

On the Traversability of Near Common Neighborhood Graph of a Graph

Abstract

The near common neighborhood graph of a graph G , denoted by $ncn(G)$ is defined as the graph on the same vertices of G , two vertices are adjacent in $ncn(G)$, if there is at least one vertex in G not adjacent to both the vertices. In this research paper, the conditions for $ncn(G)$ to be disconnected are discussed and also we present characterization for graph $ncn(G)$ to be hamiltonian and eulerian.

Keywords: Near common neighborhood graph; Hamiltonian graph; Eulerian graph

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1 Introduction

Let G be a graph. The near common neighborhood graph of G denoted by $ncn(G)$ is a graph with the same vertices as G in which two vertices u and v are adjacent if there is at least one vertex $w \in V(G)$ not adjacent to both of u and v [Al-Kenani et al., 2016].

A cycle in graph G that contains every vertex of G is called hamiltonian cycle of G . Thus a hamiltonian cycle of G is a spanning cycle of G . A hamiltonian graph is a graph that contains a hamiltonian cycle.

An euler trail in a graph is a trail that contains every edge of that graph. An euler tour is a closed euler trail. An eulerian graph is a graph that has an euler tour.

The graphs cosiderered in this paper are simple, undirected and without loops with vertex set $v_i \in V(G)$, $i = 1, 2, 3, \dots n$. Basic terminologies are referred from [Harary, 1969].

[Al-Kenani et al., 2016] studied near common neighborhood of a graph and obtained results for paths, cycles and complete graphs. In this paper the conditions for $ncn(G)$ to be disconnected are discussed and also we present theorems stating necessary and sufficient conditions for a graph $ncn(G)$ to possess hamiltonian and eulerian cycle.

2 Prelimnaries

The below mentioned some important results are used through out the paper for proving the theorems.

Proposition 2.1. [Al-Kenani et al., 2016] For any path P_n ,

$$ncn(P_n) = \begin{cases} \overline{K_n}, & \text{if } n = 2, 3 \\ 2K_2, & \text{if } n = 4 \\ K_n, & \text{if } n \geq 7. \end{cases}$$

Proposition 2.2. [Al-Kenani et al., 2016] For any path C_n ,

$$ncn(C_n) = \begin{cases} \overline{K_n}, & \text{if } n = 3, 4 \\ C_5, & \text{if } n = 5 \\ K_n, & \text{if } n \geq 7. \end{cases}$$

Proposition 2.3. [Al-Kenani et al., 2016]. For any complete graph K_n and totally disconnected graph $\overline{K_n}$, we have

$$\begin{aligned} 1.ncn(K_n) &= \overline{K_n} \\ 2.ncn(\overline{K_n}) &= K_n, n \geq 3 \end{aligned}$$

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Theorem 2.1. [Singh, 2010]. Let G be a simple graph with $p \geq 3$ and $\delta \geq p/2$. Then G is Hamiltonian.

Theorem 2.2. [Singh, 2010] A nonempty connected graph is Eulerian if and only if it has no vertices of odd degree.

Remark 2.1. [Singh, 2010] The complete graph K_p , for $p \geq 3$, is always Hamiltonian.

3 Results

Theorem 3.1. If G is a graph with n vertices, then $ncn(G)$ is disconnected if one of the following conditions holds

1. G is $P_n, C_n, n \leq 4$
2. G is $K_n, n \geq 3$
3. G has $\Delta(G) = n - 1$
4. G is a graph with two complete graphs connected by bridge
5. G is $K_n \bullet P_t, n \geq 3, t \leq 3$

Proof. The proof of the theorem is constructed by considering the following cases.

Case 1. Suppose $G=P_n$ or $C_n, n \leq 4$.

We consider the two cases.

Subcase 3.1. Suppose $G=P_n, n \leq 4$.

If $n = 2, 3, 4$, then by Proposition 2.1, $ncn(G)$ is disconnected.

Subcase 3.2. Suppose $G=C_n, n \leq 4$.

If $n = 3, 4$, then from Proposition 2.2, $ncn(G)$ is disconnected.

Case 2. Suppose $G=K_n, n \geq 3$.

Then from Proposition 2.3, $ncn(G)$ is disconnected.

Case 3. Let G be a graph with vertex set $V(G) = \{v_i | i \in N\}$ vertices and $\Delta(G) = n - 1$. Near common neighbourhood graph $ncn(G)$ is a graph with same vertices v_i as G . The vertices v_i and $v_j, j = 1, 2, 3, \dots, n, i \neq j$ of $ncn(G)$ are adjacent if there is at least one vertex $w \in V(G)$ not adjacent to both v_i and v_j .

Since $\Delta(G) = n - 1$ in G (i.e., v_i is adjacent to all other vertices of G), there does not exist any nonadjacent vertex for v_i and thus v_i cannot be connected to any vertex of $ncn(G)$. This results $ncn(G)$ into disconnected.

Case 4. Let G be a graph with two complete graphs K_m and K_n connected by bridge. Let $v_i \in V(G), i = 1, 2, 3, \dots, m$ be the vertex set of K_m and $v_j \in V(G), j = m + 1, m + 2, m + 3, \dots, n$ be the vertex set of K_n , where m and n are the vertices of bridge.

The near common neighbourhood graph $ncn(G)$ has the same vertices as G . Two vertices of $ncn(G)$ are adjacent if there is at least one vertex $w \in V(G)$ in G not adjacent to both the vertices.

As G consists of two complete graphs, vertices v_i of K_m and v_j of K_n are respectively mutually adjacent. Nonadjacent vertices for all the vertices $v_i \in K_m$ exists in K_n and for $v_j \in K_n$ exists in K_m . Thus the vertices v_i of K_m and v_j of K_n are mutually connected in $ncn(G)$. This produces the disconnected graph with two components. Further, the end points of bridge are also mutually adjacent to all the vertices of K_m and K_n respectively. Hence nonadjacent vertex does not exists for end points of bridge. This produces the graph $ncn(G)$ into disconnected.

Case 5. Let G be a $K_n \bullet P_t$, $n \geq 3$ and $t \leq 3$.

$ncn(G)$ has the same vertices as G . Two vertices of $ncn(G)$ are adjacent if there is at least one vertex $w \in V(G)$ not adjacent to both the vertices.

Let $v_i, i = 1, 2, 3, \dots, n, n+1, n+2$ be the vertex set of $K_n \bullet P_t$. The vertices of K_n are $v_1, v_2, v_3, \dots, v_n$ and vertices of P_t are v_n, v_{n+1}, v_{n+2} . The vertex v_n is the common vertex which connects K_n and P_t .

Subcase 5.1 Suppose $t = 2$. i.e., P_t is path with two vertices, then $G = K_n \bullet P_2$. Since $\Delta(G) = n - 1$, there exists at least one vertex which is adjacent to all the other vertices G . From theorem 3.1 (Case 3), $ncn(G)$ is disconnected.

Subcase 5.2 Suppose $t = 3$. i.e., $t = n, n+1, n+2$.

In G , all the pairs of vertices of K_n have the nonadjacent vertex as v_{n+2} and can be mutually connected in $ncn(G)$. Also the vertices of P_t, v_{n+1} and v_{n+2} have the nonadjacent vertices in K_n and can be connected in $ncn(G)$. As there does not exists any nonadjacent vertices for the pair of common vertex n and the vertices of P_t in G , they cannot be connected in $ncn(G)$. This produces the the graph $ncn(G)$ with two components. Thus $ncn(G)$ is disconnected. □.

Theorem 3.2. For any graph G , $ncn(G)$ is hamiltonian if and only if

1. G contains all the vertices of $\deg < n - 1$ except for $C_4 \bullet P_2$
2. $G = P_n$ or $C_n, n \geq 5$
3. G is $K_n \bullet P_t, n \geq 3, t \geq 4$.

Proof. Let G be graph with vertex set $V(G)$. $ncn(G)$ is the graph with the same set of vertices as G . Two vertices of $ncn(G)$ are adjacent if there is at least one vertex $w \in V(G)$ in G not adjacent to both the vertices.

Suppose $ncn(G)$ is hamiltonian.

In light of the above theorem 3.1 that $ncn(G)$ is disconnected if G is $P_n, C_n, n \leq 4, K_n, n \geq 3, G$ has $\Delta(G) = n - 1, G$ is a graph with two complete graphs connected by bridge and G is $K_n \bullet P_t, n \geq 3, t \leq 3$.

Now we consider the graphs for which $ncn(G)$ is connected.

Case 1. Suppose G contains all the vertices of $\deg < n - 1$.

Let G be a graph $v_i \in V(G)$ vertices with degree of $v_i < n - 1$ (v_i is not adjacent to all the vertices).

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$ncn(G)$ be the graph with same set of vertices as G .

As degree of v_i is $< n - 1$ in G , there exists at least one nonadjacent vertex for any pair of vertices of G . Hence by definition of $ncn(G)$, those vertices in $ncn(G)$ can be connected which produces connected $ncn(G)$ graph.

Further, since for each pair of vertices of G there exists a nonadjacent vertex, $ncn(G)$ contains a cycle and $\delta \geq n/2$. From Theorem 2.1 $ncn(G)$ is hamiltonian.

Next suppose $G = C_4 \bullet P_2$. Let $V_i, i = 1, 2, 3, 4, 5$ be the vertex set of $C_4 \bullet P_2$ with one common vertex between C_4 and P_2 . Among the four vertices of C_4 of G , three vertices (except the common vertex) can be connected mutually adjacent as they have nonadjacent vertex (not common vertex) in P_2 .

Similarly, a vertex of P_2 which is not common can be connected with only three vertices of C_4 in $ncn(G)$ as there exists a nonadjacent vertex for these each pair of vertices. Whereas the common vertex can be connected only with a vertex of P_2 in $ncn(G)$ which is not common, since there exists a nonadjacent vertex in C_4 for this pair and does not exists nonadjacent vertex for the pair of vertices with C_4 .

This results $ncn(G)$ into connected graph with one pendent vertex and thus does not contain hamiltonian cycle.

Thus $ncn(C_4 \bullet P_2)$ is connected but not hamiltonian.

Case 2. Suppose G is P_n or $C_n, n \geq 5$.

Let G be a P_n or $C_n, n \geq 5$. From proposition 2.1 and proposition 2.2, $ncn(P_n)$ and $ncn(C_n), n = 5, 6$ are connected which contains a cycle and $\delta \geq n/2$. For $n \geq 7$, $ncn(G)$ is K_n . From theorem 2.1 and remark 2.1, $ncn(G)$ is hamiltonian.

Case 3. Suppose G is $K_n \bullet P_t, n \geq 3, t \geq 4$.

Let G be a $K_n \bullet P_t, n \geq 3, t \geq 4$, where n is the number of vertices of K_n and t is the number of vertices of P_t .

Let $v_i \in V(G), i = 1, 2, 3, \dots, n$ be the vertex set of K_n and $v_j \in V(G), j = n, n+1, n+2, \dots, t$ be the vertex set of P_t , where n is the common vertex of K_n and P_t .

$ncn(G)$ has the same vertex set as G . Two vertices of $ncn(G)$ are adjacent if there exists at least $w \in V(G)$ not adjacent to both the vertices.

As there is increase in the number of vertices (path length) in P_t of G , there exists a nonadjacent vertex for each pair of vertices of K_n and vertices of P_t , which produces connected graph $ncn(G)$ with cycles and also $\delta \geq n/2$. From theorem 2.1, $ncn(G)$ is hamiltonian.

Converse is obvious.

□

Theorem 3.3. For any graph G , $ncn(G)$ is eulerian if only if

1. $ncn(G)$ is K_n , $n = 2s + 1$, $s = 1, 2, \dots, n$.
2. $G = C_n$, $n = 5, 6$

Proof. Let G is a graph with vertex set $v_i \in G$, $i = 1, 2, 3, \dots, n$.

$ncn(G)$ be a graph has the same vertices as G . Two vertices of $ncn(G)$ are adjacent if there exists a vertex $w \in V(G)$ not adjacent to both the vertices. Suppose $ncn(G)$ is eulerian, then degree of each v_i of $ncn(G)$ is even. From theorems 3.1 and 3.2, $ncn(G)$ is disconnected if G is P_n, C_n , $n \leq 4$, K_n , $n \geq 3$, G has $\Delta(G) = n - 1$, G is a graph with two complete graphs connected by bridge, G is $K_n \bullet P_t$, $n \geq 3$, $t \leq 3$ and is connected only if G contains all the vertices of $\deg < n - 1$, $G = P_n$ or C_n , $n \geq 5$ and G is $K_n \bullet P_t$, $n \geq 3$, $t \geq 4$. From propositions 2.1 and 2.2, $ncn(P_n \text{ or } C_n)$ is K_n , $n \geq 7$, $ncn(C_n)$, $n = 5, 6$ is a graph with each vertex degree even (2-regular and 4-regular graph) and $ncn(K_n \bullet P_t)$, $n \geq 3$, $t \geq 5$ is k_n .

The degree of each v_i of graph G is even only if $G = K_n$, $n = 2s + 1$, $n \geq 3$, $s = 1, 2, 3, \dots, n$, $G = C_n$, $n \geq 3$ and G is r -regular graph, $r = 2t$, $t = 1, 2, 3, \dots, r$. From theorem 2.2, $ncn(G)$ is eulerian.

Conversely, $ncn(G)$ is a graph with same vertices as G .

From theorem 3.1, $ncn(G)$ is disconnected if G is P_n, C_n , $n \leq 4$, K_n , $n \geq 3$, G has $\Delta(G) = n - 1$, G is a graph with two complete graphs connected by bridge and G is $K_n \bullet P_t$, $n \geq 3$, $t \leq 3$ in all other cases it is connected.

We consider the following cases.

Case 1. Suppose $ncn(G)$ is K_n , $n = 1, 2, 3, \dots, n$, then from theorem 3.2, if $ncn(G)$ is connected and it is K_n for $G = P_n, C_n$, $n \geq 5$ and $k_n \bullet P_t$, $n \geq 3$, $t \geq 5$.

Subcase 1.1 Suppose $G = P_n$ or C_n , then from propositions 2.1 and 2.2, $ncn(G)$ is K_n , $n \geq 7$. The degree of each vertex of K_n , $n = 2s + 1$, $s = 1, 2, 3, \dots, n$. From theorem 2.2, $ncn(G)$ is eulerian.

Subcase 1.2. Suppose $G = K_n \bullet P_t$, $n \geq 3$, $t \geq 5$, then from theorem 3.2, if $ncn(G)$ is connected and it is K_n for $n \geq 3$, $t \geq 5$. The degree of k_n is even if n is odd.

From theorem 2.2, $ncn(G)$ is eulerian.

Case 2. Suppose $G = C_n$, $n = 5, 6$.

Subcase 2.1 Suppose $G = C_n$, $n = 5$, then from proposition 2.2, $ncn(G)$ is C_5 or 2-regular graph. From theorem 2.2 $G(n)$ is eulerian.

Subcase 2.2 Suppose $G = C_n$, $n = 6$, then from proposition 2.2, $ncn(G)$ is 2-regular graph. Hence from theorem 2.2, $G(n)$ is eulerian.

Case 3. Suppose $G = K_n \bullet P_t$, $n \geq 3$, $t \geq 5$.

From theorem 3.2, if $ncn(G)$ is connected and it is K_n for $n \geq 3$, $t \geq 5$.

The degree of k_n is even if n is odd. From theorem 2.2, $ncn(G)$ is eulerian.

4 Conclusion

In this paper the study on near common neighborhood graph of a graph is extended and various general conditions for which $ncn(G)$ is disconnected are discussed. It is disconnected for the graphs $P_n, C_n, n \leq 4, K_n, n \geq 3, G$ has $\Delta(G) = n - 1, G$ is a graph with two complete graphs connected by bridge and G is $K_n \bullet P_t, n \geq 3, t \leq 3$ in all other cases it is connected. Further we have also obtained the necessary and sufficient condition for $ncn(G)$ to possess hamiltonian and eulerian cycles. It contains hamiltonian cycle whenever it is connected except for $C_4 \bullet P_2$. It contains eulerian cycle if $ncn(G)$ is K_n with odd vertices and G is $C_n, n = 5, 6$.

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