

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Restrained star edge coloring of graphs and its application in optimal & safe storage practices

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Abstract

In this paper we introduce the concept of restrained star edge coloring of graphs by restraining the conditions of the star coloring of graphs. The restrained star edge coloring of graphs is a path based graph coloring which is said to be proper if all the bichromatic subgraphs of the graph are in the form of a galaxy. The minimum requirement for this coloring is its restrained star chromatic index, denoted as $\chi'_{rs}(G)$. This paper exclusively explains, the restrained star edge coloring of several families of graphs including path, cycle, wheel, etc., and provides the exact value of its respective restrained star chromatic index, χ'_{rs} with the usage of appropriate illustrations. In addition to this, an application of this coloring in the optimal utilization of storage spaces and in ensuring safe storage practices is also briefly elaborated.

Keywords: Graph coloring, Path coloring, Star edge coloring, Restrained star edge coloring, Restrained Star Chromatic Index
2020 AMS subject classifications: 05C15, 05C38, 05C90 [1]

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1 Introduction

A graph G is considered to be properly vertex (edge) colored if two adjoining vertices (edges) are colored uniquely. Star coloring introduced by Grünbaum [1973] is a path-based coloring in which no path of length 3 should be bicolored. Following this, Fertin et al. [2004] observed that in a star vertex coloring any bichromatic subgraph is in the form of galaxy. Where a galaxy is a forest composing only of stars according to Gallian [2022]. This coloring was introduced along with the acyclic coloring of graphs and several results of star coloring in comparison with the acyclic coloring of graphs can be noted in Albertson et al. [2004] and Brause et al. [2022].

The edge version of the same introduced by Deng [2007] states that a star-edge coloring of a graph is a proper edge coloring if no path of length 4 is bicolored. Following its entry several results and theorems have emerged in said topic and can be seen in Bezegová et al. [2016], Casselgren et al. [2021], Evangeline Lydia and Vijaya Xavier Parthipan [2022] and Lei and Shi [2020]. However, it can be noted that the property "every bichromatic subgraph of a properly star colored graph consists of tree composing only of stars" was not observed in the case of star edge coloring of graphs. Having been inspired by the same we define the concept of restrained star edge coloring of graphs, which is a variation of the star edge coloring problem with an additional constraint as follows.

Definition 1.1. *Given a graph G, the proper edge coloring of G in which all its bichromatic subgraphs are in the form of a galaxy is referred to as the restrained star edge coloring of the graph. Where, a galaxy is a graph whose disjoint components are composed only of stars.*

Moreover, this can be achieved only when no path of length 3 in the graph is bicolored.

Definition 1.2. *The smallest integer $k \in \mathbb{N}$ for which a proper coloring of the graph G is obtained is termed as the restrained star chromatic index and is expressed as $\chi'_{rs}(G)$.*

In this paper we are using the abbreviated term 'RSE coloring' to denote the restrained star edge coloring of graphs. The next section is dedicated for some preliminaries and pre-existing results which are required for the understanding of the upcoming sections. In section 3, we present some properties and results of this coloring including the definite value of the χ'_{rs} of some graph families, and in section 4, we discuss one of the numerous possible application of this coloring in ensuring safe storage practices of hazardous chemicals.

2 Preliminaries

This division includes a few fundamentals that are essential in proving the theorems in the upcoming sections. It is to be noted that all the graphs considered in this paper are undirected, connected and finite.Chartrand and Zhang [2019]

A graph is said to be tree if there exists exactly one path between any pair of vertices. A tree where all the vertices except the two endpoints have degree 2 is a path graph . A path graph whose endpoints are adjacent, such that all its vertices have degree 2 is known as a cycle graph. A cycle graph is said to be a wheel if it has a vertex in the middle such that all the other vertices of the cycle are adjacent to this vertex.

A complete bipartite graph is a graph that can be separated into two partitions in such a way that each vertex of the partition-1 is adjacent to every other vertex in partition-2 and no two vertices of the same partition are connected by an edge. A Petersen graph is a non planar graph consisting of 10 vertices and 15 edges whose chromatic index is 4.

Lemma 2.1. *Deng [2007], Wang et al. [2019]. A cycle of order $n \geq 3$ has a proper 3-star edge coloring except when $n \neq 5$ and 4-star edge coloring when $n = 5$*

Lemma 2.2. *Wang et al. [2019]. The star chromatic index of a wheel graph with order $n \geq 4$ is described as*

$$\chi'_s(W_n) = \begin{cases} n+2 \text{ if } n = 5 \\ n+1 \text{ if } n = 4, 6, 7 \\ n-1 \text{ if } n \geq 8 \end{cases}$$

Lemma 2.3. *Chartrand and Zhang [2019]. The chromatic index of a complete bipartite graph is*

$$\chi'(k_{m,n}) = max\{m, n\}$$

Lemma 2.4. *(Vizing's Theorem) Chartrand and Zhang [2019]. For any graph G with maximum degree Δ, satisfies the inequality,*

$$\Delta \leq \chi'(G) \leq \Delta + 1$$

3 Restrained star edge coloring

In this section we provide some results on the RSE coloring and its corresponding restrained star chromatic index, χ'_{rs}. In addition the RSE coloring for certain families of graphs is also discussed.

Remark 3.1. *Any graph G satisfies the inequality,* $\chi(G) \leq \chi'_s(G) \leq \chi'_{rs}(G)$ *where* $\chi(G), \chi'_s(G)$ *and* $\chi'_{rs}(G)$ *represents the chromatic index, star chromatic index and restrained star chromatic index of the graph G respectively.*

Remark 3.2. *For every graph G with order,* $n \geq 4$, $\chi'_{rs}(G) \geq 3$

Example 3.3. *The Petersen's graph exhibits a proper 5-RSE coloring.*

From Remark 3.1 we obtain the inequalities $3 \leq \chi' \leq 4$ and $\chi'(G) = 4 \leq \chi'_{rs}(G)$. The structure of the Petersen's graph consists of K_5 without the outer edges, whose vertices are joined to vertices of an outer cycle C_5 by an edge. Now the steps involved in coloring the Petersen's graph can be separated into three parts namely, coloring the C_5, the inner K_5 without the outer edges and finally coloring the inner edges joining C_5 and K_5. It is evident that 5 distinct colors are required to properly color a 5-cycle. Thus, $5 \leq \chi'_{rs}(G)$. The inner K_5 without its outer edges is equal to a C_5, since the vertices of the outer and inner cycle are separated by edges, they can be colored using the same 5 colors. By making sure that the parallel colors are distinct, the edge joining these two cycles can be colored by the remaining color that is not used by the 4 adjacent edges. As per these coloring no path of length 3 is bicolored, such that the coloring is proper.

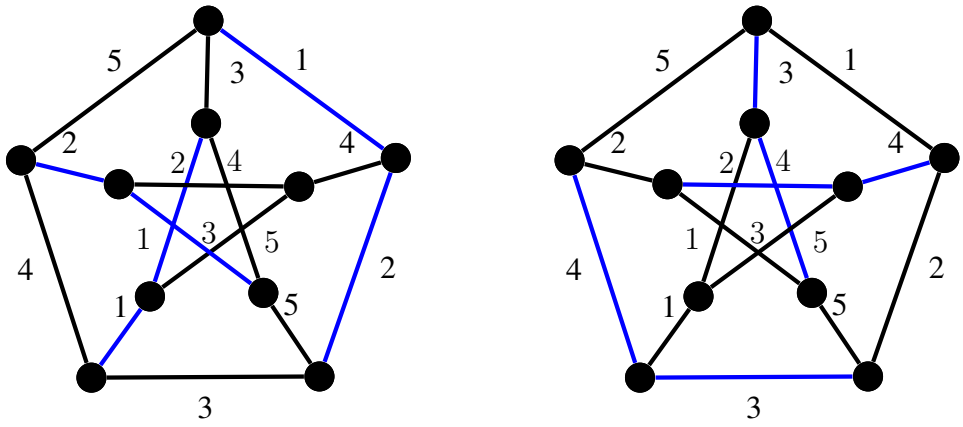

Figure 1: Some of the bichromatic subgraphs of RSE colored Petersen's graph is highlighted

Remark 3.4. *Every bichromatic subgraph of the graph G that is properly RSE colored results in a forest composing only of stars of orders 2 and 3.*

Proof. Now a forest is a graph that contains disjoint union of trees. Assume a graph of order n, the following two cases are possible as per the assignment of the colors.

Case 1. Suppose if a color pair occur in consecutive adjacent edges, then its bichromatic subgraph will be of the form S_3 and since as per our definition of RSE coloring, 2 colors cannot consecutively occur more than once, the order of the star graph cannot be more than 3.

Case 2. Suppose if the pair of colors did not occur in consecutive adjacent edges, then its bichromatic subgraph will consist of disjoint star graphs of order 2, S_2 and since we are considering the edges, the minimum order of the bichromatic subgraph cannot be less than 2.

Therefore, if some pair of colors occur as adjacent pairs in one position and individually in another position, then its bichromatic subgraph will consist of disjoint star graphs of order 2 and 3 only. □

Proposition 3.5. *For any path graph P_n with $n \geq 4$, $\chi'_{rs}(P_n) = 3$*

Proof. Let us consider a path of order 4, consisting of three edges. Thus, it is obvious that 3 unique colors are required to properly color the graph as per the definition. Suppose the colors used are c_1, c_2, c_3, As the order of the graph increases, it is evident that in each subsequent cases it is possible to color the graph such that each color is repeated only at a distance of three. Since these three colors are distinct, it is possible to color the graph P_n using these colors. □

Theorem 3.6. *For any cycle graph C_n of order $n \geq 3$,*

$$\chi'_{rs}(C_n) = \begin{cases} 3 \text{ if } 3|n \\ 5 \text{ if } n = 5 \\ 4 \text{ otherwise} \end{cases}$$

Proof. We shall examine the possibilities for χ'_{rs} by taking two cases on the order of the cycle, C_n.

Case 1. In the case of a cycle of length 5, let the edges be $\{e_1, e_2, e_3, e_4, e_5\}$. It can be noted from Remark 3.1 that $\chi'_{rs}(C_5) \geq 3$. Thus, $3 \leq \chi'_{rs}(C_5) \leq 5$. Now, $\chi'_{rs}(C_5) = 5$ as the usage of 4 colors will result in at least one bichromatic subgraph becoming a P_4 which is not a star.

Case 2. When $n \neq 5$, we examine the coloring of the graph with the utilization of three cases.

Subcase 2.1. Suppose $n = 3k$, where $k = \{1, 2, 3, \dots\}$. Then for all values of k, the edge set $\{e_1, e_2, e_3, \dots, e_{3k}\}$ can be separated into k paths of distance 3

consisting of edges $\{e_{3i+1}, e_{3i+2}, e_{3i+3}\}$ where $i = \{0, 1, 2, \ldots, (k-1)\}$ and even if the color class$\{c_1, c_2, c_3\}$ is repeated no path of length 3 will become bicolored, thus ensuring all its bichromatic subgraphs form a galaxy. Thus, $\chi'_{rs}(G) = 3$, as per the coloring defined below.

$$c(e_i) = \begin{cases} c_1 \text{ if } i \equiv 1 \ (mod\ 3) \\ c_2 \text{ if } i \equiv 2 \ (mod\ 3) \\ c_3 \text{ if } i \equiv 0 \ (mod\ 3) \end{cases} \quad \forall\, i \in \mathbb{N}\ and\ i \leq n$$

Subcase 2.2. Suppose $n = 3k + 1$, then for any value of $k \in \mathbb{N}$, the edge set will consist of k paths of distance 3 consisting of edges consisting of edges $\{e_{3i+1}, e_{3i+2}, e_{3i+3}\}$ where $i = \{0, 1, 2, \ldots (k-1)\}$ along with 1 additional edge e_k which cannot be colored using the same color class. Thus, we require an additional color c_4 to reach a proper coloring, therefore $\chi'_{rs}(G) = 4$, as per the coloring defined below.

$$c(e_i) = \begin{cases} c_1 \text{ if } i \equiv 1 \ (mod\ 3) \\ c_2 \text{ if } i \equiv 2 \ (mod\ 3) \\ c_3 \text{ if } i \equiv 0 \ (mod\ 3) \end{cases} \quad \forall\, i \in \mathbb{N}\ and\ i \leq n-1, c(e_i) = c_4\ for\ i = n$$

Subcase 2.3. Suppose $n = 3k + 2$, then for any value of $k \in \mathbb{N}$, the edge set can be divided into k paths of distance 3 along with 2 additional edges. Depending on the nature of n, two possibilities remain.

Subcase 2.3.1. Suppose n is even, then separate the edge set to equal partition of length $n/2$. The edge partition would be $\{e_1, e_2, \ldots, e_l\}$ and $\{e_{l+1}, e_{l+2}, \ldots, e_{2l}\}$ where, $l = 2k$ color the first edge of both the partition $\{e_1, e_{l+1}\}$ using the color c_1. The remaining edges $\{e_1, e_2 \ldots, e_{2l}\}/\{e_1, e_{l+1}\}$ of the partitions can be colored using the rest of colors in the color class $\{c_2, c_3, c_4\}$ by repeating the colors between edges of distance more than 3, thus $\chi'_{rs}(G) = 4$. The coloring can be defined as follows,

$$c(e_i) = \begin{cases} c_2 \text{ if } i \equiv 2 \ (mod\ 3) \\ c_3 \text{ if } i \equiv 0 \ (mod\ 3) \\ c_4 \text{ if } i \equiv 1 \ (mod\ 3) \end{cases} \quad \forall\, i = \{2, 3 \ldots, l, l+2 \ldots, 2l\}$$

and $c(e_i) = c_1\ for\ i = 1, l+1$

Subcase 2.3.2. Suppose n is odd, then separate the edge set into two subsequent paths of length 4 consisting of edges $\{e_{4i+1}, e_{4i+2}, e_{4i+3}, e_{4i+4}\}$ for $i = \{0, 1\}$ and $k-2$ subsequent paths of length 3 consisting of edges $\{e_{3i}, e_{3i+1}, e_{3i+2}\}$ where $i =$

$\{3, 4, 5, \ldots k\}$. The paths of length 4 are colored using the colors $\{c_1, c_2, c_3, c_4\}$ and the paths of length 3 are colored using the colors $\{c_1, c_2, c_3\}$ by ensuring that the colors are repeated only at a distance more than 3, thus $\chi'_{rs}(G) = 4$ □

Theorem 3.7. *For any Wheel graph W_n of order $n \geq 4$ and $k \in \mathbb{N}$,*

$$\chi'_{rs}(W_n) = \begin{cases} n+4 & \textit{if } n = 6 \\ n+2 & \textit{if } n = 3k+1 \\ n+3 & otherwise \end{cases}$$

Proof. The process of coloring a wheel can be divided to two parts namely, coloring the edges of the outer cycle and coloring the edges connecting the inner vertex with the cycle. Now, since the a wheel graph of n vertices consist of a cycle of length $n-1$ and $n-1$ edges incident from the center vertex with the inner vertex being adjacent to all other vertices, the colors required to color the edges connecting the inner vertex and the outer cycle cannot be from the same color class.

Case 1. Let $n = 6$, it is trivial that 5 distinct colors are needed to color the edges incident with the center vertex and 5 other distinct colors are necessary to color the outer cycle as each edge is at a length of at most 3, then $\chi'_{rs}(G) = n + 4$.

Case 2. Let $n \neq 6$, then we observe the following cases.

Subcase 2.1 Suppose $n = 3k, k \in \mathbb{N}$, then the graph contains $3k-1$ edges connecting the outer cycle with center vertex that can be properly colored only with the usage of $3k-1$ colors, say $\{c_i : where\ i = 1, 2 \ldots, 3k-1\}$. Now the outer cycle can be colored using 4 additional colors as defined in theorem 3.6 using the colors $\{c_j : where\ j = 3k \ldots, 3k+3\}$. Thus, $3k+3$ colors are required in other words $n + 3$ colors.

Subcase 2.2 Suppose $n = 3k + 1, k \in \mathbb{N}$, then the graph contains $3k$ edges connecting the outer cycle with center vertex that can be properly colored only with the usage of $3k$ colors, say $\{c_i : where\ i = 1, 2 \ldots, 3k\}$. Now the outer cycle can be colored using 3 additional colors as defined in theorem 3.6 using the colors $\{c_j : where\ j = 3k+1, \ldots, 3k+3\}$. Thus, $3k+3$ colors are required in other words $\chi'_{rs}(G) = n + 2$.

Subcase 2.3 Suppose $n = 3k + 2, k \in \mathbb{N}$, then the graph contains $3k + 1$ edges connecting the outer cycle with center vertex that can be properly colored only with the usage of $3k+1$ colors, say $\{c_i : where\ i = 1, 2 \ldots, 3k+1\}$. Now the

outer cycle can be colored using 4 additional colors as defined in 3.6 using the colors $\{c_j : where\ j = 3k+2, \ldots, 3k+5\}$. Thus, $3k+5$ colors are required, thus $\chi'_{rs}(G) = n+3$. □

Proposition 3.8. *A complete bipartite graph $k_{m,n}$ with $m, n \geq 2$ has a proper mn-RSE coloring.*

Proof. Consider a complete bipartite graph consisting of $m+n$ vertices and mn edges such that $m > n$. A complete bipartite graph expresses a chromatic index of $max\{m, n\}$, which is in turn equal to its maximum degree Δ. Then, according to our hypothesis $\chi'(G) = m$. Therefore, with the usage of Remark 3.1 we acquire the inequality $m \leq \chi'_{rs}(k_{m,n}) \leq mn$.

Suppose $\chi'_{rs}(G) < mn$, then it implies it is possible to star color the graph properly with $mn-1$ colors. As per the structure of a complete bipartite graph considered, there exist at least one path of P_3 between every two vertices. This would result in at least one path of length 3 becoming bicolored.

Thus, it is not possible to find a proper coloring using $mn-1$ colors, resulting in a contradiction. Which in turn would imply that $\chi'_{rs}(G) = mn$, and the coloring using mn distinct colors is the remaining possibility for a proper restrained star coloring of the graph. □

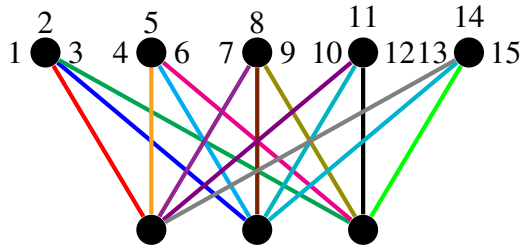

Figure 2: The RSE coloring of $k_{5,3}$

4 Application of Restrained star edge coloring

In this section, we explain one of the many possible applications of RSE coloring of graphs in the proper utilization of storage spaces. Out of all kinds of goods that are stored, the storage of chemicals must be handled with extreme caution as possible misplacing could end up in severe damages. Certain incompatible chemicals should not be stored together as their neutralization reactions might

result in production of heat and gas, which in turn might even cause fire to erupt. Such cases have also been reported in areas where the precautionary methods were not followed. And some of these fires are hard to put off due to its chemical reaction with the atmosphere. Thus, the importance of ensuring safe storage of such chemicals is extremely significant.

Generally, for the safety of the environment and our well being, it is absolutely necessary that these chemicals that are incompatible ought to be stored in separate divisions (rooms) if possible but such kind of storage spaces is not available in all laboratories that is low on space such as the chemistry laboratories in schools and colleges. In such cases, it is said that the chemicals should be placed at least at a distance of 3 meters apart. This is where the concept of RSE coloring can be used to ensure that such incompatible chemicals are placed at a distance of at least 3 meters as according to its definition, every bichromatic subgraph is form a galaxy. Therefore, by converting the area available to a graph and by finding its χ'_{rs} we can find the optimal placement of such chemicals, thus minimizing the risk of fire.

Firstly, we assign color labels to the chemicals such that chemicals that cannot be stored together are labelled using the same color. This is applicable as per the basic definition of graph coloring which would ensure that a particular color cannot be positioned together with the same color. Thus, the chemicals labelled by the color say c_1 cannot be placed together with another chemical with the same labelling. After this we compare the storage area available to square grid of length 1.5 meters and construct the graph by plotting its vertices and edges. Then by finding the RSE-coloring of the graph thus constructed we can find the optimal and safe way to store these chemicals.

Algorithm:

Step 1: Compare the area to a grid and figure out the area available for storage

Step 2: Select the area satisfying the requirement and ignore the remaining areas

Step 3: Plot the vertices of the graph by assigning a vertex to every adjacent corner in such a way that each square contains 2 vertices.

Step 4: Join the vertices to get the edges of the graph.

Step 5: Find the RSE coloring of the graph obtained from the previous steps.

Step 6: Convert the coloring obtained by assigning the said color to the squares of the grid.

Step 7: This colored squares would provide the optimal way to store the chemicals in a safe manner.

Example:

Consider the area map as given in the Figure 3, firstly we compare the area with grids of length 1.5 meters and discard the squares whose area is less than that of 2.25 metres. Then two squares will make a distance of 3 meters and two squares diagonally will have a distance of 4 metres between them.

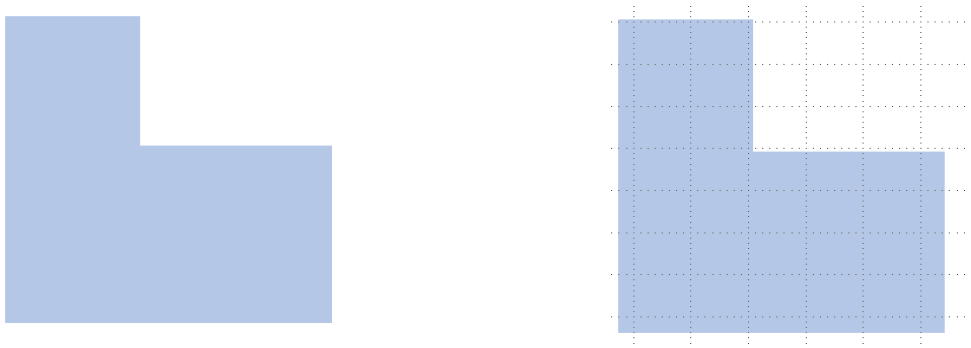

Figure 3: The area map and its comparison with a grid of length 1.5m

Plot the vertices on the corner of each grid in a way that each grid would have two diagonal vertices as illustrated in Figure 3. Then by joining these two vertices of each square we get the edges. And by joining all the vertices we get a connected graph.

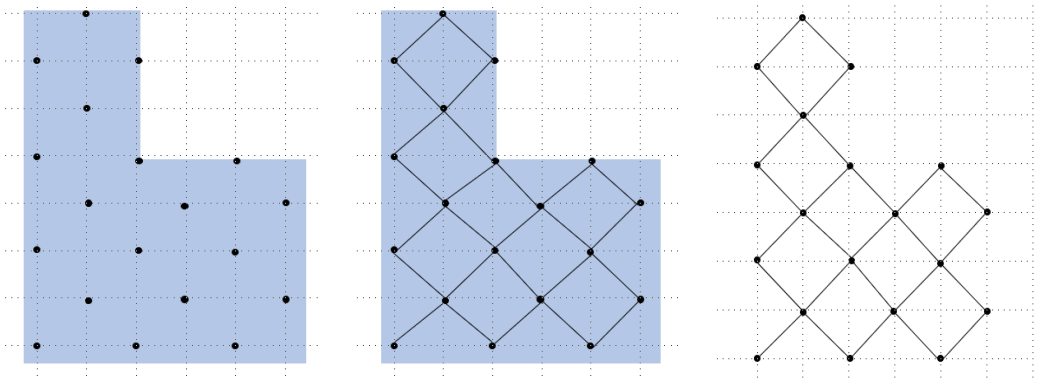

Figure 4: The process of plotting the graph

Then by finding the RSE coloring of this graph, we can find the optimal way to store the chemicals in such a way that every incompatible chemicals are kept at a minimum distance of 3 meters.

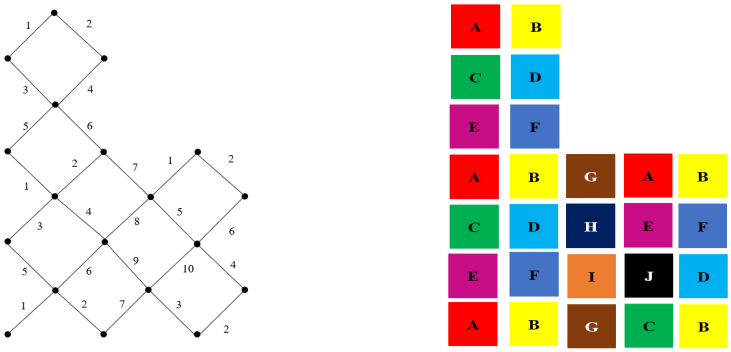

Figure 5: The RSE coloring and the optimal storage as per the area available

It can be noted that each chemical denoted by a particular color say (red) is incompatible to a chemical of the same color (in this case, red). The color that is distinct from that particular color (red) is used to denote either a chemical that is compatible with the chemical of that particular color or empty spaces if no other such chemicals are available. In this we have used the RSE coloring and its respective χ'_{rs} to find the optimal way of storage of these hazardous chemicals. However it can be noted that in the case of smaller scale the same results can be achieved with the help of restrained star vertex coloring and its respective restrained star chromatic number as well.

5 Results and Discussions

The RSE coloring of graphs is found to be an alluring topic with boundless prospects. It is to be noted that it is a challenging task to find the exact value of χ'_{rs} when the graph under consideration is complex, especially if the said graphs is obtained as a result of some graph operations. In such cases this paper would serve as a basis for understanding the said coloring of the basics of such complex graphs and would help in further generalizations of the RSE coloring.

6 Conclusions

Thus, in this paper the RSE coloring of few families of graphs including path, cycle, wheel, complete bipartite graph is defined and the definite values of its respective χ'_{rs} is provided. In addition to this, a possible application of RSE coloring in ensuring safe storage of chemicals is discussed in section 4. It was observed that for any graph G its star chromatic index is less than or equal to restrained star chromatic index, in other words $\chi'_s(G) \leq \chi'_{rs}(G)$.

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Short-time Fourier transform based ensemble classifiers for detection of atrial fibrillation from ECG datasets

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Abstract

Atrial Fibrillation (Afib) is a common cardiac arrhythmia characterized by irregular and often rapid heart rate, leading to inefficient blood pumping from the atria. It increases the risk of stroke, heart failure, and other heart-related complications. Afib is often associated with symptoms like palpitations, shortness of breath, and fatigue. Diagnosis typically involves Electrocardiography (ECG) to detect irregular electrical activity in the heart. Treatment options range from medication to procedures like catheter ablation, aimed at restoring normal heart rhythm and reducing associated risks. Soft computing methods can aid in automating the classification of cardiovascular diseases, assisting clinicians in diagnosing arrhythmias. In this research paper, ensemble classifiers are employed for the classification of Atrial Fibrillation based on ECG datasets. When utilizing the Catboost Classifier in conjunction with the STFT-based GEO implementation, the results indicate an average perfect classification rate of approximately 99%, an error rate of 1%, and a kappa coefficient of 0.9689% for detection of Afib.

Keywords: Adaboost, Cardiovascular Disease, Catboost, ECG, XGboost.

2010 AMS subject classification: 92C50, 92C55.[§]

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1. Introduction

In the empire of medical diagnostics, accurate and timely detection of cardiac arrhythmias holds immense importance for effective patient care. One such significant arrhythmia is atrial fibrillation (Afib), characterized by irregular and often rapid heartbeats. Afib detection plays a crucial role in the diagnosis and treatment of cardiovascular conditions, as its early identification can prevent potentially serious complications [1]. To address the challenges associated with Afib detection, researchers and medical professionals are increasingly turning to advanced signal processing techniques and machine learning approaches. Here are some notable accomplishments within the field of Afib detection. Attia et al. [2] employed deep learning techniques (AFCODE) for the detection of Afib, this AFCODE study achieved an accuracy of 84% in detecting Afib. Clifford et al. [3] utilized the different learning techniques for CinC Challenge 2017. The winning entry in the 2017 CinC Challenge achieved an accuracy of 92.7% in distinguishing Afib from other rhythms. Bumgardner et al. [4] applied the Deep Learning-based detection of Afib using a Smartwatch. This study reported an accuracy of 90.2% in detecting Afib from smartwatch ECG data. Hannun et al. [5] employed the Convolutional Neural Networks (CNN) for Afib Detection, the CNN model achieved an accuracy of 92% in Afib detection. Kiranyaz et al. [6] engaged the Afib detection from 12-lead ECG Recordings. This research achieved an accuracy of 98.8% in classifying Afib from 12-lead ECG recordings.

Smith et al. [7] utilized the deep learning-based detection of Afib and achieved an accuracy of 92.5% in detecting atrial fibrillation using a CNN on ECG data. Chen et al. [8] applied the deep learning model for the detection of atrial fibrillation, this study reported an accuracy of 95.2% in atrial fibrillation detection with a deep neural network applied to single-lead ECG signals. Kim et al. [9] demonstrated an accuracy of 93.8% in detecting atrial fibrillation using a machine learning model applied to data from wearable ECG devices. Li et al. [10] employed a random forest and CNN model for the detection of atrial fibrillation from ECG data. Achieved an accuracy of 91.7% in atrial fibrillation detection using a combination of random forest and CNN on ECG data. Wang et al. [11] utilized Recurrent Neural Networks (RNN) with Support Vector Machines (SVM) for the detection of Afib from ECG signals. This model reported an accuracy of 94.3% in detecting atrial fibrillation using a hybrid model combining RNN and support vector machines.

Rajpurkar et al. [12] employed the deep learning model for the classification of cardiovascular signals, in this research demonstrated an accuracy of 98 % for AF detection using deep learning techniques. Shashikumar et al. [13] utilized the deep learning model for the detection of Afib from single-lead ECG recordings. Achieved an accuracy of 92.5% in detecting Afib from short single-lead ECG recordings. Asgari et al.

[14] employed Afib detection using data from wearable smartwatches with a reported accuracy of 91.7%. Li et al. [15] utilized the deep learning model for the detection of Afib, in this study achieved an accuracy of 94.5% in identifying Afib from ECG signals. Liu et al. [16] employed the deep learning model for the classification of Afib diseases using mobile health observation. Reported an accuracy of 94% in Afib detection using deep learning models. Xiong et al. [17] employed the CNN model for the detection of Afib disease from ECG data. This model reported an accuracy of 95.3% in detecting AF from ECG signals. Lee et al. [18] employed the machine learning classification model for the detection of Afib from photoplethysmography datasets. Achieved an accuracy of 91.2% in AF detection from photoplethysmography datasets. Ma et al. [19] proposed a novel technique of LSTM-RNN algorithm for the detection of Afib from ECG datasets. Reported an accuracy of 93.7% in AF detection using LSTM-based recurrent neural networks.

Maturo and Verde [20] employed the integration of random forest and functional data analysis for the classification of the electrocardiogram data. This model reported a training accuracy of 97.28% and a testing accuracy of 93.64%. Maturo and Verde [21] proposed novel techniques of functional random forest with functional principal components and functional KNN algorithms for ECG data. Reported the highest accuracy of 94% in ECG data detection using functional random forest with functional principal components technique. Maturo and Verde [22] utilized the functional classification trees, functional random forest, and functional bagging, in this investigation achieved a best training accuracy of 98.51% and testing accuracy of 97.57% in the detection of electrical power demand using a functional random forest classifier.

This introduction delves into the pivotal role of feature extraction and optimization algorithms in the proposal of Afib detection. Feature extraction is a fundamental step in analyzing electrocardiogram (ECG) signals to unveil distinctive patterns indicative of Afib. In this study, explores the significance of Afib detection, the role of feature extraction techniques like the Short-Time Fourier Transform (STFT), and the feature selection offered by nature-inspired optimization algorithms such as the Spider Monkey Optimization (SMO) and Golden Eagle Optimization (GEO) algorithms for enhancing the accuracy and efficiency of Afib detection models. Finally, Ensemble classifiers are employed as a classifier for the detection of Afib. By merging scientific inquiry with computational innovation, the medical community is poised to achieve advancements in the realm of Afib detection, ultimately leading to improved patient outcomes and

enhanced cardiac health management. Figure 1 displays the block diagram of the proposed process.

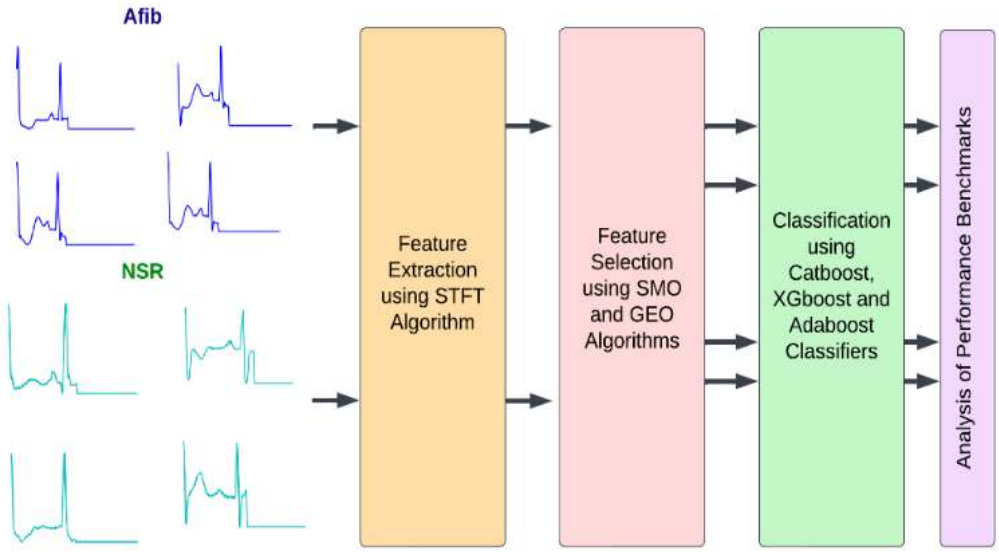

Figure 1. Block diagram of the proposed process

2. Materials and methods

The dataset consists of meticulously sourced ECG data from electrocardiograms (ECGs), obtained from Kaggle (https://www.kaggle.com) [23]. Its primary goal is to enable accurate detection of atrial fibrillation, a crucial cardiac arrhythmia marked by irregular and accelerated heartbeats. This dataset holds a wealth of visual insights derived from ECG recordings, providing a fertile ground for developing and evaluating advanced algorithms and models tailored for precise atrial fibrillation (Afib) detection. Sourced from Kaggle, a renowned platform for sharing diverse datasets, the dataset revolves around ECG datasets that capture the intricate electrical dynamics of the heart. These datasets visually represent the heart's rhythm, unveiling potential deviations that could indicate Afib.

In this paper, The Afib category includes 50 recordings (65000 Samples per Recordings), each sampled at a frequency of 250 Hz, with a sampling interval of 0.004 seconds. Within the Afib category, there are a total of 13,000 epochs, and each epoch contains 250 samples. The NSR category includes 50 recordings (65000 Samples per Recordings), each sampled at a frequency of 128 Hz, with a sampling interval of 0.078 seconds. Within the NSR category, there are a total of 25, 400 epochs and each epoch

contains 128 samples. Table.1. provides an overview of the dataset particulars in this study.

Category	Total Recordings	Samples per Recordings	No. of Epochs	Samples per epochs
Afib	50	65000	13000	250
NSR	50	65000	25400	128

Table 1. Overview of the dataset

To extract features from cardiac signals, they are divided into epochs, allowing for the convenient computation of statistical features and the tracking of cardiac signal evolution. In this study, feature extraction serves to streamline the representation of extensive cardiac data efficiently. The STFT algorithm is employed to extract features from both Afib and NSR signals.

3. Feature extraction using short-time Fourier transform (STFT) algorithm

Atrial fibrillation (Afib) is a common cardiac arrhythmia marked by irregular and rapid heartbeats. Precise Afib detection is crucial for accurate diagnosis and effective treatment. In this proposal, feature extraction holds a pivotal role in signal processing and machine learning approaches for Afib detection. An effective technique used for this purpose is the Short-Time Fourier Transform (STFT). The STFT is a versatile method for time-frequency analysis, providing insights into signal frequency dynamics over time. It breaks down a signal into frequency components, yielding a comprehensive view of its frequency variations in a two-dimensional spectrogram matrix. Initially, the input electrocardiogram (ECG) signal, reflecting heart electrical activity, is divided into overlapping windows of fixed length. These short windows capture local signal variations that hint at irregular heartbeats in Afib cases [24].

$$X(\tau, f) = \int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-j2\pi ft}dt \quad (1)$$

Where $X(\tau, f)$ represents the STFT coefficient at time τ and frequency f, $x(t)$ indicates the windowed signal, $w(t-\tau)$ represents the window function centered at τ, $e^{-j2\pi ft}$ indicates the complex exponential for frequency f. The magnitudes of the FFT results for each segment are used to construct the spectrogram. The x-axis represents time, the y-axis represents frequency, and the color or intensity represents the magnitude of the

corresponding frequency component. Calculate the magnitude or power of the STFT coefficients [25]. This can be done using:

$$S(\tau, f) = |X(\tau, f)|^2 \quad (2)$$

Where $S(\tau, f)$ represents the power spectral density at time τ and frequency f. The resulting spectrogram can be used for feature extraction. Various features can be extracted from the spectrogram to capture different aspects of the signal's frequency content over time.

After feature extraction, within the Afib category, there are a total of 1250 epochs, and each epoch contains 250 samples. Within the NSR category, there are a total of 2441 epochs and each epoch contains 128 samples. In this paper statistical measures are analyzed such as mean, variance (VAR), skewness, kurtosis, Pearson correlation coefficient, sample entropy (SamEn), and P-value. A significance level of $P = 0.01$ defines the threshold for the study's significance. If $P \leq 0.01$ is observed, it indicates a strong level of significance. Conversely, when $P \geq 0.01$ is observed, it suggests a lack of significance. Table.2. presents an investigation into the Average Statistical Metrics derived from the STFT Feature Extraction Algorithm. Table 2. reveals that the statistical parameters exhibit skewness and a flat kurtosis nature. Figure 2. represents the histogram difference analysis of STFT features for Afib and NSR classes. The Normal probability analysis of STFT features for Afib and NSR classes is represented in Figure 3.

Class	**Mean**	**VAR**	**Skewness**	**Kurtosis**	**PCC**	**SamEn**	**P-value**
AFib	0.003	0.004	-0.032	10.078	0.007	8.008	0.419
NSR	0.045	0.0001	0.059	-1.168	0.995	3.055	0.250

Table 2. Average statistical measures for STFT feature extraction algorithm

Table.2. reveals that the statistical parameters display skewed distributions, with a flattened kurtosis nature. The Pearson Correlation Coefficient (PCC) demonstrates negligible correlation among intra-class features. Sample Entropy highlights significant deviations between classes. P values indicates the lack of significant.

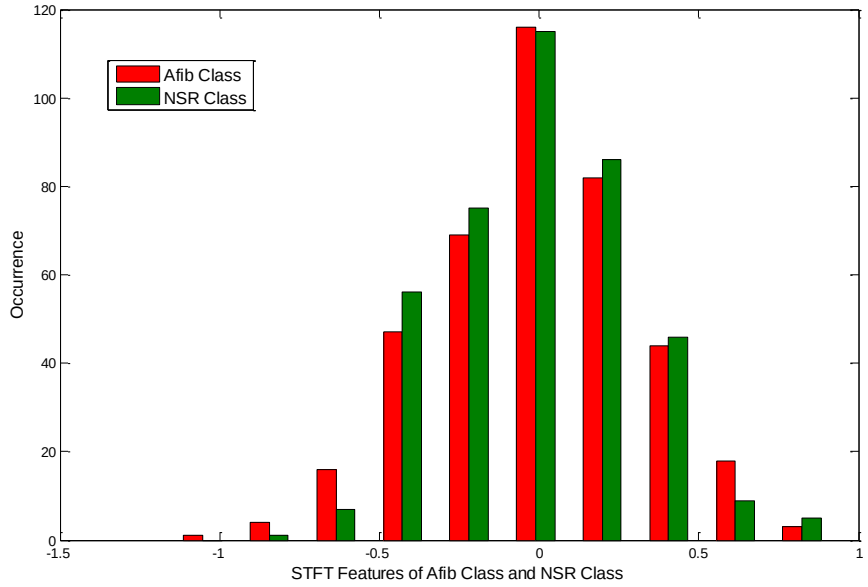

Figure 2. Histogram difference analysis STFT features of Afib and NSR classes.

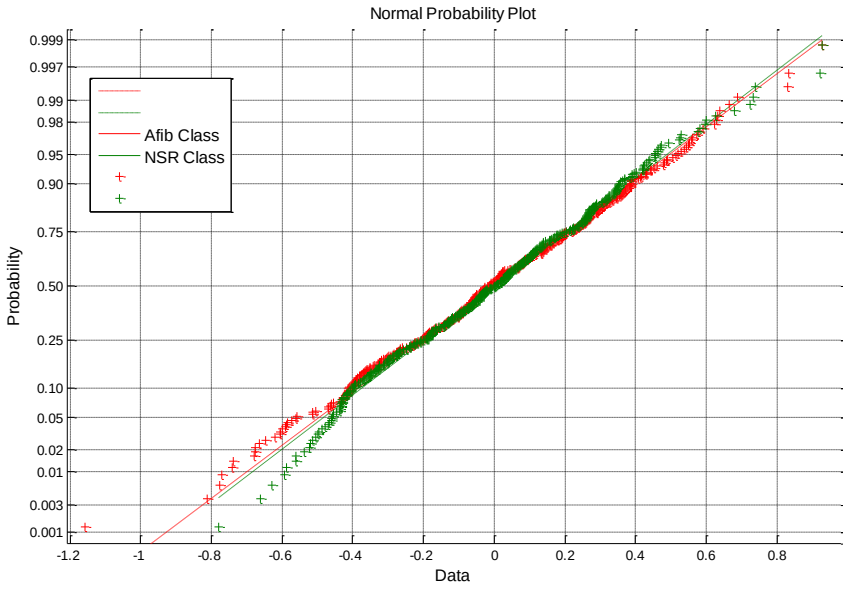

Figure 3. Normal probability analysis of STFT features of Afib and NSR classes.

From figure 2 and figure 3 represent the high overlapping, Outliers, and Nonlinearity of the classes. Consequently, the feature selection process is anticipated to enhance overall performance efficiency.

4. Feature selection using nature-inspired optimization algorithms

Detecting atrial fibrillation (Afib), a common cardiac arrhythmia characterized by irregular and often rapid heartbeats is a critical task in modern healthcare. Signal processing and machine learning techniques have emerged as invaluable tools in Afib detection, with feature selection playing a pivotal role in enhancing the accuracy and efficiency of these methods. Nature-inspired optimization algorithms have gained prominence in recent years as effective means of feature selection, aiding in the identification of the most relevant and informative features from complex datasets. Two such algorithms, the Spider Monkey Optimization (SMO) Algorithm and the Golden Eagle Optimization (GEO) Algorithm, have shown promise in addressing the intricacies of Afib detection.

4. 1 Spider monkey optimization (SMO) algorithm

The SMO algorithm takes its cues from the intricate conduct of spider monkeys as they navigate their surroundings in pursuit of sustenance and necessities. These monkeys showcase extraordinary social acumen and adaptability, characteristics that have been computationally leveraged to tackle intricate optimization challenges, including the selective extraction of features to detect Afib. The SMO algorithm functioning closely emulates the foraging behavior of spider monkeys. It employs a population of potential solutions, representing different feature subsets. These solutions evolve over generations through processes akin to monkey social behaviors, such as exploration, exploitation, and communication. Here are the general steps involved in the Spider Monkey Optimization algorithm [26]:

Step 1: **Initialization:** Generate an initial population of spider monkeys (solution candidates) randomly or through some heuristic method.

$$SM_{pq} = SM_{minq} + G(0,1) * \left(SM_{maxq} - SM_{minq}\right) \quad (3)$$

Where SM_{maxq} and SM_{minq} indicates the upper bound and lower of the dimension and $G(0,1)$ represents the random number.

Step 2: **Ranking:** Rank the monkeys in the population based on their fitness values. This can be done in ascending or descending order, depending on whether you are minimizing or maximizing the objective function.

Step 3: **Leader Selection:** Choose the monkey with the best fitness as the leader. This monkey will guide the exploration and exploitation phases of the algorithm.

Step 4: **Exploration Phase:** Select a subgroup of monkeys, known as exploratory monkeys, to explore new solutions. These monkeys will randomly move in the search space to discover potential solutions.

Step 5: **Exploitation Phase:** Select another subgroup of monkeys, known as exploitative monkeys. These monkeys will exploit the information provided by the leader and the exploratory monkeys to refine solutions in promising areas of the search space.

Step 6: **Update Leader:** Recalculate the fitness of the leader monkey based on the new solutions discovered during the exploration and exploitation phases. If a monkey with better fitness is found during this process, update the leader.

$$SM_{newpq} = SM_{pq} + G(0,1) * \left(GL_q - SM_{pq}\right) + G(0,1) * \left(SM_{tq} - LL_{kq}\right) \quad (4)$$

Where GL_q represents the local leader stage and LL_{kq} indicates the local leader stage.

Step 7: **Stopping Criterion & Termination:** Check if a stopping criterion is met, such as a maximum number of iterations, a convergence threshold, or a predetermined runtime. If the stopping criterion is met, end the algorithm and return the best solution found.

Just as spider monkeys explore their surroundings for various food sources, the algorithm explores different feature combinations to determine their efficacy in enhancing Afib detection accuracy. Through a process of exploitation, promising feature subsets are fine-tuned to improve performance. The algorithm incorporates a communication mechanism similar to the social interactions observed in spider monkey groups. This communication enables sharing of knowledge and solutions among different members of the population, fostering the discovery of optimal feature subsets. The fitness of each potential solution is evaluated based on its ability to effectively discriminate between Afib and normal heart rhythms. This is typically done using a machine learning classifier trained on the selected features.

4. 2 Golden eagle optimization (GEO) algorithm

The Golden Eagle Optimization (GEO) Algorithm takes its inspiration from the majestic hunting strategies of the golden eagle. Renowned for their exceptional visual acuity, strategic hunting tactics, and precise prey capture, these eagles serve as a model for the GEO algorithm's optimization approach. This algorithm harnesses these attributes to tackle intricate problems, such as feature selection for the detection of atrial fibrillation (Afib). The GEO algorithm closely mirrors the hunting conduct of golden eagles, employing a population of potential solutions in a dynamic manner. It incorporates mechanisms that are informed by the eagle's adept hunting prowess. Figure 4 represents the general steps involved in the Golden Eagle Optimization Algorithm [27]:

Much like the methodical surveying undertaken by golden eagles to pinpoint their prey, the GEO algorithm scrutinizes various feature subsets to identify those that yield the highest discriminatory potency in Afib detection. This approach mirrors the eagle's attentive assessment of its surroundings for potential targets. Similarly, as populations of golden eagles adapt to shifts in their environment, the GEO algorithm maintains a diverse collection of solutions that evolves across generations. This diversity assures a

comprehensive exploration of feature combinations and augments the prospects of discovering optimal subsets. The algorithm introduces selective pressure, akin to the eagles' unwavering focus on securing the most suitable prey. This pressure underscores the significance of the most adept solutions, reinforcing their prominence in contributing to subsequent generations of solutions. This strategy aligns with the eagles' meticulous selection of their quarry.

In the empire of Afib detection, both the Spider Monkey Optimization (SMO) and Golden Eagle Optimization (GEO) algorithms contribute to the evolution of feature selection techniques. By mimicking the behaviors of spider monkeys and golden eagles, respectively, these algorithms offer novel avenues for navigating the intricate feature space landscape. This endeavor ultimately enhances the precision and efficiency of models employed in Afib detection. It's imperative, however, to acknowledge that while these algorithms exhibit promise, their application must be subject to rigorous validation and meticulous calibration to harmonize with the unique attributes of Afib datasets and the exigencies of detection requirements.

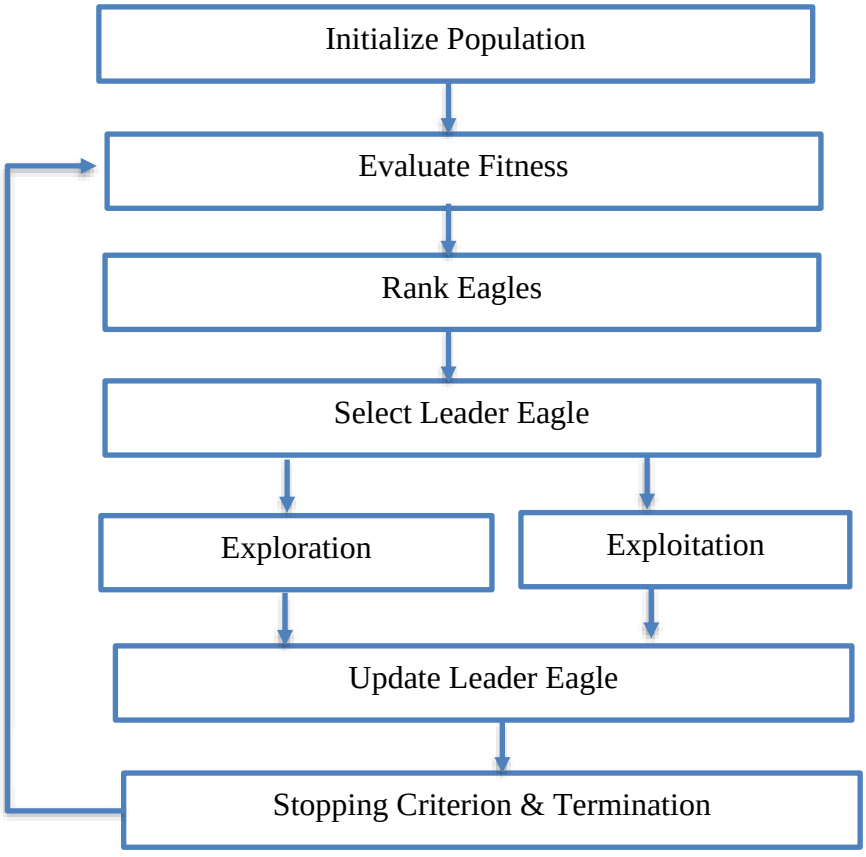

Figure 4. Steps of GEO algorithm.

After feature extraction, within the Afib category, there are a total of 400 epochs, and each epoch contains 250 samples. Within the NSR category, there are a total of 781 epochs and each epoch contains 128 samples. Table.3. presents an investigation into the

Average Statistical Metrics derived from the STFT features with SMO and GEO Feature Selection Algorithms. Table 3. reveals that the statistical parameters exhibit skewness and a flat kurtosis nature. Figure 5. represents the histogram difference analysis of STFT features with SMO algorithm for Afib and NSR classes. The Normal probability analysis of STFT features with SMO algorithm for Afib and NSR classes are represented by Figure 6. Figure 7. represents the histogram difference analysis of STFT features with GEO algorithm for Afib and NSR classes. The Normal probability analysis of STFT features with GEO algorithm for Afib and NSR classes are represented by Figure 8.

STFT Features with Spider Monkey Optimization (SMO) Algorithm							
Class	Mean	VAR	Skewness	Kurtosis	PCC	SamEn	P-value
AFib	0.143	0.0002	0.173	-1.142	0.996	5.481	0.00081
NSR	0.131	0.0004	0.361	-1.041	0.966	2.982	0.00010
STFT Features with Golden Eagle Optimization (GEO) Algorithm							
Class	Mean	VAR	Skewness	Kurtosis	PCC	SamEn	P-value
AFib	0.099	0.000003	0.166	-1.009	0.915	3.079	0.00022
NSR	0.781	0.00005	0.722	1.699	0.647	2.182	0.00030

Table 3. Average statistical measures for STFT features with SMO algorithm and STFT features with GEO Algorithm

Table 3 reveals that the statistical parameters display skewed distributions, with a flattened kurtosis nature. The Pearson Correlation coefficient demonstrates less correlation among intra-class features. Sample Entropy highlights less significant deviations between classes. P values indicates the highly of significant.

From figure 5, 6, 7 and 8 represents the less overlapping, less Outliers and less Nonlinearity of the classes. Compared to SMO and GEO algorithms, the GEO algorithms gives the better solutions.

Consequently, the selection of suitable classifiers is anticipated to enhance overall performance efficiency.

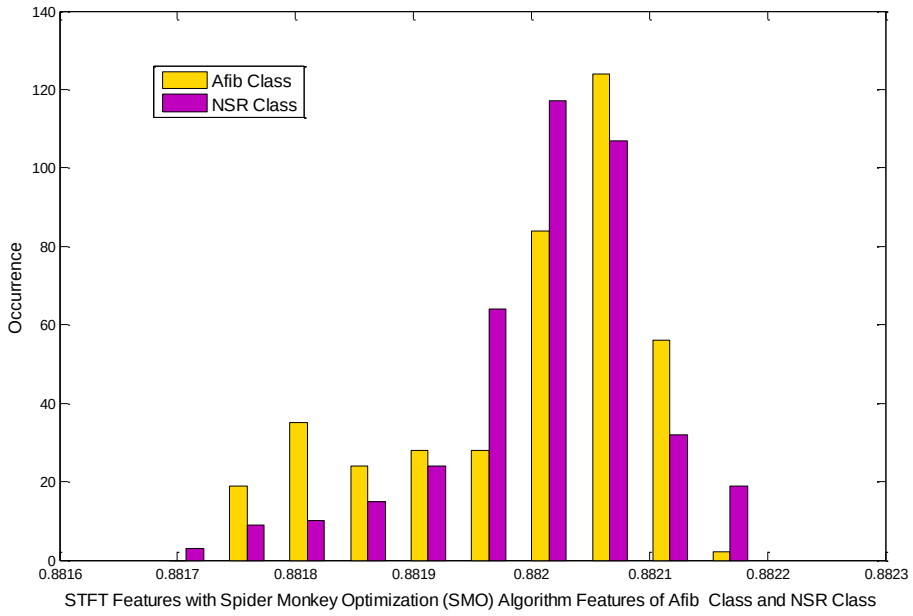

Figure 5. Histogram difference analysis STFT features with SMO features of Afib and NSR classes.

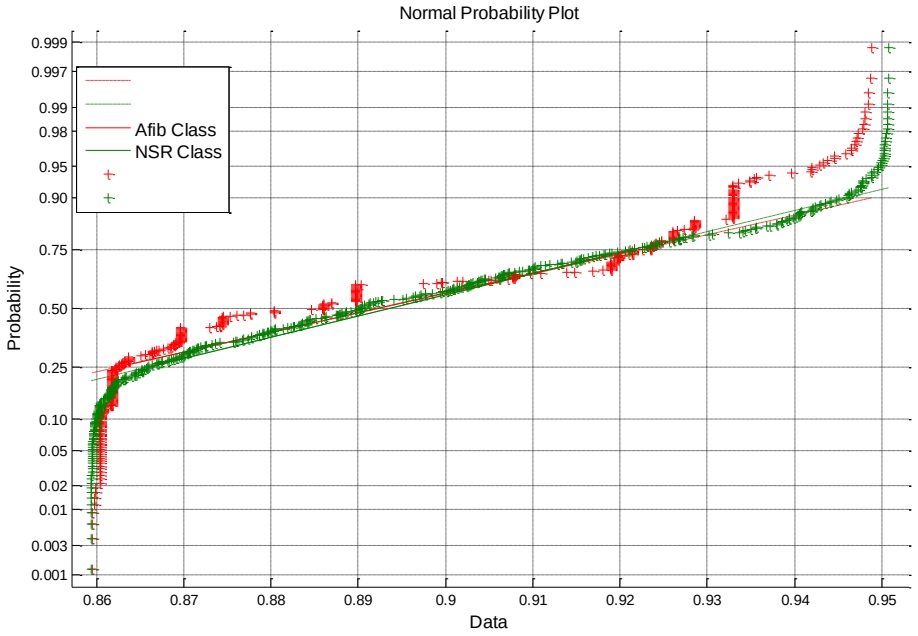

Figure 6. Normal probability analysis of STFT features with smo features of Afib and NSR classes.

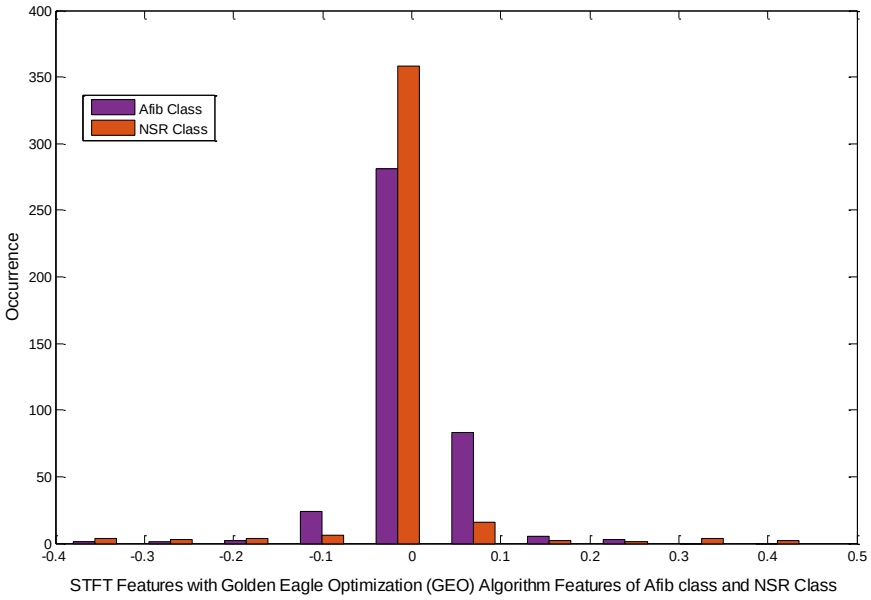

Figure 7. Histogram difference analysis STFT features with GEO features of Afib and NSR classes.

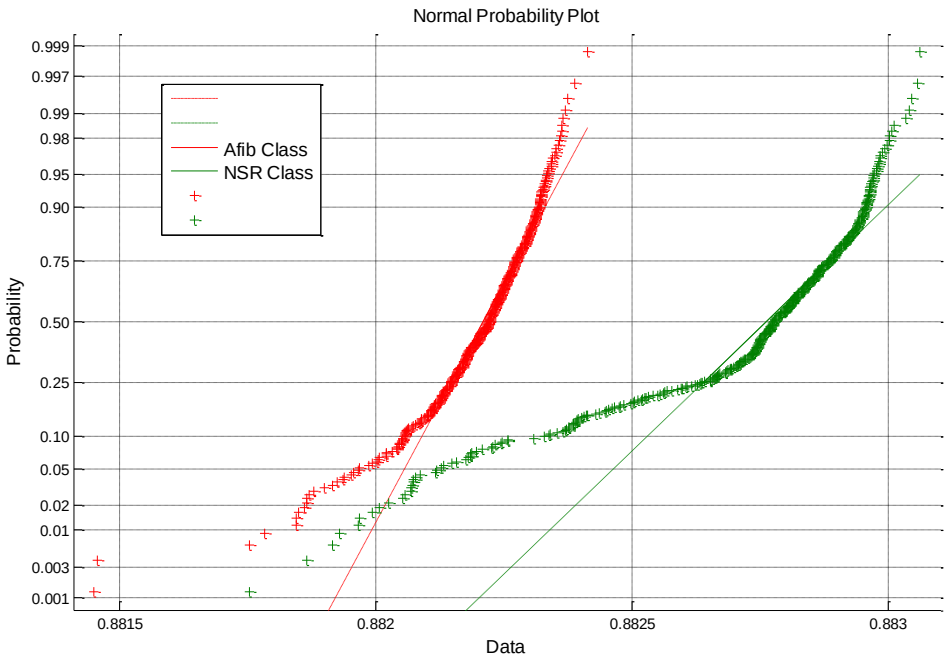

Figure 8. Normal probability analysis STFT features with geo features of Afib and NSR classes.

5. Classifications using ensemble classifiers

The features obtained through feature extraction, along with the subsequently reduced feature set, undergo analysis and comparison using ensemble classifiers. The detection of atrial fibrillation (Afib) holds paramount importance within the realm of cardiac health assessment. The precise identification of this irregular heart rhythm plays a pivotal role in enabling timely intervention and ultimately enhancing patient outcomes. To bolster the accuracy and reliability of Afib classification, the application of ensemble classifiers emerges as a powerful strategy. These ensemble classifiers represent a potent category of machine learning algorithms that harness the collective strengths of multiple individual models. In this proposal, we delve into three prominent ensemble classifiers: The Catboost Classifier, the XGboost Classifier, and the Adaboost Classifier.

5. 1 Catboost classifier

The Catboost Classifier, an advanced machine learning algorithm at the forefront of innovation, stands out as a valuable asset in the quest for precise detection of atrial fibrillation (Afib). Afib, characterized by irregular and often accelerated heart rhythms, necessitates accurate identification to facilitate effective medical intervention. Harnessing the distinct capabilities of the Catboost Classifier opens a pathway toward enhanced diagnostic outcomes. The Catboost Classifier distinguishes itself through its role as a resilient ensemble algorithm, meticulously crafted not only to adeptly manage categorical features but also to optimize the intricate process of gradient boosting. The general equation for Catboost can be represented as [28]:

$$Y_{pred} = \sum X + shrinkage \tag{5}$$

Where Y_{pred} represents the predicted output (class probability) for a given instance, X indicates the prediction of an individual base model (usually decision trees) and shrinkage represents the regularization term that controls the contribution of each base model. It exhibits a remarkable ability to seamlessly integrate categorical feature information, effectively mitigating the intricate preprocessing challenges frequently associated with such data. A standout feature within the Catboost algorithm is its introduction of the concept of ordered boosting. This innovative technique empowers the sequential training of models, strategically capitalizing on the complex interplay between features and the target variable. This intelligent approach allows the algorithm to harness the relationships between these factors, leading to more refined and accurate classification results.

5. 2 XGboost classifier

The XGboost Classifier stands out as a notable exemplar within the ensemble methods framework, firmly rooted in the foundational principles of gradient boosting. Its underlying architecture is characterized by an ensemble of decision trees, each systematically trained in succession. This coherent arrangement ensures that subsequent trees within the ensemble are dedicated to rectifying the misclassifications committed by their precursors. The XGboost algorithm has gained widespread recognition, largely attributable to its exceptional computational efficiency, along with its commendable versatility in accommodating an extensive range of data types. Additionally, the algorithm integrates crucial regularization mechanisms, strategically employed as safeguards to adeptly mitigate the perils associated with overfitting. Within the realm of atrial fibrillation detection, this robust classifier assumes particular importance, wielding its prowess to effectively contribute to accurate and reliable diagnosis. The equation for XGboost can be represented as [29]:

$$Y_{pred} = \sum X + \gamma * \sum \tag{6}$$

Where γ represents the learning rate, controlling the step size during optimization and $\sum$ indicates the regularization terms applied to each tree to prevent overfitting.

5. 2 Adaboost classifier

The Adaboost (Adaptive Boosting) Classifier presents a distinctive approach designed to enhance the precision of its predictive outcomes. This ensemble technique embarks on an iterative journey, progressively refining the classification abilities of individual "weak learners." Instances that encountered misclassification in prior iterations are granted amplified importance through higher weights, thus directing the classifier's attention towards the intricate nuances posed by these challenging samples. The equation for Adaboost can be represented as [30]:

$$Y_{pred} = sign\left(\sum(\alpha * base_model(x))\right) \tag{7}$$

Where $sign()$ indicates the function that maps positive values to +1 and negative values to -1 and α indicates the weight for the prediction of each base model. As the iterations unfold, succeeding "weak learners" are strategically crafted to rectify the previously identified misclassifications. This strategic orchestration ultimately leads to the formation of a potent final model, poised to tackle complex tasks such as the detection of atrial fibrillation.

6. Results and discussion

To conduct a comprehensive analysis of Afib classification, we examine various metrics. These metrics encompass Accuracy, Error Rate and Kappa Coefficient Measures. A 10-fold cross-validation approach is utilized, where in each iteration, one subset (representing 10% of the data) is used for testing, while the remaining nine subsets (90%) are used for training. This folding process is repeated 10 times, and subsequently, the average performance metrics across all the folds are computed and compared. These metrics are expressed through the following mathematical formulas [31]:

$$Accuracy = \frac{TN + TP}{TN + TP + FN + FP} * 100 \tag{8}$$

$$Error\ Rate = \frac{FN + FP}{TN + TP + FN + FP} * 100 \tag{9}$$

$$Kappa\ Coefficient = \frac{accuracy - expected\ accuracy}{1 - expected\ accuracy} \tag{10}$$

The confusion matrices analysis on multiple ensemble classifiers utilizing SMO and GEO feature extraction methods for distinguishing between Afib and NSR cardiac detection represented by Table 4. From Table.4. STFT feature based GEO feature selection algorithm with Catboost classifier achieved the higher accuracy of 99%, 1% of error rate and 0.9689 Kappa Coefficient for detection of Afib category. Figure 9. indicates the graphical analysis of Multiple Ensemble Classifiers. Table 5 highlights the superior outcome achieved by the proposed approach when compared to prior studies.

Models	**Classifiers**	**Confusion Matrices**				**Performance Metrics**		
		TP	**TN**	**FP**	**FN**	**Accuracy**	**Error Rate**	**Kappa Coefficient**
STFT	Catboost	285	612	169	115	76	24	0.4815
	XGboost	248	594	187	152	71	29	0.3734
	Adaboost	238	528	253	162	64	35	0.2553
STFT with SMO	Catboost	357	704	77	43	90	10	0.7780
	XGboost	342	683	98	58	87	13	0.7121
	Adaboost	321	645	136	79	82	18	0.6066
	Catboost	**392**	**773**	**8**	**8**	**99**	**1**	**0.9689**

STFT with GEO	XGboost	360	732	49	40	93	7	0.8339
	Adaboost	338	714	67	62	89	10	0.7558

Table 4. The confusion matrices average analysis on multiple ensemble classifiers utilizing SMO and GEO feature extraction methods for Afib cardiac detection

S.No	**References**	**Models**	**Accuracy in %**
1	Attia et al. [2]	AFCODE	84
2	Bumgardner et al. [4]	Deep learning model	90.02
2	Li et al. [10]	Random forest and CNN	91.7
3	Wang et al. [11]	RNN with SVM	94.3
4	Maturo and Verde [20]	Integration of random forest and functional data analysis	Training accuracy :97.28 and testing accuracy: 93.64
5	**In this paper**	STFT feature based GEO feature selection algorithm with Catboost classifier	99

Table 5. Comparison of the proposed approach with prior studies

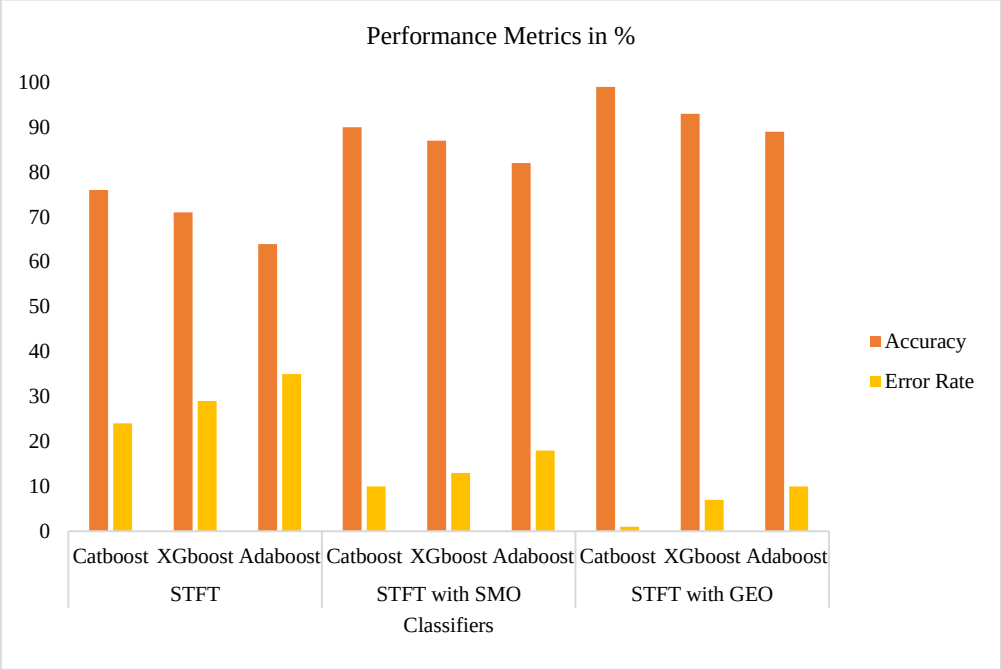

Figure 6. Normal probability analysis of STFT features with SMO features of Afib and NSR classes.

7. Conclusion

In this study, we introduce a novel approach for classifying Atrial Fibrillation (Afib) from ECG datas. When utilizing the Catboost Classifier in conjunction with the STFT-based GEO implementation, the results indicate an average perfect classification rate of approximately 99%, an error rate of 1%, and a kappa coefficient of 0.9689% for detection of Afib. Future plans involve integrating optimization techniques within post-classification processes and exploring a range of deep learning methods to enhance the precision of cardiovascular disease classification from ECG signals.

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Polynomial collocation methods based on successive integration technique for solving system of neutral delay differential equations

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Abstract

This paper presents a new approach to using polynomials such as Hermite, Bernoulli, Chebyshev, Fibonacci and Bessel to solve a system of linear and nonlinear neutral delay differential equations. The proposed method is based on the truncated polynomial expansion of the function together with collocation points and successive integration techniques. This method reduces the given equation to system of non-linear equations with unknown polynomial coefficients which can be easily calculated. The convergence of the proposed method is discussed with several mild conditions. Numerical examples are considered to demonstrate the efficiency of the method. The numerical results reveal that the proposed new approach gives better results than other conventional methods. It demonstrates the reliability and efficiency of this method for solving a system of linear and nonlinear neutral delay differential equations.

Keywords: Polynomials; Collocation method; successive integration technique; neutral delay differential equations.

2010 AMS subject classification: 65D30, 42C05, 39A13[♥]

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1.Introduction

Delay Differential Equations (DDEs) are a differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. The terms involving previous times are called delay terms. The delay terms may be constant, state-dependent and time-dependent. Neutral delay differential equations (NDDEs) are another type of DDEs in which the highest-order derivative of the unknown function occurs with delay. DDEs arise in the fields of signal processing, digital images, control systems, epidemiology, chemical kinetics, etc. Some notable applications of DDEs and NDDEs are in chemical kinetics [1], climate model [2], SIR epidemic model [3], iterative survival model of red blood cells [4], immunology model [5] and cell growth model [6].

DDEs and NDDEs have been studied by many authors and developed various analytical and numerical methods. Some of them are Adams predictor corrector algorithm [7], Homotopy perturbation method [8], Reproducing kernel Hilbert space method [9], Variational iteration method [10], Elzaki transform method [11], Haar wavelet series method [12], Higher order derivative Runge Kutta method [13], Composite Runge Kutta methods and new one-step techniques [14], Analytical algorithm [15], Hybrid Multistep block method [16] and Generalized Rational multi-step method [17] for solving DDEs and NDDEs.

The Collocation method based on various polynomials is a powerful technique for solving differential equations. Mustafa Gulsu *et al.* [18,19] have proposed collocation method based on Hermite and Chebyshev polynomials for solving DDEs with variable coefficients under mixed conditions. Suvip Yiizbas *et al.* [20] have presented Bessel polynomial operational matrix method for solving NDDEs. Ali H Bhrawy *et al.* [21] have proposed a Legendre-Gauss collocation method for solving NDDEs with proportional delay. Tohidi *et al.* [22] have presented Bernoulli operational matrix for solving DDEs. Ayse Betul Koc *et al.* [23] have presented a matrix method based on Fibonacci polynomial for solving DDEs. Birol Ibis *et al.* [24] have applied Hermite polynomials for solving NDDEs with proportional delays.

The above-mentioned collocation methods using different polynomials are based on operational matrices. In this study, we propose a new approach of using Polynomial Collocation methods based on Successive Integration Technique for solving systems of both linear and non-linear NDDEs.

This paper is organized as follows: In Section 2, the basic definitions of different polynomials are given. In Section 3, the description of the method for solving NDDEs is provided. In Section 4, the convergence analysis of the proposed method is discussed. In Section 5, illustrative examples are provided.

2. Basic definition of polynomials

2.1 Hermite polynomial

The Hermite polynomial $H_n(t)$ of order n is defined on the interval $(-\infty, \infty)$. There are different ways to define for Hermite polynomial, one of them is the so-called Rodrigues' formula

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2} \tag{1}$$

From Eqn. (1), the recurrence relation for the polynomials can be derived as

$$H_n(t) = 2t\, H_{n-1}(t) - H'_{n-1}(t) \tag{2}$$

$H_0(t)$ can be obtained from Eqn. (1) and the remaining terms are determined by using the recursion relation Eqn. (2).

Thus, we have the following sequence of polynomials:

$H_0(t) = 1$

$H_1(t) = 2t$

$H_2(t) = 4t^2 - 2$

$H_3(t) = 8t^3 - 12t$

$H_4(t) = 16t^4 - 48t^2 + 12$

The n^{th} order Hermite polynomial $H_n(t)$ has a leading coefficient 2^n.

2.2 Bernoulli polynomial

The Bernoulli polynomial is named after Jacob Bernoulli which combines the Bernoulli numbers and binomial coefficients. The generating function for the Bernoulli polynomial of order n is defined by

$$\sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} = \frac{xe^{xt}}{e^x - 1} \tag{3}$$

The recursion formula for Bernoulli polynomial is:

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k(t) = nt^{n-1}, \qquad n = 2, 3, \ldots \tag{4}$$

$B_0(t)$ can be obtained from Eqn. (3) and the remaining terms are determined by using the recursion relation Eqn. (4).

Thus, we have few terms of the Bernoulli polynomials as:

$$B_0(t) = 1$$

$$B_1(t) = t - \frac{1}{2}$$

$$B_2(t) = t^2 - t + \frac{1}{6}$$

$$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$$

$$B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}$$

2.3 Chebyshev polynomial

The Chebyshev polynomial related to cosine functions on the interval $[-1, 1]$ of order n is defined as

$$T_n(\cos t) = \cos(nt) \tag{5}$$

The recursion relation of Chebyshev polynomial is:

$$T_{n+1}(t) = 2t\, T_n(t) - T_{n-1}(t) \tag{6}$$

$T_0(t)$ and $T_1(t)$ can be obtained from Eqn. (5). Then the remaining terms are determined from Eqn. (6). Thus, we have the following sequence of polynomials:

$$T_0(t) = 1$$

$$T_1(t) = t$$

$$T_2(t) = 2t^2 - 1$$

$$T_3(t) = 4t^3 - 3t$$

$$T_4(t) = 8t^4 - 8t^2 + 1$$

2.4 Fibonacci polynomial

The Fibonacci polynomials are a polynomial sequence which can be considered of Fibonacci numbers. The Fibonacci polynomials are defined by a recurrence relation

$$F_n(t) = \begin{cases} 0, & if\ n = 0 \\ 1, & if\ n = 1 \\ tF_{n-1}(t) + F_{n-2}(t), & if\ n \geq 2. \end{cases}$$

The first few Fibonacci polynomials are:

$$F_0(t) = 0$$

$$F_1(t) = 1$$

$$F_2(t) = t$$

$$F_3(t) = t^2 + 1$$

$$F_4(t) = t^3 + 2t$$

2.5 Bessel polynomial

The Bessel polynomial is defined by

$$y_n(t) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\,k!}\left(\frac{t}{2}\right)^k$$

The recursion equation for the Bessel polynomial is: $y_n(t) = (2n-1)ty_{n-1}(t) + y_{n-2}(t)$

The first few Bessel polynomials are:

$$y_0(t) = 1$$

$$y_1(t) = t + 1$$

$$y_2(t) = 3t^2 + 3t + 1$$

$$y_3(t) = 15t^3 + 15t^2 + 6t + 1$$

3. Description of the proposed method

Consider the n[th] order NDDE of the form

$$y^{(n)}(t) = f\big(t, y(t), y'(t), \dots, y^{(n-1)}(t), y(t-\tau_0), y'(t-\tau_1), \dots, y^{(n)}(t-\tau_n)\big), \quad t > t_0 \tag{7}$$

with initial conditions

$$y^{(i)}(t_0) = \emptyset(t),\ i = 1, 2, 3, \dots, (n-1) \ \text{ for } \ t \le t_0 \tag{8}$$

Here $\emptyset(t)$ is the initial function and τ is the delay term.

Let P(t) represent any orthogonal polynomials. For the proposed method, we assume that

$$y^{(n)}(t) \approx B^T P(t) = \sum_{j=0}^{N} c_j P_j(t) \tag{9}$$

where N being any positive integer,

$$B^T = (c_0, c_1, \dots c_N)$$

$$P(t) = \left(P_0(t), P_1(t) \dots P_N(t)\right)^T$$

Here T stands for transpose of the matrix.

Our aim is to determine the polynomial coefficients $c_j's$. For this, we integrate Eqn. (9) with respect to t from t_0 to t,

$$\left.\begin{aligned} y^{(n-1)}(t) &= y(t_0) + \int_{t_0}^{t} B^T P(t)\, dt \\ y^{(n-2)}(t) &= y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T P(t)\, dt \\ &\vdots \\ y'(t) &= \sum_{i=0}^{N-1} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \dots \int_{t_0}^{t} B^T P(t)\, dt \\ y(t) &= \sum_{i=0}^{N} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \dots \int_{t_0}^{t} B^T P(t)\, dt \end{aligned}\right\} \quad (10)$$

Now, for delay terms

$$\left.\begin{aligned} y^{(n)}(t-\tau_n) &= B^T P(t-\tau_n) \\ y^{(n-1)}(t-\tau_{n-1}) &= y(t_0) + \int_{t_0}^{t} B^T P(t-\tau_{n-1})\, dt \\ y^{(n-2)}(t-\tau_{n-2}) &= y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T P(t-\tau_{n-2})\, dt \\ &\vdots \\ y'(t-\tau_1) &= \sum_{i=0}^{N-1} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \dots \int_{t_0}^{t} B^T P(t-\tau_1)\, dt \\ y(t-\tau_0) &= \sum_{i=0}^{N} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \dots \int_{t_0}^{t} B^T P(t-\tau_0)\, dt \end{aligned}\right\} \quad (11)$$

Then we substitute (10) and (11) in (7) and use the collocating points $t_i = \frac{i}{N}$, where i = 0, 1…N. This yields a system of linear or nonlinear equations subject to the linear and nonlinear terms in Eqn. (7). On solving this system of equations, we get the respective polynomial coefficients c_j's from which the solution of the NDDE (7) can be obtained.

In similar manner, the above proposed method can be applied to solve system of linear and nonlinear NDDEs.

4. Convergency analysis

Consider the first order NDDE of the form

$$\big(y(t) + a(t)y(q_0 t)\big)' = b(t)y(t) + c(t)y(q_1 t) + f(t), t \in [0, T] \quad (12)$$

with initial value condition

$$y(0) = y_0. \quad (13)$$

Here $a(t), b(t)$ and $c(t)$ are given analytical functions.

The convergence of the proposed method will be provided under several mild conditions such as the solution boundedness of the Eqn. (12). Some definitions and lemmas are provided in order to clarify the main convergence theorem of this section.

Definition 4.1 [21]

A function $\xi: [-1, 1] \to \Re$ belongs to the Sobolev space $\mathcal{W}^{m,p}$, if its j^{th} weak derivative $\xi^{(j)}$ lies in $L^p[-1, 1]$ for all $0 \le j \le m$ with the norm

$$\|\xi\|_{\mathcal{W}^{m,p}} = \sum_{j=0}^{m} \left\|\xi^{(j)}\right\|_{L^p}$$

where $\|\xi\|_{L^p}$ denotes the usual Lebesgue norm

$$\|\xi\|_{L^p} = \left(\int_{-1}^{1} \|\xi\|^p dx\right)^{\frac{1}{p}}$$

and $\|\xi\|$ stands for any finite dimensional norm in $\Re^n$.

Lemma 4.1 [21]

For a given function $\xi \in \mathcal{W}^{m,\infty}$, there exists a polynomial $y_N(t)$ of degree $\le$ N such that

$$\|\xi(t) - y_N(t)\|_{L^\infty} \le C C_0 N^{-m}, \ \ t \in [-1, 1]$$

where $C_0 = \|\xi\|_{\mathcal{W}^{m,p}}$, C is a constant independent of N and m is the order of smoothness of ξ. Here $y_N(t)$ with the smallest norm $\|\xi(t) - y_N(t)\|_{L^\infty}$ is known as the N^{th} order best polynomial approximation of $\xi(t)$ in the norm of L^∞.

Note that if $\xi \in C^\infty$, then $m = \infty$. This implies that $y_N(t)$ converges to ξ at a spectral rate, that is it would be faster than any given polynomial rate. Moreover, let us denote the set of continuous functions in a linear space on $[0, T]$ by $C[0, T]$ and the uniform norm in $C[0, T]$ by

$$\|f\|_\infty = \max_{0\le t\le x}|f(t)|, \qquad f \in C[0,T].$$

Now integrating the Eqn. (12) in the interval $[0,T]$ and using the initial condition (13), we get

$$\big(y(t) + a(t)y(q_0t)\big) - (y_0 + a(0)y_0) = \int_0^t b(x)y(x) + c(x)y(q_1x)dx$$

Taking $\theta = q_0x$ we rewrite the above equation in the following form

$$y(t) = g(t) - a(t)y(q_0t) + \int_0^t b(x)y(x)\,dx + \int_0^{q_1t} \hat{c}(\theta)y(\theta)\,d\theta \quad (14)$$

where

$$g(t) = (y_0 + a(0)y_0) + \int_0^t f(x)\,dx$$

and

$$\hat{c}(\theta) = \frac{1}{q_1}c\left(\frac{\theta}{q_1}\right).$$

In the following Theorem 4.1, we show that the approximate solution expressed in terms of the orthogonal polynomials converges to the exact solution under several mild conditions.

Theorem 4.1

Let $y(t)$ and $y_N(t)$ be the exact and numerical solutions of Eqn. (12). Also, assume the approximations of $a(t), b(t), g(t)$ and $\hat{c}(t)$ be $a_N(t), b_N(t), g_N(t)$ and $\hat{c}_N(t)$ respectively. Moreover, suppose that $\|a(t)\|_\infty \le A$, $\|b(t)\|_\infty \le B$, $\|\hat{c}(t)\|_\infty \le C$ and $\|y_N(t)\|_\infty \le Y_N$ where $t \in [0,T]$.

Then, $\lim_{N\to\infty} y_N(t) = y(t)$ subject to the condition $A + TB + TC \ll 1$.

Proof:

Suppose that the unknown functions $a(t), b(t), \hat{c}(t)$ and $g(t)$ are approximated in terms of any orthogonal polynomials. Then the numerical solution is an approximated polynomial in the form of $y_N(t)$. We need to find an upper bound for the error between $y(t)$ and $y_N(t)$ for Eqn. (14).

According to the assumptions, Eqn. (14) can be written as

$$y_N(t) = g_N(t) - a_N(t)y(q_0t) + \int_0^t b_N(x)y(x)\,dx + \int_0^{q_1t} \hat{c}_N(\theta)y(\theta)\,d\theta \quad (15)$$

Now, Eqn. (14) – Eqn. (15) yields

$$\begin{aligned}\|y(t)-y_N(t)\|_\infty &\\ &= \Big\| g(t)-g_N(t)+\big(-a(t)y(q_0t)+a_N(t)y(q_0t)\big)\\ &+\int_0^t b(x)y(x)\,dx-\int_0^t b_N(x)y(x)\,dx+\int_0^{q_1t}\hat{c}(\theta)y(\theta)\,d\theta\\ &-\int_0^{q_1t}\hat{c}_N(\theta)y(\theta)\,d\theta\Big\|_\infty\end{aligned}$$

By using the triangle inequality, we get

$$\begin{aligned}\|y(t)-y_N(t)\|_\infty &\\ &\le \|g(t)-g_N(t)\|_\infty+\|-a(t)y(q_0t)+a_N(t)y(q_0t)\|_\infty\\ &+\left\|\int_0^t b(x)y(x)\,dx-\int_0^t b_N(x)y(x)\,dx\right\|_\infty\\ &+\left\|\int_0^{q_1t}\hat{c}(\theta)y(\theta)\,d\theta-\int_0^{q_1t}\hat{c}_N(\theta)y(\theta)\,d\theta\right\|_\infty\end{aligned}$$

Since $0\le q_1\le 1$ and $t\in[0,T]$, the above inequality reduces to

$$\begin{aligned}\|y(t)-y_N(t)\|_\infty \le \|g(t)-g_N(t)\|_\infty &\\ +\|-a(t)y(q_0t)+a_N(t)y(q_0t)\|_\infty T\|b(t)y(t)-b_N(t)y(t)\|_\infty &\\ +T\|\hat{c}(t)y(t)-\hat{c}_N(t)y(t)\|_\infty &\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\|y(t)-y_N(t)\|_\infty \le \|g(t)-g_N(t)\|_\infty+\big\|-a(t)\big(y(t)-y_N(t)\big)\big\|_\infty &\\ +\big\|-\big(a(t)-a_N(t)\big)y_N(t)\big\|_\infty+T\big\|b(t)\big(y(t)-y_N(t)\big)\big\|_\infty &\\ +T\big\|\big(b(t)-b_N(t)\big)y_N(t)\big\|_\infty+T\big\|\hat{c}(t)\big(y(t)-y_N(t)\big)\big\|_\infty &\\ +T\big\|\widehat{c}(t)-\hat{c}_N(t))y_N(t)\big\|_\infty &\end{aligned}$$

Using the assumptions

$$\|-a(t)\|_\infty\le A,\quad \|b(t)\|_\infty\le B,$$

$$\|\hat{c}(t)\|_\infty\le C\,,\qquad \|y_N(t)\|_\infty\le Y_N\,.$$

in the above inequality, we get

$$\begin{aligned}\|y(t)-y_N(t)\|_\infty &\le \|g(t)-g_N(t)\|_\infty + A\left\|\left(y(t)-y_N(t)\right)\right\|_\infty + Y_N\|(a(t)-a_N(t))\|_\infty \\ &+ TB\left\|\left(y(t)-y_N(t)\right)\right\|_\infty + TY_N\|(b(t)-b_N(t))\|_\infty \\ &+ TC\|(y(t)-y_N(t))\|_\infty + TY_N\|\hat{c}(t)-\hat{c}_N(t)\|_\infty\end{aligned}$$

$$\begin{aligned}\|y(t)-y_N(t)\|_\infty - A\left\|\left(y(t)-y_N(t)\right)\right\|_\infty - TB\left\|\left(y(t)-y_N(t)\right)\right\|_\infty \\ - TC\left\|\left(y(t)-y_N(t)\right)\right\|_\infty \\ \le \|g(t)-g_N(t)\|_\infty + Y_N\left\|\left(a(t)-a_N(t)\right)\right\|_\infty + TY_N\|(b(t)-b_N(t))\|_\infty \\ + TY_N\|\hat{c}(t)-\hat{c}_N(t)\|_\infty\end{aligned}$$

$$\begin{aligned}\|y(t)-y_N(t)\|_\infty(1-A-TB-TC) \\ \le \|g(t)-g_N(t)\|_\infty + Y_N\left\|\left(a(t)-a_N(t)\right)\right\|_\infty + TY_N\|(b(t)-b_N(t))\|_\infty \\ + TY_N\|\hat{c}(t)-\hat{c}_N(t)\|_\infty\end{aligned}$$

Let us introduce the following notations:

$$G_1(t) = g(t) - g_N(t), \qquad G_2(t) = a(t) - a_N(t),$$

$$G_3(t) = b(t) - b_N(t), \qquad G_4(t) = \hat{c}(t) - \hat{c}_N(t).$$

Then the above equation becomes

$$\|y(t)-y_N(t)\|_\infty \le \frac{\|G_1(t)\|_\infty + Y_N\|G_2(t)\|_\infty + TY_N\|G_3(t)\|_\infty + TY_N\|G_4(t)\|_\infty}{(1-A-TB-TC)}$$

If $A + TB + TC \ll 1$, then

$$\lim_{N\to\infty}\|y(t)-y_N(t)\|_\infty = 0$$

i.e., $\lim_{N\to\infty}\|y_N(t)\|_\infty = \lim_{N\to\infty}\|y(t)\|_\infty$

This is possible because of the smoothness of $a(t), b(t), \hat{c}(t)$ and $g(t)$. Now Lemma 4.1 implies

$$\lim_{N\to\infty}\|a(t)-a_N(t)\|_\infty = 0,$$

$$\lim_{N\to\infty}\|b(t)-b_N(t)\|_\infty = 0,$$

$$\lim_{N\to\infty}\|\hat{c}(t)-\hat{c}_N(t)\|_\infty = 0 \text{ and}$$

$$\lim_{N\to\infty}\|g(t)-g_N(t)\|_\infty = 0$$

This completes the proof.

5. Numerical examples

In this section, numerical examples of linear and non-linear system of NDDEs and an example of NDDE which arises in the modelling of E.coli growth are given to demonstrate the accuracy and effectiveness of the proposed collocation method based on successive integration technique.

Example 1 [15]

Consider the 2 – dimensional linear system of NDDEs with constant delay

$$y_1'(t) = y_1'(t-1) + 4y_2(t), \qquad 0 \le t \le 2$$

$$y_2'(t) = y_1(t) - 4y_1(t-1), \qquad 0 \le t \le 2$$

with history function

$$y_1(t) = e^{-2t}, \quad y_2(t) = \frac{1}{2}\left(e^{-2(t-1)} - e^{-2t}\right). \qquad t \in [-1,0].$$

The exact solution is $y_1(t) = e^{-2t}, \quad y_2(t) = \frac{1}{2}\left(e^{-2(t-1)} - e^{-2t}\right)$.

The numerical results of the proposed method using by Hermite polynomial for different values of N are given in Table 1. The numerical results of the proposed method using Bessel polynomial, Bernoulli polynomial, Chebyshev polynomial, Hermite polynomial and Fibonacci polynomials for different values of N at time t = 1 are given in Table 2. The solution graph obtained by using the proposed method with N = 10 is presented in Figure 1.

Time t	**Hermite polynomial collocation method**					
	N = 5		**N = 7**		**N = 10**	
	y_1	y_2	y_1	y_2	y_1	y_2
0.2	1.11 e-07	1.13 e-08	2.63 e-11	2.08 e-11	6.06 e-17	9.91 e-17
0.4	6.24 e-08	3.99 e-08	5.27 e-13	2.83 e-11	1.03 e-16	5.85 e-17
0.6	1.77 e-07	3.15 e-07	5.31 e-11	5.69 e-11	1.95 e-17	8.14 e-17
0.8	3.95 e-07	3.32 e-07	2.25 e-11	5.97 e-11	5.97 e-17	9.52 e-17
1.0	1.43 e-08	0.00 e+00	1.81 e-10	0.00 e+00	3.61 e-16	0.00 e+00
1.2	2.91 e-08	4.88 e-08	6.77 e-12	1.02 e-11	6.45 e-18	2.16 e-17
1.4	5.12 e-08	3.15 e-08	8.31 e-12	6.81 e-13	3.55 e-17	3.71 e-17
1.6	1.02 e-08	6.77 e-08	1.03 e-11	1.99 e-11	2.12 e-17	6.73 e-18
1.8	1.26 e-07	1.64 e-07	2.17 e-11	9.06 e-12	2.91 e-17	2.17 e-17
2.0	1.45 e-07	2.22 e-08	2.19 e-11	6.84 e-11	3.34 e-17	1.28 e-16

Table 1: Absolute errors for Example 1

Polynomials	N = 5		N = 7		N = 10	
	y_1	y_2	y_1	y_2	y_1	y_2
Bessel	9.77 e-07	1.55 e-06	2.23 e-10	4.39 e-10	2.68 e-16	7.25 e-16
Bernoulli	8.30 e-08	2.43 e-07	7.02 e-12	2.75 e-11	1.06 e-18	8.13 e-18
Chebyshev	3.66 e-07	7.09 e-07	6.84 e-11	1.68 e-10	6.18 e-17	2.19 e-16
Hermite	1.45 e-07	2.22 e-08	2.19 e-11	6.84 e-11	3.34 e-17	1.28 e-16
Fibonacci	1.58 e-06	2.28 e-06	5.80 e-10	9.64 e-10	1.37 e-15	2.84 e-15

Table 2: Absolute errors for Example 1

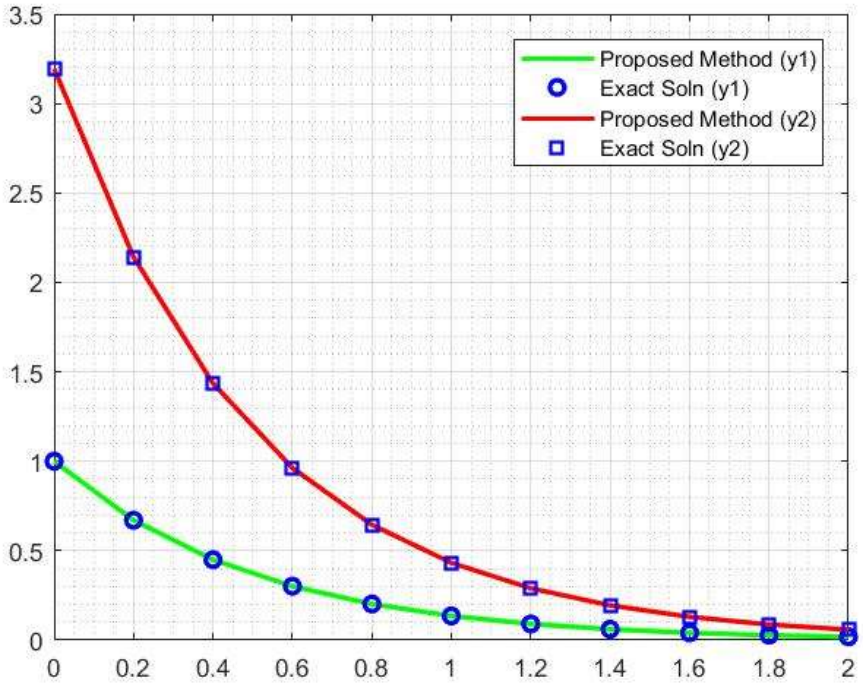

Figure 1: Solution graph for Example 1

Example 2 [15]

Consider the third-order non-linear system of NDDE with both constant and proportional delay.

$$y_1'''(t) = y_1'''(t-2)y_1\left(\frac{t}{3}\right) + \left(y_1(t)\right)^{\frac{2}{3}} + 2t + e^{-t}$$

$$y_2'''(t) = \frac{1}{2}y_2'''\left(\frac{t}{2}\right) + y_2'(t-1)y_1\left(\frac{t}{3}\right), \qquad t \geq 1$$

with history function $y_1(t) = e^t$ and $y_2(t) = t^2, \quad t \in [-2,0]$.

The given initial conditions are

$$y_1(0) = 1, \quad y_1'(0) = 1, \quad y_1''(0) = 1$$

$$y_2(0) = 0, \quad y_2'(0) = 0, \quad y_2''(0) = 2$$

For this example, the numerical results are obtained by using the proposed method on the above mentioned five polynomials. The solution graph by the proposed method is given in Figure 2.

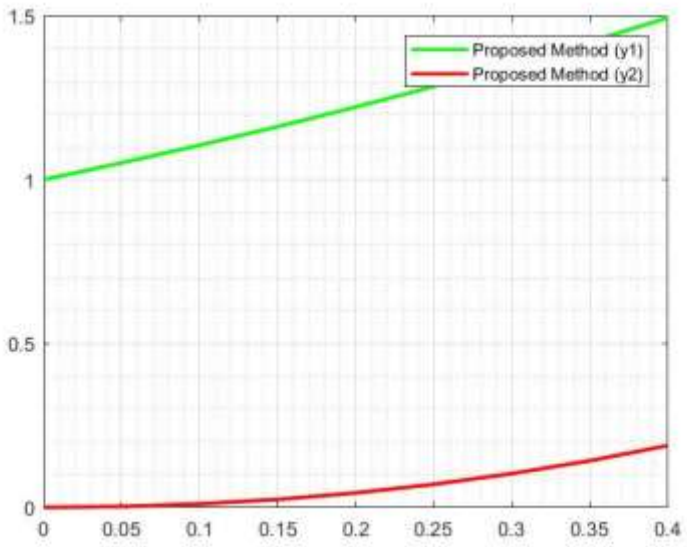

Figure 2: Solution graph for Example 2

Example 3 [6]

Consider the following NDDEs arise in modelling the growth patterns of Escheria coli (E. coli) cells, which clearly exhibit the presence of a time delay in their cell division stage.

$$y'(t) = \rho_0\, y(t) + \rho_1\, y(t-\tau) + \rho_2\, y'(t-\tau) \quad t \geq 0$$

For this example, the numerical results are obtained by using the proposed method based on the above-mentioned five polynomials. The numerical simulation by the proposed method at N = 5 is compared with the graphical solution given in [6]. These have been shown in Figure 3 and 4.

Parameters	Biological Interpretation	Values
$\tau > 0$	The average cell-division time	20.2229
$\rho_0 \geq 0$	The rate of cell-death	-0.0057
$\rho_1 \geq 0$	The rate of commitment to the cell-division process	0.0131
$0 \leq \rho_2 \geq 2$	The gradual dispersal of synchronization of cell-division	1.8407

Table 3: The biological interpretation of the parameters and their values of Example 3

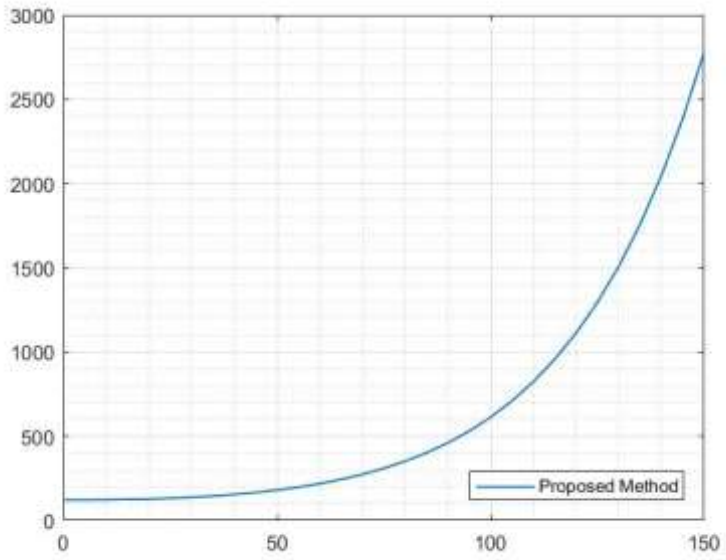

Figure 3: Solution graph for Example 3

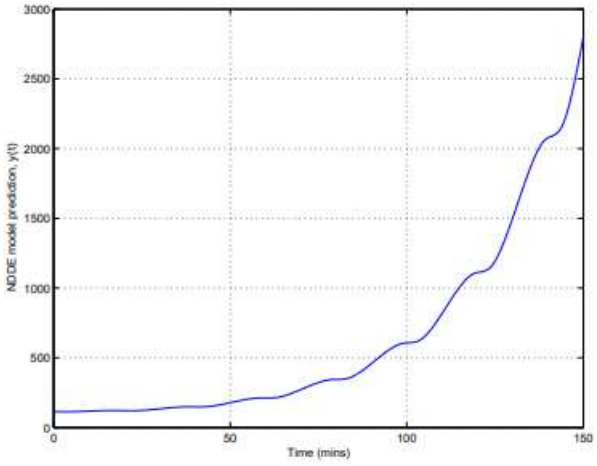

Figure 4: Solution graph of Example 3 in [6]

6. Conclusion

In this paper, a new application of the Polynomial collocation method based on successive integration techniques is presented for solving a system of neutral delay differential equations. The convergence analysis of the proposed method has been discussed. Numerical examples of linear and non-linear systems of neutral delay differential equations are considered to demonstrate the efficiency of the proposed method.

The Tables show that the proposed polynomial collocation method based on successive integration techniques gives results with reasonable accuracy. Also, it is observed that accuracy increases as N increases. The proposed method is computationally simple. Hence it is concluded that the proposed method is suitable for solving linear and non-linear systems of neutral delay differential equations in real-world problems that exist in different fields of science and engineering.

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