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Anti-fuzzy Bi-ideals in Hypersemigroups

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Abstract

This study aims at defining the concept of anti-fuzzy bi-ideals of hypersemigroups. This study also defines the hypersemigroup bi-ideals in terms of anti-fuzzy bi-ideals additionally.
Keywords: Hypergroupoid, hypersemigroup, anti-fuzzy bi-ideal, regular, intra-regular.
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1 Introduction

In 1965, Zadeh [7] has presented the idea of fuzzy set which has seen a great success in a variety of domains. Good and Hughes [2] are the first two authors who have proposed the notion of the bi-ideal of the fuzzy set. Kuroki [5] was the first one to suggest fuzzy ideals in semigroups. According to Biswas [1], the idea of anti-fuzzy subgroup was first proposed. Shabir and Nawaz [6] has carried out the research on anti-fuzzy principles of semi-groups and gave some intriguing findings. Kehayopulu [4] was the first one to examine fuzzy sets in hypersemigroups. In this study, the bi-ideals of hypersemigroups in terms of anti-fuzzy bi-ideals is defined and the concept of anti-fuzzy bi-ideals of a hypersemigroup is established.

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2 Preliminaries

An hypergroupoid is a nonempty group $\mathcal{S}^{\mathcal{H}}$ with an hyperoperation $\circ : \mathcal{S}^{\mathcal{H}} \times \mathcal{S}^{\mathcal{H}} \to \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) | (\mathfrak{f}, \mathfrak{g}) \to \mathfrak{f} \circ \mathfrak{g}$ on $\mathcal{S}^{\mathcal{H}}$ and an operation $* : \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) \times \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) \to \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) | (\mathcal{R}, \mathcal{P}) \to \mathcal{R} * \mathcal{P}$ on $\mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$ (introduced by the operation on $\mathcal{S}^{\mathcal{H}}$) such that $\mathcal{R} * \mathcal{P} = \bigcup_{(\mathfrak{f},\mathfrak{g}) \in \mathcal{R} \times \mathcal{P}} (\mathfrak{f} \circ \mathfrak{g})$ for every $\mathcal{R}, \mathcal{P} \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})(\mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$ being the set of nonempty subsets of $\mathcal{S}^{\mathcal{H}})$.

The purpose of the operation $*$ is clearly stated. If $(\mathcal{R}, \mathcal{P}) \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) \times \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$, then $\mathcal{R} * \mathcal{P} = \bigcup_{(\mathfrak{f},\mathfrak{g}) \in \mathcal{R} \times \mathcal{P}} (\mathfrak{f} \circ \mathfrak{g})$. For every $(\mathfrak{f}, \mathfrak{g}) \in \mathcal{R} \times \mathcal{P}$, we get $(\mathfrak{f}, \mathfrak{g}) \in \mathcal{S}^{\mathcal{H}} \times \mathcal{S}^{\mathcal{H}}$, then $(\{ \circ \}) \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$. Thus we get $\mathcal{R} * \mathcal{P} \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$. If $(\mathcal{R}, \mathcal{P}), (\mathcal{F}, \mathcal{G}) \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}}) \times \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$ such that $(\mathcal{R}, \mathcal{P}) = (\mathcal{F}, \mathcal{G})$, then $\mathcal{R} * \mathcal{P} = \bigcup_{(\mathfrak{f},\mathfrak{g}) \in \mathcal{R} \times \mathcal{P}} (\mathfrak{f} \circ \mathfrak{g}) = \bigcup_{(\mathfrak{f},\mathfrak{g}) \in \mathcal{F} \times \mathcal{G}} (\mathfrak{f} \circ \mathfrak{g}) = \mathcal{F} * \mathcal{G}$.

Since the hyperoperation '$\circ$ 'and the operation '$*$ 'are interdependent, hypergroupoids may alternatively be expressed by $(\mathcal{S}^{\mathcal{H}}, \circ)$ rather than $(\mathcal{S}^{\mathcal{H}}, \circ, *)$. Suppose $\mathcal{S}^{\mathcal{H}}$ is an hypergroupoid, then $\forall\, \mathrm{m}, \mathrm{n} \in \mathcal{S}^{\mathcal{H}}$, we get $\mathrm{m} \circ \mathrm{n} = \{\mathrm{m}\} * \{\mathrm{n}\}$. That is, $\{\mathrm{m}\} * \{\mathrm{n}\} = \bigcup_{x \in \{\mathrm{m}\}, y \in \{\mathrm{n}\}} x \circ y = \mathrm{m} \circ \mathrm{n}$.

An hypergroupoid $\mathcal{S}^{\mathcal{H}}$ is said to be hypersemigroup if $(\mathrm{m} \circ \mathrm{n}) * \{\mathrm{o}\} = \{\mathrm{m}\} * (\mathrm{n} \circ \mathrm{o})$ for every $\mathrm{m}, \mathrm{n}, \mathrm{o} \in \mathcal{S}^{\mathcal{H}}$. Since $\mathrm{m} \circ \mathrm{n} = \{\mathrm{m}\} * \{\mathrm{n}\}$ for any $\mathrm{m}, \mathrm{n} \in \mathcal{S}^{\mathcal{H}}$, an hypergroupoid $\mathcal{S}^{\mathcal{H}}$ is an hypersemigroup if and only if for any $\mathrm{m}, \mathrm{n}, \mathrm{o} \in \mathcal{S}^{\mathcal{H}}$, we have $(\{\mathrm{m}\} * \{\mathrm{n}\}) * \{\mathrm{o}\} = \{\mathrm{m}\} * (\{\mathrm{n}\} * \{\mathrm{o}\})$.

According to Zadeh [7], we claim that the hypergroupoid $(\mathcal{S}^{\mathcal{H}}, \circ)$'s fuzzy subset (or fuzzy set) f if ψ maps $\mathcal{S}^{\mathcal{H}}$ into the real closed interval [0,1] of real integers, i.e., $\psi : \mathcal{S}^{\mathcal{H}} \to [0, 1]$.

The subset of $\mathcal{S}^{\mathcal{H}} \times \mathcal{S}^{\mathcal{H}}$ classified as follows is denoted by $\mathcal{A}_{\mathfrak{h}}$:

$\mathcal{A}_{\mathfrak{h}} = \{(a, b) \in \mathcal{S}^{\mathcal{H}} \times \mathcal{S}^{\mathcal{H}} | \mathfrak{h} \in a \circ b\}$ for any element $\mathfrak{h}$ of $\mathcal{S}^{\mathcal{H}}$.

We designate by $\psi \circ \mathfrak{q}$, the fuzzy subset of $\mathcal{S}^{\mathcal{H}}$ defined by any two fuzzy subsets of ψ and $\mathfrak{q}$,

$$\psi \circ \mathfrak{q} : \mathcal{S}^{\mathcal{H}} \to [0, 1] \quad | \quad \mathfrak{h} \to \begin{cases} \bigwedge_{(a,b) \in \mathcal{A}_{\mathfrak{h}}} \max\{f(a), g(b)\} & \text{if} \quad \mathcal{A}_{\mathfrak{h}} \neq \phi \\ 1 \quad \text{if} \quad \mathcal{A}_{\mathfrak{h}} = \phi. \end{cases}$$

The collection of all fuzzy subsets of $\mathcal{S}^{\mathcal{H}}$ is indicated by $\mathcal{F}(\mathcal{S}^{\mathcal{H}})$ and the power relation $\leq$ on $\mathcal{F}(\mathcal{S}^{\mathcal{H}})$ classified by: $\psi \leq \amalg$ if and only if $\psi(\mathrm{m}) \leq \amalg(\mathrm{m})$ for every $\mathrm{m} \in \mathcal{S}^{\mathcal{H}}$.

For an hypersemigroup $\mathcal{S}^{\mathcal{H}}$, the following definitions apply to the fuzzy sets "0"and "1":

$0 : \mathcal{S}^{\mathcal{H}} \to [0, 1] \quad | \quad \mathrm{m} \to 0(\mathrm{m}) = 0$ and $1 : \mathcal{S}^{\mathcal{H}} \to [0, 1] \quad | \quad \mathrm{m} \to 1(\mathrm{m}) = 1$.

An hypersemigroup $\mathcal{S}^{\mathcal{H}}$ is allegedly regular suppose $\forall\, \mathfrak{h} \in \mathcal{S}^{\mathcal{H}}\ \exists\, u \in \mathcal{S}^{\mathcal{H}} \ni \mathfrak{h} \in (\mathfrak{h} \circ u) * \{\mathfrak{h}\}$.

An hypersemigroup $\mathcal{S}^{\mathcal{H}}$ is allegedly intra-regular suppose $\forall\, \mathfrak{h} \in \mathcal{S}^{\mathcal{H}}\ \exists\, a, b \in \mathcal{S}^{\mathcal{H}} \ni \mathfrak{h} \in (a \circ \mathfrak{h}) * (\mathfrak{h} \circ b)$.

The same symbol is used to denote both the hyperoperation on $\mathcal{S}^{\mathcal{H}}$ and the multiplication among its two fuzzy subsets (no confusion is possible).

Proposition 2.1. [3] *Assume $(\mathcal{S}^{\mathcal{H}}, \circ)$ be an hypergroupoid, $x \in \mathcal{S}^{\mathcal{H}}$ and $\mathcal{R}, \mathcal{P} \in \mathfrak{M}^*(\mathcal{S}^{\mathcal{H}})$. Then we obtain the following:*

(1) $x \in \mathcal{R} * \mathcal{P} \Longleftrightarrow x \in \mathfrak{f} \circ \mathfrak{g}$ *for some* $\mathfrak{f} \in \mathcal{R}, \mathfrak{g} \in \mathcal{P}$.

(2) If $\mathfrak{f} \in \mathcal{R}$ *and* $\mathfrak{g} \in \mathcal{P}$, *then* $\mathfrak{f} \circ \mathfrak{g} \subseteq \mathcal{R} * \mathcal{P}$.

3 Anti-fuzzy bi-ideals

Unless otherwise stated, let $\mathcal{S}^{\mathcal{H}}$ denote a hypersemigroup in this section.

Definition 3.1. *An anti-fuzzy subsemigroup of $\mathcal{S}^{\mathcal{H}}$ is a fuzzy subset of ψ of $\mathcal{S}^{\mathcal{H}}$ suppose $\psi(\gamma \circ \zeta) \leq \max\{\psi(\gamma), \psi(\zeta)\}$ for all $\gamma, \zeta \in \mathcal{S}^{\mathcal{H}}$.*

Definition 3.2. *An anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$ is a fuzzy subset of ψ of $\mathcal{S}^{\mathcal{H}}$ if*

(i) $\psi(\gamma \circ \zeta) \leq \max\{\psi(\gamma), \psi(\zeta)\}$

(ii) $\psi((\gamma \circ \zeta) * \{k\}) \leq \max\{\psi(\gamma), \psi(k)\}$ *for all* $\gamma, \zeta, k \in \mathcal{S}^{\mathcal{H}}$.

Lemma 3.1. *A bi-ideal of $\mathcal{S}^{\mathcal{H}}$ is not an empty subset of $\mathcal{A}$ of $\mathcal{S}^{\mathcal{H}}$ iff the fuzzy subset $\psi_{\mathcal{A}}$ of $\mathcal{S}^{\mathcal{H}}$ classified as*

$$\psi_{\mathcal{A}} : \mathcal{S}^{\mathcal{H}} \to \{\mathfrak{r}, \rho\}, \imath \to \begin{cases} \mathfrak{r} & \text{if } \imath \in \mathcal{A} \\ \rho & \text{if } \imath \notin \mathcal{A} \end{cases}$$

where $0 \leq \mathfrak{r}, \rho \leq 1$ and $\mathfrak{r} \leq \rho$ is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$.

Proof: Assume $\mathcal{A}$ is a bi-ideals of S. Let $\gamma, \zeta \in \mathcal{A}$. Then $\gamma \circ \zeta \in \mathcal{A}$.

(1) Suppose $\gamma, \zeta \in \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \psi_{\mathcal{A}}(\zeta) = \mathfrak{r}$ and so $\psi_{\mathcal{A}}(\gamma \circ \zeta) \leq \max\{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(\zeta)\}$.

(2) If $\gamma \in \mathcal{A}, \zeta \notin \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \mathfrak{r}, \psi_{\mathcal{A}}(\zeta) = \rho$ and then $\psi_{\mathcal{A}}(\gamma \circ \zeta) = \rho \leq \max\{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(\zeta)\}$.

(3) If $\gamma \notin \mathcal{A}, \zeta \in \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \rho, \psi_{\mathcal{A}}(\zeta) = \mathfrak{r}$ and so $\psi_{\mathcal{A}}(\gamma \circ \zeta) = t \leq \max\{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(\zeta)\}$.

(4) If $\gamma, \zeta \notin \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \psi_{\mathcal{A}}(\zeta) = \rho$ and then $\psi_{\mathcal{A}}(\gamma \circ \zeta) = \rho \leq \max\{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(\zeta)\}$ for all$\gamma, \zeta \in \mathcal{S}^{\mathcal{H}}$.

Let $\gamma, k \in \mathcal{A}$ amd $\zeta \in \mathcal{S}^{\mathcal{H}}$. Then $(\gamma \circ \zeta) * \{k\} \in \mathcal{A}$.

(1) If $\gamma, k \in \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \psi_{\mathcal{A}}(k) = \mathfrak{r}$ and so $\psi_{\mathcal{A}}((\gamma \circ \zeta) * \{k\}) \leq \max\{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(k)\}$.

(2) If $\gamma \in \mathcal{A}, k \notin \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \mathfrak{r}, \psi_{\mathcal{A}}(k) = \rho$ and then $\psi_{\mathcal{A}}((\gamma \circ \zeta) * \{k\}) \leq$ max$\{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(k)\}$.

(3) If $\gamma \notin \mathcal{A}, k \in \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \rho, \psi_{\mathcal{A}}(k) = \mathfrak{r}$ and so $\psi_{\mathcal{A}}((\gamma \circ \zeta) * \{k\}) \leq$ max$\{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(k)\}$.

(4) If $\gamma, k \notin \mathcal{A}$, then $\psi_{\mathcal{A}}(\gamma) = \psi_{\mathcal{A}}(k) = \rho$ and then $\psi_{\mathcal{A}}((\gamma \circ \zeta) * \{k\}) \leq$ max$\{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(k)\}$ for all$\gamma, \zeta \in \mathcal{S}^{\mathcal{H}}$.

Thus, an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$ is $\psi_{\mathcal{A}}$.
Alternatively, assume ψ_A is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$.
Consider $\gamma, \zeta, k \in \mathcal{A}$.
Then $\psi_{\mathcal{A}}(\gamma) = \psi_{\mathcal{A}}(\zeta) = \psi_{\mathcal{A}}(k) = \mathfrak{r}$.
Therefore, ψ_A is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$, we obtain
$\psi_{\mathcal{A}}(\gamma \circ \zeta) \leq$ max$\{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(\zeta)\}$ =max$\{\mathfrak{r}, \mathfrak{r}\} = \mathfrak{r}$.
This implies $\gamma \circ \zeta \in \mathcal{A}$.
Also $\psi_{\mathcal{A}}((\gamma \circ \zeta) * \{k\}) \leq \max\{\psi_{\mathcal{A}}(\gamma), \psi_{\mathcal{A}}(k)\} = \max\{\mathfrak{r}, \mathfrak{r}\} = \mathfrak{r}$.
This implies $(\gamma \circ \zeta) * \{k\}) \in \mathcal{A}$.
Hence $\mathcal{A}$ is a bi-ideal of $\mathcal{S}^{\mathcal{H}}$. □

Definition 3.3. *Assume ψ be a fuzzy subset of $\mathcal{S}^{\mathcal{H}}$.*
Then $\mathcal{L}(\psi; \rho) = \left\{\gamma \in \mathcal{S}^{\mathcal{H}} | \psi(\gamma) \leq \rho\right\}$ where $0 < \rho \leq 1$ is said to be the lower level cut of ψ.

The characterization of an anti-fuzzy bi-ideal is then provided in terms of its level bi-ideals.

Theorem 3.1. *A fuzzy subset ψ of $\mathcal{S}^{\mathcal{H}}$ is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$ iff the collection $\mathcal{L}(\psi; \rho) = \left\{\gamma \in \mathcal{S}^{\mathcal{H}} | \psi(\gamma) \leq \rho\right\}$ where $0 < \rho \leq 1$ is a bi-ideal of $\mathcal{S}^{\mathcal{H}}$.*

Proof: Assume ψ is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$.
Consider $\gamma, \zeta \in \mathcal{L}(\psi; \rho)$.
Then $\psi(\gamma) \leq \rho$ and $\psi(\gamma) \leq t$.
As ψ is the anti-fuzzy bi-ideal of the $\mathcal{S}^{\mathcal{H}}, \psi(\gamma \circ \zeta) \leq \max\{\psi(\gamma), \psi(\zeta)\} \leq \max\{\rho, \rho\} = \rho$.
This suggests $\gamma \circ \zeta \in \mathcal{L}(\psi; \rho)$.
Let $\gamma, k \in \mathcal{L}(\psi; \rho)$ and $\zeta \in \mathcal{S}^{\mathcal{H}}$.

Then $\psi(\gamma) \leq \rho$ and $\psi(k) \leq \rho$.
Now $\psi((\gamma \circ \zeta) * \{k\}) \leq \max\{\psi(\gamma), \psi(k)\} \leq \max\{\rho, \rho\} = \rho$.
This suggests $(\gamma \circ \zeta) * \{k\} \in \mathcal{L}(\psi; \rho)$.
Therefore, $\mathcal{L}(\psi; \rho)$ a bi-ideal of $\mathcal{S}^{\mathcal{H}}$.
Alternatively, suppose $\mathcal{L}(\psi; \rho)$ is a bi-ideal of $\mathcal{S}^{\mathcal{H}} \forall 0 < \rho \leq 1$.
If $\exists \gamma, \zeta \in \mathcal{S}^{\mathcal{H}}$ such that $\psi(\gamma \circ \zeta) > \max\{\psi(\gamma), \psi(\zeta)\} = \rho$, then $\rho \in (0, 1], \gamma, \zeta \in \mathcal{L}(\psi; \rho)$ but $\gamma \circ \zeta \notin \mathcal{L}(\psi; \rho)$, which is incongruous.
Thus $\psi(\gamma \circ \zeta) \leq \max\{\psi(\gamma), \psi(\zeta)\} \forall \gamma, \zeta \in \mathcal{S}^{\mathcal{H}}$.
Suppose $\exists\ \gamma, \zeta, k \in \mathcal{S}^{\mathcal{H}} \ni \psi((\gamma \circ \zeta) * \{k\}) > \max\{\psi(\gamma), \psi(k)\} = \rho_0$, then $\rho_0 \in (0, 1], \gamma, \zeta \in \mathcal{L}(\psi; \rho_0)$ but $(\gamma \circ \zeta) * \{k\} \notin \mathcal{L}(\psi; \rho)$, which is incongruous.
Thus $\psi((\gamma \circ \zeta) * \{k\}) \leq \max\{\psi(\gamma), \psi(k)\}$ for all $\gamma, \zeta, k \in \mathcal{S}^{\mathcal{H}}$.
Hence ψ is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$. □

Lemma 3.2. *A fuzzy subset of $\mathcal{S}^{\mathcal{H}}$ is ψ. Then $\psi$is an anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$ iff $\psi \leq \psi \circ 0 \circ \psi$.*

Proof: Let $h \in \mathcal{S}^{\mathcal{H}}$.
Then $\psi(h) \leq (\psi \circ 0 \circ \psi)(h)$.
If $\mathcal{A}_h = \phi$, then $(\psi \circ 0 \circ \psi)(h) = ((\psi \circ 0) \circ \psi)(h) = 1 \geq \psi(h)$.
If $\mathcal{A}_h \neq \phi$, then $(\psi \circ 0 \circ \psi)(h) = \bigwedge_{(\zeta,k) \in \mathcal{A}_h} \max\{(\psi \circ 0)(\zeta), \psi(k)\}$.
It suffices to demonstrate this max$\{(\psi \circ 0)(\zeta), \psi(k)\} \geq \psi(h)$ for every
$(\zeta, k) \in \mathcal{A}_h$. (1)
For this purpose, let us assume $(\zeta, k) \in \mathcal{A}_h$.
If $\mathcal{A}_\zeta = \phi$, then $(\psi \circ 0)(\zeta) = 1 \geq \psi(k)$ and $\max\{(\psi \circ 0)(\zeta), \psi(k)\} = 1 \geq \psi(h)$.
Now assume that $\mathcal{A}_\zeta \neq \phi$.
Then $(\psi \circ 0)(\zeta) = \bigwedge_{(r,s) \in \mathcal{A}_\zeta} \max\{\psi(r), 0(s)\} = \bigwedge_{(r,s) \in \mathcal{A}_\zeta} \psi(r)$. (2)
Let's think about the following scenarios:

(i) Let $\psi(h) \leq (\psi \circ 0)(\zeta)$. Then $\psi(h) \leq (\psi \circ 0)(\zeta) \leq$max$\{(\psi \circ 0)(\zeta), \psi(k)\}$. Thus (1) is proved.

(ii) Let $\psi(h) > (\psi \circ 0)(\zeta)$.

Then there exists $(r, s) \in \mathcal{A}_\zeta$ such that $\psi(h) > \psi(r)$. (3)
If $\psi(h) \leq \psi(r)$ for every $(r, s) \in \mathcal{A}_\zeta$, then $\psi(h) \leq \bigwedge_{(r,s) \in \mathcal{A}_\zeta} \psi(r)$.
Then by (2), we get $\psi(h) \leq (\psi \circ 0)(\zeta)$ it is unattainable.
Since $(\zeta, k) \in \mathcal{A}_\zeta$, we have $\zeta, k \in \mathcal{S}^{\mathcal{H}}$ and $h \in \zeta \circ k$.
Since $(\mathfrak{r}, \mathfrak{s}) \in \mathcal{A}_\zeta$, we have $\mathfrak{r}, \mathfrak{s} \in \mathcal{S}^{\mathcal{H}}$ and $\zeta \in \mathfrak{r} \circ \mathfrak{s}$.
Then we get $h \in \zeta \circ k = \{\zeta\} * \{k\} \subseteq (r \circ s) * \{k\}$.
Because ψ is a anti-fuzzy bi-ideal of $\mathcal{S}^{\mathcal{H}}$, we obtain
$\psi(h) \leq \max\{\psi(r), \psi(k)\}$. (4)

If $\psi(r) \geq \psi(k)$, then $\psi(h) \leq \psi(r)$ which is not possible by (3).
Thus we have $\psi(r) < \psi(k)$. (5)
Then $\max\{\psi(r), \psi(k)\} = \psi(k)$ and by (4), we have $\psi(h) \leq \psi(k)$. (6)
Since $(r, s) \in \mathcal{A}_\zeta$ and by (2), we have $\psi(r) \geq (\psi \circ 0)(\zeta)$. (7)
By (5) and (7), we have $\psi(k) > \psi(r) \geq (\psi \circ 0)(\zeta)$.
Then $\max\{(\psi \circ 0)(\zeta), \psi(k)\} = \psi(k) \geq \psi(h)$ [by (6)] and (1) is proved.
Conversely, assume that $a, b, c \in \mathcal{S}^{\mathcal{H}}$ and $p \in (a \circ b) * \{c\}$.
Then $\psi(p) \leq \max\{\psi(a), \psi(c)\}$.
Since $p \in (a \circ b) * \{c\}$, by Proposition 2.1, there exists $\gamma \in a \circ b$ such that $p \in \gamma \circ c$.
Since $\gamma \in a \circ b$, we have $(a, b) \in \mathcal{A}_\gamma$.
Since $p \in \gamma \circ c$, we have $(\gamma, c) \in \mathcal{A}_p$.
Since $(\gamma, c) \in \mathcal{A}_p$, $\mathcal{A}_p$ is a non-empty set and we also have

$$\begin{aligned}(\psi \circ 0 \circ \psi)(p) &= \bigwedge_{(u,v)\in\mathcal{A}_p} \max\{(\psi \circ 0)(u), \psi(v)\} \\ &\leq \max\{(\psi \circ 0)(\gamma), \psi(c)\}.\end{aligned}$$

Since $(a, b) \in \mathcal{A}_\gamma$, we have $\mathcal{A}_\gamma \neq \phi$ and we also have

$$\begin{aligned}(\psi \circ 0)(\gamma) &= \bigwedge_{(c,w)\in\mathcal{A}_\gamma} \max\{\psi(c), 0(w)\} \\ &\leq \max\{\psi(a), 0(b)\} = \psi(a).\end{aligned}$$

Then we get $(\psi \circ 0 \circ \psi)(p) \leq \max\{\psi(a), \psi(c)\}$.
On the otherhand, since $p \in (a \circ b) * \{c\} \subseteq \mathcal{S}^{\mathcal{H}} * \mathcal{S}^{\mathcal{H}} \subseteq \mathcal{S}^{\mathcal{H}}$ and $\psi \leq \psi \circ 0 \circ \psi$, we obtain $\psi(p) \leq (\psi \circ 0 \circ \psi)(p)$.
Thus we have $\psi(p) \leq (\psi \circ 0 \circ \psi)(p) \leq \max\{\psi(a), \psi(c)\}$. □

Theorem 3.2. *If a hypersemigroup $\mathcal{S}^{\mathcal{H}}$ is regular, then we get $\psi \circ 0 \circ \psi = \psi$ for any anti-fuzzy bi-ideal ψ of $\mathcal{S}^{\mathcal{H}}$.*

Proof: Assume $\mathcal{S}^{\mathcal{H}}$ is regular and ψ is its anti-fuzzy bi-ideal.
Let $h \in \mathcal{S}^{\mathcal{H}}$.
Given that $\mathcal{S}^{\mathcal{H}}$ is regular, $u \in \mathcal{S}^{\mathcal{H}}$ exists like that $h \in (h \circ u) * \{h\}$.
This implies $(h, uh) \in \mathcal{A}_h$ and then $\mathcal{A}_h \neq \phi$.
Then

$$\begin{aligned}(\psi \circ 0 \circ \psi)(h) &= \bigwedge_{(p,q)\in\mathcal{A}_h} \max\{(\psi(p), (0 \circ \psi)(q)\} \\ &\leq \max\{\psi(h), (0 \circ \psi)(uh)\} \\ &= \max\left\{\psi(h), \bigwedge_{(\mathfrak{r},\mathfrak{s})\in\mathcal{A}_{uh}} \max\{0(r), \psi(s)\}\right\} \\ &\leq \max\left\{\psi(h), \max\{0(u), \psi(h)\}\right\} \\ &= \max\left\{\psi(h), \max\{0, \psi(h)\}\right\} \\ &= \max\{\psi(h), \psi(h)\} \\ &= \psi(h).\end{aligned}$$

Thus $(\psi \circ 0 \circ \psi)(h) \leq \psi(h)$.

By the above lemma, we obtain $\psi(h) \leq (\psi \circ 0 \circ \psi)(h)$.
Hence $\psi \circ 0 \circ \psi = \psi$. $\square$

4 Conclusion

In this work, we defined anti-fuzzy bi-ideal in hypersemigroup and discussed about the characterization of an anti-fuzzy bi-ideals in hypersemigroup. Also we defined the lower level cut of the fuzzy subset and we analysed the characterization of an anti-fuzzy bi-ideal in terms of lower level cut of the fuzzy subset. Moreover, in this study we proved an anti-fuzzy bi-ideal is equivalently denoted by $\psi \leq \psi \circ 0 \circ \psi$. Also we proved $\psi = \psi \circ 0 \circ \psi$ if the hypersemigroup is regular. In the forthcoming work, the regular and intra-regular hypersemigroups will be defined using anti-fuzzy bi-ideal concepts.

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