Ratio Mathematica Volume 50, 2023

Isolate g-eccentric domination in fuzzy graph

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Abstract

In a fuzzy graph $G(\rho, \mu)$, a dominating set $D \subseteq P(G)$ is said to be g-eccentric if at least one g-eccentric vertex a of every vertex b in $P - D$ exists in D. If the induced fuzzy sub graph $< D >$ has at least one isolated vertex, then a g-eccentric dominating set D of G is said to be an isolate g-eccentric dominating set. The isolate g-eccentric domination number is defined as the smallest cardinality over all isolate g-eccentric dominating sets of G. This article introduces the isolate g-eccentric point set, isolate g-eccentric dominating set, and their numbers in fuzzy graphs. In some standard fuzzy graphs, bounds are found for an isolate g-eccentric domination number.

Keywords: Fuzzy graph(FG); g-Distance; Isolate g-eccentric number; Isolate domination number; g-eccentric domination number; Isolate g-eccentric domination number.
2020 AMS subject classifications: 05C72. [1]

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[1]Received on May 19, 2023. Accepted on December 16, 2023. Published on December 31, 2023. doi: 10.23755/rm.v50i0.1548. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Rosenfeld [1975] created the idea of fuzzy graphs(FG). The g-eccentric node and related concepts were introduced by Linda and Sunitha [2012]. Janakiraman et al. [2010] initiated the eccentric domination in graph. Arriola [2015] introduced the concept of isolate domination in graph. Isolate domination in graphs discussed by Hamid and Balamurugan [2016]. The isolate eccentric domination in graphs introduced by Bhanumathi and Niroja [2018]. Ismayil and S.Muthupandiyan [2020] initiated the g-eccentric domination in FG. Ismayil and Muthupandiyan [2020] initiated the complmentary nil g-eccentric domination in FG. Muthupandiyan and Ismayil [2021] initiated the idea of perfect g-eccentric domination in FG.

The isolate g-eccentric point set, isolate g-eccentric dominating set and associated its numbers in FG are introduced in this article. Additionally, the bounds for various well-known FG's isolate g-eccentric domination number are determined. There are a few theorems that are presented and demonstrated based on isolate g-eccentric domination in FG. Graph and FG theoretic terminologies can refer Harary [1992] and Rosenfeld [1975] respectively. Here after, the notation G means connected simple FG.

A FG $G(\rho, \mu)$ on $G^*(P, Q)$ is characterized with two functions ρ on P and μ on $Q \subseteq P \times P$, where $\rho : P \to [0, 1]$ (P is a vertex set on G^*, ρ is a vertex set on G) and $\mu : Q \to [0, 1]$ (Q is an edge set on G^* , μ is an edge set on G) such that $\mu(a, b) \leq \rho(a) \wedge \rho(b), \forall a, b \in P$. We anticipate that μ will be reflexive and symmetric functions, and ρ will be a non-empty finite fuzzy set. We indicate the crisp graph $G^*(\rho^*, \mu^*)$ of the FG $G(\rho, \mu)$ where $\rho^* = \{a \in P : \rho(a) > 0\}$ and $\mu^* = \{(a, b) \in Q : \mu(a, b) > 0\}$. The order and size of a FG $G(\rho, \mu)$ are defined by $p = \sum_{a \in P} \rho(a)$ and $q = \sum_{(ab) \in Q} \mu(a, b)$ respectively.

A path P of length n is a sequence of different vertices $a_0, a_1, ..., a_n$ such that $\mu(a_{i-1}, a_i) > 0$, $i = 1, 2, .., n$ and strength of the path P is $s(P) = min\{\mu(a_{i-1}, a_i), i = 1, 2 \cdots n\}$. An edge (a, b) is said to be strong (or strong arc) if its weight is greater than or equal to maximum strength of all paths between a and b. Symbolically $\mu(a, b) \geq CONN_{G-(a,b)}(a, b)$.

The strong neighbourhood of a node a is defined as $N_s(a) = \{b \in P : (a, b)$ is a strong arc $\}$. The minimum strong degree of a FG $G(\rho, \mu)$ is $\delta_s(G) = min\{d_s(b) : b \in \rho^*\}$ and maximum strong degree of G is $\Delta_s(G) = max\{d_s(b) : b \in \rho^*\}$.

If there is no shorter strong path from a to b, the strong path π from a to b is referred to as a geodesic in a FG. The length of a geodesic from a to b is the geodesic distance (g-distance) from a to b and is indicated by $d_g(a, b)$. The expression $e_g(a) = max\{d_g(a, b), b \in P\}$ describes the geodesic eccentricity (g-eccentricity) of a node $a \in P$ in a connected FG. The g-radius and g-diameter of the vertices of G are both defined in terms of g-eccentricity, with the former being denoted by $r_g(G) = min\{e_g(a), a \in P\}$ and the latter by $d_g(G) = max\{e_g(a) : a \in P\}$. Let $a, b \in P(G)$ be any two vertices in a FG $G(\rho, \mu)$, a vertex a at a g-distance $e_g(b)$ from b is a g-eccentric point of b. The g-eccentric set of a vertex a is defined and denoted by $E_g(b) = \{a : d_g(a, b) = e_g(b)\}$. The set $S \subseteq P$ in a FG is called g-eccentric point set if $\forall\ a \in P - S$, $\exists$ at least one g-eccentric point $b \in S$ of a.

A set $D \subseteq P(G)$ in a FG $G(\rho, \mu)$ is called a dominating set (D-set) for $\forall$ $b \in P - D$, $\exists$ at least a vertex $a \in D$, such that (a, b) is a strong arc. A D-set $D \subseteq P(G)$ of a graph G is a g-eccentric dominating set (g-ED set), if for every $b \in P - D$, there exists at least a g-eccentric vertex a of b in D. A g-ED set D is a minimal g-ED set if any subset $D' \subset D$ is not a g-ED set. The g-ED number is denoted and defined by $\gamma_{ged}(G) = min\{|D_i|/D_i,$ is a minimal g-ED set$\}$. The upper g-ED number is denoted and defined by $\Gamma_{ged}(G) = max\{|D_i|/D_i,$ is a minimal g-ED set$\}$. A vertex a is called isolate if $\mu(a, b) = 0$, for all $b \in P(G)$.

2 Isolate g-eccentric point set in a fuzzy graph

In a FG $G(\rho, \mu)$ an isolate g-eccentric point set and its numbers are introduced with suitable examples and certain points are observed in this part. We stated and demonstrated the theorems based on the isolated g-eccentric point set.

Definition 2.1. *A g-eccentric point set $S \subseteq P(G)$ of a FG $G(\rho, \mu)$ is an isolate g-eccentric point set (Ig-EP set) if the induced fuzzy sub graph $< S >$ contains at least an isolate vertex. An Ig-EP set S is a minimal Ig-EP set if any subset $S' \subset S$ is not an Ig-EP set. The Ig-EP number is denoted and defined by $e_{og}(G)$= min $\{|S_i|/S_i$ is a minimal Ig-EP set$\}$. The upper Ig-EP number is denoted and defined by $E_{og}(G) = max\{|S_i|/S_i$ is a minimal Ig-EP set$\}$.*

Example 2.1.

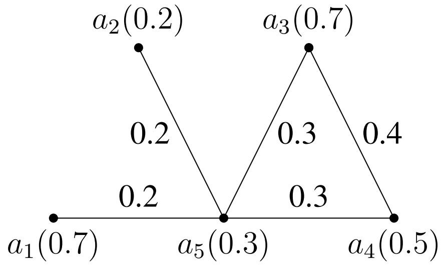

Figure 1: Ig-EP - set in FG

The FG given in figure 1, the Ig-EP sets are $S_1 = \{a_1, a_2\}$, $S_2 = \{a_2, a_3, \}$, $S_3 = \{a_2, a_4\}$, $S_4 = \{a_1, a_3\}$ and $S_5 = \{a_1, a_4\}$. The Ig-EP number is $e_{og}(G) = 0.7$ and the upper Ig-EP number is $E_{og}(G) = 1.4$.

Observation 2.1.

(i) If S is a minimal Ig-EP set, then $S' \subset S$ is not an Ig-EP set.

(ii) In a fuzzy tree, every Ig-EP set has at least one pendent vertex.

(iii) For any FG $G(\rho, \mu)$, $e_{og}(G) \leq E_{og}(G)$.

(iv) The complement of an Ig-EP set is need not be an Ig-EP set.

(v) If S is an Ig-EP set, then $S' \supset S$ is need not be an Ig-EP set.

Example 2.2. *In a FG given in example 2.1, the set $S_1 = \{a_1, a_3\}$ is an Ig-EP set, the complement of $S_1 = \{a_2, a_4, a_5\}$ is a g-eccentric point set (g-EP- set) not an IgEP-set.*

Example 2.3. *Add the vertex a_5 in the set S_1, S_2, S_3, S_4 and S_5 in example 2.1, all the sets are not an IgEP-set.*

Result 2.1.

(i) $e_{og}(K_\rho) = \rho_0$, where $\rho_0 = min\{\rho(a), a \in P(K_\rho)\}$.

(ii) $e_{og}(P_\rho) = \sum \rho(a)$, where a is a pendent vertex .

Theorem 2.1. *Let $G(\rho, \mu)$ be a FG and let S be an Ig-EP set. Then S is a minimal Ig-EP $\Leftrightarrow \forall\ a \in S$, satisfies one of the following condition:*

1. *a has no Ig-EP in S.*

2. *There exists some $b \in P - S$ such that $E_g(b) \cap S = \{a\}$.*

Proof. Assume that S is a minimal Ig-EP set of a FG $G(\rho, \mu)$. Then $\forall\ a \in S, S - \{a\}$ is not an Ig-EP set. Then $\exists$ some node $b \in P - S \cup \{a\}$ which is not an isolate g-eccentric vertex in $S - \{a\}$ or $\exists\ b \in P - S \cup \{a\}$, i.e b has no Ig-EP in $S - \{a\}$.
Case(i) If $b = a$, then a has no Ig-EP in S.
Case (ii) If $b \neq a$, then assume that b has no Ig-EP in $S - \{a\}$ but b has an Ig-EP in S. Then a is the only IgEP of b in S. That is $E_g(b) \cap S = \{a\}$.
Conversly, assume that S is an Ig-EP- set and $\forall\ a \in S$ one of the condition holds. Suppose that S is not a minimal Ig-EP set, $\exists$ a vertex $a \in S$ such that $S - \{a\}$ is an Ig-EP set. Therefore, a is strong neighbor to at least one node b in$S - \{a\}$ and a has an Ig-EP in $S - \{a\}$. Hence, condition (i) does not hold.
Also, if $S - \{a\}$ is an Ig-EP set, then $\forall\ b$ in $P - S$ is strong neighbor to at least one vert ex in $S - \{a\}$ and b has an Ig-EP in $S - \{a\}$. Therefore, condition (ii) does not hold. this is a contradiction to our assumption that $\forall\ a \in D$ satisfies one of the condition. Hence, S is minimal Ig-EP set. $\square$

3 Isolate g-eccentric domination in a fuzzy graph

This section studies an Ig-ED set, a novel domination parameter as well as an Ig-ED number for some FG. A few observations and theorems are discussed.

Definition 3.1. *A g-ED set $D \subseteq P(G)$ of a FG $G(\rho, \mu)$ is called an Ig-ED set (Ig-ED-set)if the induced fuzzy sub graph $< D >$ contains at least an isolate node. An Ig-ED set D is a minimal Ig-ED set if a subset $D' \subset D$ is not a Ig-ED set. The Ig-ED number is denoted and defined by $\gamma_{oged}(G)$= min$\{|D_i|/D_i$ is a minimal Ig-ED set$\}$. The upper Ig-ED number is denoted and defined by $\Gamma_{oged}(G) = max\{|D_i|/D_i$ is a minimal Ig-ED set$\}$.*

Example 3.1.

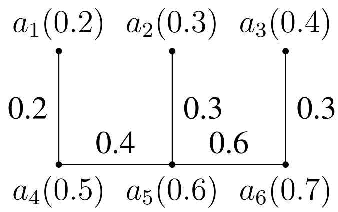

Figure 2: Ig-ED - set in FG

The FG in figure 2, the Ig-ED-sets are $D_1 = \{a_1, a_2, a_3\}$ and $D_2 = \{a_1, a_3, a_5\}$. Hence, $\gamma_{oged}(G) = 0.9$ and $\Gamma_{oged}(G) = 1.2$.

Example 3.2. *The FG given in figure 1, the D-set* $D_1 = \{a_5\}$*, ID- set* $D_2 = \{a_1, a_2, a_4\}$*, gED-set* $D_3 = \{a_2, a_3, a_5\}$ *and Ig-ED set* $D_4 = \{a_1, a_2, a_3\}$*. Then,* $\gamma(G) = 0.3, \gamma_{od}(G) = 1.4, \gamma_{ged}(G) = 1.2$ *and* $\gamma_{oged}(G) = 1.6$.

Observation 2: For any FG, $G(\rho, \mu)$

(i) $\gamma(G) \leq \gamma_o(G) \leq \gamma_{oged}(G)$

(ii) $\gamma_{ged}(G) \leq \gamma_{oged}(G)$

(iii) $\rho_0 \leq \gamma_{oged}(G) \leq p$.

(iv) Every subset $D' \subset D$ is not an Ig-ED set.

(v) Every super set $D" \supset D$ is need not be an Ig-ED set.

(vi) The complement of an Ig-ED set is need not be an Ig-ED set.

Example 3.3. *Let us consider the FG figure 1 and in example 3.2, the set* $D_4 = \{a_1, a_2, a_3\}$ *is an Ig-ED set. Add one more vertex* a_5 *in* D_4 *then the set* D_4 *is not an Ig-ED set.*

Example 3.4.

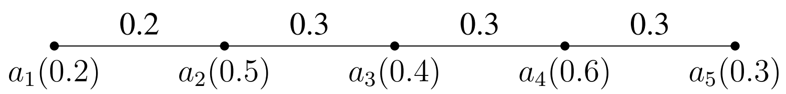

Figure 3: Path FG

The FG given in figure 3, $D = \{a_1, a_3, a_5\}$ is a Ig-ED-set. The Complement $P - D = \{a_2, a_4\}$ of D is ID-set but not a Ig-ED- set.

Result 3.1.

(i) If $G = \bar{K}_{\rho_1} + K_{\rho_2} + K_{\rho_3} + \bar{K}_{\rho_4}$ where $\mid \rho_1^* \mid = 2$, $\mid \rho_2^* \mid = 1$, $\mid \rho_3^* \mid = 1$ and $\mid \rho_4^* \mid = 2$, then $\gamma(G) = \mid \rho_2 \mid + \mid \rho_3 \mid$, $\gamma_{od}(G) = min\{\mid \rho_1 \mid + \mid \rho_3 \mid, \mid \rho_2 \mid + \mid \rho_4 \mid\}$ and $\gamma_{ged}(G) = \gamma_{oged}(G) = \mid \rho_1 \mid + \mid \rho_4 \mid$.

(ii) If $G = K_{\rho_1} + K_{\rho_2} + K_{\rho_3} + K_{\rho_4}$ where $\mid \rho_1^* \mid = \mid \rho_4^* \mid = n, n > 2 \mid \rho_2^* \mid = 1$, $\mid \rho_3^* \mid = 1$, then $\gamma(G) = \gamma_{od}(G) = \gamma_{ged}(G) = \gamma_{oged}(G) \leq 2$.

(iii) If $G = \bar{K_{\rho_1}} + K_{\rho_2} + K_{\rho_3} + \bar{K_{\rho_4}}$ where $\mid \rho_1^* \mid = q$, $\mid \rho_2^* \mid = 1$, $\mid \rho_3^* \mid = 1$ and $\mid \rho_4^* \mid = p$,and $q \leq p$, then $\gamma(G) = \mid \rho_2^* \mid + \mid \rho_3^* \mid$, $\gamma_{od}(G) = \mid \rho_1^* \mid + min\{\mid \rho_2^* \mid, \mid \rho 3^* \mid\}$, $\gamma_{ged}(G) \leq 4$ and $\gamma_{oged}(G) \leq min\{\mid \rho_1^* \mid, \mid \rho_4^* \mid\} + 2$.

(iv) $\gamma_{oged}(k_\rho) = \rho_0$.

(v) $\gamma_{oged}(G) = p$ if and only if $G(\sigma, \mu) = \bar{K_\sigma}$.

(vi) $\gamma_{oged}(K_{\rho_1,\rho_2}) = p$, $\mid \rho_1^* \mid = 1$, $\mid \rho_2 \mid = \emptyset$.

4 Bounds on isolate g-eccentric domination in fuzzy graph

This section discusses the bounds for an Ig-ED number of a FG under certain conditions.

Theorem 4.1. *Let P_ρ be a path FG, $\mid \rho^* \mid = n$ then*

$$\gamma_{oged}(P_\rho) \leq \begin{cases} \frac{p}{3} + 1, \text{ n=3k+1} \\ \frac{p}{3} + 2, \text{ n =3k+2} . \end{cases}$$

Proof. Let P_ρ be a path FG, $\mid \rho^* \mid = n$.
Case(i) $n = 3k$.
Let $D = \{a_1, a_4, a_7, ..., a_{3k-2}, a_{3k}\}$ be the minimum g-ED set in P_ρ. Then $\langle D \rangle$ contains at least one isolated vertex. Hence D is an Ig-ED set. Therefore, $\gamma_{oged}(P_\rho) \leq \frac{p}{3} + 1$.
Case(ii) $n = 3k + 1$.
Let $D = \{a_1, a_4, a_7, ..., a_{3k-2}, a_{3k+1}\}$ be the minimum g-ED set in P_ρ. Then $\langle D \rangle$ contains at least one isolated vertex. Hence $\langle D \rangle$ is an Ig-ED set. Therefore,$\gamma_{oged}(P_\rho) \leq \frac{p}{3} + 1$.
Case(iii) $n = 3k + 2$.
Let $D = \{a_1, a_4, a_5, a_8, ..., a_{3k-1}, a_{3k+1}a_{3k+2}\}$ be the minimum g-ED set in P_ρ. Then $\langle D \rangle$ contains at least one isolated node. Hence D is an Ig-ED set. Therefore, $\gamma_{oged}(P_\rho) \leq \frac{p}{3} + 2$.
Hence, From cases (i), (ii) and (iii) we conclude that

$$\gamma_{oged}(P_\rho) \leq \begin{cases} \frac{p}{3} + 1, \text{ n=3k, 3k+1} \\ \frac{p}{3} + 2, \text{ n=3k+2 .} \end{cases} .$$

□

Theorem 4.2. *Let C_ρ be a cycle FG, $\mid \rho^* \mid = n$ then*

$$\gamma_{oged}(C_\rho) \leq \begin{cases} \frac{p}{3}, \text{ n is even} \\ \frac{p}{3} + 1, \text{ n is odd} . \end{cases}$$

Proof. Proof of (i): If $n = 4$, any two adjacent nodes of C_ρ is a g-ED set but not an Ig-ED set.
Let $n = 2k$ and $k > 2$.
Let the cycle $C_\rho = \{a_1a_2a_3..., a_2ka_1\}$. Each vertex of C_ρ has exactly one g-eccentric vertex (unique g-eccentric node FG).
Case(i) k is odd.
Let $D = \{a_1, a_3, ..., a_k, a_{k+2}, ..., a_{2k-1}\}$ be the minimum g-ED set and $\langle D\rangle$ contains an isolated node. Hence $\langle D\rangle$ is an Ig-ED set. Therefore, $\gamma_{oged}(C_\rho) \leq \frac{p}{3}$.
Case (ii) k is even.
Let $D = \{a_1, a_3, ..., a_{k-1}, a_{k+2}, ..., a_{2k}\}$ be the minimum g-ED set and $\langle D\rangle$ contains an isolated node. Hence $\langle D\rangle$ is an Ig-ED set. Therefore, $\gamma_{oged}(C_\rho) \leq \frac{p}{3}$.
Proof of (ii)
Let $n = 2k + 1$, is odd, each vertex of C_ρ has exactly two g-eccentric node. If $b_i \in P(G)$ has b_{i+k}, b_{i+k+1} is a g-eccentric vertex.
Case (i): $n = 3k, n-$ odd implies m is odd.
Let $D = \{a_1, a_4, x_7, ..., a_k, a_{k+3}, ..., a_{2k-1}\}$ be the minimum g-ED set and $\langle D\rangle$ contains an isolated node. Hence $\langle D\rangle$ is an Ig-ED set. Therefore, $\gamma_{oged}(C_\rho) \leq \frac{p}{3}$.
Case (ii): $n = 3k + 1$, $n-$ odd implies k is even.
Let $D = \{a_1, a_4, ..., a_{k+1}, a_{k+3}, a_{k+6}, ..., a_{2k-1}\}$ be the minimum g-ED set and $\langle D\rangle$ contains an isolated node. Hence $\langle D\rangle$ is an Ig-ED set. Therefore, $\gamma_{oged}(C_\rho) \leq \frac{p}{3}$.
Case (iii): $n = 3k + 2$, $3k-$ odd implies k is odd.
Let $D = \{a_1, a_4, ..., a_{k-1}, a_k, a_{k+3}, ..., a_{2k+1}\}$ be the minimum g-ED set and $\langle D\rangle$ contains an isolated node. Hence $\langle D\rangle$ is an Ig-ED set. Therefore, $\gamma_{oged}(C_\rho) \leq \frac{p}{3} + 1$.
From the cases (i), (ii) and (iii) we obtain $\gamma_{oged}(C_\rho) \leq \begin{cases} \frac{p}{3}, \text{ n is even} \\ \frac{p}{3} + 1, \text{ n is odd}. \end{cases}$ □

Theorem 4.3. *Let G be a FG of order p with n nodes. Then $\gamma_{oged}(G \circ K_{\rho_1}) \leq n$, where $|\ \rho_1^*\ | = 1$.*

Proof. Let $V(G) = \{a_1, a_2, a_3, ...a_n\}$. Let b'_t be the pendent vertex adjacent to b_t in $G \circ K_\rho$. Then $\{b'_1, b'_2, ...b'_n\}$ is an Ig-ED set for $G \circ K_{\rho_1}$. Hence, $\gamma_{oged}(G \circ K_\rho) \leq n$. □

5 Discussion and conclusion

The geodesic distance and geodesic eccentricity(g-eccentricity) in FG are already defined and discussed in Ismayil and Muthupandiyan [2020] using strong arc. In this article, Ig-EP set, Ig-ED set, and their corresponding numbers in FG are introduced. For some standard FG, these numbers ($\gamma_{oged}(G)$ & $\gamma_{og}(G)$) are obtained and bounds for $\gamma_{oged}(G)$ are also derived. Monophonic Ig-ED set and monophonic connected Ig-ED set and their number in FG may be extended using strong arcs.

References

B. H. Arriola. Isolate domination in the join and corona of graphs. *Applied Mathematical Sciences*, 9(31):1543–1549, 2015.

M. Bhanumathi and R. Niroja. Isolate eccentric domination in graphs. *International Journal of Advanced Research Trends in Engineering and Technology (IJARTET)*, 5(12):951–955, 2018.

I. S. Hamid and S. Balamurugan. Isolate domination in graphs. *Arab Journal of Mathematical Sciences*, 22(2):232–241, 2016.

F. Harary. *Graph Theory*. Addition – Wesley Publishing Company Reading,Mass, 1992.

A. M. Ismayil and S. Muthupandiyan. Complementary nil g-eccentric domination in fuzzy graphs. *Advances in Mathematics: Scientific Journal*, 9(4):1719–1728, 2020.

A. M. Ismayil and S.Muthupandiyan. g-eccentric domination in fuzzy graphs. *Our Heritage*, 68(4):191–200, 2020.

T. Janakiraman, M. Bhanumathi, and S. Muthammai. Eccentric domination in graphs. *International Journal of Engineering Science, Advanced Computing and Bio-Technology*, 1(2):1–16, 2010.

J. Linda and M. Sunitha. On g-eccentric nodes g-boundary nodes and g-interior nodes of a fuzzy graph. *Int Jr Math Sci Appl*, 2:697–707, 2012.

S. Muthupandiyan and A. M. Ismayil. Perfect g-eccentric domination in fuzzy graphs. *Journal of Mathematical Control Science and Applications*, 7(2):413–424, 2021.

A. Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.