Ratio Mathematica Volume 43, 2022

Generalized Continuous Integral Wavelet Transform and Its Properties

Manoj Kumar*

Abstract

In this paper, the generalized admissibility condition has been obtained and corresponding generalized continuous integral wavelet transforms have been defined. Two new theorems on generalized continuous integral wavelet transforms are established.
Keywords: $L^2(\mathbb{R})-$space, basic wavelet, generalized admissibility condition, generalized continuous wavelet transforms.
2020 AMS subject classifications: 40A30, 42C15. [1]

*Applied Sciences and Humanities Department, Institute of Engineering and Technology, Lucknow-226021, India; manojkumar@ietlucknow.ac.in.
[1] Received on June 3, 2022. Accepted on December 16, 2022. Published on December 30, 2022. doi: 10.23755/rm.v43i0.793. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Wavelet Analysis has wide applications in Science, Engineering and Technology. Its applications have very important role in data analysis, signals and image processions. Wavelet transforms and wavelet expansions are two mean features of this analysis.

A function $\varphi \in L^2(\mathbb{R})$ is called a basic wavelet if it make certain the admissibility condition,

$$C_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty.$$

Let $\varphi_1, \varphi_2 \in L^2(\mathbb{R})$. This admissibility condition is extended as

$$C_{\varphi_1,\varphi_2} = \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1(\xi)}\hat{\varphi}_2(\xi)|}{|\xi|} d\xi < \infty.$$

We notice that if $\varphi_1 = \varphi_2 = \varphi$ then C_{φ_1,φ_2} reduces to C_φ.

In this paper, the admissibility condition C_{φ_1,φ_2} has generalized and new more general admissibility condition $C_{{}^c\varphi_1, {}^d\varphi_2}$ has been obtained as following

$$C_{{}^c\varphi_1, {}^d\varphi_2} = \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1\left(\frac{\xi}{cd}\right)}\hat{\varphi}_2\left(\frac{\xi}{d}\right)|}{\sqrt{cd}|\xi|} d\xi < \infty,$$

where $1 \leq c < 2,\ 1 \leq d < 2$.

It's worth noting that

(1) if $c = d = 1$ then $C_{{}^c\varphi_1, {}^d\varphi_2}$ reduces to C_{φ_1,φ_2};

(2) if $c = d = 1,\ \varphi_1 = \varphi_2 = \varphi$ then $C_{{}^c\varphi_1, {}^d\varphi_2}$ reduces to C_φ.

To conduct a more in-depth investigation in this area, the generalized integral wavelet transform is defined and inverse integral wavelet transform is obtained with the help of generalized admissibility condition. The aim of this paper is also to study the property of generalized continuous integral wavelet transform and a function K of $L^p(\mathbb{R}),\ 1 < p < \infty$.

2 Definitions and Preliminaries

Let $\varphi_1, \varphi_2 \in L^2(\mathbb{R})$ and the Fourier transforms of φ_1 and φ_2 be

$$\hat{\varphi_1}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x}\varphi_1(x)dx \text{ and } \hat{\varphi_2}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x}\varphi_2(x)dx$$

respectively. Suppose that φ_1 and φ_2 satisfy the condition

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}_1(\xi)\hat{\varphi}_2(\xi)|}{|\xi|} d\xi < \infty, \qquad \text{(Daubechies [1992]).}$$

Define

$$\varphi_i^{a,b}(x) = |a|^{-\frac{1}{2}} \varphi_i \left(\frac{x-b}{a} \right), \ i = 1, 2.$$

$\varphi_i^{a,b}, \ i = 1, 2$, are obtained by first dilating by $a \neq 0$ and then translating by b. The normalization has been taken in such a way that $\|\varphi_i^{(a,b)}\|_{L^2(\mathbb{R})} = \|\varphi_i\|_{L^2(\mathbb{R})}$. We assume that $\|\varphi_i\|_2 = 1$.

The continuous integral wavelet transform of $f \in L^2(\mathbb{R})$ by φ_1 is defined by

$$\begin{aligned} (W_{\varphi_1} f)(a,b) &= |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x) \overline{\varphi_1 \left(\frac{x-b}{a} \right)} dx \\ &= \int_{-\infty}^{\infty} f(x) \overline{\varphi_1^{a,b}(x)} dx. \\ &= \langle f, \varphi_1^{a,b} \rangle. \end{aligned}$$

The admissibility condition is generalized as; if

$$C_{\varphi_1,\varphi_2} = \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1(\xi)} \hat{\varphi}_2(\xi)|}{|\xi|} d\xi < \infty, \tag{1}$$

then for any $f \in L^2(\mathbb{R})$, we have the L^2- inversion formula

$$f(x) = C_{\varphi_1,\varphi_2}^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_{\varphi_1} f)(a,b) \varphi_2^{(a,b)}(x) \frac{dadb}{a^2}. \tag{2}$$

Define the functions ${}^c\varphi_1, \ {}^d\varphi_2 \in L^2(\mathbb{R})$ as follows

$${}^c\varphi_1(x) = {}_1^c\varphi_{b,a}(x) = \left(\frac{|a|}{c} \right)^{-\frac{1}{2}} \varphi_1 \left(\frac{cx-b}{a} \right) \tag{3}$$

and

$${}^d\varphi_2(x) = {}_2^d\varphi_{b,a}(x) = \left(\frac{|a|}{d} \right)^{-\frac{1}{2}} \varphi_2 \left(\frac{dx-b}{a} \right), \text{ where } 1 \leq c, d < 2. \tag{4}$$

We observe that

$$\|{}^c\varphi_1\|_{L^2(\mathbb{R})} = \|\varphi_1\|_{L^2(\mathbb{R})} = 1$$

and

$$\|{}^{d}\varphi_2\|_{L^2(\mathbb{R})} = \|\varphi_2\|_{L^2(\mathbb{R})} = 1.$$

The Fourier transform of ${}^{c}\varphi_1$ is given by

$$\begin{aligned}
{}^{c}\hat{\varphi}_1(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x}\, {}^{c}_{1}\varphi_{b,a}(x)dx \\
&= \int_{-\infty}^{\infty} e^{-i\xi x} \left(\frac{|a|}{c}\right)^{-\frac{1}{2}} \varphi_1\left(\frac{cx-b}{a}\right) dx, \qquad \frac{cx-b}{a} = u \\
&= \int_{-\infty}^{\infty} e^{-i\xi\left(\frac{b+au}{c}\right)} \left(\frac{|a|}{c}\right)^{-\frac{1}{2}} \varphi_1(u)\frac{a}{c}du \\
&= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}} e^{-i(\frac{b}{c})\xi} \int_{-\infty}^{\infty} e^{-i(\frac{a\xi}{c})u}\varphi_1(u)du \\
&= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}} e^{-i(\frac{b}{c})\xi} \hat{\varphi}_1\left(\frac{a\xi}{c}\right). \qquad (5)
\end{aligned}$$

In the same way, the Fourier transform of ${}^{d}\varphi_2$ is given by

$${}^{d}\hat{\varphi}_2(\xi) = \frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}} e^{-i(\frac{b}{d})\xi} \hat{\varphi}_2\left(\frac{a\xi}{d}\right). \qquad (6)$$

In this paper, the admissibility condition C_{φ_1,φ_2} has generalized and new more possible general admissibility condition $C_{{}^{c}\varphi_1,{}^{d}\varphi_2}$ has been obtained as following

$$C_{{}^{c}\varphi_1,{}^{d}\varphi_2} = \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1\left(\frac{\xi}{cd}\right)}\hat{\varphi}_2\left(\frac{\xi}{d}\right)|}{\sqrt{cd}|\xi|} d\xi < \infty,$$

where $1 \leq c < 2,\ 1 \leq d < 2.$

It is important to note that

(1) if $c = d = 1$ then $C_{{}^{c}\varphi_1,{}^{d}\varphi_2}$ reduces to C_{φ_1,φ_2},

(2) if $c = d = 1,\ \varphi_1 = \varphi_2 = \varphi$ then $C_{{}^{c}\varphi_1,{}^{d}\varphi_2}$ reduces to

$$C_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty \qquad (7)$$

.

which is the admissibility condition for a function φ to be a wavelet. If $\varphi \in L^1(\mathbb{R})$, then $\hat{\varphi}$ is continuous and the condition (7) is satisfied only if

$$\hat{\varphi}(0) = 0 \text{ or } \int_{-\infty}^{\infty} \varphi(x)dx = 0.$$

There are following two advantages of selecting two different functions φ_1 and φ_2:

1. φ_1 & φ_2 may have different aspects;

2. $C_{\varphi_1,\varphi_2} = \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1(\xi)}\hat{\varphi}_2(\xi)|}{|\xi|} d\xi < \infty$ even φ_1 & φ_2 both may not satisfy admissibility conditions.

Similarly

(a) ${}^c\varphi_1$ & ${}^d\varphi_2$ may have different aspects;

(b) $C_{{}^c\varphi_1,{}^d\varphi_2} = \int_{-\infty}^{\infty} \frac{|\overline{\hat{\varphi}_1\left(\frac{\xi}{cd}\right)}\hat{\varphi}_2\left(\frac{\xi}{d}\right)|}{\sqrt{cd}|\xi|} d\xi < \infty$ even ${}^c\varphi_1$ & ${}^d\varphi_2$ both may not satisfy admissibility conditions.

The continuous integral wavelet transform related to the basic wavelet ${}^c\varphi_1$ is given by

$$\begin{aligned}
\left(W_{{}^c\varphi_1} f\right)(b,a) &= \left(\frac{|a|}{c}\right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x)\overline{\varphi_1\left(\frac{cx-b}{a}\right)} dx \\
&= \langle f,\, {}^c\varphi_1\rangle \\
&= \frac{1}{2\pi}\langle \hat{f},\, {}^c\hat{\varphi}_1\rangle \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, {}_1^c\hat{\varphi}_{b,a}(\xi) d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\overline{{}_1^c\hat{\varphi}_{b,a}(\xi)} d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}} e^{i\left(\frac{b}{c}\right)\xi}\overline{\hat{\varphi}_1\left(\frac{a\xi}{c}\right)} d\xi, \quad \text{(by (5))} \\
&= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{b}{c}\right)\xi}\hat{f}(\xi)\overline{\hat{\varphi}_1\left(\frac{a\xi}{c}\right)} d\xi, \\
\text{i.e. } \left(W_{{}^c\varphi_1} f\right)(b,a) &= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}}\mathfrak{F}^{-1}[\hat{f}(\xi)\overline{\hat{\varphi}_1\left(\frac{a\xi}{c}\right)}](\frac{b}{c}).
\end{aligned}$$

Thus the Fourier transform of $W_{{}^c\varphi_1}$ is given by

$$\mathfrak{F}[\left(W_{{}^c\varphi_1 f}\right)(b,a)] = \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}}\hat{f}(\xi)\overline{\hat{\varphi}_1\left(\frac{a\xi}{c}\right)}. \tag{8}$$

Similarly, the integral wavelet transform related to the basic wavelet ${}^d\varphi_2$ is given by

$$\begin{aligned}\left(W_{{}^d\varphi_2}g\right)(b,a) &= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\frac{b}{d})\xi}\hat{g}(\xi)\overline{\hat{\varphi}_2\left(\frac{a\xi}{d}\right)}d\xi \\ &= \frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}}\mathfrak{F}^{-1}[\hat{g}(\xi)\overline{\hat{\varphi}_2\left(\frac{a\xi}{d}\right)}](\frac{b}{d}).\end{aligned}$$

Therefore

$$\mathfrak{F}[\left(W_{{}^d\varphi_2}g\right)(b,a)] = \frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}}\hat{g}(\xi)\overline{\hat{\varphi}_2\left(\frac{a\xi}{d}\right)}. \tag{9}$$

Throughout this paper, we choose ${}^c\varphi_1$ and ${}^d\varphi_2$ such that

(i) ${}^c\varphi_1,\ {}^d\varphi_2 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$;

(ii) $x\,{}^c\varphi_1, x\,{}^d\varphi_2 \in L^1(\mathbb{R})$;

(iii) ${}^d\varphi_2 \in L_2^s(\mathbb{R}),\ s > 0$,

where $L_2^s(\mathbb{R})$ is the Liouville space with the norm

$$\|\varphi\|_{L_2^s(\mathbb{R})} = \|(1+|\xi|)^{\frac{s}{2}}\hat{\varphi}(\xi)\|_{L^2(\mathbb{R})}, \qquad \text{(Nikol'skii [1995]).}$$

In this paper, two new theorems has been established in the following forms:

Theorem 2.1. *Let ${}^c\varphi_1$ and ${}^d\varphi_2$ be the basic wavelets which define the integral wavelet transforms $W_{{}^c\varphi_1}$ and $W_{{}^d\varphi_2}$ respectively. Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(W_{{}^c\varphi_1}f\right)(b,a)\overline{\left(W_{{}^d\varphi_2}g\right)(b,a)}\frac{da\,db}{a^2} = C_{{}^c\varphi_1,{}^d\varphi_2}\langle f,g\rangle$$

for all $f, g \in L^2(\mathbb{R})$, where

$$C_{{}^c\varphi_1,{}^d\varphi_2} = \int_{-\infty}^{\infty}\frac{|\overline{\hat{\varphi}_1\left(\frac{\xi}{cd}\right)}\hat{\varphi}_2\left(\frac{\xi}{d}\right)|}{\sqrt{cd}|\xi|}d\xi.$$

Furthermore, for any $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ at which f is continuous,

$$f(x) = \frac{1}{C_{{}^c\varphi_1,{}^d\varphi_2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(W_{{}^c\varphi_1}f\right)(b,a)\,{}_2^d\varphi_{(b,a)}\frac{da\,db}{a^2}.$$

Proof.

$$\begin{aligned}
& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(W_{{}^c\varphi_1}f\right)(b,a)\overline{\left(W_{{}^d\varphi_2}g\right)(b,a)}\frac{dadb}{a^2} \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dadb}{a^2}\left(W_{{}^c\varphi_1}f\right)(b,a)\frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i(\frac{b}{d})\xi}\overline{\hat{g}(\xi)}\hat{\varphi}_2\left(\frac{a\xi}{d}\right)d\xi \\
= & \int_{-\infty}^{\infty}\frac{da}{a^2}\int_{-\infty}^{\infty}\frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}}\frac{1}{2\pi}\overline{\hat{g}(\xi)}\hat{\varphi}_2\left(\frac{a\xi}{d}\right)d\xi\int_{-\infty}^{\infty}\left(W_{{}^c\varphi_1}f\right)(b,a)e^{-i(\frac{b}{d})\xi}db \\
= & \int_{-\infty}^{\infty}\frac{da}{a^2}\int_{-\infty}^{\infty}\frac{a|a|^{-\frac{1}{2}}}{\sqrt{d}}\frac{1}{2\pi}\overline{\hat{g}(\xi)}\hat{\varphi}_2\left(\frac{a\xi}{d}\right)\frac{a|a|^{-\frac{1}{2}}}{\sqrt{c}}\hat{f}(\xi)\overline{\hat{\varphi}_1\left(\frac{a\xi}{cd}\right)}d\xi \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi\int_{-\infty}^{\infty}\frac{\overline{\hat{\varphi}_1\left(\frac{a\xi}{cd}\right)}\hat{\varphi}_2\left(\frac{a\xi}{d}\right)}{\sqrt{cd}|a|}da,\ a\xi=u \\
= & \frac{1}{2\pi}\langle\hat{f},\hat{g}\rangle\int_{-\infty}^{\infty}\frac{\overline{\hat{\varphi}_1\left(\frac{u}{cd}\right)}\hat{\varphi}_2\left(\frac{u}{d}\right)}{\sqrt{cd}|u|}du \\
= & \langle f,g\rangle\int_{-\infty}^{\infty}\frac{|\overline{\hat{\varphi}_1\left(\frac{\xi}{cd}\right)}\hat{\varphi}_2\left(\frac{\xi}{d}\right)|}{\sqrt{cd}|\xi|}d\xi \\
= & C_{{}^c\varphi_1,{}^d\varphi_2}\langle f,g\rangle,
\end{aligned}$$

where

$$\begin{aligned}
C_{{}^c\varphi_1,{}^d\varphi_2} &= \int_{-\infty}^{\infty}\frac{|\overline{\hat{\varphi}_1\left(\frac{\xi}{cd}\right)}\hat{\varphi}_2\left(\frac{\xi}{d}\right)|}{\sqrt{cd}|\xi|}d\xi,\ \ \frac{\xi}{d}=v \\
&= \int_{-\infty}^{\infty}\frac{|\overline{\hat{\varphi}_1\left(\frac{v}{c}\right)}\hat{\varphi}_2(v)|}{\sqrt{cd}d|v|}d\,dv \\
&= \int_{-\infty}^{\infty}\frac{|\overline{\hat{\varphi}_1\left(\frac{\xi}{c}\right)}\hat{\varphi}_2(\xi)|}{\sqrt{cd}|\xi|}d\xi. \qquad (10)
\end{aligned}$$

This is the generalized admissibility condition for the functions ${}^c\varphi_1$ and ${}^d\varphi_2$ to be the basic wavelets.

For $c=d=1$ and ${}^c\varphi_1 = {}^d\varphi_2 = \varphi$, the condition (10)reduces to the condition

$$C_\varphi = \int_{-\infty}^{\infty}\frac{|\hat{\varphi}(\xi)|^2}{|\xi|}d\xi$$

which is the admissibility condition for the single function φ to be a wavelet, (Chui [1992]).

Furthermore,

$$\begin{aligned}
C_{^c\varphi_1, ^d\varphi_2}\langle f, g\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(W_{^c\varphi_1} f\right)(b,a)\overline{\left(W_{^d\varphi_2} g\right)(b,a)}\frac{dadb}{a^2} \\
\Rightarrow \langle f, g\rangle &= \frac{1}{C_{^c\varphi_1, ^d\varphi_2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(W_{^c\varphi_1} f\right)(b,a)\overline{\left(\left(\frac{|a|}{d}\right)^{-\frac{1}{2}}\int_{-\infty}^{\infty} g(x)\overline{\varphi\left(\frac{dx-b}{a}\right)}dx\right)}\frac{dadb}{a^2} \\
&= \frac{1}{C_{^c\varphi_1, ^d\varphi_2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(W_{^c\varphi_1} f\right)(b,a)\left(\int_{-\infty}^{\infty}\overline{g(x)}\, {}^d_2\varphi_{b,a}(x)dx\right)\frac{dadb}{a^2} \\
&= \frac{1}{C_{^c\varphi_1, ^d\varphi_2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(W_{^c\varphi_1} f\right)(b,a){}^d_2\varphi_{b,a}(x)\overline{g(x)}\frac{dadb}{a^2}dx \\
&= \frac{1}{C_{^c\varphi_1, ^d\varphi_2}}\langle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(W_{^c\varphi_1} f\right)(b,a){}^d_2\varphi_{b,a}(x)\frac{dadb}{a^2}, g\rangle,\ \forall\ f, g \in L^2(\mathbb{R}).
\end{aligned}$$

Therefore

$$f(x) = \frac{1}{C_{^c\varphi_1, ^d\varphi_2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(W_{^c\varphi_1} f\right)(b,a){}^d_2\varphi_{b,a}(x)\frac{dadb}{a^2}.$$

The Theorem 2.1 has thus been proven.□

Theorem 2.2. *If K is a function in $L^p(\mathbb{R}),\ 1 < p < \infty$ such that*

$$\hat{K}(\xi) = \frac{1}{C_{^c\varphi_1, ^d\varphi_2}}\int_{|a|\geq|\xi|} {}^d\hat{\varphi}_2(a)\overline{^c\hat{\varphi}_1(a)}\frac{da}{|a|}. \tag{11}$$

Then it satisfies the following properties:

(i) K *is continuous and bounded.*

(ii) $|K(x)| \leq \frac{C}{1+\alpha x^2}$, *where* $0 < C < \infty,\ 1 \leq \alpha < 2$.

(iii) $\|K\|_q = O(1), \frac{1}{p} + \frac{1}{q} = 1$.

Proof.

(i) Define the function K which Fourier transform is given by

$$\hat{K}(\xi) = \frac{1}{C_{^c\varphi_1, ^d\varphi_2}}\int_{|a|\geq|\xi|} {}^d\hat{\varphi}_2(a)\overline{^c\hat{\varphi}_1(a)}\frac{da}{|a|}.$$

Therefore

$$\begin{aligned}
|\hat{K}(\xi)| &\leq \frac{1}{|C_{^c\varphi_1, ^d\varphi_2}|}\int_{|a|\geq|\xi|} |{}^d\hat{\varphi}_2(a)||{}^c\hat{\varphi}_1(a)|\frac{da}{|a|} \\
&< \infty
\end{aligned}$$

i.e.,

$$|\hat{K}(\xi)| < C.$$

Also by Cauchy-Schwartz's inequality, we have

$$\begin{aligned}
|\hat{K}(\xi)| &\leq \frac{1}{C_{{}^c\varphi_1,{}^d\varphi_2}} \left(\int_{-\infty}^{\infty} |{}^d\hat{\varphi}_2(a)|^2 da\right)^{\frac{1}{2}} \left(\int_{|a|\geq|\xi|} \frac{|{}^c\hat{\varphi}_1(a)|^2}{|a|^2} da\right)^{\frac{1}{2}} \\
&= \frac{1}{C_{{}^c\varphi_1,{}^d\varphi_2}} \left(\int_{|a|\geq|\xi|} \left(\frac{|{}^c\hat{\varphi}_1(a)|}{|a|^{1+s}}\right)^2 da\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |a|^{2s}|{}^d\hat{\varphi}_2(a)|^2 da\right)^{\frac{1}{2}} \\
&\leq \frac{1}{C_{{}^c\varphi_1,{}^d\varphi_2}} \frac{1}{|\xi|^{1+s}} \left(\int_{|a|\geq|\xi|} |{}^c\hat{\varphi}_1(a)|^2 da\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} (1+|a|^2)^s|{}^d\hat{\varphi}_2(a)|^2 da\right)^{\frac{1}{2}} \\
&\leq C|\xi|^{-1-s}\|{}^c\hat{\varphi}_1\|_{L^2(\mathbb{R})}\|{}^d\hat{\varphi}_2\|_{L_2^s(\mathbb{R})} \\
&\leq C(1+|\xi|)^{-1-s}.
\end{aligned}$$

Thus the function K is continuous and bounded.

(ii) We have

$$\begin{aligned}
x^2|K(x)| &\leq \int_{-\infty}^{\infty} |\hat{K}''(\xi)|d\xi \\
\Rightarrow\ \alpha x^2|K(x)| &\leq \int_{-\infty}^{\infty} |\alpha\hat{K}''(\xi)|d\xi,\ 1<\alpha<2. \qquad (12)
\end{aligned}$$

Since the function K is bounded, therefore, to prove our result (ii) , it is sufficient to prove that the right hand side of above inequality (12) is bounded. The Fourier transform of ${}^c\varphi_1$ is given by

$${}^c\hat{\varphi}_1(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x}\ {}^c\varphi_1(x)dx.$$

Differentiating both sides w.r.t. ξ and applying Leibnitz's rule for differentiation under the sine of integral, we have

$$\begin{aligned}
\frac{d}{d\xi}\ {}^c\hat{\varphi}_1(\xi) = {}^c\hat{\varphi}_1'(\xi) &= \int_{-\infty}^{\infty} \frac{\partial}{\partial\xi} e^{-i\xi x}\ {}^c\varphi_1(x)dx \\
&= \int_{-\infty}^{\infty} e^{-i\xi x}(-ix)\ {}^c\varphi_1(x)dx \\
&= -i\int_{-\infty}^{\infty} e^{-i\xi x}\left(x\ {}^c\varphi_1(x)\right)dx.
\end{aligned}$$

Since $x\ {}^c\varphi_1 \in L^1(\mathbb{R})$ therefore ${}^c\hat{\varphi}_1$ is differentiable. Similarly ${}^d\hat{\varphi}_2$ is also differentiable.

Since

$$\hat{K}(\xi) = \frac{1}{C_{{}^c\varphi_1,{}^d\varphi_2}} \int_{|a|\geq|\xi|} {}^d\hat{\varphi}_2(a)\overline{{}^c\hat{\varphi}_1(a)}\frac{da}{|a|}.$$

Applying the above process again, we have

$$\hat{K}'(\xi) \quad = \quad \frac{1}{C_{{}^c\varphi_1,{}^d\varphi_2}}\left(\frac{{}^d\hat{\varphi}_2(-\xi)\overline{{}^c\hat{\varphi}_1(-\xi)}}{(-\xi)} - \frac{{}^d\hat{\varphi}_2(\xi)\overline{{}^c\hat{\varphi}_1(\xi)}}{\xi}\right).$$

Again differentiating w. r. t. ξ, we have

$$\begin{aligned}
\hat{K}''(\xi) \quad &= \quad \frac{1}{C_{{}^c\varphi_1,{}^d\varphi_2}}[\frac{1}{(-\xi)^2}\{(-\xi)\left(-\ {}^d\hat{\varphi}_2(-\xi)\overline{{}^c\hat{\varphi}_1'(-\xi)} -\ {}^d\hat{\varphi}_2'(-\xi)\overline{{}^c\hat{\varphi}_1(-\xi)}\right)\\
&\quad + {}^d\hat{\varphi}_2(-\xi)\overline{{}^c\hat{\varphi}_1(-\xi)}\} - \frac{1}{\xi^2}\{\xi\left({}^d\hat{\varphi}_2'(\xi)\overline{{}^c\hat{\varphi}_1(\xi)} +\ {}^d\hat{\varphi}_2(\xi)\overline{{}^c\hat{\varphi}_1'(\xi)}\right)\\
&\quad -\ {}^d\hat{\varphi}_2(\xi)\overline{{}^c\hat{\varphi}_1(\xi)}\}]
\end{aligned}$$

$$\begin{aligned}
&= \quad \frac{1}{C_{{}^c\varphi_1,{}^d\varphi_2}}[\left(\frac{\overline{{}^c\hat{\varphi}_1(-\xi)}\ {}^d\hat{\varphi}_2'(-\xi) +\ {}^d\hat{\varphi}_2(-\xi)\overline{{}^c\hat{\varphi}_1'(-\xi)}}{\xi}\right) + \frac{\overline{{}^c\hat{\varphi}_1(-\xi)}\ {}^d\hat{\varphi}_2(-\xi)}{\xi^2}\\
&\quad - \left(\frac{\overline{{}^c\hat{\varphi}_1(\xi)}\ {}^d\hat{\varphi}_2'(\xi) +\ {}^d\hat{\varphi}_2(\xi)\overline{{}^c\hat{\varphi}_1'(\xi)}}{\xi}\right) + \frac{\overline{{}^c\hat{\varphi}_1(\xi)}\ {}^d\hat{\varphi}_2(\xi)}{\xi^2}]\\
&= \quad \frac{1}{C_{{}^c\varphi_1,{}^d\varphi_2}}[\frac{1}{\xi^2}\left({}^d\hat{\varphi}_2(\xi)\ \overline{{}^c\hat{\varphi}_1(\xi)} +\ {}^d\hat{\varphi}_2(-\xi)\ \overline{{}^c\hat{\varphi}_1(-\xi)}\right)\\
&\quad -\frac{1}{\xi}\{\left({}^d\hat{\varphi}_2'(\xi)\ \overline{{}^c\hat{\varphi}_1(\xi)} +\ {}^d\hat{\varphi}_2(\xi)\ \overline{{}^c\hat{\varphi}_1'(\xi)}\right)\\
&\quad - \left({}^d\hat{\varphi}_2'(-\xi)\ \overline{{}^c\hat{\varphi}_1(-\xi)} +\ {}^d\hat{\varphi}_2(-\xi)\ \overline{{}^c\hat{\varphi}_1'(-\xi)}\right)\}].
\end{aligned}$$

Since ${}^c\hat{\varphi}_1(0) = {}^d\hat{\varphi}_2(0) = 0$ and ${}^c\hat{\varphi}_1'$, ${}^d\hat{\varphi}_2'$ are continuous and bounded, using Taylor's formula at the origin, we have

$$|{}^c\hat{\varphi}_1(\xi)| \leq C|\xi|$$

and

$$|{}^d\hat{\varphi}_2(\xi)| \leq C|\xi|.$$

Using the above inequalities and the boundedness of the functions ${}^c\hat{\varphi}_1'$, ${}^d\hat{\varphi}_2'$, we have

$$\int_{|\xi|<1} |\alpha\hat{K}''(\xi)|d\xi \leq C. \tag{13}$$

Again, since ${}^{c}\hat{\varphi}_1$ and ${}^{d}\hat{\varphi}_2$ are in the space $L^2(\mathbb{R})$ therefore using the Cauchy-Schwartz inequality, we have

$$\int_{|\xi|>1} |\alpha \hat{K}''(\xi)| d\xi \leq C. \tag{14}$$

By using the inequalities (12), (13) and (14), the proof of (ii) of the Theorem 2.2 has been completed.

(iii)

$$\begin{aligned}
\|K\|_q^q = \int_{-\infty}^{\infty} |K(x)|^q dx \ &\leq \ \int_{-\infty}^{\infty} \frac{C^q}{(1+\alpha x^2)^q} dx, \ 1<\alpha<2 \\
&\leq \ 2C^q \int_0^{\infty} \frac{dx}{(1+\alpha x^2)^q}, \ 1<q<\infty
\end{aligned}$$

$$\begin{aligned}
&\leq \ 2C^q \int_0^{\infty} \frac{dx}{(1+x^2)^q} \\
&= \ 2C^q \left(\int_0^1 \frac{dx}{(1+x^2)^q} + \int_1^{\infty} \frac{dx}{(1+x^2)^q} \right) \\
&\leq \ 2C^q \left(\int_0^1 \frac{dx}{(1+x^2)^q} + \int_1^{\infty} \frac{dx}{x^{2q}} \right).
\end{aligned}$$

Since $\int_0^1 \frac{dx}{(1+x^2)^q}$ and $\int_1^{\infty} \frac{dx}{x^{2q}}$ are convergent at 0 and ∞ respectively. Therefore

$$\|K\|_q^q = O(1).$$

Consequently

$$\|K\|_q = O(1).$$

In this way, the proof of the Theorem 2.2 has been completed. □

3 Conclusions

The generalized integral wavelet transform is defined and inverse integral wavelet transform is obtained with the help of generalized admissibility condition in the form of Theorem 2.1. Also, the property of generalized continuous integral wavelet transform and a function K of $L^p(\mathbb{R}), \ 1<p<\infty$, have been studied in Theorem 2.2.

Acknowledgements

This paper and the research behind it would not have been possible without the exceptional support of my supervisor, Prof. Shyam Lal. I am also grateful for the insightful comments offered by the anonymous peer reviewers at this paper.

References

C. K. Chui. *An introduction to wavelets (Wavelet analysis and its applications)*. Academic Press, USA, 1992.

I. Daubechies. *Ten Lectures on Wavelets*. SIAM, Philadelphia, USA, 1992.

S. M. Nikol'skii. *Approximation of Functions of Several Variables and Imbedding Theorems*. Springer-Verlag, Switzerland, 1995.