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sc-graphs

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Abstract

Any graph operation of a graph drive only one graph structure. But eccentric graph operation of almost all the graphs drives more than one graph structure, which are interesting and marvelous one. The eccentric graph of G is denoted by $ecc(G)$. Only few families of prime cordial eccentric graphs are investigated. A graph G is separation cordial graph, if it admits separation cordial labeling. In this paper, we investigate some new families of separation cordial eccentric graphs such as $ecc(C_n \odot K_1)$, $ecc(P(C_n(2)))$ and $ecc(S(m, n))$.
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1 Introduction

Labeling of graph is one of the interesting and active area in the field of graph theory. Rosa [1967] introduced the concept of graceful. Graham and Sloane [1980] introduced the concept of harmonious. Graceful labeling and harmonious labeling are the origin of the graph labeling methods in graph theory. Cahit [1987] introduced another important type of labeling, which contains aspects of both graceful and harmonious, is called cordial. Separation cordial graph was introduced by Elachini V. Lal [2015a] and presented the separation cordial labeling of some standard graph. Elachini V. Lal [2015b] investigated some star and bistar related separation cordial graphs.
Eccentricity related study of graphs are interesting and marvelous, one such eccentricity related study of graphs is eccentric graphs. The special type of graph operation, which is called eccentric graph of graph was defined by Akiyama et al. [1985] and they also contributed few interesting characterizations of such graphs. Any graph operation of a graph drive only one graph structure. But eccentric graph operation of almost all the graphs drives more than one graph structure, which are interesting and marvelous. Such a interesting graph structures of eccentric graphs of some particular classes of graphs are found and characterized by Kaspar et al. [2018]. Some few families of prime cordial eccentric graphs are investigated by Poulomi Ghosh and Anita Pal [2017].
Graphs discussed here are simple, finite and undirected graphs. For graph theoretical terminologies we refer Harary [1972]. For all standard terminologies and notations in Number theory we refer Tom M Apostol [1998] and we refer Gallian [2019] for the literature on graph labeling. In this paper, we obtained some new families of separation cordial eccentric graphs such as $ecc(C_n \odot K_1)$, $ecc(P(C_n(2)))$ and $ecc(S(m, n))$.
We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1. *A graph labeling is the assignment of unique identifiers to the edges and vertices of a graph.*

Definition 1.2. *A separation cordial labeling (or sc-labeling) of a graph G with vertex set V is a bijection f from V to 1, 2,$\cdots$, $|V(G)|$ such that if each edge uv assigned the label 1 if $f(u) + f(v)$ is an odd number and label 0 if $f(u) + f(v)$ is an even number then the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph G is separation cordial graph (or sc-graph), if it admits separation cordial labeling.*

Definition 1.3. *Let G be a graph with vertex set V and edge set E. The eccentric graph of G is denoted by $ecc(G) = (V, E')$, where $E' = \{(u, v) : d_G(u, v) = ecc(u)$ or $d_G(u, v) = ecc(v)$ for $u, v \in V\}$.*

Definition 1.4. *Double star $S(m,n)$ graph where $m,n \geqslant 1$ is the graph consisting of the union of two starts $K_{1,m}$ and $K_{1,n}$ together with a line joining their centers.*

Definition 1.5. *Shee and Ho [1996] Let $G_1, G_2, \cdots, G_n$ be $n(\geqslant 2)$ copies of a graph G. It is denote by $G(n)$ the graph obtained by adding an edge to G_i and G_{i+l}, $i = 1, 2, \cdots, n-1$, and call $G(n)$ the path-union of n copies of the graph G.*

2 sc-graphs

We will present some new families of separation cordial eccentric graphs.

Theorem 2.1. *$ecc(C_n \odot K_1)$ is sc-graph for $n \geqslant 3$.*

Proof. $C_n \odot K_1$ has $2n$ vertices and $2n$ edges. $v_1, v_2, v_3, \cdots, v_n$ are vertices of C_n; $w_1, w_2, w_3, \cdots, w_n$ are pendent vertices and $v_1w_1, v_2w_2, v_3w_3, \cdots, v_nw_n$ are pendent edges of $C_n \odot K_1$

Case 1 : when n is even.

$ecc(C_n \odot K_1)$ is $\frac{n}{2}P_4$ and the vertices of $ecc(C_n \odot K_1)$ are $v_i, w_{\frac{n}{2}+i}, w_i, v_{\frac{n}{2}+i}$, for $i = 1, 2, \cdots, \frac{n}{2}$.

Then, $|V(ecc(C_n \odot K_1))| = 2n$ and $|E(ecc(C_n \odot K_1))| = \frac{3n}{2}$. Define $f : V(ecc(C_n \odot K_1)) \to \{1, 2, \cdots, 2n\}$ as follows.

Sub case 1(a) : n = 4.

$f(v_1) = 1$, $f(v_2) = 6$, $f(v_{\frac{n+2}{2}}) = 4$, $f(v_{\frac{n+4}{2}}) = 8$, $f(w_1) = 2$, $f(w_2) = 7$, $f(w_{\frac{n+2}{2}}) = 3$, $f(w_{\frac{n+4}{2}}) = 5$. Now $e_f(0)$ = $e_f(1)$ = 3, then $|e_f(0)–e_f(1)| \leqslant 1$. Thus, $ecc(C_n \odot K_1)$ is sc-graph for n = 4.

Sub case 1(b) : $n \equiv 0(mod4)$ and $n > 4$.

$f(v_1) = 1$, $f(v_2) = 6$, $f(v_{\frac{n+2}{2}}) = 4$, $f(v_{\frac{n+4}{2}}) = 8$, $f(w_1) = 2$, $f(w_2) = 7$, $f(w_{\frac{n+2}{2}}) = 3$, $f(w_{\frac{n+4}{2}}) = 5$, $f(v_{i+2}) = f(v_i) + 8$, for $i = 1, 2, \cdots, \frac{n-4}{2}$, $f(w_{i+2}) = f(w_i) + 8$, for $i = 1, 2, \cdots, \frac{n-4}{2}$, $f(v_{i+2}) = f(v_i) + 8$, for $i = \frac{n+2}{2}, \frac{n+4}{2}, \cdots, n-2$, $f(w_{i+2}) = f(w_i) + 8$, for $i = \frac{n+2}{2}, \frac{n+4}{2}, \cdots, n-2$.

Now $e_f(0)$ = $e_f(1)$ = $\dfrac{3n}{4}$, then $|e_f(0)–e_f(1)| \leqslant 1$. Thus, $ecc(C_n \odot K_1)$ is sc-graph for $n \equiv 0(mod4)$ and $n > 4$.

Sub case 1(c) : $n \equiv 2(mod4)$ and $n > 4$.

$f(v_1) = 1$, $f(v_2) = 6$, $f(v_{\frac{n+2}{2}}) = 4$, $f(v_{\frac{n+4}{2}}) = 8$, $f(w_1) = 2$, $f(w_2) = 7$, $f(w_{\frac{n+2}{2}}) = 3$, $f(w_{\frac{n+4}{2}}) = 5$, $f(v_{i+2}) = f(v_i) + 8$, for $i = 1, 2, \cdots, \frac{n-4}{2}$, $f(w_{i+2}) = f(w_i) + 8$, for $i = 1, 2, \cdots, \frac{n-4}{2}$, $f(v_{i+2}) = f(v_i) + 8$, for $i = \frac{n+2}{2}, \frac{n+4}{2}, \cdots, n-3$, $f(w_{i+2}) = f(w_i) + 8$, for $i = \frac{n+2}{2}, \frac{n+4}{2}, \cdots, n-3$, $f(v_n) = 2n$ and $f(w_n) = 2n - 3$.

Now $e_f(0) = \frac{3n+2}{4}$ and $e_f(1) = \frac{3n-2}{4}$, then $|e_f(0)–e_f(1)| \leqslant 1$. Thus, $ecc(C_n \odot K_1)$ is sc-graph for $n \equiv 2(mod4)$ and $n > 4$. Therefore, $ecc(C_n \odot K_1)$ is sc-graph for n is even.

Case 2 : when n is odd.

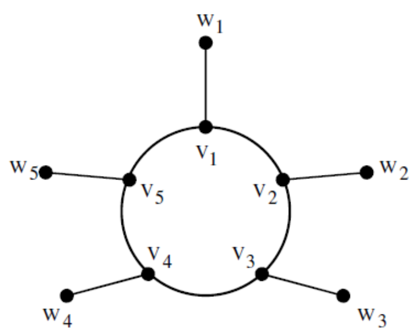
$ecc(C_n \odot K_1)$ is a subdivision of outer edges of $C_n(1, \frac{n-1}{2})$, where $w_1, w_2, w_3, \cdots, w_n$ are vertices of $C_n(1, \frac{n-1}{2})$ and $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, \cdots, v_n, v_1, v_2, v_3, \cdots, v_{\frac{n+1}{2}}$ are the subdivided vertices of the edges $w_1w_2, w_2w_3, w_3w_4, \cdots, w_nw_1$ of $C_n(1, \frac{n-1}{2})$. Then, $|V(ecc(C_n \odot K_1))| = 2n$ and $|E(ecc(C_n \odot K_1))| = 3n$. Define $f : V(ecc(C_n \odot K_1)) \to \{1, 2, \cdots, 2n\}$ as follows.

$f(w_i) = i$, for $i = 1, 2, \cdots, n$, $f(v_{\frac{n+1}{2}+i}) = n + i$, for $i = 1, 2, \cdots, \frac{n-1}{2}$, $f(v_i) = \frac{3n-1}{2} + i$, for $i = 1, 2, \cdots, \frac{n+1}{2}$.

Now $e_f(0) = \frac{3n-1}{2}$ and $e_f(1) = \frac{3n+1}{2}$, when $n \equiv 1(mod4)$ and $e_f(0) = \frac{3n+1}{2}$ and $e_f(1) = \frac{3n-1}{2}$, when $n \equiv 3(mod4)$. Then $|e_f(0)–e_f(1)| \leqslant 1$. Therefore, $ecc(C_n \odot K_1)$ is sc-graph for n is odd.

Hence, $ecc(C_n \odot K_1)$ is sc-graph for $n \geqslant 3$. □

Example 2.1. *$C_5 \odot K_1$, $ecc(C_5 \odot K_1)$ and sc-labeling of $ecc(C_5 \odot K_1)$ are shown in below figure.*



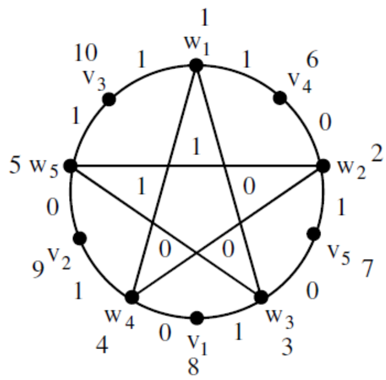


Figure 2.1 : $C_5 \odot K_1$, $ecc(C_5 \odot K_1)$ and sc-labeling of $ecc(C_5 \odot K_1)$

Theorem 2.2. *$ecc(P(C_n(2)))$ is sc-graph for $n \geqslant 3$.*

Proof. $P(C_n(2))$ has $2n$ vertices and $2n+1$ edges. The vertices of $P(C_n(2))$ are $v_1, v_2, v_3, \cdots, v_n, w_1, w_2, w_3, \cdots, w_n$.

Case 1 : when n is even.

$ecc(P(C_n(2)))$ is $S(n-1, n-1)$ and the vertices of $ecc(P(C_n(2)))$ are two centers $w_{\frac{n+2}{2}}, v_{\frac{n+2}{2}}$, and two set of pendent vertices $v_1, v_2, v_3, \cdots, v_n$ and $w_1, w_2, w_3, \cdots, w_n$.

$|V(ecc(P(C_n(2))))| = 2n$ and $|E(ecc(P(C_n(2))))| = 2n-1$ and define $f : V(ecc(P(C_n(2)))) \to \{1, 2, \cdots, 2n\}$ as follows.

$f(v_{\frac{n+2}{2}}) = 1, f(v_1) = 2$ and $f(w_{\frac{n+2}{2}}) = 3$, $f(v_i) = i+2$, for $i = 2, 3, \cdots, \frac{n}{2}$, $f(v_i) = i+1$, for $i = \frac{n+4}{2}, \frac{n+6}{2}, \cdots, n$, $f(w_i) = i+n+1$, for $i = 1, 2, \cdots, \frac{n}{2}$, $f(w_i) = i+n$, for $i = \frac{n+4}{2}, \frac{n+6}{2}, \cdots, n$. Now $e_f(0) = n-1$ and $e_f(1) = n$, then $|e_f(0) – e_f(1)| \leqslant 1$. Thus, $ecc(P(C_n(2)))$ is sc-graph for n is even.

Case 2 : when n is odd.

$ecc(P(C_n(2)))$ is $\overline{K_{n-2}} + \overline{K_2} + \overline{K_2} + \overline{K_{n-2}}$ and the vertices of $ecc(P(C_n(2)))$ are $w_1, w_2, \cdots, w_{\frac{n-1}{2}}, w_{\frac{n+5}{2}}, \cdots, w_n$, $v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}$, $w_{\frac{n+1}{2}}, w_{\frac{n+3}{2}}$, $v_1, v_2, \cdots, v_{\frac{n-1}{2}}$, $v_{\frac{n+5}{2}}, \cdots, v_n$. $|V(ecc(P(C_n(2))))| = 2n$ and $|E(ecc(P(C_n(2))))| = 4n-4$. Define $f : V(ecc(P(C_n(2)))) \to \{1, 2, \cdots, 2n\}$ as follows.

$f(v_{\frac{n+1}{2}}) = 1$ and $f(v_{\frac{n+3}{2}}) = 2$, $f(w_{\frac{n+1}{2}}) = 3$ and $f(w_{\frac{n+3}{2}}) = 4$, $f(v_i) = n+2+i$, for $i = 1, 2, \cdots, \frac{n-1}{2}$, $f(v_i) = n+i$, for $i = \frac{n+5}{2}, \frac{n+7}{2}, \cdots, n$, $f(w_i) = 4+i$, for $i = 1, 2, \cdots, \frac{n-1}{2}$, $f(w_i) = 2+i$, for $i = \frac{n+5}{2}, \frac{n+7}{2}, \cdots, n$.

Now $e_f(0) = e_f(1) = 2n-2$, then $|e_f(0) – e_f(1)| \leqslant 1$. Thus, $ecc(P(C_n(2)))$ is sc-graph for n is odd.

Hence, $ecc(P(C_n(2)))$ is sc-graph for $n \geqslant 3$. □

Example 2.2. *$P(C_5(2))$, $ecc(P(C_5(2)))$ and sc-labeling of $ecc(P(C_5(2)))$ are shown in below figure.*

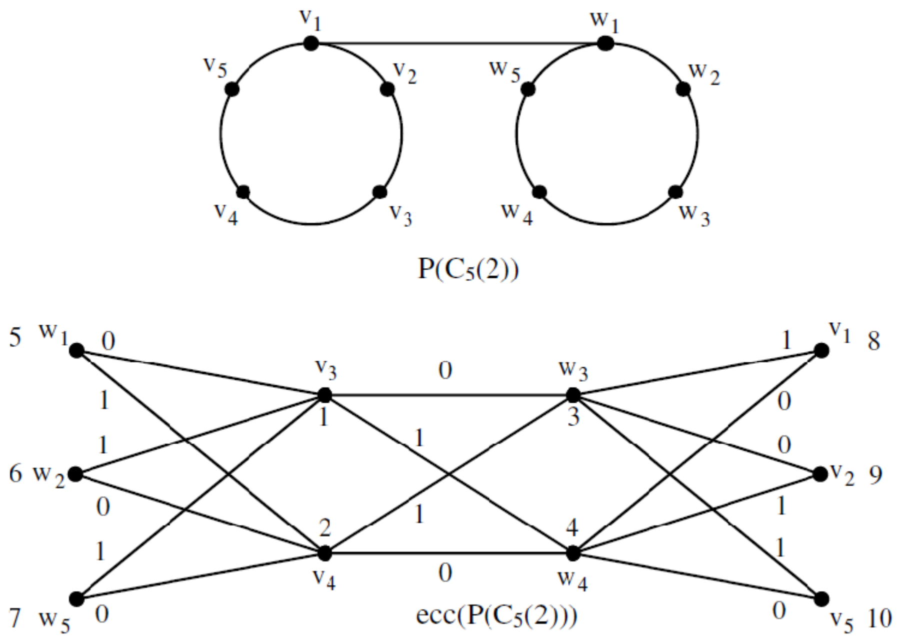


Figure 2.2: $P(C_5(2))$, $ecc(P(C_5(2)))$ and sc-labeling of $ecc(P(C_5(2)))$

Theorem 2.3. *$ecc(S(m,n))$ is sc-graph for $m, n \geqslant 2$.*

Proof. $S(m,n)$ is a double star for $m, n \geqslant 2$.
$S(m,n)$ has $m+n+2$ vertices and $m+n+1$ edges. The vertices of S(m,n) are $v, v_1, v_2, \cdots, v_m,\ w, w_1, w_2\ , \cdots, w_n$. $ecc(S(m,n))$ is $K_1 + \overline{K_n} + \overline{K_m} + K_1$, the vertices of $ecc(S(m,n))$ are $v, w_1, w_2, \cdots, w_n,\ w, v_1, v_2,\ , \cdots, v_n$. In $ecc(S(m,n))$, v is adjacent to $w_1, w_2, \cdots, w_n$; w is adjacent to $v_1, v_2, \cdots, v_m$, and, also $w_1, w_2, \cdots, w_n$ and $v_1, v_2, \cdots, v_m$ are mutually adjacent.

$|V(ecc(S(m,n)))| = m+n+2$ and $|E(ecc(S(m,n)))| = nm+m+n$. Define $f : V(ecc(S(m,n))) \rightarrow 1, 2, \cdots, m+n+2$ as follows.

$f(v) = 1$ and $f(w) = 2$, $f(w_i) = i+2$, for $i = 1, 2, \cdots, m$, $f(v_i) = i+n+2$, for $i = 1, 2, \cdots, n$. Now $e_f(0) = e_f(1) = \frac{mn+m+n}{2}$, when m and n are even. $e_f(0) = \frac{mn+m+n-2}{2}$ and $e_f(1) = \frac{mn+m+n+2}{2}$, when either m is even and n is odd or m is odd and n is even. $e_f(0) = \frac{mn+m+n+2}{2}$ and $e_f(1) = \frac{mn+m+n-2}{2}$, when m and n are odd. Thus, $|e_f(0) - e_f(1)| \leqslant 1$.
Hence, $ecc(S(m,n))$ is sc-graph for $m, n \geqslant 2$. □

Example 2.3. $S(4,5)$, $ecc(S(4,5))$ *and sc-labeling of* $ecc(S(4,5))$ *are shown in below figure.*

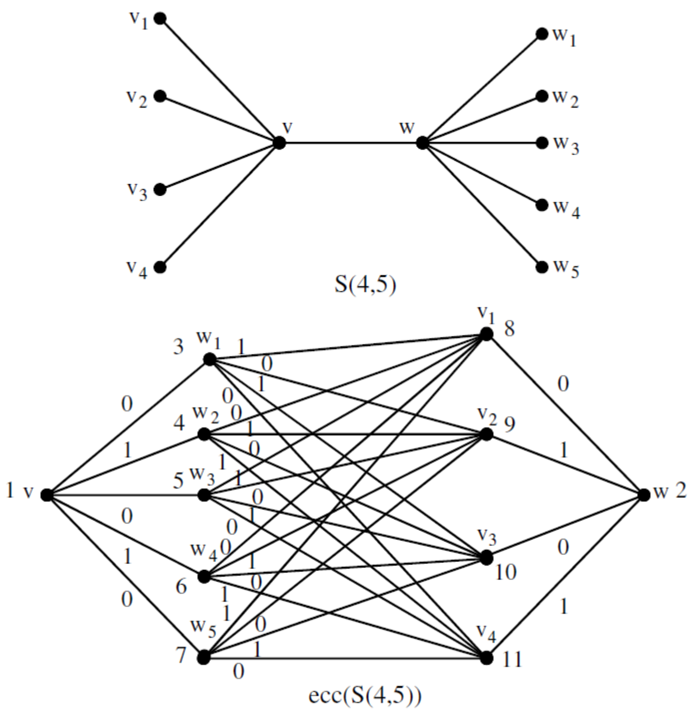


Figure 2.3 : $S(4,5)$, $ecc(S(4,5))$ and sc-labeling of $ecc(S(4,5))$

3 Conclusions

Interesting and marvelous graph structures of some more new families of eccentric graph are derived and verified all are separation cordial eccentric graphs. We discussed only the eccentric graphs of graphs, which are obtained by some more graph operations of graphs. The construction of these eccentric graphs are difficult but interesting one. We presented $ecc(C_n \odot K_1)$, $ecc(P(C_n(2)))$ and $ecc(S(m,n))$ are separation cordial eccentric graphs.

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