

On a class of sets between a-open sets and $g\delta$ -open sets

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Abstract

In this paper, a new class of sets called Da-open sets are introduced and investigated with the help of $g\delta$ -open and δ -closed sets. Relationships between this new class and other related classes of sets are established and as an application Da-continuous functions have been defined to study its properties in terms of Da-open sets. Finally, some properties of Da-closed graph and (D.a)-closed graphs are investigated.

Keywords : a-open set, δ -open set, $g\delta$ -open, Da-open set, Da-closed set.

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1 Introduction

The concept of generalized open sets introduced by Levine[8] plays a significant role in General Topology. The study of generalized open sets and its properties found to be useful in computer science and digital topology[8,9]. Since Professor El- Naschie has recently shown in [6,7] that the notion of fuzzy topology may be relevant to quantum particle physics in connection with string theory and ϵ^∞ theory. So, the fuzzy topological version of the notions and results introduced in this paper are very important. Ekici [1] introduced the notion of a-open sets in topological spaces. Dontchev et al. [17] introduced $g\delta$ -closed sets and $g\delta$ -continuity. The concept of δ -open set and introduced by Velicko [13].

In this paper, we introduce and study the class of Da-open sets which is strictly placed between the class of all a-open sets and the class of all $g\delta$ -open subsets of X .

Also Da-continuous functions, Da-closed graphs and (D,a)-closed graphs have been defined to study its properties in terms of Da-open sets.

2 Prerequisites, Definitions and Theorems

Definition 2.1 In (X, σ) , let $N \subseteq X$. A set N is called

(i) a-closed if $cl(int(cl_\delta(N))) \subseteq N$.

(ii) δ -closed[15] if $N = cl_\delta(N)$

where $cl_\delta(N) = \{p \in X : int(cl(O)) \cap N \neq \emptyset, O \in \tau \text{ and } p \in O\}$.

(iii) generalized δ -closed (briefly, $g\delta$ -closed)[6] if $Cl(N) \subset G$ whenever $N \subset G$ and G is δ -open in X .

(iv) generalized δ -closed (briefly, $g\delta$ -closed)[6] if $Cl(N) \subset G$ whenever $N \subset G$ and G is δ -open in X .

The complements of the above mentioned closed sets are their respective open sets.

The a-closure[14](resp, $g\delta$ -closure) of a set N is the intersection of all a-closed(resp, $g\delta$ -closed) sets containing N and is denoted by $a-Cl(N)$ (resp, $Cl_{g\delta}(N)$).

The a-interior[14](resp, $g\delta$ -interior) of a set N is the union of all a-open(resp, $g\delta$ -open) sets contained in N and is denoted by $a-Int(N)$ (resp, $Int_{g\delta}(N)$).

Theorem 2.2 Let C and D be subsets of a topological space (X, σ) . Then

(i) If C is $g\delta$ -closed, then $Cl_{g\delta}(C) = C$.

(ii) If $C \subset D$, then $Cl_{g\delta}(C) \subset Cl_{g\delta}(D)$.

(iv) $x \in Cl_{g\delta}(C)$ if and only if for each $g\delta$ -open set O containing x , $O \cap C \neq \emptyset$,

- (v) $Cl_{g\delta}(C) \cup Cl_{g\delta}(D) \subseteq Cl_{g\delta}(A \cup D)$.
 (vi) $Cl_{g\delta}(C \cap D) \subseteq Cl_{g\delta}(C) \cap Cl_{g\delta}(D)$.

3 Da-Open Sets.

Definition 3.1 A subset M of a topological space (X, σ) is said to be:

- (1) a Da-open set if $M \subset Int_{g\delta}(Cl_{\delta}(Int_{g\delta}(M)))$.
 (2) a Da-closed set if $Cl_{g\delta}(Int_{\delta}(Cl_{g\delta}(M))) \subset M$.

The collection of all Da-open (resp, Da-closed) sets in (X, σ) is denoted by $DaDaO(X)$ (resp, $DaC(X)$).

Theorem 3.2 Let (X, σ) be a space. Then for any $N \subset X$,

- (i) N is δ -open implies a-open[4].
 (ii) N is δ -open implies $g\delta$ -open[5].
 (iii) N is g -open implies $g\delta$ -open[6].
 (iv) N is a-open implies Da-open.
 (v) N is $g\delta$ -open implies Da-open.

Proof: (iv) Since δ -open set is $g\delta$ -open, $Int_{\delta}(N) \subset Int_{g\delta}(N)$.

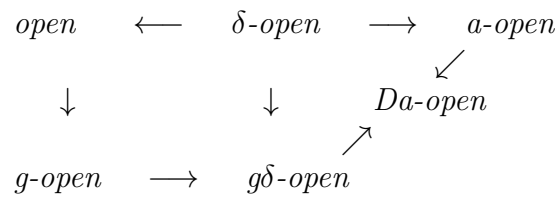
Now, let N is a-open, then $N \subset Int(Cl(Int_{\delta}(N)))$. Therefore,
 $N \subset Int(Cl(Int_{\delta}(N))) = Int_{\delta}(Cl(Int_{\delta}(N))) \subset Int_{g\delta}(Cl_{\delta}(Int_{g\delta}(N)))$.
 Hence N is Da-open in X .

(v) Suppose N is $g\delta$ -open. Then $Int_{g\delta}(N) = N$ [14].

Therefore, $Int_{g\delta}(N) \subset Cl_{\delta}(Int_{g\delta}(N))$. Then

$N = Int_{g\delta}(N) = Int_{g\delta}(Int_{g\delta}(N)) \subset Int_{g\delta}(Cl_{\delta}(Int_{g\delta}(N)))$. Hence N is Da-open.

Remark 3.3 The following implications hold for a subset N of a topological space (X, σ) .



None of these implications is reversible

Example 3.4 Consider $X = \{p, q, r, s\}$ with topology

$\sigma = \{X, \phi, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{p, q, r\}\}$, then

$aO(X) = \{X, \phi, \{q\}, \{p, r\}, \{p, q, r\}\}$

$G\delta O(X) = \{X, \phi, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}\}$.

$DaO(X) = \{X, \phi, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}, \{p, q, s\}, \{q, r, s\}\}$.

Therefore, $\{b, c, d\} \in DaO(X)$ but $\{q, r, s\} \notin aO(X)$ and $\{q, r, s\} \notin G\delta O(X)$.

Lemma 3.5 *If $\exists$ a $M \in G\delta O(X)$ such that $M \subset N \subset Int_{g\delta}(Cl_\delta(M))$, then N is Da-open.*

Proof: Since M is $g\delta$ -open, $Int_{g\delta}(M) = M$. Therefore,
 $Int_{g\delta}(Cl_\delta(Int_{g\delta}(N))) \supset Int_{g\delta}(Cl_\delta(Int_{g\delta}(M))) = Int_{g\delta}(Cl_\delta(M)) \supset N$.
Hence N is Da-open.

Converse of the Lemma 3.5 is not true as shown in Example 3.6.

Example 3.6 In Example 3.4, $\{p, q, r\} \in DaO(X)$ and $\{p, r\} \in G\delta O(X)$ but $\{p, r\} \subseteq \{p, q, r\} \not\subseteq Int_{g\delta}(Cl_\delta(\{p, r\})) = \{p, r\}$.

Lemma 3.7 *For a family $\{B_\lambda : \lambda \in \Lambda\}$ of subsets of a topological space (X, σ) , the following hold:*

- (1) $Cl_{g\delta}(\cap \{B_\lambda : \lambda \in \Lambda\}) \subset \cap \{Cl_{g\delta}(B_\lambda) : \lambda \in \Lambda\}$
- (2) $Cl_{g\delta}(\cup \{V_\lambda : \lambda \in \Lambda\}) \supset \cup \{Cl_{g\delta}(B_\lambda) : \lambda \in \Lambda\}$
- (3) $Cl_\delta(\cap \{B_\lambda : \lambda \in \Lambda\}) \subset \cap \{Cl_\delta(B_\lambda) : \lambda \in \Lambda\}$
- (4) $Cl_\delta(\cup \{B_\lambda : \lambda \in \Lambda\}) \supset \cup \{Cl_\delta(B_\lambda) : \lambda \in \Lambda\}$

Theorem 3.8 *If $\{G_\alpha : \alpha \in \Lambda\}$ is a collection of Da-open sets in a space (X, σ) , then $\bigcup_{\alpha \in \Lambda} G_\alpha$ is a Da-open set in (X, σ) :*

Proof: Since each G_α is Da-open, $G_\alpha \subset Int_{g\delta}(Cl_\delta(Int_{g\delta}(G_\alpha)))$ for each $\alpha \in \Lambda$ and hence $\bigcup_{\alpha \in \Lambda} G_\alpha \subset \bigcup_{\alpha \in \Lambda} Int_{g\delta}(Cl_\delta(Int_{g\delta}(G_\alpha))) \subset Int_{g\delta}(Cl_\delta(Int_{g\delta}(\bigcup_{\alpha \in \Lambda} G_\alpha)))$. Thus $\bigcup_{\alpha \in \Lambda} G_\alpha$ is Da-open.

Corollary 3.9 *If $\{F_\alpha : \alpha \in \Lambda\}$ is a collection of Da-closed sets in a space (X, σ) , then $\bigcap_{\alpha \in \Lambda} F_\alpha$ is a Da-closed set in (X, σ)*

Remark 3.10 M and $N \in DaO(X) \not\Rightarrow M \cap N \in DaO(X)$ as seen from Example 3.4, where both $M = \{q, r, s\}$ and $N = \{p, q, s\} \in DaO(X)$ but $M \cap N = \{q, s\} \notin DaO(X)$.

Corollary 3.11 *If $M \in DaO(X)$ and $B \in aO(X)$, then $M \cup B \in DaO(X)$.*

Proof: Follows from Theorem and Theorem

Corollary 3.12 *If $M \in DaO(X)$ and $B \in g\delta O(X)$, then $M \cup B \in DaO(X)$.*

Proof: Follows from Theorem and Theorem

Definition 3.13 *In (X, σ) , let $M \subset X$.*

- (1) *The Da-interior of M , denoted by $Int_a^D(M)$ is defined as $Int_a^D(M) = \bigcup \{G : G \subseteq M \text{ and } M \in DaO(X)\}$;*
- (2) *The Da-closure of M , denoted by $Cl_a^D(M)$ is defined as $Cl_a^D(A) = \bigcap \{F : M \subseteq F \text{ and } F \in DaC(X)\}$.*

Theorem 3.14 In (X, τ) , let $M, N, F \subset X$. Then:

- (1) $M \subset Cl_a^D(M) \subset aCl(M)$, $Cl_a^D(M) \subset Cl_{g\delta}(M)$;
- (2) $Cl_a^D(M)$ is a Da-closed set;
- (3) If F is a Da-closed set, and $F \supset M$, then $F \supset Cl_a^D(M)$;
i.e., $Cl_a^D(M)$ is the smallest Da-closed set containing M .
- (4) M is Da-closed set if and only if $Cl_a^D(M) = M$;
- (5) $Cl_a^D(Cl_a^D(M)) = Cl_a^D(M)$;
- (6) $M \subseteq N$ implies $Cl_a^D(M) \subseteq Cl_a^D(N)$;
- (7) $p \in Cl_a^D(M)$ if and only if for each Da-open set V containing p , $V \cap M \neq \emptyset$;
- (8) $Cl_a^D(M) \cup Cl_a^D(N) \subset Cl_a^D(M \cup N)$;
- (9) $Cl_a^D(M \cap N) \subset Cl_a^D(M) \cap Cl_a^D(N)$.

Proof: (1) It follows from Theorem 3.2(iii) and (iv)

(2) It follows from Definition 3.13 and Corollary 3.9

(3) Let F be a Da-closed set, containing M . $Cl_a^D(M)$ is the intersection of Da-closed sets containing M , and F is one among these; hence $F \supset Cl_a^D(M)$.

(4) Let M be Da-closed, then by Definition 3.13(2), $Cl_a^D(M) = M$.

Conversely, let $Cl_a^D(M) = M$. Then by (2) above, M is Da-closed.

(5) It follows from (2) and (4).

(6) Obvious.

$$\begin{aligned}
 (7) \quad x \notin Cl_a^D(M) &\Leftrightarrow (\exists G \in DaC(X))(M \subset G)(x \notin G) \\
 &\Leftrightarrow (\exists G \in DaC(X))(M \subset G)(x \in G^c) \\
 &\Leftrightarrow (\exists G^c \in DaO(X))(M \cap G^c = \emptyset)(x \in G^c) \\
 &\Leftrightarrow (\exists G^c \in DaO(X, x))(M \cap G^c = \emptyset) \\
 &\text{i.e., } (\exists U (=G^c) \in DaO(X, x))(M \cap U = \emptyset)
 \end{aligned}$$

(8) and (9) follows from (6).

Remark 3.15 (1) $Cl_a^D(M) \cup Cl_a^D(N) \neq Cl_a^D(M \cup N)$, in general, as seen from Example 3.4 where $M = \{a\}$, $N = \{c\}$ and $M \cup N = \{a, c\}$. Then $Cl_a^D(M) = \{a\}$, $Cl_a^D(N) = \{c\}$, $Cl_a^D(M) \cup Cl_a^D(N) = \{a, c\}$ and $Cl_a^D(M \cup N) = \{a, c, d\}$;

(2) $Cl_a^D(M \cap N) \neq Cl_a^D(M) \cap Cl_a^D(N)$, in general, as seen from Example 3.4 where, $M = \{a, b, c\}$, $N = \{d\}$ and $M \cap N = \emptyset$. Then $Cl_a^D(M) = X$, $Cl_a^D(N) = \{d\}$, $Cl_a^D(M) \cap Cl_a^D(N) = \{d\}$ and $Cl_a^D(M \cap N) = \emptyset$

Lemma 3.16 In (X, σ) , let $M \subset X$. Then

- (1) $Cl_a^D(X \setminus M) = X \setminus Int_a^D(M)$,
- (2) $Int_a^D(X \setminus M) = X \setminus Cl_a^D(M)$.

Theorem 3.17 In (X, σ) , let $M, N, G \subset X$,

- (1) $aInt(M) \subseteq Int_a^D(M) \subseteq M$, $Int_{g\delta}(M) \subseteq Int_a^D(M)$;
- (2) $Int_a^D(M)$ is a Da-open set;
- (3) If G is a Da-open set, and $G \subset M$, then $G \subset Int_a^D(M)$;
i.e., $Int_a^D(M)$ is the largest Da-open set contained in M .
- (4) M is Da-open set if and only if $Int_a^D(M) = M$;

- (5) $Int_a^D(Int_a^D(M)) = Int_a^D(M)$;
- (6) $M \subseteq N$ implies $Int_a^D(M) \subseteq Int_a^D(N)$;
- (7) $x \in Int_a^D(M)$ if and only if there exist Da-open W such that $x \in W \subseteq M$;
- (8) $Int_a^D(M \cap N) \subseteq Int_a^D(M) \cap Int_a^D(N)$.
- (9) $Int_a^D(M) \cup Int_a^D(N) \subseteq Int_a^D(M \cup N)$.

Remark 3.18 (8) $Int_a^D(M \cap N) \neq Int_a^D(M) \cap Int_a^D(N)$, in general, as seen from Example 3.4, where $M = \{a, b, d\}$, $N = \{b, c, d\}$ and $M \cap N = \{b, d\}$. Then $Int_a^D(M) = \{a, b, d\}$, $Int_a^D(N) = \{b, c, d\}$, $Int_a^D(M) \cap Int_a^D(N) = \{b, d\}$ and $Int_a^D(M \cap N) = \{b\}$.

(9) $Int_a^D(M) \cup Int_a^D(N) \neq Int_a^D(M \cup N)$, in general, as seen from Example 3.4, where $M = \{a, b, c\}$, $N = \{d\}$ and $M \cup N = X$. Then $Int_a^D(M) = \{a, b, c\}$, $Int_a^D(N) = \phi$, $Int_a^D(M) \cup Int_a^D(N) = \{a, b, c\}$ and $Int_a^D(M \cup N) = X$.

Lemma 3.19 In (X, σ) , let $M \subset X$. Then

- (1) M is Da-open if and only if $M = M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))$.
- (2) M is Da-closed if and only if $M = M \cup Cl_{g\delta}(Int_\delta(Cl_{g\delta}(M)))$.

Proof: (1) Let M be an Da-open. Then,

$$M \subseteq Int_{g\delta}(Cl_\delta(Int_{g\delta}(M))) \Rightarrow M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M))) = M.$$

Conversely, let $M = M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M))) \Rightarrow M \subseteq Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))$.

(2) It follows from (1)

Lemma 3.20 In (X, σ) , let $M \subset X$. Then

- (i) $M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))$ is Da-open
- (ii) $M \cup Cl_{g\delta}(Int_\delta(Cl_{g\delta}(M)))$ is Da-closed.

Proof: (i) $Int_{g\delta}(Cl_\delta(Int_{g\delta}(M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))))) = Int_{g\delta}(Cl_\delta(Int_{g\delta}(M) \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))))) = Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))$. This implies that $M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M))) = M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))))) \subseteq Int_{g\delta}(Cl_\delta(Int_{g\delta}(M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M))))))$. Therefore $M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))$ is Da-open.

(ii) From (i) we have $X \setminus (M \cup Cl_{g\delta}(Int_\delta(Cl_{g\delta}(M)))) = (X \setminus M) \cap Cl_{g\delta}(Int_\delta(Cl_{g\delta}(X \setminus M)))$ is Da-open that further implies $M \cup Cl_{g\delta}(Int_\delta(Cl_{g\delta}(M)))$ is Da-closed.

Lemma 3.21 In (X, σ) , let $M \subset X$. Then

- (i) $Int_a^D(M) = M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))$.
- (ii) $Cl_a^D(M) = M \cup Cl_{g\delta}(Int_\delta(Cl_{g\delta}(M)))$.

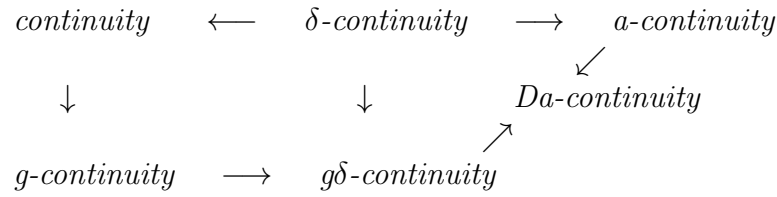
Proof: (i) Let $N = Int_a^D(M)$, then $N \subseteq M$. Since N is Da-open, $N \subseteq Int_{g\delta}(Cl_\delta(Int_{g\delta}(N))) \subseteq Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))$. Then $N \subseteq M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M))) \subseteq M$. Therefore, by Theorem 3.16 it follows that $M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))$ is a Da-open set contained in M . But $Int_a^D(M)$ is the largest Da-open set contained in M it follows that $M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M))) \subseteq Int_a^D(M) = N$. Then it follows that $N = M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))$. Therefore, $Int_a^D(M) = M \cap Int_{g\delta}(Cl_\delta(Int_{g\delta}(M)))$.

(ii) It follows from (i)

4 Da-Continuous functions.

Definition 4.1 A function $f:(X,\sigma) \rightarrow (Y,\eta)$ is said to be a Da-continuous if $f^{-1}(U)$ is Da-open in (X,σ) for each open set U of (Y,η) . Clearly, $f:(X,\sigma) \rightarrow (Y,\eta)$ is Da-continuous if and only if $f^{-1}(F)$ is Da-closed in (X,τ) for each $F \in DaC(Y)$.

Remark 4.2 The following implications hold for a function $f:(X,\sigma) \rightarrow (Y,\eta)$:



Example 4.3 Consider (X,σ) as in Example 3.4 and $\eta=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,b,c\}\}$. Define $f:(X,\sigma) \rightarrow (X,\eta)$ by $f(p)=s, f(q)=p, f(r)=q$ and $f(s)=r$. Then f is Da-continuous but neither a-continuous nor $g\delta$ -continuous since $\{p,q,r\}$ is open in (X,η) , $f^{-1}(\{p,q,r\})=\{q,r,s\} \in DaO(X)$ but $\{q,r,s\} \notin aO(X)$ and $\{q,r,s\} \notin g\delta O(X)$. The other Examples are shown in [5,6]

Theorem 4.4 Let $f:(X,\sigma) \rightarrow (Y,\eta)$ be a function. Then the following conditions are equivalent:

- (1) f is Da-continuous;
- (2) For each $p \in X$ and $D \in O(Y,\eta)$ containing $f(p)$, there exists $C \in DaO(X,\sigma)$ containing p such that $f(C) \subset D$;
- (3) For each subset N of Y , $Cl_{g\delta}(Int_{\delta}(Cl_{g\delta}(f^{-1}(N)))) \subseteq f^{-1}(Cl(N))$;
- (4) For each subset N of Y , $f^{-1}(Int(N)) \subseteq Int_{g\delta}(Cl_{\delta}(Int_{g\delta}(f^{-1}(N))))$;
- (5) For each subset N of Y , $Cl_a^D(f^{-1}(N)) \subseteq f^{-1}(Cl(N))$;
- (6) For each subset M of X , $f(Cl_a^D(M)) \subseteq Cl(f(M))$;
- (7) For each subset N of Y , $f^{-1}(Int(N)) \subseteq Int_a^D(f^{-1}(N))$;

Proof: (1) $\longleftrightarrow$ (2) is obvious

(1) $\rightarrow$ (3) Let $N \subseteq Y$. Then by (1), $f^{-1}(Cl(N)) \in DaC(X) \Rightarrow f^{-1}(Cl(N)) \supseteq Cl_{g\delta}(Int_{\delta}(Cl_{g\delta}(f^{-1}(Cl(N)))) \supseteq Cl_{g\delta}(Int_{\delta}(Cl_{g\delta}(f^{-1}(N))))$.

(3) $\rightarrow$ (4). Replace Y by $Y \setminus N$ in (3), we have $Cl_{g\delta}(Int_{\delta}(Cl_{g\delta}(f^{-1}(Y \setminus N)))) \subseteq f^{-1}(Cl(Y \setminus N))$, and therefore $f^{-1}(Int(N)) \subseteq Int_{g\delta}(Cl_{\delta}(Int_{g\delta}(f^{-1}(N))))$ for each $N \subseteq Y$.

(4) $\rightarrow$ (1). Obvious

(1) $\rightarrow$ (5). Let $N \subseteq Y$ and $f^{-1}(Cl(N))$ be Da-closed in X . Then $Cl_a^D(f^{-1}(N)) \subseteq Cl_a^D(f^{-1}(Cl(N))) = f^{-1}(Cl(N))$.

(5) $\rightarrow$ (1). Let $N \subseteq Y$ be a closed set. Then by (5), $Cl_a^D(f^{-1}(N)) \subseteq f^{-1}(Cl(N)) = f^{-1}(N) \Rightarrow Cl_a^D(f^{-1}(N)) = f^{-1}(N)$.

Therefore $f^{-1}(N) \in DaC(X)$.

(5)→(6). Let $M \subseteq X$. Then $f(M) \subseteq Y$. By (5), we get
 $f^{-1}(Cl(f(M))) \supseteq Cl_a^D(f^{-1}(f(M))) \supseteq Cl_a^D(M)$.
 Therefore, $f(Cl_a^D(M)) \subseteq f(f^{-1}(Cl(f(M)))) \subseteq Cl(f(M))$.

(6)→(5). Let $N \subseteq Y$ and $M=f^{-1}(N) \subseteq X$. Then by (6),
 $f(Cl_a^D(f^{-1}(N))) \subseteq Cl(f(f^{-1}(N))) \subseteq Cl(N)$. $\implies Cl_a^D(f^{-1}(N)) \subseteq f^{-1}(Cl(N))$.

(5)→(7). Replace N by $Y \setminus$ in (5), we get
 $Cl_a^D(f^{-1}(Y \setminus N)) \subseteq f^{-1}(Cl(Y \setminus N))$. $\implies Cl_a^D(X \setminus f^{-1}(N)) \subseteq f^{-1}(Y \setminus Int(N))$
 Therefore $f^{-1}(int(N)) \subseteq Int_a^D(f^{-1}(N))$ for each $N \subseteq Y$.

(7)→(1). Let $G \subseteq Y$ be open. Then $f^{-1}(G)=f^{-1}(Int(G)) \subseteq Int_a^D(f^{-1}(G))$
 $\implies Int_a^D(f^{-1}(G))=f^{-1}(G)$. By Theorem Hence, $f^{-1}(V)$ is Da-open in X . Therefore, f is Da-continuous.

Definition 4.5 Two non-empty subsets A and B of a topological space (X, σ) are said to be Da-separated if there exist two Da-open sets G and H , such that $A \subset G, B \subset H, A \cap H = \emptyset$ and $B \cap G = \emptyset$.

Definition 4.6 Two non-empty subsets A and B of a topological space (X, σ) are said to be strongly Da-separated if there exist two Da-open sets U and V , such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

Definition 4.7 A topological space (X, σ) is said to be
 (1) Da- T_2 if any two distinct points are strongly Da-separated in (X, σ)
 (2) Da- T_1 if every pair of distinct points is Da-separated in (X, σ) .

Remark 4.8 The following implications are hold for a topological space (X, σ)

$$\begin{array}{ccccc} a-T_2 & \longrightarrow & Da-T_2 & \longleftarrow & T_2 \\ \downarrow & & \downarrow & & \downarrow \\ a-T_1 & \longrightarrow & Da-T_1 & \longleftarrow & T_1 \end{array}$$

Theorem 4.9 If an injective function $f: (X, \sigma) \rightarrow (Y, \eta)$ is Da-continuous and (Y, η) is T_1 , then (X, σ) is Da- T_1 .

Proof: Let (Y, σ) be T_1 and $p, q \in X$ with $p \neq q$. Then there exist open subsets G, H in Y such that $f(p) \in G, f(q) \notin G, f(p) \notin H$ and $f(q) \in H$. Since f is Da-continuous, $f^{-1}(G)$ and $f^{-1}(H) \in DaO(X)$ such that $p \in f^{-1}(G), q \notin f^{-1}(G), p \notin f^{-1}(H)$ and $q \in f^{-1}(H)$. Hence, (X, σ) is Da- T_1 .

Corollary 4.10 If an injective function $f: (X, \tau) \rightarrow (Y, \sigma)$ is Da-continuous and (Y, σ) is T_2 , then (X, τ) is Da- T_2 .

Theorem 4.11 *If $f:(X,\tau) \rightarrow (Y,\sigma)$ is Da-continuous and $g:(Y,\sigma) \rightarrow (Z,\eta)$.
Proof: Obvious*

Remark 4.12 *The composition of two Da-continuous functions need not be Da-continuous as show by the following example.*

Example 4.13 *Let (X, τ) be topological space, where $X=\{a,b,c\}$ and topology $\tau=\{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$*

Definition 4.14 *A function $f:X \rightarrow Y$ has a Da-closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in DaO(X, x)$ and $V \in O(Y, y)$ such that $(U \times V) \cap G(f) = \phi$.*

Lemma 4.15 *Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function. Then the graph $G(f)$ is Da-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in DaO(X, x)$ and $V \in O(Y, y)$ such that $f(U) \cap V = \phi$.*

Proof: Let f be Da-closed graph, then for each $(c, d) \in (X \times Y) \setminus G(f)$, $\exists U \in DaO(X, c)$ and $V \in O(Y, d)$ such that $(U \times V) \cap G(f) = \phi \implies$ for each $f(c) \in f(U)$ and $d \in V$. Since $d \neq f(c)$, for each $c \in U$ and $d \in V$. Therefore $f(U) \cap V = \phi$.
Conversely Let $(c, d) \in (X \times Y) \setminus G(f)$. Then by hypothesis, there exist $U \in DaO(X, c)$ and $V \in O(Y, d)$ such that $f(U) \cap V = \phi \implies f(c) \neq d$ for each $c \in U$ and $d \in V$. Therefore $(U \times V) \cap G(f) = \phi$

Theorem 4.16 *If $f:(X,\tau) \rightarrow (Y,\sigma)$ is an injective function with the Da-closed graph, then X is Da- T_1*

Proof: Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. The injectivity of f gives $f(x_1) \neq f(x_2)$. Thus there exist a Da-open set U and an open set V containing x_1 and $f(x_2)$, respectively such that $f(U) \cap V = \phi$. Therefore $x_2 \notin U$ and it follows that X is Da- T_1 .

Theorem 4.17 *If $f:(X,\tau) \rightarrow (Y,\sigma)$ is a surjective function with the Da-closed graph, then Y is T_1*

Proof: Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since f is surjective, there exists a point x in X such that $f(x) = y_2$. By Lemma 4.13, there exists Da-open set U and an open set V containing x and y_1 , respectively such that $f(U) \cap V = \phi$. It follows that $y_2 \notin V$. Hence Y is T_1 .

Corollary 4.18 *If $f:(X,\tau) \rightarrow (Y,\sigma)$ is a surjective function with the Da-closed graph, then Y is Da- T_1*

Proof: Follows from Theorem 4.15 and the fact that every T_1 space is Da- T_1

Corollary 4.19 *If $f:(X,\tau) \rightarrow (Y,\sigma)$ is a bijective function with the Da-closed graph, then both X and Y is Da- T_1*

Proof: Follows from Theorems 4.15 and 4.16

Definition 4.20 A function $f: X \rightarrow Y$ has a (D, a) -closed-closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in DaO(X, x)$ and $V \in aO(Y, y)$ such that $(U \times aCl(V)) \cap G(f) = \phi$.

Theorem 4.21 Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function and

- (i) f is (D, a) -closed graph;
- (ii) For each $(c, d) \in (X \times Y) \setminus G(f)$, $\exists U \in DaO(X, c)$ and $V \in aO(Y, d)$ such that $f(U) \cap aCl(V) = \phi$.
- (iii) For each $(c, d) \in (X \times Y) \setminus G(f)$, $\exists U \in DaO(X, c)$ and $V \in DaO(Y, d)$ such that $(U \times Cl_a^D(V)) \cap G(f) = \phi$
- (iv) For each $(c, d) \in (X \times Y) \setminus G(f)$, $\exists U \in DaO(X, c)$ and $V \in DaO(Y, d)$ such that $f(U) \cap Cl_a^D(V) = \phi$. Then

- (1) $(i) \longleftrightarrow (ii)$
- (2) $(i) \longrightarrow (iii)$
- (3) $(ii) \longrightarrow (iv)$
- (4) $(iii) \longleftrightarrow (iv)$

Proof: (2)(i) $\longrightarrow$ (iii): By (i), for each $(c, d) \in (X \times Y) \setminus G(f)$, there exist $U \in DaO(X, c)$ and $V \in aO(Y, d)$ such that $(U \times aCl(V)) \cap G(f) = \phi$. Since a -open set is Da -open, then $Cl_a^D(V) \subseteq Cl_a(V)$. Thus $(U \times Cl_a^D(V)) \cap G(f) = \phi$.

(3)(ii) $\longrightarrow$ (iv): Let $(c, d) \in (X \times Y) \setminus G(f)$. Then $\exists U \in DaO(X, c)$ and $V \in aO(Y, d)$ such that $f(U) \cap aCl(V) = \phi$. Since a -open set is Da -open and $Cl_a^D(V) \subseteq Cl_a(V)$, $f(U) \cap Cl_a^D(V) \subseteq f(U) \cap Cl_a(V) = \phi$.

(4)(i) $\longleftrightarrow$ (iv): From (1).

Theorem 4.22 Let $A \subseteq X$. Then $x \in aCl(A)$ if and only if $U \cap A \neq \Phi$, for every a -open set U containing x .

Theorem 4.23 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a surjective function with the (D, a) -closed-closed graph, then Y is $a-T_2$.

Proof: Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. The surjectivity of f gives $x_1 \in X$ such that $f(x_1) = y_1$ and hence $(x_1, y_2) \in (X \times Y) \setminus G(f)$. The Da -closedness of $G(f)$ provides $E \in DaO(X, x_1)$ and $F \in aO(Y, y_2)$ such that $f(E) \cap aCl(F) = \phi$. Now $x_1 \in E$ implies $f(x_1) = y_1 \in f(E)$. This and the fact that $f(E) \cap aCl(F) = \phi$ guarantee that $y_1 \notin aCl(F)$. By Theorem 4.15, there exists $D \in aO(Y, y_1)$ such that $D \cap F = \phi$. Hence Y is $a-T_2$.

Corollary 4.24 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a surjective function with (D, a) -closed graph, then Y is $a-T_1$

Proof: Follows from Theorem 4.21 and the fact that every $a-T_2$ space is $a-T_1$

Corollary 4.25 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a surjective function with the (D, a) -closed graph, then Y is $Da-T_2$.

Proof: Follows from Theorem 4.21 and the fact that every $a-T_2$ space is $Da-T_2$

Corollary 4.26 *If $f:(X,\tau) \rightarrow (Y,\sigma)$ is a surjective function with the (D,a) -closed graph, then Y is $Da-T_1$*

Proof: Follows from Corollary 4.22 and the fact that every $a-T_1$ space is $Da-T_1$

5 Da-Continuous functions.

Definition 5.1 *A function $f:(X,\sigma) \rightarrow (Y,\eta)$ is said to be almost Da-continuous if for each point $p \in X$ and each open subset V of Y containing $f(p)$, there exists $U \in DaO(X,p)$ such that $f(U) \subset \text{int}(Cl(V))$.*

Theorem 5.2 *Let $f:(X,\sigma) \rightarrow (Y,\eta)$ be a function. Then the following conditions are equivalent:*

- (1) f is almost Da-continuous;
- (2) For every $N \in RO(Y)$, $f^{-1}(N) \in DaO(X)$.
- (3) For every $M \in RC(Y)$, $f^{-1}(M) \in DaC(X)$.
- (4) For each subset C of X , $f(Cl_a^D(C)) \subseteq Cl_\delta(f(C))$.
- (5) For each subset D of Y , $Cl_a^D(f^{-1}(D)) \subseteq f^{-1}(Cl_\delta(D))$.
- (6) For every $G \in \delta C(Y)$, $f^{-1}(G) \in DaC(X)$.
- (7) For every $H \in \delta O(Y)$, $f^{-1}(H) \in DaO(X)$.
- (8) For every $N \in O(Y)$, $f^{-1}(\text{Int}(Cl(N))) \in DaO(X)$
- (9) For every $M \in C(Y)$, $f^{-1}(Cl(\text{Int}(M))) \in DaC(X)$.
- (10) For every $N \in \beta O(Y)$, $Cl_a^D(f^{-1}(N)) \subseteq f^{-1}(Cl(N))$.
- (11) For every $M \in \beta C(Y)$, $f^{-1}(\text{Int}(M)) \subseteq \text{Int}_a^D(f^{-1}(M))$
- (12) For every $M \in SC(Y)$, $f^{-1}(\text{Int}(M)) \subseteq \text{Int}_a^D(f^{-1}(M))$
- (13) For every $N \in SO(Y)$, $Cl_a^D(f^{-1}(N)) \subseteq f^{-1}(Cl(N))$.
- (14) For every $M \in PO(Y)$, $f^{-1}(M) \subseteq \text{Int}_a^D(f^{-1}(\text{Int}(Cl(M))))$
- (15) for each $p \in X$ and each open set N of Y containing $f(p)$ there exists $M \in DaO(X,p)$ such that $f(M) \subset sCl(N)$
- (16) for each $p \in X$ and each regular open set N of Y containing $f(p)$ there exists $M \in DaO(X,p)$ such that $f(M) \subset N$
- (17) for each $p \in X$ and each δ -open set N of Y containing $f(p)$ there exists $M \in DaO(X,p)$ such that $f(M) \subset N$

Proof: (1) $\rightarrow$ (2) Suppose that $N \in RO(Y)$ and let $p \in f^{-1}(N)$, then $f(p) \in N$. Since N is an open set and f is an almost Da-continuous function, there exists $M \in DaO(X,p)$ such that $f(M) \subset \text{int}(Cl(N)) = N$. Thus $p \in M \subset f^{-1}(f(M)) \subset f^{-1}(N)$ and hence we obtain $f^{-1}(N) \in DaO(X)$.

. (2) $\rightarrow$ (3) It follows from the fact that $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$.

(3) $\rightarrow$ (4) Suppose that D is a δ -closed set in Y containing $f(C)$. Observe that $D = Cl_\delta(D) = \bigcap \{F : D \subset F \text{ and } F \in RC(Y)\}$ and so $f^{-1}(D) = \bigcap \{f^{-1}(F) : D \subset F \text{ and } F \in RC(Y)\}$. Now, by (3), $f^{-1}(D) \in DaC(X)$ and $C \subset f^{-1}(D)$. Hence $Cl_a^D(C) \subset f^{-1}(D)$, and it follows that $f(Cl_a^D(C)) \subset D$. Since this is true for any δ -closed set D containing $f(C)$, we have $f(Cl_a^D(C)) \subset Cl_\delta(f(C))$.

(4) $\longrightarrow$ (5) Let D be a subset of Y , then $f^{-1}(D)$ is a subset of X . By part (4), $f(Cl_a^D(f^{-1}(D))) \subset Cl_\delta(f(f^{-1}(D))) \subset Cl_\delta(D)$ so that $Cl_a^D(f^{-1}(D)) \subseteq f^{-1}(Cl_\delta(D))$.
 (5) $\longrightarrow$ (6) Suppose that $M \in \delta C(Y)$, then $Cl_a^D(f^{-1}(D)) \subseteq f^{-1}(Cl_\delta(D)) = f^{-1}(D)$. In consequence, $Cl_a^D(f^{-1}(D)) = f^{-1}(D)$ and hence $f^{-1}(D) \in DaC(X)$.

(6) $\longrightarrow$ (7) Obvious.

(7) $\longrightarrow$ (1) Let $p \in X$ and let O be any open set in Y such that $f(p) \in O$. Put $D = Int(Cl(O))$ and $C = f^{-1}(D)$. Since $D \in \delta O(Y)$ and by part (7), $C = f^{-1}(D) \in DaO(X)$. Now, $f(p) \in O = int(O) \subset int(Cl(O)) = D$, it follows that $p \in f^{-1}(D) = C$ and $f(C) = f(f^{-1}(D)) \subset D = Int(Cl(O))$.

(2) $\longleftrightarrow$ (8) Let N be an open subset of Y . Since $Int(Cl(N))$ is regular open, $f^{-1}(Int(Cl(N)))$ is Da -open. The converse is similar.

(3) $\longleftrightarrow$ (9) It is similar to (8) $\longleftrightarrow$ (2).

(3) $\longrightarrow$ (10): Let N be any β -open set of Y . Since $Cl(N) \in RC(Y)$, by Theorem 2.2, $f^{-1}(Cl(N))$ is Da -closed in X and $f^{-1}(N) \subset f^{-1}(Cl(N))$. Therefore, $Cl_a^D(f^{-1}(N)) \subset f^{-1}(Cl(N))$.

(10) $\longrightarrow$ (11):

(11) $\longrightarrow$ (12):

(12) $\longrightarrow$ (13):

(13) $\longrightarrow$ (3):

(2) $\longleftrightarrow$ (14): Let N be any preopen set of Y . Since $int(Cl(N)) \in RO(Y)$, by Theorem 2.2, we have $f^{-1}(int(Cl(N))) \in DaO(X)$ and hence $f^{-1}(N) \subset f^{-1}(int(Cl(N))) = Int_a^D(f^{-1}(int(Cl(N))))$.

Conversely, let N be any regular open set of Y . Since $N \in PO(Y)$, $f^{-1}(N) \subset Int_a^D(f^{-1}(int(Cl(N)))) = Int_a^D(f^{-1}(V))$ and hence $f^{-1}(N) \in DaO(X)$.

(1) $\longrightarrow$ (15): Let $p \in X$ and N be an open set of Y containing $f(p)$. By (1), there exists $M \in DaO(X, p)$ such that $f(M) \subset int(Cl(N))$. Since N is a preopen set, then by Lemma 1.4, $f(M) \subset sCl(N)$.

(15) $\longrightarrow$ (16): Let $p \in X$ and N be a regular open set of Y containing $f(p)$. Then N is an open set of Y containing $f(p)$. By (15), there exists $M \in DaO(X, p)$ such that $f(M) \subset sCl(N)$. Since N is a preopen set, then by Lemma 1.4, $f(M) \subset int(Cl(N)) = N$.

(16) $\longrightarrow$ (17): Let $p \in X$ and V be a δ -open set of Y containing $f(p)$. Then, there exists an open set G containing $f(p)$ such that $G \subset int(Cl(G)) \subset N$. Since $int(Cl(G))$ is a regular open set of Y containing $f(p)$, by part (16), there exists $M \in DaO(X, p)$ such that $f(M) \subset int(Cl(G)) \subset N$.

(17) $\longrightarrow$ (1). Let $p \in X$ and N be an open set of Y containing $f(p)$. Then $int(Cl(N))$ is a δ -open set of Y containing $f(p)$. By (17), there exists $M \in DaO(X, p)$ such that $f(M) \subset int(Cl(N))$. Therefore, f is almost continuous

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