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A Wavelet Collocation Method For Some Fractional Models

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Abstract

This article presents an effective numerical approach based on the operational matrix of fractional order integration of Haar wavelets for dealing with the fractional models of the mixing and the Newton law of cooling problems. A general procedure of obtaining the fractional integration operational matrix of Haar wavelets which converts the fractional models into a system of algebraic equations is derived so that the computation is very simple and it is much effective than the conventional numerical methods. The reliability and the applicability of the current numerical technique for fractional models are examined by comparing the achieved results with the precise solutions.
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1 Introduction

Consideration of derivatives and integrals of any order makes fractional calculus as an extension of integer order calculus. In recent years, several fractional models have drawn much attention in diverse disciplines of science and technology, such as, economics, medicine, earthquake, colored noise, signal processing, electromagnetism, electrochemistry, fluid-dynamic traffic model, physical chemistry, finance, social science, electric circuit, mechanical system and so on. Besides modeling, the solution techniques and their reliabilities are most important to catch critical points at which a sudden divergence, convergence or bifurcation starts. Therefore, high accuracy solutions are always needed. So, in the last few years, many researchers have proposed several efficient numerical methods for the numerical solutions of fractional models such as, Laplace transform [Podlubny [1999]], Fractional Differential Transform Method (FDTM)[Arikoglu and Ozkol [2009]], Variational Iteration Method (VIM) [Das [2009]], Power Series Method [Odibat and Shawagfeh [2007]], Adomian decomposition method (ADM) [El-Wakil et al. [2006]], Fractional Difference Method (FDM) [Meerschaert and Tadjeran [2006]] and so on.

Wavelet theory has drawn a lot of attention due to widespread applications, particularly in the field of image processing, signal processing, time-frequency analysis, edge extrapolation and fast algorithms. A special kind of function known as a wavelet has a number of convenient features including compact support, orthogonality, exact representation of polynomials to a certain degree and ability to represent functions at different levels of resolution. Therefore, for solving fractional models, numerous operational matrices based on different wavelets are employed, such as Haar wavelets [Chen et al. [2012], Mohammadi et al. [2019], Shah et al. [2017]], Bernoulli wavelets [Rahimkhani and Moeti [2018]], Legendre wavelets [ur Rehman and Khan [2011]], Euler wavelets [Aruldoss et al. [2021]], Chebyshev wavelets [Farooq et al. [2019]], Muntz-Legendre polynomial [Rahimkhani et al. [2018]], Chelyshkov polynomials [Talaei and Asgari [2018]], laguerre polynomials [Ji and Hou [2020]], Bernstein polynomials [Kadkhoda [2020]], Lagrange polynomials [Sabermahani et al. [2018]], spectral method [Talaei and Asgari [2018]], wavelet based numerical inversion Laplace transform [Aruldoss and Balaji [2022]] and so on.

Since the Haar wavelets are constructed from piecewise constant functions, they are the mathematically simplest wavelets among all wavelet families. The advantages of Haar wavelet technique are its ease of use, less computation costs and more precise outcomes.

The main objective of the current work is to get numerical solutions for fractional models of mixing problems and Newton law of cooling problems, using a numerical scheme based on Haar wavelets.

This paper is organised as follows. Some essential terminology and mathematical background information for fractional calculus are discussed in Section 2. Functional approximation using Haar wavelets and the operational matrix for fractional integration of Haar wavelets are covered in section 3. In Section 4, we demonstrate the applicability and the high accuracy of the present numerical algorithm for the fractional models of mixing and Newton law of cooling problems. Finally, the conclusion is made in Section 5.

2 Preliminaries

Here, we cover fractional order integral and differential operators along with some fundamental fractional calculus concepts.

Definition 2.1 (Diethelm [2004], Miller and Ross [1993], Oldham and Spanier [1974], Podlubny [1999])**.** *The Riemann-Liouville integral of fractional order* $\lambda \geq 0$*, of the function* $h(x) \in L^1([0,\infty))$*, is given by*

$$I^{\lambda}h(x) = \begin{cases} \frac{1}{\Gamma(\lambda)}\int_0^x (x-\upsilon)^{\lambda-1}h(\upsilon)d\upsilon, & \lambda > 0. \\ \\ h(x), & \lambda = 0. \end{cases}$$

Let $h(x)$, $g(x) \in L^1([0,\infty))$, a, $b \in \mathbb{R}$, $\nu > -1$, $\lambda \geq 0$.
Then the following properties are attained.

(i) $I^{\lambda}\left(ah(x) + bg(x)\right) = aI^{\lambda}h(x) + bI^{\lambda}g(x)$.

(ii) $I^{\lambda}x^{\nu} = \dfrac{\Gamma(\nu+1)}{\Gamma(\nu+\lambda+1)}x^{\lambda+\nu}$.

Definition 2.2 (Diethelm [2004], Miller and Ross [1993], Oldham and Spanier [1974], Podlubny [1999])**.** *The Caputo fractional derivative of fractional order* $\lambda \geq 0$*, of the function* $h(x) \in L^1([0,\infty))$*, is given by*

$$D^{\lambda}h(x) = I^{m-\lambda}\left(D^m h(x)\right) = \begin{cases} \frac{1}{\Gamma(m-\lambda)}\int_0^x \frac{h^{(m)}(\upsilon)}{(x-\upsilon)^{\lambda-m+1}}d\upsilon, & m-1 < \lambda < m,\ m \in \mathbb{N}. \\ \\ \frac{d^m}{dx^m}h(x), & \lambda = m \in \mathbb{N}. \end{cases}$$

Let $h(x)$, $g(x) \in L^1([0,\infty))$, a, $b \in \mathbb{R}$, $\lambda \geq 0$.

Then the following properties are attained.

(i) $D^{\lambda}k = 0, \quad where\ k\ is\ a\ constant.$

(ii) $D^{\lambda}\left(ah(x) + bg(x)\right) = a\left(D^{\lambda}h(x)\right) + b\left(D^{\lambda}g(x)\right).$

(iii) $D^{\lambda}(I^{\lambda}h(x)) = h(x).$

(iv) $I^{\lambda}(D^{\lambda}h(x)) = h(x) - \sum_{k=0}^{m-1} h^{(k)}(0^{+})\frac{x^k}{k!}, \qquad m-1 < \lambda \leq m,$

where $m \in \mathbb{N}$, $x > 0$ and $h^{(k)}(0^{+}) := \lim_{x\to 0^{+}} D^{(k)}h(x)$, $k = 0, 1, 2, \ldots, m-1$.

3 Haar wavelets approximations of functions

The Haar wavelet and its fractional integration operational matrix are presented here.

3.1 Haar wavelets

Alfred Haar, a Hungarian mathematician, developed the Haar functions in 1910. A square wave with magnitude of 1 in some interval and zero in all other intervals is referred to as the orthogonal set of Haar functions. The first curve is $h_1(x) = 1$ on the entire interval $0 \leq x \leq T$, $T \in \mathbb{R}^{+}$. The mother wavelet or the fundamental square wave which also spans the entire interval $[0, T)$ is represented by the second curve. By performing the two operations of translation and dilation on $h_1(x)$, all other subsequent curves are produced. The scaling function $h_1(x)$ for the family of Haar wavelets is defined as

$$h_1(x) = \begin{cases} \frac{1}{\sqrt{2^{J+1}}}, & for\ \ x \in [0, T), \\ \\ 0, & otherwise. \end{cases}$$

The haar wavelet family for $x \in [0, T)$is defined as

$$h_i(x) = \sqrt{\frac{m}{2^{J+1}}} \begin{cases} 1, & for\ \ x \in [\frac{kT}{m}, \frac{(k+0.5)T}{m}\}, \\ \\ -1, & for\ \ x \in [\frac{(k+0.5)T}{m}, \frac{(k+1)T}{m}\}, \\ \\ 0, & otherwise. \end{cases}$$

where $m = 2^l$, $l = 1\ to\ J, J \in N$, indicates the level of resolution and the translation parameter. The maximum level of resolution is J.The index i is calculated using $i = m + k + 1$. the minimal value of i is 2 and the maximal value of i is $N = 2^{J+1}$.

3.2 Square integrable functions approximation

With regard to Haar wavelets, every function in $f(t)$ from $L^2([0,T))$ can be expressed as

$$f(t) = \sum_{i=1}^{\infty} c_i h_i(t), \tag{1}$$

where the following inner product resolves the coefficients c_i.

$$\langle f(t), h_i(t) \rangle = \int_0^T f(t) h_i(t) dt.$$

If the series (1) is truncated, $f(t)$ can be roughly calculated as

$$f(t) \approx \sum_{i=1}^{N} c_i h_i(t) = CH(t), \tag{2}$$

where $N = 2^{J+1}$, $C = [c_1, c_2, ..., c_N]_{1\times N}$ is the coefficient vector, $H(t) = [h_1(t), h_2(t), ..., h_N(t)]'$.
Discretizing (2) at $t_i = \frac{(2i-1)T}{2N}$, $i = 1, 2, 3, \dots, N$, we attain

$$F \simeq CH_{N\times N}, \tag{3}$$

where $F = [f(t_1), f(t_2), ..., f(t_N)]_{1\times N}$ and $H_{N\times N} = [H(t_1), H(t_2), ..., H(t_N)]$ is a Haar wavelet coefficient matrix of order N. Especially, for $J = 2$ and $T = 1$, the Haar wavelet coefficient matrix becomes

$$H_{8\times 8} = \begin{pmatrix} 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 \\ 0.3536 & 0.3536 & 0.3536 & 0.3536 & -0.3536 & -0.3536 & -0.3536 & -0.3536 \\ 0.5000 & 0.5000 & -0.5000 & -0.5000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5000 & 0.5000 & -0.5000 & -0.5000 \\ 0.7071 & -0.7071 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7071 & -0.7071 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7071 & -0.7071 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7071 & -0.7071 \end{pmatrix}$$

3.3 Haar wavelets operational matrix of fractional integration

According to [Kilicman and Al Zhour [2007]], the N Block pulse functions(BPFs) on $[0, T)$ are given by,

$$b_m(t) = \begin{cases} 1, & [\frac{(m-1)T}{N}, \frac{mT}{N}), \\ 0, & otherwise, \end{cases}$$

where $m = 1, 2, 3, \ldots, N$, $N \in \mathbb{N}$.
For $t \in [0, T)$,

$$b_n(t)b_m(t) = \begin{cases} b_m(t), & m = n. \\ 0, & m \neq n. \end{cases}$$

and

$$\int_0^T b_n(t)b_m(t)dt = \begin{cases} \frac{T}{N}, & m = n. \\ 0, & m \neq n. \end{cases}$$

Every function $f(t) \in L^2(0, T)$ can be written in terms of BPFs as

$$f(t) \simeq \sum_{m=1}^{N} f_m b_m(t) = f' B_N(t),$$

where $f = [f_1, f_2, ..., f_N]'$, $f_m = \dfrac{N}{T} \displaystyle\int_{\frac{(m-1)T}{N}}^{\frac{mT}{N}} f(t)b_m(t)dt$ and $B_N(t) = [b_1(t), b_2(t), ..., b_N(t)]'$.

The relationship between the Haar wavelets and BPfs is given by

$$H(t) = H_{N \times N} B_N(t). \tag{4}$$

According to [Kilicman and Al Zhour [2007]], the integration I^λ, with fractional order $\lambda \geq 0$ of $B_N(t)$ is approximated as

$$I^\lambda \left(B_N(t)\right) \simeq E^\lambda_{N \times N} B_N(t), \tag{5}$$

where $E^\lambda_{N \times N}$ is the Block pulse operational matrix of $I^\lambda \left(B_N(t)\right)$ given by

$$E^\lambda_{N \times N} = \left(\frac{T}{N}\right)^\lambda \frac{1}{\Gamma(\lambda + 2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \ldots & \xi_{N-1} \\ 0 & 1 & \xi_1 & \xi_2 & \ldots & \xi_{N-2} \\ 0 & 0 & 1 & \xi_1 & \ldots & \xi_{N-3} \\ 0 & 0 & 0 & 1 & \ldots & \xi_{N-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \xi_1 \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

with $\xi_r = (r+1)^{\lambda+1} - 2r^{\lambda+1} + (r-1)^{\lambda+1}$.
The integration I^λ with fractional order $\lambda \geq 0$ of $H(t)$ is approximated as,

$$I^\lambda\left(H(t)\right) \simeq P^\lambda_{N\times N} H(t), \tag{6}$$

where $P^\lambda_{N\times N}$ is referred to as the Haar wavelets operational matrix of N for fractional integration with order $\lambda \geq 0$.
Using (4) and (5), we arrive

$$I^\lambda\left(H(t)\right) = I^\lambda\left(H_{N\times N} B_N(t)\right) = H_{N\times N} I^\lambda\left(B_N(t)\right) \simeq H_{N\times N} E^\lambda_{N\times N} B_N(t). \tag{7}$$

Thus, combining (6) and (7), we attain

$$P^\lambda_{N\times N} H(t) \simeq I^\lambda\left(H(t)\right) \simeq H_{N\times N} E^\lambda_{N\times N} B_N(t) = H_{N\times N} E^\lambda_{N\times N} \left(H_{N\times N}\right)^{-1} H(t)$$

and so

$$P^\lambda_{N\times N} \simeq H_{N\times N} E^\lambda_{N\times N} \left(H_{N\times N}\right)^{-1}. \tag{8}$$

Especially, for $J = 2$, $T = 1$ and $\lambda = 0.5$, the Haar wavelets operational matrix for fractional integration becomes,

$$P^{0.5}_{8\times 8} = \begin{pmatrix} 0.7523 & -0.2203 & -0.1102 & -0.0580 & -0.0551 & -0.0290 & -0.0224 & -0.0189 \\ 0.2203 & 0.3116 & -0.1102 & 0.1623 & -0.0551 & -0.0290 & 0.0878 & 0.0391 \\ 0.0580 & 0.1623 & 0.2203 & -0.0350 & -0.0779 & 0.1148 & -0.0275 & -0.0045 \\ 0.1102 & -0.1102 & 0 & 0.2203 & 0 & 0 & -0.0779 & 0.1148 \\ 0.0189 & 0.0391 & 0.1148 & -0.0045 & 0.1558 & -0.0247 & -0.0026 & -0.0009 \\ 0.0224 & 0.0878 & -0.0779 & -0.0275 & 0 & 0.1558 & -0.0247 & -0.0026 \\ 0.0290 & -0.0290 & 0 & 0.1148 & 0 & 0 & 0.1558 & -0.0247 \\ 0.0551 & -0.0551 & 0 & -0.0779 & 0 & 0 & 0 & 0.1558 \end{pmatrix}$$

4 Illustrative Examples

The following fractional models provide an insight into the applicability and the effectiveness of the proposed numerical approach based on Haar wavelets.

Example 4.1 (Kreyszig [2007])**.** *Mixing Problem: A tank holds 1000 litres of water with 200 kilograms of salt dissolved in it. The tank receives 50 litres of brine every minute, each containing (1+cos t) kilograms of dissolved salt. The mixture runs out evenly since it is constantly being stirred. Find the tank salt content at any moment t by solving for $y(t)$.*
Solution:
First step. Modeling. The fractional order time rate of change $\frac{d^\lambda y(t)}{dt^\lambda}$ of the amount of salt $y(t)$ in the tank at any time t equals the inflow of salt minus the outflow of

the salt. The salt inflow rate is $50(1 + cos\ t)kg/min$*. Since the tank always contains 1000 litres of brine, the salt outflow rate is* $0.05y(t)$.

$$\text{Thus } \frac{d^{\lambda}y(t)}{dt^{\lambda}} = 50(1 + cos\ t) - 0.05y(t), \quad 0 < \lambda \leq 1, \quad 0 \leq t < T, \tag{9}$$

with the initial condition $y(0) = 200$*. When* $\lambda = 1$*, the exact solution of* (9) *is*

$$y(t) = 1000 + 2.494cos(t) + 49.88sin(t) - 802.5e^{-0.05t}.$$

Second step. Numerical solution.

$$Suppose \quad \frac{d^{\lambda}y(t)}{dt^{\lambda}} \approx CH(t). \tag{10}$$

$$Then, \quad y(t) = I^{\lambda}\left(y^{\lambda}(t)\right) \approx CP^{\lambda}H(t) + y(0). \tag{11}$$

Using (10), (11) in (9), we have

$$CH(t) + 0.05\left(CP^{\lambda}H(t) + 200\right) - 50(1 + cost) = 0. \tag{12}$$

We can attain the coefficient vector C *by solving the equation(12) at the collocation points. Using the coefficient vector* C *in (11), we get the numerical solutions of (9) for any time* t *in* $[0, T)$.

t	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 0.95$	$\lambda = 1$	Absolute error $\lambda = 1$
0.0625	245.9747	223.5968	211.6746	206.5021	205.6014	7.1053e-03
0.1875	262.5291	242.6841	227.3894	218.5203	216.7208	1.5601e-02
0.3125	270.0022	254.4603	239.8020	229.8287	227.6261	2.3793e-02
0.4375	274.5230	263.3066	250.4973	240.5658	238.2246	3.1552e-02
0.5625	277.1876	270.1939	259.8753	250.7210	248.4280	3.8758e-02
0.6875	278.4453	275.5373	268.1011	260.2539	258.1535	4.5299e-02
0.8125	278.5408	279.5608	275.2559	269.1170	267.3255	5.1073e-02
0.9375	277.6367	282.4078	281.3853	277.2637	275.8764	5.5989e-02

Table 1: The achieved absolute errors and numerical solutions of Example 4.1 using the proposed strategy for $J = 2$ and $T = 1$.

Example 4.2 (Kreyszig [2007]). *Mixing Problem: A tank holds 500 litres of water with 80 kilograms of salt dissolved in it. 20 kilograms of salt are dissolved in 20 litres of water, and this inflow occurs once every minute. A uniform mixture at a rate of* $20\ litres/min$ *exists. Calculate the tank's salt content* $y(t)$ *at time t.*
Solution:
First step. Modeling. The fractional order time rate of change $\frac{d^{\lambda}y(t)}{dt^{\lambda}}$ *of the amount*

of salt $y(t)$ in the tank at any time t equals the inflow of salt minus the outflow of the salt. The salt inflow rate is $20kg/min$. Since the tank always contains 500 litres of brine, the salt outflow rate is $0.04y(t)$.

$$Thus, \qquad \frac{d^{\lambda}y(t)}{dt^{\lambda}} = 20 - 0.04y(t), \quad 0 < \lambda \leq 1, \quad 0 \leq t < T, \tag{13}$$

with the initial condition $y(0) = 80$. *When* $\lambda = 1$, *the exact solution of* (13) *is*

$$y(t) = 500 - 420e^{-0.04t}.$$

Second step. Numerical solution.

$$Suppose \qquad \frac{d^{\lambda}y(t)}{dt^{\lambda}} \approx CH(t). \tag{14}$$

$$Then, \qquad y(t) = I^{\lambda}\left(y^{\lambda}(t)\right) \approx CP^{\lambda}H(t) + y(0). \tag{15}$$

Using (14), (15) in (13), we have

$$CH(t) + 0.04\left(CP^{\lambda}H(t) + 80\right) - 20 = 0. \tag{16}$$

We can attain the coefficient vector C by solving the equation(16) at the collocation points. Using the coefficient vector C in (15), we get the numerical solutions of (13) for any time t in $[0, T)$.

t	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 0.95$	$\lambda = 1$	Absolute error $\lambda = 1$
0.0625	88.6354	84.4211	82.1845	81.2159	81.0474	1.3070e-03
0.1875	91.8360	88.0451	85.1477	83.4755	83.1369	1.2962e-03
0.3125	93.4299	90.3737	87.5417	85.6353	85.2160	1.2854e-03
0.4375	94.5790	92.2397	89.6813	87.7390	87.2848	1.2747e-03
0.5625	95.4957	93.8406	91.6570	89.8011	89.3432	1.2641e-03
0.6875	96.2661	95.2620	93.5132	91.8294	91.3914	1.2535e-03
0.8125	96.9350	96.5516	95.2761	93.8289	93.4293	1.2430e-03
0.9375	97.5286	97.7393	96.9628	95.8028	95.4571	1.2326e-03

Table 2: The achieved absolute errors and numerical solutions of Example 4.2 using the proposed strategy for $J = 2$ and $T = 1$.

Example 4.3 (Kreyszig [2007]). *Newton's law of cooling: A room that is* $22°C$ *in temperature has a thermometer in it that reads* $5°C$. *Find the thermometer reading at any time t if the temperature is* $12°C$ *one minute later.*
Physical details:
According to the experiments, the fractional order time rate of change $\frac{dV^{\lambda}}{dt^{\lambda}}$ *of the*

temperature V of a body is directly related to the difference between V and the temperature V_A of the surrounding medium.

$$Thus \ \frac{d^\lambda V}{dt^\lambda} = k(V - V_A), \tag{17}$$

where $0 < \lambda \leq 1$, $0 \leq t < T$, k is the constant of proportionality, which is assumed to be -0.5306282 when $0 < \lambda \leq 1$, with the initial condition $V(0) = 5$. When $\lambda = 1$, the exact solution of (17) *is*

$$V(t) = 22 - 17e^{-0.5306282t}.$$

$$Suppose \quad \frac{d^\lambda V}{dt^\lambda} \approx CH(t). \tag{18}$$

$$Then, \quad V(t) = I^\lambda \left(\frac{d^\lambda V}{dt^\lambda} \right) \approx CP^\lambda H(t) + V(0). \tag{19}$$

Using (18), (19) in (17), we have

$$CH(t) - 0.5306282 \left(CP^\lambda H(t) + 5 - 22 \right) = 0. \tag{20}$$

We can attain the coefficient vector C by solving the equation(20) at the collocation points. Using the coefficient vector C in (19), we get the numerical solutions of (17) for any time t in $[0, T)$.

t	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 0.95$	$\lambda = 1$	Absolute error $\lambda = 1$
0.0625	8.7029	7.1024	6.1026	5.6305	5.5457	8.8512e-03
0.1875	9.7989	8.6288	7.5086	6.7620	6.6021	7.9085e-03
0.3125	10.2629	9.4660	8.5301	7.7714	7.5906	7.0503e-03
0.4375	10.5811	10.0819	9.3698	8.6915	8.5157	6.2697e-03
0.5625	10.8246	10.5743	10.0892	9.5369	9.3814	5.5603e-03
0.6875	11.0226	10.9861	10.7199	10.3170	10.1915	4.9161e-03
0.8125	11.1896	11.3405	11.2814	11.0390	10.9496	4.3317e-03
0.9375	11.3343	11.6517	11.7867	11.7087	11.6590	3.8020e-03

Table 3: The achieved absolute errors and numerical solutions of Example 4.3 using the proposed strategy for $J = 2$ and $T = 1$.

Example 4.4 (Kreyszig [2007]). *Newton's law of cooling: A metal bar that is $20°C$ in temperature is submerged in boiling water. Find the temperature of the bar at any time t if the temperature of the bar is $51.5°C$, after one minute of heating.*
Physical details:

According to the experiments, the fractional order time rate of change $\frac{dV^{\lambda}}{dt^{\lambda}}$ *of the temperature* V *of a body is directly related to the difference between* V *and the temperature* V_A *of the surrounding medium.*

$$Thus \ \frac{d^{\lambda}V}{dt^{\lambda}} = k(V - V_A), \tag{21}$$

where $0 < \lambda \leq 1, 0 \leq t < T$, k *is the constant of proportionality, which is assumed to be -0.500463 when* $0 < \lambda \leq 1$, *with the initial condition* $V(0) = 20$. *When* $\lambda = 1$, *the exact solution of* (21) *is*

$$V(t) = 100 - 80e^{-0.500463t}.$$

$$Suppose \quad \frac{d^{\lambda}V}{dt^{\lambda}} \approx CH(t). \tag{22}$$

$$Then, \quad V(t) = I^{\lambda}\left(\frac{d^{\lambda}V}{dt^{\lambda}}\right) \approx CP^{\lambda}H(t) + V(0). \tag{23}$$

Using (22), (23) in (21), we have

$$CH(t) - 0.500463\left(CP^{\lambda}H(t) + 20 - 100\right) = 0. \tag{24}$$

We can attain the coefficient vector C *by solving the equation(24) at the collocation points. Using the coefficient vector* C *in (23), we get the numerical solutions of (21) for any time t in* $[0, T)$.

t	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 0.95$	$\lambda = 1$	Absolute error $\lambda = 1$
0.0625	36.6409	29.3975	24.9118	22.8043	22.4264	3.7166e-02
0.1875	41.6282	36.2681	31.1983	27.8475	27.1321	3.3425e-02
0.3125	43.7581	40.0722	35.7956	32.3651	31.5523	3.0000e-02
0.4375	45.2215	42.8837	39.5932	36.5011	35.7043	2.6869e-02
0.5625	46.3438	45.1396	42.8612	40.3155	39.6046	2.4006e-02
0.6875	47.2575	47.0320	45.7381	43.8492	43.2682	2.1392e-02
0.8125	48.0296	48.6651	48.3085	47.1323	46.7095	1.9007e-02
0.9375	48.6989	50.1026	50.6297	50.1890	46.7095	1.6833e-02

Table 4: The achieved absolute errors and numerical solutions of Example 4.4 using the proposed strategy for $J = 2$ and $T = 1$.

5 Conclusions

In this article, the Haar wavelet-based numerical scheme was successfully employed to optimize the fractional models of mixing and Newton's law of cooling problems. The functions and the components of the fractional models are discretized by utilizing the operational matrix of fractional integration of Haar wavelets. Accordingly, the present numerical algorithm is found to be appropriate for fractional models that can be expressed as differential equations of integer or fractional orders subject to initial or boundary conditions. Error analysis demonstrates that the numerical solutions quickly approach the correct solution. The numerical results demonstrate that this current numerical approach is efficient and straightforward for fractional models.

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