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Extended b-metric spaces and the related approximate fixed point results

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Abstract

In this paper, we investigate some approximate fixed point results in extended b-metric spaces using several contraction mappings such as the Mohseni-Saheli contraction, the B-contraction, and their consequences. Additionally, we prove some ϵ-fixed point results by using rational type contraction mappings, which were discussed mainly in Dass and Gupta [1975] and Jaggi [1977]. Also, a few examples are included to illustrate the results. Finally, we discuss some applications that support our main results in the field of applied mathematics.
Keywords: fixed point, b-metric space, approximate fixed point, rational contraction.
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1 Introduction

By using fixed point theory $(f.p.t)$, researchers from many fields have contributed to the progress of science and technology. Large scale problems requiring $f.p.t$ are highly esteemed for their lightning-fast solutions. As a result, in recent years, many scholars have focused on developing $f.p.t$ approaches and have provided various useful techniques for discovering $f.p$'s in complex issues. These are currently crucial in many mathematics related areas and its applications, including economics, astronomy, dynamical systems, decision theory, and parameter estimation. The father of $f.p.t$, mathematician Brouwer [1908], proposed $f.p$ theorems $(f.p.t's)$ for continuous mappings on finite dimensional spaces. Further, Banach [1922] established and confirmed the renowned Banach contraction principle. Several authors used the Banach contraction principle in numerous ways and presented numerous fixed point results $(f.p.r's)$ [see, Bianchini [1972], Chatterjea [1972], Ciric [1971], Kannan [1968], Kannan [1969], Karapinar [2011], Khan [1976], Reich [1971] & Zamfirescu [1972]]. On the other hand, the concept of b-metric space was initiated by Bakhtin [1989], which was an interesting expansion of metric spaces. Recently, more and more extensions of b-metric spaces, such as $b_v(s)$-metric spaces and b-rectangular metric spaces, were introduced, and some $f.p.t's$ are shown on these spaces [see, Mitrović and Radenović [2017], Roshan et al. [2016]].

Due to the complexity of finding an exact $f.p$, one can show the results of an approximate $f.p$ $(a.f.p)$. Because an exact $f.p$ has overly strict limitations. This is the main reason to find an $a.f.p$ (ϵ-fixed point). Assume that a selfmap, $T : M \to M$, has an $a.f.p$ (called, p_0). In which case, the point Mp_0 is "very near" to the point p_0. Here, the distance is less than ϵ, that is., $d(Mp_0, p_0) < \epsilon$. An $a.f.p$ is a point that is nearly located at its respective $f.p$. Originally, Tis et al. [2003] established the existence of $a.f.p's$, which turn out to be still guaranteed under various weakened versions of the well-known $f.p.t's$ of Brouwer, Kakutani, and Banach. Moreover, they studied $a.f.p.r's$ for contraction and nonexpansive mappings (refer to Theorems 2.1, 2.2, 3.1 and 4.1, respectively). Following that, Berinde [2006] proposed $a.f.p.r's$ (Qualitative theorems) by using various contraction operators on metric spaces (not necessarily complete). Further, he derived the diameter of the $a.f.p.r's$ (Quantitative theorems) by using two main lemmas (also, refer Berinde [2002] and Berinde [2003]). In addition to that, Dey and Saha [2012] extended these results, and they explained that the diameter of $a.f.p$ for the Reich operator tends to zero when ϵ approaches zero. Further, S. A. M. Mohsenialhosseini (refer, Mohsenialhosseini [2017], Mohsenalhosseini et al. [2011]) considered some new $a.f.p.r's$ for cyclical contraction mappings. Moreover, Mohsenalhosseini and Saheli [2021] extended these results for a family of contraction mappings and showed the existence of a common $f.p$ for Mohseni-

Saheli mapping. Also, the authours Tijani and Olayemi [2021] proved $a.f.p.r's$ by using some rational contraction mappings. In this manuscript, we derive some results, which include $a.f.p.t's$ and diameters of $a.f.p.t's$ on b-metric spaces, by using some rational contraction mappings.

On the other hand, the first researchers to investigate a generalization of the Banach $f.p$ theorem while simultaneously using a contraction condition of the rational type were Dass and Gupta [1975]. Later, Jaggi [1977], used a contraction condition of the rational type to prove a $f.p$ theorems in complete metric spaces. Moreover, rational contraction conditions have been heavily employed in both the $f.p$ and common $f.p$ locations. In this, we first provide $f.p$ solutions utilizing two contraction mappings, B-contraction mapping (refer to Marudai and Bright [2015]), the Hardy and Rogers [1973] contraction mapping, and their associated consequents. Secondly, we demonstrate the $f.p.r's$ in the setting of rational contraction mappings, which were discussed mostly in Dass and Gupta [1975] and Jaggi [1977].

The remaining parts of this manuscript are displayed as follows: In Section 2, some basic notions, such as notations, definitions and lemmas, are recalled from previous literature. In Section 3, we propose the main results of this work, where the existence and uniqueness of $a.f.p.r's$ are rigorously discussed in b-metric space. In Section 4, we go one step forward and provide some applications of $a.f.p.$ in the field of applied mathematics. Finally, in Section 5, we present some conclusions.

2 Preliminaries

In this section, some notations and basic notions, such as fundamental definitions and lemmas, from earlier research are recalled. These are then employed throughout the remainder of the main findings of this manuscript.

Definition 2.1. *Boriceanu [2009] Let M be a non-empty set and $b \geq 1$ be a given real number. A function $d : M \times M \longrightarrow \mathbb{R}_+$ is called a b-metric provided that for all $p, q, r \in M$ satisfies the following conditions.*

(i) $d(p, q) = 0$ *iff* $p = q$;

(ii) $d(p, q) = d(q, p)$;

(iii) $d(p, q) \leq b[d(p, r) + (r, q)]$

The pair (M, d) is called a b-metric space. Immediately from the notion of b-metric space we have the result, every metric space is a b-metric space with $b = 1$. But the converse does not hold.

Example 2.1. *Boriceanu [2009] Let* $M = \{0, 1, 2\}$ *and* $d(2, 0) = d(0, 2) = m \geq 2$
$d(0, 1) = d(1, 2) = d(1, 0) = d(2, 1) = 1$ *and*
$d(0, 0) = d(1, 1) = d(2, 2) = 0$
Then, $d(p, q) \leq \frac{m}{2}[d(p, r) + d(r, q)]$ *for all* $p, q, r \in M$.
if $m > 2$ *then the triangle inequality does not hold.*

Definition 2.2. *Berinde [2006] Let* (M, d) *be a metric space and* $T : M \to M$. *Then* $p \in M$ *is said to be an* ϵ*-fixed point (approximate fixed point) of* T*, if for every* $\epsilon > 0$ *implies*

$$d(p, Tp) < \epsilon.$$

Let $F_\epsilon(T) = \{p \in M : d(p, Tp) < \epsilon\}$ *denotes the set of all* ϵ*-fixed point of* T *for a given* $\epsilon > 0$.

Definition 2.3. *Berinde [2006] Let* $T : M \to M$. *Then* T *has the approximate fixed point property (a.f.p.p) if for every* $\epsilon > 0$,

$$F_\epsilon(T) \neq \emptyset.$$

Lemma 2.1. *Berinde [2006] Let* A *be a closed subset of a metric space* (M, d) *and* $T : A \to M$ *be a compact map. Then* T *has a fixed point if and only if it has the approximate fixed point property.*

Remark 2.1. *Berinde [2006] In the following, by* $\delta(A)$ *for a set* $A \neq \emptyset$ *we will understand the diameter of the set* A*, i.e.,*

$$\delta(A) = sup\{d(p, q) : p, q \in A\}.$$

Definition 2.4. *Berinde [2006] Let* (M, d) *be a metric space,* $T : M \to M$ *a operator and* $\epsilon > 0$. *We define the diameter of the set* $F_\epsilon(T)$*, i.e.,*

$$F_\epsilon(T) = sup\{d(p, q) : p, q \in F_\epsilon(T)\}.$$

Lemma 2.2. *Berinde [2006] Let* (M, d) *be a metric space,* $T : M \to M$ *an operator and* $\epsilon > 0$. *We assume that:*

(i) $F_\epsilon(T) \neq \emptyset$*; and*

(ii) for all $\theta > 0$*, there exists* $\phi(\theta) > 0$ *such that* $d(p, q) - d(Tp, Tq) \leq \theta$ *implies that* $d(p, q) \leq \phi(\theta)$*, for all* $p, q \in F_\epsilon(T)$.

Then;

$$\delta(F_\epsilon(T)) \leq \phi(2\epsilon).$$

Definition 2.5. *Mohsenalhosseini and Saheli [2021] Let (M, d) be a b-metric space with coefficient $b \geq 1$. A selfmap $T : M \to M$ is said to be a Mohseni-saheli type contraction if there exists $k_1 \in [0, \frac{1}{2})$ with $k_1(b + 1) < 1$ such that*

$$d(Tp, Tq) \leq k_1[d(p, q) + d(Tp, Tq)], \text{ for all } p, q \in M.$$

Definition 2.6. *Marudai and Bright [2015] Let (M, d) be a b-metric space with coeficient $b \geq 1$. A selfmap $T : M \to M$ is said to be a B-contraction if there exists $k_1, k_2, k_3 \in [0, 1)$ with $k_1(b + 1) + bk_2 + b(b + 1)k_3 < 1$ such that*

$$\begin{aligned} d(Tp, Tq) \leq\ & k_1[d(p, Tp) + d(q, Tq)] + k_2 d(p, q) \\ & + k_3[d(p, Tq) + d(q, Tp)], \text{ for all } \in M. \end{aligned}$$

Definition 2.7. *Bianchini [1972] Let (M, d) be a b-metric space. A selfmap $T : M \to M$ is said to be a Bianchini contraction if there exists $k \in (0, 1)$ with $kb < 1$ such that*

$$d(Tp, Tq) \leq kB(p, q),$$

where

$$B(p, q) = \max \{d(p, Tp), d(q, Tq)\}, \text{ for all } p, q \in M.$$

3 Main Results

In this section, we prove the approximate fixed point theorems using various types of contraction mappings, including the B-contraction, the Mohseni-Saheli contraction, and their consequences on b-metric spaces.

Theorem 3.1. *Let (M, d) be a metric space and let $T : M \to M$ be a contraction. Then T has an approximate fixed point (ϵ-fixed point).*

Proof. Fix $p_0 \in M$. Define a sequence $\{p_n\}$ by $p_{n+1} = Tp_n,$ for all $n \geq 0$. Then $\{p_n\}$ is a Cauchy sequence. Thus for all $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ implies $d(p_n, p_m) < \epsilon$. In particular, if $n \geq n_0, d(p_n, p_{n+1}) < \epsilon$. That is, $d(p_n, Tp_n) < \epsilon$. Therefore, $p_n \in F_\epsilon(T) \neq \emptyset$, for all $\epsilon > 0$. Hence T has an approximate fixed point (ϵ-fixed point). □

Example 3.1. *Let $M = (0, 1)$. A selfmap $T : M \to M$ is defined by $Tp = p/2$. Then T is a contraction. Since $(0, 1)$ is not complete and hence T does not have a fixed point. But $\{p_n\}$ is defined by $p_{n+1} = Tp_n$ with $p_0 = 1/2$. Note that $p_{n+1} = 1/2^{n+2}$. Then, $F_\epsilon(T) \neq \emptyset$, for all $\epsilon > 0$. Hence T has an ϵ-fixed point.*

Following is the Mohseni-saheli contraction mapping considered here to prove our first $a.f.p$ theorem in a b-metric space.

Theorem 3.2. *Let $T : M \to M$ be a Mohseni-saheli contractive type mapping in a b-metric space with $b \geq 1$. Then T has an ϵ-fixed point and*

$$\delta(F_\epsilon(T)) \leq \frac{2\epsilon(1+k_1 b)}{1-k_1-k_1 b}, \text{ for all } \epsilon > 0.$$

Proof. Given that T is a Mohseni-saheli type contraction mapping. Let $\epsilon > 0$ and $p_0 \in M$. Define a sequence $\{p_n\}$ such that $p_{n+1} = Tp_n,$ for all $n \geq 0$. Consider,

$$\begin{aligned}
d(T^n p, T^{n+1} p) &= d(T(T^{n-1}p), T(T^n p)) \\
&\leq k_1 d(T^{n-1}p, T^n p) + k_1 d(T^n p, T^{n+1} p) \\
&\leq \left(\frac{k_1}{1-k_1}\right) d(T^{n-1}p, T^n p) \\
&= \lambda d(T^{n-1}p, T^n p), \text{ where } \lambda = \frac{k_1}{1-k_1} \\
&\leq \lambda^2 d(T^{n-2}p, T^{n-1}p) \\
&\ldots \\
&\leq \lambda^n d(p, Tp)
\end{aligned}$$

Since $d(T^n p, T^{n+1} p) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $p \in M$. i.e., $\{p_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\epsilon(T) \neq \emptyset$. That is, T has an ϵ-fixed point. It shows that the condition (i) of Lemma 2.2 is satisfied. Now only to show that the condition (ii) of Lemma 2.2 holds. Let $\theta > 0$ and $p, q \in F_\epsilon(T)$. Assume that $d(p,q) - d(Tp,Tq) \leq \theta$. Show that $\phi(\epsilon) > 0$ exists.

$$\begin{aligned}
d(p,q) &\leq d(Tp,Tq) + \theta \\
&\leq k_1 d(p,q) + k_1 d(Tp,Tq) + \theta \\
&\leq k_1 d(p,q) + k_1 b[d(Tp,p) + d(p,q) + d(q,Tq)] + \theta \\
&= \left(\frac{2k_1 b\epsilon + \theta}{1-k_1-k_1 b}\right) \\
&= \gamma
\end{aligned}$$

So, for all $\theta > 0$, there exists $\phi(\theta) = \gamma > 0$ such that $d(p,q) - d(Tp,Tq) \leq \theta$ implies $d(p,q) \leq \phi(\theta)$. By Lemma 2.2, $\delta(F_\epsilon(T)) \leq \phi(2\epsilon),$ for all $\epsilon > 0$. Hence,

$$\delta(F_\epsilon(T)) \leq \frac{2\epsilon(1+k_1 b)}{1-k_1-k_1 b}, \text{ for all } \epsilon > 0.$$

□

Following is the B-contraction mapping considered here to prove our second $a.f.p.t$ in a b-metric space.

Theorem 3.3. *Let $T : M \to M$ be a B-contraction in a b-metric space with $b \geq 1$. Then T has an ϵ-fixed point and $\delta(F_\epsilon(T)) \leq \dfrac{2\epsilon(k_1 + k_3 b + 1)}{1 - k_2 - 2k_3 b}$, for all $\epsilon > 0$.*

Proof. Given that T is a B-contraction operator. Let $\epsilon > 0$ and $p_0 \in M$. Define a sequence $\{p_n\}$ such that $p_{n+1} = Tp_n,$ for all $n \geq 0$. Consider,

$$\begin{aligned}
d(T^n p, T^{n+1} p) &= d(T(T^{n-1}p), T(T^n p)) \\
&\leq k_1[d(T^{n-1}p, T^n p) + d(T^n p, T^{n+1}p)] + k_2 d(T^n p, T^{n-1}p) \\
&\qquad + k_3[d(T^{n-1}p, T^{n+1}p) + d(T^n p, T^n p)] \\
&\leq k_1[d(T^{n-1}p, T^n p) + d(T^n p, T^{n+1}p)] + k_2 d(T^{n-1}p, T^n p) \\
&\qquad + k_3 b[d(T^{n-1}p, T^n p) + d(T^n p, T^{n+1}p)] \\
&\leq \left(\frac{k_1 + k_2 + k_3 b}{1 - k_1 - k_3 b}\right) d(T^{n-1}p, T^n p) \\
&= \lambda d(T^{n-1}p, T^n p), \text{ where } \lambda = \frac{k_1 + k_2 + k_3 b}{1 - k_1 - k_3 b} \\
&\leq \lambda^2 d(T^{n-2}p, T^{n-1}p) \\
&\cdots \\
&\leq \lambda^n d(p, Tp)
\end{aligned}$$

Since $d(T^n p, T^{n+1}p) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $p \in M$. i.e., $\{p_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\epsilon(T) \neq \emptyset$. That is, T has an ϵ-fixed point. It shows that condition (i) of Lemma 2.2 is satisfied. Now only to show that the condition (ii) of Lemma 2.2 holds. Let $\theta > 0$ and $p, q \in F_\epsilon(T)$. Assume that $d(p,q) - d(Tp, Tq) \leq \theta$. Show that $\phi(\epsilon) > 0$ exists.

$$\begin{aligned}
d(p,q) &\leq d(Tp, Tq) + \theta \\
&\leq k_1[d(p, Tp) + d(q, Tq)] + k_2 d(p,q) + k_3[d(p, Tq) + d(q, Tp)] + \theta \\
&= 2k_1\epsilon + k_2 d(p,q) + 2bk_3\epsilon + 2k_3 b d(p,q) + \theta \\
&= \left(\frac{2k_1\epsilon + 2k_3 b\epsilon + \theta}{1 - k_2 - 2k_3 b}\right) \\
&= \gamma
\end{aligned}$$

So, for all $\theta > 0$, there exists $\phi(\theta) = \gamma > 0$ such that $d(p,q) - d(Tp,Tq) \leq \theta$ implies $d(p,q) \leq \phi(\theta)$. By Lemma 2.2, $\delta(F_\epsilon(T)) \leq \phi(2\epsilon),$ for all $\epsilon > 0$. Hence,

$$\delta(F_\epsilon(T)) \leq \frac{2\epsilon(k_1 + k_3 b + 1)}{1 - k_2 - 2k_3 b}, \text{ for all } \epsilon > 0.$$

□

Following is the Bianchini contraction mapping considered here to prove our third $a.f.p.t$ in a b-metric space.

Theorem 3.4. *Let $T : M \to M$ be a Bianchini contraction in a b-metric space with $b \geq 1$. Then T has an ϵ-fixed point and $\delta(F_\epsilon(T)) \leq \epsilon(k+2)$, for all $\epsilon > 0$.*

Proof. Given T is Bianchini contraction. Let $\epsilon > 0$ and $p_0 \in M$. Define a sequence $\{p_n\}$ such that $p_{n+1} = Tp_n,$ for all $n \geq 0$.

Case 1. *If $B(p,q) = d(p,Tp)$. Then, the Definition 2.7 becomes:*

$$\begin{aligned} d(Tp,Tq) &\leq kd(p,Tp) \\ \textit{Substitute } q = Tp,\, d(Tp,T^2p) &\leq kd(p,Tp) \\ \textit{Again subtitute } p = Tp,\, d(T^2p,T^3p) &\leq kd(Tp,T^2p) \\ &= k^2 d(p,Tp) \\ &\vdots \\ d(T^np,T^{n+1}p) &\leq k^n d(p,Tp). \end{aligned}$$

Case 2. *If $B(p,q) = d(q,Tq)$. Then, the Definition 2.7 becomes:*

$$\begin{aligned} d(Tp,Tq) &\leq kd(q,Tq) \\ \textit{Substitute } q = Tp,\, d(Tp,T^2p) &\leq kd(Tp,T^2p) \end{aligned}$$

This is impossible because $k \in (0,1)$. Therefore, **Case 2** does not exist. Now by **Case 1**, $d(T^np,T^{n+1}p) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $p,q \in M$. i.e., $\{p_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\epsilon(T) \neq \emptyset$. That is, T has an ϵ-fixed point. It shows that the condition (i) of Lemma 2.2 is satisfied. Now only to show that the condition (ii) of Lemma 2.2 holds. Let $\theta > 0$ and $p,q \in F_\epsilon(K)$. Assume that $d(p,q) - d(Kp,Kq) \leq \theta$. Show that $\phi(\epsilon) > 0$ exist.

$$\begin{aligned} d(p,q) &\leq d(Tp,Tq) + \theta \\ &\leq kB(p,q) + \theta \\ &= kd(p,Tp) + \theta \\ &= k\epsilon + \theta \\ &= \gamma \end{aligned}$$

So, for all $\theta > 0$, there exists $\phi(\theta) = \gamma > 0$ such that $d(p,q) - d(Tp,Tq) \leq \theta$ implies $d(p,q) \leq \phi(\theta)$. By Lemma 2.2, $\delta(F_\epsilon(T)) \leq \phi(2\epsilon),$ for all $\epsilon > 0$. Hence,

$$\delta(F_\epsilon(T)) \leq \epsilon(k+2), \text{ for all } \epsilon > 0.$$

□

Remark 3.1. *We have proved many approximate $a.f.p.r's$ by using various contraction mappings in b-metric space. The following results shows the diameters of various contraction mappings.*

(i) *Let (M, d) be a b-metric space with the coefficient $b \geq 1$ and $T: M \to M$. Then there exists $k_1 \in (0, 1)$ with $bk_1 < 1$ such that*

$$d(Tp, Tq) \leq k_1 d(p, q), \text{ for all } p, q \in M.$$

Hence T has an ϵ-fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{2\epsilon}{1 - k_1}, \text{ for all } \epsilon > 0.$$

This is called contraction mapping in b-metric space.

(ii) *Let (M, d) be a b-metric space with the coefficient $b \geq 1$ and $T: M \to M$. Then there exists $k_1 \in [0, \frac{1}{2})$ with $k_1(b + 1) < 1$ such that*

$$d(Tp, Tq) \leq k_1[d(p, Tp) + d(q, Tq)], \text{ for all } p, q \in M.$$

Hence T has an ϵ-fixed point and

$$\delta(F_\epsilon(T)) \leq 2\epsilon(k_1 + 1), \text{ for all } \epsilon > 0.$$

This is called Kannan contraction mapping in b-metric space.

(iii) *Let (M, d) be a b-metric space with the coefficient $b \geq 1$ and $T: M \to M$. Then there exists $k_1 \in [0, \frac{1}{2})$ with $k_1 b(b + 1) < 1$ such that*

$$d(Tp, Tq) \leq k_1[d(q, Tp) + d(p, Tq)], \text{ for all } p, q \in M.$$

Hence T has an ϵ-fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{2\epsilon(k_1 b + 1)}{1 - 2k_1 b}, \text{ for all } \epsilon > 0.$$

This is called Chatterjea contraction mapping in b-metric space.

(iv) *Let (M, d) be a b-metric space with the coefficient $b \geq 1$ and $T: M \to M$. Then there exists $k_1, k_2, k_3, k_4, k_5 \in [0, 1)$ with $(k_1+k_2)b+k_3+(b^2+b)k_4 < 1$ such that*

$$\begin{aligned} d(Tp, Tq) \leq{}& k_1 d(p, q) + k_2 d(p, Tp) + k_3 d(q, Tq) \\ &+ k_4 d(p, Tq) + k_5 d(q, Tp), \text{ for all } p, q \in M. \end{aligned}$$

Hence T has an ϵ-fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{\epsilon(k_2 + k_3 + k_4 b + k_5 b + 2)}{1 - k_1 - k_4 b - k_5 b}, \text{ for all } \epsilon > 0.$$

This is called Hardy-Rogers contraction mapping in b-metric space.

(v) *Let (M, d) be a b-metric space with the coefficient $b \geq 1$ and $T : M \to M$. Then there exists $k_1, k_2, k_3, k_4 \in [0, 1)$ with $b(k_1 + k_2) + k_3 + (b^2 + b)k_4 < 1$ such that*

$$\begin{aligned} d(Tp, Tq) &\leq k_1 d(p, q) + k_2 d(p, Tp) + k_3 d(q, Tq) \\ &\quad + k_4[d(p, Tq) + d(q, Tp)], \text{ for all } p, q \in M. \end{aligned}$$

Hence T has an ϵ-fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{\epsilon(k_2 + k_3 + 2k_4 b + 2)}{1 - k_1 - 2k_4 b}, \text{ for all } \epsilon > 0.$$

This is called Ciric-type contraction mapping in b-metric space.

(vi) *Let (M, d) be a b-metric space with the coefficient $b \geq 1$ and $T : M \to M$. Then there exists $k_1, k_2, k_3 \in [0, 1)$ with $b(k_1 + k_2) + k_3 < 1$ such that*

$$d(Tp, Tq) \leq k_1 d(p, q) + k_2 d(p, Tp) + k_3 d(q, Tq), \text{ for all } p, q \in M.$$

Hence T has an ϵ-fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{\epsilon(k_2 + k_3 + 2)}{1 - k_1}, \text{ for all } \epsilon > 0.$$

This is called Reich-type contraction mapping in b-metric space.

Next we prove some $a.f.p.t's$ for rational type contraction mappings in the settings of b-metric spaces.

Theorem 3.5. *Let (M, d) be a b-metric space with the coefficient $b \geq 1$ and $T : M \to M$. Then there exists $k \in [0, 1)$ such that $d(p, Tq) + d(q, Tp) \neq 0$ and*

$$d(Tp, Tq) \leq \frac{k[d(p, Tp)d(p, Tq) + d(q, Tq)d(q, Tp)]}{d(p, Tq) + d(q, Tp)}, \text{ for all } p, q \in M.$$

Then T has an ϵ-fixed point and

$$\delta(F_\epsilon(T)) \leq \frac{b\epsilon^2(k^2 + 6k + 9) + \epsilon(k + 1)}{2}, \text{ for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0$ and $p_0 \in M$. Define a sequence $\{p_n\}$ such that $p_{n+1} = Tp_n,$ for all $n \geq 0$. Consider,

$$\begin{aligned} d(T^n p, T^{n+1} p) &= d(T(T^{n-1} p), T(T^n p)) \\ &\leq k \left[\frac{d(T^{n-1} p, T^n p) d(T^{n-1} p, T^{n+1} p) + d(T^n p, T^{n+1} p) d(T^n p, T^n p)}{d(T^{n-1} p, T^{n+1} p) + d(T^n p, T^n p)} \right] \\ &\leq k d(T^{n-1} p, T^n p) \\ &\dots \\ &\leq (k)^n d(p, Tp) \end{aligned}$$

Since $d(T^np, T^{n+1}p) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $p \in M$. i.e., $\{p_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\epsilon(T) \neq \emptyset$. It shows that condition (i) of Lemma 2.2 is satisfied. Now only to show that the condition (ii) of Lemma 2.2 holds. Let $\theta > 0$ and $p, q \in F_\epsilon(T)$. Assume that $d(p,q) - d(Tp,Tq) \leq \theta$. Show that $\phi(\epsilon) > 0$ exists.

$$\begin{aligned}
d(p,q) &\leq k\left[\frac{d(p,Tp)d(p,Tq) + d(q,Tq)d(q,Tp)}{d(p,Tq) + d(q,Tp)}\right] + 2\epsilon \\
&= k\left[\frac{d(p,Tp)[bd(p,q) + bd(q,Tq)] + d(q,Tq)[bd(p,q) + bd(p,Tp)]}{bd(p,q) + bd(q,Tq) + bd(p,q) + bd(p,Tp)}\right] + 2\epsilon \\
&= k\left[\frac{\epsilon[bd(p,q) + b\epsilon] + \epsilon[bd(p,q) + b\epsilon]}{2bd(p,q) + 2b\epsilon}\right] + 2\epsilon \\
&= \frac{2bk\epsilon d(p,q) + 2kb\epsilon^2 + 4b\epsilon d(p,q) + 4b\epsilon^2}{2bd(p,q) + 2b\epsilon}
\end{aligned}$$

On simplyfying, we get

$$2b[d(p,q)]^2 \leq 2b\epsilon(1+k)d(p,q) + 2b\epsilon^2(k+2)$$

Which implies $a = 2b, b = -2b\epsilon(1+k)$ and $c = -2b\epsilon^2(k+2)$. Therefore,

$$\begin{aligned}
d(p,q) &\leq \frac{2b\epsilon(1+k) + \sqrt{4b^2\epsilon^2(1+k)^2 + 16b^2\epsilon^2(k+2)}}{4b} \\
&= \frac{2b\epsilon(1+k) + \sqrt{36b^2\epsilon^2 + 24b^2\epsilon^2 k + 4k^2b^2\epsilon^2}}{4b} \\
&= \frac{b\epsilon(1+k) + \sqrt{9b^2\epsilon^2 + 6b^2\epsilon^2 k + k^2b^2\epsilon^2}}{2b} \\
&< \frac{b\epsilon + b\epsilon k + 9b^2\epsilon^2 + 6b^2\epsilon^2 k + k^2b^2\epsilon^2}{2b} \\
&= \frac{\epsilon + \epsilon k + 9b\epsilon^2 + 6b\epsilon^2 k + bk^2\epsilon^2}{2}
\end{aligned}$$

Hence,

$$\delta(F_\epsilon(T)) < \frac{b\epsilon^2(k^2 + 6k + 9) + \epsilon(k+1)}{2}, \text{ for all } \epsilon > 0.$$

□

Example 3.2. *Let (M,d) be a b-metric space with the coefficient $b = 1$. Let $M = (0, 1/2]$ be endowed with usual metric. Let $T : M \to M$ be defined by $Tp = p/2$, for all $p \in M$. To show that the equation is satisfied. For that choose*

$k = 1/4$ and $\epsilon = 1/2$. Also $p = 1/2, q = 1/4 \in M$. Then $d(p, Tp) = 1/3 < 1/2; d(q, Tq) = 1/8 < 1/2$. We have $d(p, q) = 1/4$ and

$$\delta(F_\epsilon(T)) < \frac{b\epsilon^2(k^2 + 6k + 9) + \epsilon(k + 1)}{2} = \frac{193}{128}.$$

Therefore, it satisfies all the conditions of Theorem 3.5. Hence, it has an ϵ-fixed point.

Corolary 3.1. *Let (M, d) be a b-metric space with the coefficient $b \geq 1$ and $T : M \to M$. Then there exist $k_1, k_2 \in [0, 1)$ with $k_1 + bk_2 < 1$ such that*

$$d(Tp, Tq) \leq \frac{k_1 d(q, Tq)[1 + d(p, Tp)]}{1 + d(p, q)} + k_2 d(p, q), \text{ for all } p, q \in M.$$

Then T has an ϵ-fixed point and

$$\delta(F_\epsilon(T)) < \frac{k_2^2 - k_2 + 6\epsilon + 4\epsilon^2(1 + k_1 - k_1k_2) + 4\epsilon(k_1 - k_2 - k_1k_2)}{2(1 - k_2)}, \forall \epsilon > 0.$$

Proof. Complete this corollary by following the same procedure as in Theorem 3.5. □

Example 3.3. *Let (M, d) be a b-metric space with the coefficient $b = 1$. Let $M = (0, 1/2]$ be endowed with usual metric. Let $T : M \to M$ be defined by $Tp = p/2$, for all $p \in M$. For that choose $k_1 = 1/4, k_2 = 1/5$ and $\epsilon = 1/2$. Also $p = 1/2, q = 1/4 \in M$. Then $d(p, Tp) = 1/3 < 1/2; d(q, Tq) = 1/8 < 1/2$. We have $d(p, q) = 1/4$ and*

$$\frac{k_2^2 - k_2 + 6\epsilon + 4\epsilon^2(1 + k_1 - k_1k_2) + 4\epsilon(k_1 - k_2 - k_1k_2)}{2(1 - k_2)} \leq \frac{207}{80}.$$

Therefore, it satisfies all the conditions of corollary 3.1. Hence, it has an ϵ-fixed point.

Corolary 3.2. *Let (M, d) be a b-metric space with the coefficient $b \geq 1$ and $T : M \to M$. Then there exist $k_1, k_2 \in [0, 1)$ with $k_1 + bk_2 < 1$ and for all $p, q \in M$ with $p \neq q$ such that*

$$d(Tp, Tq) \leq \frac{k_1 d(p, Tp) d(q, Tq)}{d(p, q)} + k_2 d(p, q), \text{ for all } \epsilon > 0.$$

Then T has an ϵ-fixed point and

$$\delta(F_\epsilon(T)) < \epsilon\left(\frac{2}{1 - k_2} + k_1\right), \text{ for all } \epsilon > 0.$$

Proof. Complete this corollary by following the same procedure as in Theorem 3.5. □

4 Applications

The $a.f.p.t$ covers a wide range of applications in the field of mathematics, particularly differential geometry, numerical analysis, and so on. By reading Khuri and Louhichi [2018] and the references therein, one can find a variety of applications involving $a.f.p.r's$ in the field of applied mathematics. The examples below demonstrate how to apply $a.f.p$ findings in differential equations.

Example 4.1. *Let $T = C([0,1],\mathbb{R})$ and T is complete extended b-metric space defined by $d(p,q) = \sup_{t\in[0,1]} |p-q|^2$. Also, consider $y''(t) = 3y^2(t)/2$, $0 \leq t \leq 1$ and the initial conditions $y(0) = 4$, $y(1) = 1$. Here, the exact solution is $y(t) = 4/(1+t)^2$. We have, $y_0(t) = c_1t + c_2$. By using the initial conditions, we get $y_0(t) = 4 - 3t$. Now, define the integral operator,*

$$A(y) = y(t) + \int_0^1 G(t,s)[y'' - f(s,y,y')]ds \tag{1}$$

where

$$G(t,s) = \begin{cases} s(1-t) & 0 \leq s \leq t \\ t(1-s) & t \leq s \leq 1 \end{cases}$$

Then, the equation (1) *becomes*

$$\begin{aligned} A(y) &= y(t) + \int_0^1 G(t,s)y''(s)ds - \int_0^1 G(t,s)f(s,y,y')ds \\ &= (4-3t) - \int_0^1 G(t,s)[(-3/2)y^2(s)]ds \\ &= 4 - 3t + \frac{3}{2}\left\{\int_0^1 G(t,s)y^2(s)ds\right\} \end{aligned}$$

Consider,

$$\begin{aligned} d(Ap,Aq) &= \sup_{t\in[0,1]} |Ap - Aq|^2 \\ &= \sup_{t\in[0,1]} \left|\frac{3}{2}\int_0^1 G(t,s)p^2(s)ds - \frac{3}{2}\int_0^1 G(t,s)q^2(s)ds\right|^2 \\ &\leq \frac{9}{4}\left(\int_0^1 |G(t,s)|^2 ds\right)\left(\int_0^1 |p^2(s) - q^2(s)|^2 ds\right) \\ &\leq \frac{9}{4}\frac{t^2(1-t)^2}{3}\int_0^1 \left|p^2(s) - q^2(s)\right|^2 ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{3}{4}(\frac{1}{4})(\frac{1}{4}) \int_0^1 \left|p^2(s) - q^2(s)\right|^2 ds \\
&\leq \frac{3}{64} \sup_{t\in[0,1]} |p(s) - q(s)|^2 \\
&\leq \frac{3}{64} d(p,q)
\end{aligned}
$$

Then, Definition 2.6 gives

$$d(Ap, Aq) \leq (3/64)d(p, q) + k_1[d(p, Ap) + d(q, Aq)] + k_3[d(p, Aq) + d(q, Ap)].$$

Thus, $k_2 = 3/64$ *and* $k_1 = k_3 = 0$ *satisfies all the conditions of Theorem 3.3. Also, by Theorem 3.1,* A *has* $a.f.p$ *in* $T = C([0, 1], \mathbb{R})$*. Therefore, the given bounded value problem has* $a.f.p$ *in* T*.*

5 Conclusion

In this paper, some approximate fixed point theorems are established in a b-metric space by utilizing various types of contraction mappings. Further, some approximate fixed point theorems are newly developed for rational type contractions in a b-metric space. It is worth observing that in the limiting case $\epsilon \longrightarrow 0$, all the results established in the present paper produces more restricted approximate fixed points. The notion of approximate fixed points is consequently not less essential than that of exact fixed points. Results that will be given in the future can be demonstrated in a lesser setting to ensure the existence and uniqueness of the approximate fixed points.

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