

gpα- Kuratowski's Closure Operators in Topological Spaces

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Abstract

In this paper, we introduce and study topological properties of *gpα*-limit points, *gpα*-derived sets, *gpα*-interior and *gpα*-closure using the concept of *gpα*-open set. Further, the relationship between these concepts are investigated. Also, the Kuratowski's axioms are discussed.

Keywords: *gpα*-open set, *gpα*-closed set, *gpα*-limit point, *gpα*-derived set, *gpα*-closure, *gpα*-interior points.

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1 Introduction

Topology is an indispensable object of study in Mathematics with open sets as well as closed sets being the most fundamental concepts in topological spaces. General topology plays an important role in many branches of mathematics as well as many fields of applied sciences.

Introduction to the concept of pre-open sets and pre-closed sets were made by Mashhour et al. [7] and the idea of α -open sets was introduced by Njastad [6]. The concepts and characterizations of semi open, pre-open, α -open and semi pre open sets are studied in [4],[6], [5] and [1] respectively. The concept, generalized closed sets of Levine [5] opened the flood gates of research in generalizations of closed sets in general topology. Many researchers [4], [8], [11], [12] worked on weaker forms of closed sets. Recently Benchalli et al.[3] and Patil et al. [9] introduced and studied the concept of $\omega\alpha$ -open sets and $gp\alpha$ -open sets in topological spaces was studied.

The present author continued the study of $gp\alpha$ -closed sets and their properties. We study the $gp\alpha$ -closure, $gp\alpha$ -interior, $gp\alpha$ -neighbourhood, $gp\alpha$ -limit points and $gp\alpha$ -derived sets by using the concept of $gp\alpha$ -open sets and study their topological properties. We provide the relationship between $gp\alpha$ -derived set (resp. $gp\alpha$ -limit points, $gp\alpha$ -interior) and pre-derived set (resp. pre-limit points and pre-interior). Also, we studied the Kuratowski's closure axioms with respect to $gp\alpha$ -open sets.

2 Preliminaries

Throughout the paper, let X and Y ($\text{Resp}(X, \tau)$ and (Y, σ)) always denotes non-empty topological spaces on which no separation axioms are assumed unless explicitly mentioned.

The following definitions are useful in the sequel:

Definition 2.1. A subset A of X is said to be a

- (a) $\omega\alpha$ -closed [3] if $\alpha\text{-cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is ω -open in X .
- (b) $gp\alpha$ -closed [9] if $p\text{-cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is α -open in X .

3 $gp\alpha$ -Interior and $gp\alpha$ -Closure in topological spaces

This section deals with the properties of $gp\alpha$ -interior and $gp\alpha$ -closure are studied and some of their properties are investigated.

Definition 3.1. [9] Consider a subset A of X , the gp α -interior of A , gp α -int(A), and is defined as gp α -int(A) = $\cup\{G : A \subseteq G : G \text{ is gp}\alpha\text{-open in } X\}$.

Definition 3.2. [9] Consider a subset A of X , the gp α -closure of A , gp α -cl(A), and is defined as gp α -cl(A) = $\cap\{G : A \subseteq G : G \text{ is gp}\alpha\text{-closed in } X\}$.

Theorem 3.1. If A is gp α -closed then gp α -cl(A) = A .

Proof: Let A be gp α -closed in X . Since $A \subseteq A$, and A is gp α -closed in X . Then $A \in \{G : A \subseteq G \text{ and } G \text{ is gp}\alpha\text{-closed in } X\}$, that is $A = \cap\{G : A \subseteq G \text{ and } G \text{ is gp}\alpha\text{-closed}\}$. Hence gp α -cl(A) $\subseteq A$. But $A \subseteq \text{gp}\alpha\text{-cl}(A)$ is always true. Therefore gp α -cl(A) = A .

In general the equality of Theorem 3.1 is not true.

Example 3.1. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{b\}$. Then we can observe that gp α -cl(A) = $\{b, c\}$ which is gp α -closed but A is not gp α -closed in X .

Theorem 3.2. Let $A \subseteq X$ and A is gp α -closed in X . Then gp α -cl(A) is the smallest gp α -closed set containing A .

Proof: Let A be gp α -closed in X . Since $A \subseteq A$ is always true and A is gp α -closed in X . Hence $A \subseteq A$ and $A \in \text{gp}\alpha\text{-C}(X)$. Therefore gp α -cl(A) is the smallest gp α -closed set containing A .

However, the converse of the Theorem 3.2 is not true in general.

Example 3.2. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, c\}\}$. Consider $A = \{a, c\}$, then gp α -C(X) = X , which is the smallest gp α -closed set containing A . But A is not gp α -closed in X .

Remark 3.1. For subsets A, B of X , then gp α -cl($A \cap B$) $\subseteq$ gp α -cl(A) $\cap$ gp α -cl(B).

Remark 3.2. For any subset $A, B \subseteq X$ then we have the following properties:

- (i) gp α -int(X) = X and gp α -int(ϕ) = ϕ .
- (ii) gp α -int(A) $\subseteq A$.
- (iii) If B is any gp α -open set contained in A then $B \subseteq \text{gp}\alpha\text{-int}(A)$.
- (iv) if $A \subseteq B$, gp α -int(A) $\subseteq$ gp α -int(B).
- (v) gp α -int(gp α -int(A)) = gp α -int(A).

Remark 3.3. For any $A, B \subseteq X$, gp α -int($A \cup B$) = gp α -int(A) $\cup$ gp α -int(B).

Converse of the Remark 3.3 need not be true as seen from the following example.

Example 3.3. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$.

Let $A = \{a\}$ and $B = \{c\}$.

Then $gp\alpha\text{-int}(A \cup B) = \{a, c\}$ and $gp\alpha\text{-int}(A) \cup gp\alpha\text{-int}(B) = \{a\} \cup \{\phi\} = \{a\}$.

Hence $gp\alpha\text{-int}(A \cup B) \neq gp\alpha\text{-int}(A) \cup gp\alpha\text{-int}(B)$.

But, $gp\alpha\text{-int}(A) \cup gp\alpha\text{-int}(B) \subset gp\alpha\text{-int}(A \cup B)$

Example 3.4. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$.

Let $A = \{a, b, c\}$. Then $gp\alpha\text{-interior point of } A$ is 'a', that is $gp\alpha\text{-int}(A) = \{a\}$.

Let $B = \{a, b, c, d\}$. Then $gp\alpha\text{-interior points of } B$ are a, b, c, d, that is $gp\alpha\text{-int}(B) = \{a, b, c, d\}$.

Theorem 3.3. For any subset A of X , every pre-interior point of A is $gp\alpha\text{-interior point of } A$.

Proof: For any pre-interior point x of A . Then there exists pre-open set G containing x such that $G \subseteq A$. Since every pre-open set is $gp\alpha\text{-open}$ [9], then G is $gp\alpha\text{-open in } X$. Hence x is a $gp\alpha\text{-interior point of } A$.

Corolary 3.1. For any subset A of a space X , every $\alpha\text{-interior point of } A$ is $gp\alpha\text{-interior point of } A$.

Proof: It follows from the fact that every $\alpha\text{-open set is } gp\alpha\text{-open in } X$ [9].

4 $gp\alpha\text{-Neighbourhood points in topological spaces}$

This section deals with the properties of $gp\alpha\text{-neighbourhood points in topological spaces}$.

Definition 4.1. [9] A subset N of X is said to be $gp\alpha\text{-neighbourhood of a point } x \in X$, if there exists an $gp\alpha\text{-open set } G$ containing x such that $x \in G \subset N$.

Theorem 4.1. Every neighbourhood N of X is a $gp\alpha\text{-neighbourhood of } X$.

Proof: Follows from the definition of neighbourhood and every open set is $gp\alpha\text{-open}$ [9].

From the following example converse of the Theorem 4.1 is not true.

Example 4.1. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a, b\}, \{a, b, c\}\}$.

Let $a \in X$. Consider $A = \{a, d\}$. Since A is $gp\alpha\text{-neighbourhood of the point } a$, but A is not a neighbourhood of the point a .

Theorem 4.2. If $N \subseteq X$ is $gp\alpha\text{-open}$ then, N is $gp\alpha\text{-neighbourhood of each of its points}$.

Proof: Suppose, N is $gp\alpha\text{-open}$ and x be any arbitrary point in N , so $x \in N \subseteq N$. Since, x is any arbitrary point and from Definition 4.1, there exists, $gp\alpha\text{-open set}$

N . Thus $x \in N \subseteq N$. So, N is a neighbourhood of each of its points.

Converse of the Theorem 4.2 need not true in general, as seen from the Example 4.2

Example 4.2. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$. Here the gp α -open sets are: $X, \phi, \{a\}, \{b\}, \{c\}, \{c, d\}, \{a, c, d\}$. Then the set $A = \{a, b\}$ is gp α -neighbourhood of the points a and b , but the set $A = \{a, b\}$ is not gp α -open in X .

Theorem 4.3. Let A be gp α -closed set in X and $x \in A^c$. Then there exists gp α -neighbourhood N of x such that $N \cap A = \phi$.

Proof: Let A be gp α -closed in X and $x \in A^c$. Then $A^c = X \setminus A$ which is gp α -open in X . Then from Theorem 4.2, A^c is gp α -neighbourhood of each of its points. Hence, for every point $x \in A^c$, there exists gp α -neighbourhood N of x such that $N \subset A^c$, that is $N \cap A = \phi$.

Theorem 4.4. Let $x \in X$ and $gp\alpha N(x)$ be the collection of all gp α -neighbourhoods of x . Then the following results holds:

- (i) $gp\alpha N(x) \neq \phi, \forall x \in X$.
 - (ii) $N \in gp\alpha N(x)$ implies $x \in N$.
 - (iii) Let $N \in gp\alpha N(x)$ and $N \subseteq M$, then $M \in gp\alpha N(x)$.
 - (iv) $N \in gp\alpha N(x)$ and $M \in gp\alpha N(x)$ then $N \cap M \in gp\alpha N(x)$.
- Proof:** (i) We have X is always gp α -open in X . Hence $X \in gp\alpha N(x)$ of every point $x \in X$. Therefore $gp\alpha N(x) \neq \phi$ for every point $x \in X$.
- (ii) From definition of $N \in gp\alpha N(x)$, it follows that $x \in N$.
- (iii) Let $N \in gp\alpha N(x)$ and $N \subseteq M$. Then there exists gp α -open set G such that $x \in G \subseteq N$. Since $N \subseteq M$, then $x \in G \subseteq M$. So by definition of 4.1, M is a gp α -neighbourhood point of x . Hence $M \in gp\alpha N(x)$.
- (iv) Let $N \in gp\alpha N(x)$ and $M \in gp\alpha N(x)$. Then, there exist gp α -open sets U and V such that $x \in U \subseteq N$ and $x \in V \subseteq M$. That is $x \in U \cap V \subseteq N \cap M$. Therefore, for every point $x \in X$, there exists gp α -open set $U \cap V$ such that $x \in U \cap V \subseteq N \cap M$. So, $N \cap M$ is a gp α -neighbourhood of a point x . Hence, intersection of two gp α -neighbourhood of a point is again a gp α -neighbourhood of point.

Theorem 4.5. Let A be any subset of X , Then, A is gp α -closed if and only if $gp\alpha - d(A) \subseteq A$.

Proof: Let A be gp α -closed in X , then A^c is gp α -open in X such that $x \in A^c$. Then for each point $x \in X$ and from Definition 4.1, there exist gp α -open set G such that $x \in G \subset N \setminus A$. Then $A \cap (X \setminus A) = \phi$. Therefore gp α -neighbourhood of G contains no points of A . Hence x is not a gp α -limit point of A . Thus no point

of $X \setminus A$ is a $gp\alpha$ -limit point of A , that is A contains all the $gp\alpha$ -limit points. Therefore A contains the $gp\alpha$ -derived points. Hence $gp\alpha d(A) \subset A$. Conversely, suppose $gp\alpha d(A) \subset A$ and let $x \in A^c$. So $x \notin A$. Hence $x \notin gp\alpha d(A)$. Therefore x is not a limit point of A . Then, there exists $gp\alpha$ -open set G such that $G \cap (A \setminus \{x\}) = \emptyset$, that is $G \subset X \setminus A$. Therefore $\forall x \in X \setminus A$, there exists $gp\alpha$ -open set G such that $x \in G \subset X \setminus A$. Therefore $X \setminus A$ is $gp\alpha$ -open in X and hence A is $gp\alpha$ -closed.

5 $gp\alpha$ -Kuratowski's closure operators in topological spaces

Theorem 5.1. If $P-C(X, \tau)$ is closed under finite union, then $gp\alpha-C(X, \tau)$ is closed under finite union, where $P-C(X, \tau)$ and $gp\alpha-C(X, \tau)$ are the families of pre-closed sets and $gp\alpha$ -closed sets in (X, τ) respectively.

Proof: Let A and B are $gp\alpha$ -closed sets in X and $A \cup B \subseteq G$, where G is $\omega\alpha$ -open in X . Then $A \subseteq G$ and $B \subseteq G$. Since A and B are $gp\alpha$ -closed, then $pcl(A) \subseteq G$ and $pcl(B) \subseteq G$. That is, $pcl(A) \cup pcl(B) \subseteq G$. Thus, from hypothesis, $pcl(A) \cup pcl(B) \subseteq G$. Hence $A \cup B$ is $gp\alpha$ -closed in X .

Definition 5.1. Let $\tau_{gp\alpha}^*$ be the topology on X generated by $gp\alpha$ -closure in the usual manner, $\tau_{gp\alpha}^* = \{G \subset X : gp\alpha-cl(X \setminus G) = X \setminus G\}$.

Definition 5.2. Let $\tau_{g^*p}^*$ be the topology on X generated by g^*p -closure in the usual manner, that is

$$\tau_{g^*p}^* = \{G \subset X : g^*p-cl(X \setminus G) = X \setminus G\}$$

Theorem 5.2. Let $A \subseteq X$. Then the following statements holds:

(i) $\tau \subseteq \tau_{gp\alpha}^*$

(ii) $\tau \subseteq \tau^p \subseteq \tau_{gp\alpha}^*$

(τ^p is family of pre-open sets.[13])

Proof: (i) Let $A \in \tau$, then A^c is closed in X . We have $A^c \subseteq gp\alpha-cl A^c \subseteq cl A^c$. Since, A^c is closed, then $cl(A^c) = A^c$. Hence $A^c \subseteq gp\alpha-cl(A^c) \subseteq A^c$. Therefore $gp\alpha-cl(A^c) \subseteq A^c$. But $A^c \subseteq gp\alpha-cl(A^c)$ is always true. Thus $gp\alpha-cl(A^c) = A^c$. Hence $A \in \tau_{gp\alpha}^*$.

(ii) Since, every pre-closed set is $gp\alpha$ -closed in X , proof follows.

Theorem 5.3. For any subset A of a topological space X , $\tau \subseteq \tau_{gp\alpha}^* \subseteq \tau_{g^*p}^*$.

Proof: Let us consider $A \in \tau$, then A^c is closed in X . Then $A^c \subseteq gp\alpha-cl(A^c) \subseteq cl(A^c)$. Since, A^c is closed, then $cl(A^c) = A^c$. Therefore $A^c \subseteq gp\alpha-cl(A^c) \subseteq A^c$. Hence, $gp\alpha-cl(A^c) \subseteq A^c$. But $(A^c) \subseteq gp\alpha-cl(A^c)$ is always true. Hence $(A^c) = gp\alpha-cl(A^c)$. Thus $A \in \tau_{gp\alpha}^*$ and hence $\tau \subseteq \tau_{gp\alpha}^*$.

Let $A \in \tau_{gp\alpha}^*$. Then $gp\alpha cl(A^c) = A^c$. But $A^c \subseteq g^*pcl(A^c) \subseteq gp\alpha cl(A^c) = A^c$ from [9]. Hence $g^*pcl(A^c) = A^c$. Hence $A \in \tau_{g^*p}^*$. Thus $A \in \tau_{gp\alpha}^*$ implies that $A \in \tau_{g^*p}^*$. Hence $\tau \subset \tau_{gp\alpha}^* \subset \tau_{g^*p}^*$.

Theorem 5.4. The following statements are equal for the space X :

(i) every gp α -closed set is pre-closed.

(ii) $\tau^p = \tau_{gp\alpha}^*$.

(iii) for each $x \in X$, $\{x\}$ is $\omega\alpha$ -open or pre-open.

Proof: (i) $\rightarrow$ (ii) Let $G \in \tau_{gp\alpha}^*$. Then from [9] and by the definition 5.1, we have $gp\alpha-cl(A) = A$. Hence $X \setminus G = gp\alpha-cl(X \setminus G) = p-cl(X \setminus G)$. Therefore $X \setminus G$ is pre-closed and so G is pre-open. Therefore $\tau_{gp\alpha}^* \subseteq \tau^p$ and from [8], $\tau^p \subseteq \tau_{gp\alpha}^*$. Hence $\tau^p = \tau_{gp\alpha}^*$.

(ii) $\rightarrow$ (iii) Let $\{x\} \in X$. By [9], we have $x \setminus \{x\} = gp\alpha-cl(X \setminus \{x\})$ is true only when $\{x\}$ is not $\omega\alpha$ -closed. Hence $\{x\} \in \omega\alpha-C(X, \tau)$ or $x \in \tau^p$.

(iii) $\rightarrow$ (i) Let A be gp α -closed in X and $x \in p-cl(A)$. Then we have $x \in A$.

care I: If $\{x\}$ is $\omega\alpha$ -closed. Suppose $x \notin A$, then $p-cl(A) \setminus A$ contains a $\omega\alpha$ -closed set $\{x\}$, which is contradiction. Hence $x \in A$.

care II: If $\{x\}$ is pre-open. Since $x \in pcl(A)$, then $\{x\} \cap A = \phi$. Hence, we have $pcl(A) = A$ and thus A is pre-closed. Therefore $gp\alpha-C(X, \tau) \subset p-C(X, \tau)$.

Theorem 5.5. Every gp α -closed set closed if and only if $\tau = \tau_{gp\alpha}^*$.

Proof: Suppose every gp α -closed set is closed. Let A be gp α -closed then, $gp\alpha-cl(A) = cl(A)$. Thus $\tau = \tau_{gp\alpha}^*$.

Conversely, let A be gp α -closed then from [9], $A = gp\alpha-cl(A)$. Hence $X \setminus A \in \tau_{gp\alpha}^*$. Hence, A is closed in X .

Theorem 5.6. Every gp α -closed set is pre-closed if and only if $\tau^p = \tau_{gp\alpha}^*$.

Proof: Suppose that every gp α -closed set is pre-closed. Let A be gp α -closed in X . Then from hypothesis, $gp\alpha-cl(A) = pcl(A)$. Thus $\tau^p = \tau_{gp\alpha}^*$.

Conversely, let A be gp α -closed in X . Then $A = gp\alpha-cl(A)$. Thus $X \setminus A \in \tau_{gp\alpha}^*$. Hence, A is pre-closed in X .

Remark 5.1. Let A be any subset of X . Then $gp\alpha-int(A)$ is the largest gp α -open set contained in A if A is gp α -open.

6 Characterizations of gp α -closed sets in topological spaces

Definition 6.1. [9] A point $x \in X$ is a gp α -limit point of a subset A of X , if and only if every gp α -neighbourhood of x contains a point of A distinct from x . That is, $[N \setminus \{x\}] \cap A \neq \phi$ for each gp α -neighbourhood of N of x .

Definition 6.2. [9] The set of all $gp\alpha$ -limit points of A is a $gp\alpha$ -derived set of A and is denoted by $gp\alpha-d(A)$.

Example 6.1. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$.

Let $A = \{c\}$. Then the only limit point with respect to the set $A = c$ is point b .

Therefore $d(A) = \{b\}$.

But $gp\alpha$ -limit point with respect to the set A is ϕ . Therefore $gp\alpha-d(A) = \phi$.

Theorem 6.1. Let τ_1 and τ_2 be any two topologies on a set X such that $gp\alpha-O(X, \tau_1) \subseteq gp\alpha-O(X, \tau_2)$. Then for every subset A of X , every $gp\alpha$ -limit point of A with respect to τ_2 is $gp\alpha$ -limit point of A with respect to τ_1 .

Proof: Let x be a $gp\alpha$ -limit point of A with respect to τ_2 . Then by definition of $gp\alpha$ -limit point $(G \cap A) \setminus \{x\} \neq \phi$, this is true for every $G \in gp\alpha-O(X, \tau_2)$ and $x \in G$. But by hypothesis, $gp\alpha-O(X, \tau_1) \subseteq gp\alpha-O(X, \tau_2)$. Hence $(G \cap A) \setminus \{x\} \neq \phi$ for every $G \in gp\alpha-O(X, \tau_1)$ such that $x \in G$. Hence x is a $gp\alpha$ -limit point of A with respect to the topology τ_1 .

Theorem 6.2. Let A and B be any two subsets of (X, τ) . Then the following assertions are valid:

(i) $gp\alpha-d(A) \subseteq d_p(A)$, where d_p is a Pre-derived set [13].

(ii) $gp\alpha-d(gp\alpha-d(A)) \setminus A \subseteq gp\alpha-d(A)$.

(iii) $gp\alpha-d(A \cup gp\alpha-d(A)) \subseteq A \cup gp\alpha-d(A)$.

Proof: (i) It clearly observed from the fact that every pre-open set is $gp\alpha$ -open in X .

(ii) Let $x \in gp\alpha-d(gp\alpha-d(A)) \setminus A$ and $G \in gp\alpha-O(X, \tau)$ with $x \in G$. Then $G \cap (gp\alpha-d(A)) \setminus \{x\} \neq \phi$. Let $y \in G \cap (gp\alpha-d(A) \setminus \{x\})$. Then $y \in G$ and $y \in gp\alpha-d(A) \setminus \{x\}$. That is $y \in G$ and $y \in gp\alpha-d(A)$. Hence $G \cap (A \setminus \{y\}) \neq \phi$. Let $z \in G \cap (A \setminus \{y\})$, then $x \neq z$ as $x \notin A$. Thus $G \cap (A \setminus \{x\}) \neq \phi$.

(iii) Let $x \in gp\alpha-d(A \cup gp\alpha-d(A))$. If $x \in A$, then $x \in gp\alpha-d(A)$. Therefore $x \in A \cup gp\alpha-d(A)$. On the contrary assume that $x \notin A$. Then $G \cap (A \cup gp\alpha-d(A)) \setminus \{x\} \neq \phi$, is true for all $G \in gp\alpha-d(A)$ and $x \in G$. Therefore $(G \cap A) \setminus \{x\} \neq \phi$ or $G \cap (gp\alpha-d(A)) \setminus \{x\} \neq \phi$. Thus $x \in gp\alpha-d(A)$.

If $G \cap (gp\alpha-d(A)) \setminus \{x\} \neq \phi$, then will get $x \in gp\alpha-d(gp\alpha-d(A))$. Since $x \notin A$, then $x \in gp\alpha-d(gp\alpha-d(A)) \setminus A$. Therefore $gp\alpha-d(A \cup gp\alpha-d(A)) \subset A \cup gp\alpha-d(A)$.

Remark 6.1. We can see the following implification with respect to $gp\alpha$ -open sets.

Example 6.2. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$.

pre-open sets are $X, \{\phi\}, \{a\}, \{c\}, \{d\}, \{a, c\}, \{b, c\}, \{c, d\}, \{d, e\}, \{a, c, d\},$

$\{a, c, e\}, \{a, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}.$

$gp\alpha$ -open sets $X, \{\phi\}, \{a\}, \{c\}, \{d\}, \{a, b\}, \{b, d\}, \{c, d\}, \{a, c\},$

$\{b, c\}, \{d, e\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c\}, \{a, c, e\}, \{a, d, e\},$

$\{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}$.

(i) Let $A = \{b, c, d\}$. Then pre-limit point of the set A is $\{b, e\}$ and gp α -limit point of A is $\{e\}$.

Hence $D_p(A) = \{b, e\}$ and $gp\alpha-d(A) = \{e\}$.

Hence we can observe that $gp\alpha-d(A) \subset D_p(A)$.

(ii) Let $A = \{a, c, d\}$. Then $gp\alpha-d(A) = \{b, e\}$.

Consider $gp\alpha-d(gp\alpha-d(A)) = gp\alpha-d(\{b, e\}) = \{e\}$.

Then $gp\alpha-d(gp\alpha-d(A)) \setminus A = \{e\} \setminus \{a, c, d\} = \{e\}$,

thus $\{e\} \subseteq gp\alpha-d(A)$

that is $\{e\} \subseteq \{b, e\}$

(iii) Let $A = \{a, c, d\}$, then $gp\alpha-d(A) = \{b, e\}$.

Consider $A \cup gp\alpha-d(A) = \{a, c, d\} \cup \{b, e\} = X$.

But $gp\alpha-d(X) = \{b, e\}$.

Therefore $gp\alpha-d(A \cup gp\alpha-d(A)) = gp\alpha-d(X) = \{b, e\}$.

Now consider $A \cup gp\alpha-d(A) = \{a, c, d\} \cup \{b, e\} = X$.

Hence $gp\alpha-d(A \cup gp\alpha-d(A)) \neq A \cup gp\alpha-d(A)$, that is $\{b, e\} \neq X$, but $gp\alpha-d(A \cup gp\alpha-d(A)) \subset A \cup gp\alpha-d(A)$.

Remark 6.2. if $A \subseteq B$, then $gp\alpha-d(A) \subseteq gp\alpha-d(B)$.

Example 6.3. Consider the Example 6.2,

Let $A = \{b, c, d\}$. Then we have $gp\alpha-d(A) = \{e\}$.

Let $B = \{a, e\}$. Then gp α -limit point of the set B is $\{\phi\}$, that is $gp\alpha-d(B) = \{\phi\}$.

Thus we can observe that $gp\alpha-d(B) \subset gp\alpha-d(A)$ but not $B \subseteq A$.

Remark 6.3. $gp\alpha-d(A \cap B) \subseteq gp\alpha-d(A) \cap gp\alpha-d(B)$.

Example 6.4. From the Example 6.2,

Let $A = \{b, c, d\}$ and $B = \{b, c\}$ are any two subsets of X .

Then $gp\alpha-d(A) = \{e\}$ and $gp\alpha-d(B) = \{\phi\}$.

Also $gp\alpha-d(A \cap B) = \{\phi\}$.

Therefore $gp\alpha-d(A \cap B) \subseteq gp\alpha-d(A) \cap gp\alpha-d(B)$

Theorem 6.3. Let A be any subset of X and $x \in X$. Then the following statements are equal:

(i) $\forall x \in X, A \cap G \neq \phi$ where G is gp α -open in X .

(ii) $x \in gp\alpha-cl(A)$.

Proof: Let A be any subset of X .

(i) $\rightarrow$ (ii): On the contrary assume that $x \notin gp\alpha-cl(A)$. Then there exists gp α -closed set F such that $A \subseteq F$ and $x \notin F$. Then $X \setminus F$ is gp α -open in X containing a point x . Hence $A \cap (X \setminus F) \subseteq A \cap (X \setminus A) = \phi$, which is contradiction to the

assumption. Hence $x \in \text{gp}\alpha\text{-cl}(A)$.

(ii) $\rightarrow$ (i): It follows from the definition 3.2.

Corollary 6.1. For any subset A of a space X , $\text{gp}\alpha\text{-d}(A) \subset \text{gp}\alpha\text{-cl}(A)$.

Theorem 6.4. Let A be any subset of X , then $\text{gp}\alpha\text{-cl}(A) = A \cup \text{gp}\alpha\text{-d}(A)$.

Proof: Let $x \in \text{gp}\alpha\text{-cl}(A)$. On the contrary assume that $x \notin A$. Let G be any $\text{gp}\alpha$ -open set containing a point x . Then $(G \setminus \{x\}) \cap A \neq \emptyset$. Therefore x is $\text{gp}\alpha$ -limit point of A and hence x is $\text{gp}\alpha$ -derived set of A , that is $x \in \text{gp}\alpha\text{-d}(A)$.

Hence $\text{gp}\alpha\text{-cl}(A) \subset A \cup \text{gp}\alpha\text{-d}(A)$.

From the corollary 6.1, we have $\text{gp}\alpha\text{-d}(A) \subset \text{gp}\alpha\text{-cl}(A)$ and $A \subseteq \text{gp}\alpha\text{-cl}(A)$ is always true. Hence $A \cup \text{gp}\alpha\text{-d}(A) \subset \text{gp}\alpha\text{-cl}(A)$.

Therefore $\text{gp}\alpha\text{-cl}(A) = A \cup \text{gp}\alpha\text{-d}(A)$.

Theorem 6.5. Let A be $\text{gp}\alpha$ -open set in X and B be any subset of X . Then $A \cap \text{gp}\alpha\text{-cl}(B) \subseteq \text{gp}\alpha\text{-cl}(A \cap B)$.

Proof: Let $x \in A \cap \text{gp}\alpha\text{-cl}(B)$. Then $x \in A$ and $x \in \text{gp}\alpha\text{-cl}(B)$. From the theorem 6.4, we have $\text{gp}\alpha\text{-cl}(B) = B \cup \text{gp}\alpha\text{-d}(B)$.

If $x \in B$ then $x \in A \cap B$. Then $A \cap B \subseteq \text{gp}\alpha\text{-cl}(A \cap B)$.

If $x \notin B$ then $x \in \text{gp}\alpha\text{-d}(B)$. From the definition of $\text{gp}\alpha$ -limit point, we have $G \cap B \neq \emptyset$ for every $\text{gp}\alpha$ -open set G containing x . Therefore $G \cap (A \cap B) = (G \cap A) \cap B \neq \emptyset$. Hence $x \in \text{gp}\alpha\text{-d}(A \cap B) \subseteq \text{gp}\alpha\text{-cl}(A \cap B)$. Therefore $A \cap \text{gp}\alpha\text{-cl}(B) \subseteq \text{gp}\alpha\text{-cl}(A \cap B)$.

However the equality does not holds in general

Example 6.5. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \emptyset, \{a\}\}$.

Let $A = \{a, b\}$ and $B = \{a, c\}$.

Then $A \cap \text{gp}\alpha\text{-cl}(B) = \{a, b\}$

and $\text{gp}\alpha\text{-cl}(A \cap B) = X$

This implies $A \cap \text{gp}\alpha\text{-cl}(B) \neq \text{gp}\alpha\text{-cl}(A \cap B)$. But, $A \cap \text{gp}\alpha\text{-cl}(B) \subset \text{gp}\alpha\text{-cl}(A \cap B)$.

Theorem 6.6. Let A and B be any subsets of a space X . If A is $\text{gp}\alpha$ -closed then $\text{gp}\alpha\text{-cl}(A \cap B) \subseteq A \cap \text{gp}\alpha\text{-cl}(B)$.

Proof: If A is $\text{gp}\alpha$ -closed then $\text{gp}\alpha\text{-cl}(A) = A$.

Therefore $\text{gp}\alpha\text{-cl}(A \cap B) \subseteq \text{gp}\alpha\text{-cl}(A) \cap \text{gp}\alpha\text{-cl}(B) = A \cap \text{gp}\alpha\text{-cl}(B)$. Hence $\text{gp}\alpha\text{-cl}(A \cap B) \subseteq A \cap \text{gp}\alpha\text{-cl}(B)$.

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