Uniqueness of an entire function sharing a polynomial with its linear differential polynomial

Imrul Kaish*
Nasir Uddin Gazi†

Abstract
In this paper we consider an entire function when it shares a polynomial with its linear differential polynomial. Our result is an improvement of a result of P.Li.

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1 Introduction, Definitions and Results

Let \( f \) be a non-constant meromorphic function defined in the open complex plane \( \mathbb{C} \) and \( a = a(z) \) be a polynomial. We denote by \( E(a; f) \) the set of zeros of \( f - a \), counted with multiplicities and by \( \overline{E}(a; f) \) the set of distinct zeros of \( f - a \).

If for two non-constant meromorphic functions \( f \) and \( g \), we have \( E(a; f) = E(a; g) \), we say that \( f \) and \( g \) share a CM and if \( \overline{E}(a; f) = \overline{E}(a; g) \), we say that \( f \) and \( g \) share a IM.

We denote by \( S(r, f) \) any function satisfying \( S(r, f) = o\{T(r, f)\} \), as \( r \to \infty \), possibly outside of a set with finite measure.

For an entire function \( f \), we define \( \text{deg}(f) \) in the following way:

\[ \text{deg}(f) = \infty, \text{ if } f \text{ is a transcendental entire function and } \text{deg}(f) \text{ is the degree of the polynomial, if } f \text{ is a polynomial.} \]

The investigation of uniqueness of an entire function sharing two values introduced by L. A. Rubel and C. C. Yang [Rubel and Yang, 1977] in 1977. Following is their result.

**Theorem A.** [Rubel and Yang, 1977] Let \( f \) be a non-constant entire function. If \( E(a; f) = E(a; f^{(1)}) \) and \( E(b; f) = E(b; f^{(1)}) \), for distinct finite complex numbers \( a \) and \( b \), then \( f \equiv f^{(1)} \).

In 1979 E. Mues and N. Steinmetz [Mues and Steinmetz, 1979] tried to improve Theorem A by considering IM sharing of values. They proved the following theorem.

**Theorem B.** [Mues and Steinmetz, 1979]. Let \( f \) be a non-constant entire function and \( a, b \) be two distinct finite complex values. If \( \overline{E}(a; f) = \overline{E}(a; f^{(1)}) \) and \( \overline{E}(b; f) = \overline{E}(b; f^{(1)}) \), then \( f \equiv f^{(1)} \).

In 1986 G. Jank, E. Mues and L. Volkmann [Jank et al., 1986] considered an entire function sharing a nonzero value with its derivatives and they proved the following result.

**Theorem C.** [Jank et al., 1986] Let \( f \) be a non-constant entire function and \( a \) be a non-zero finite value. If \( \overline{E}(a; f) = \overline{E}(a; f^{(1)}) \subset \overline{E}(a; f^{(2)}) \), then \( f \equiv f^{(1)} \).

H. Zhong [Zhong, 1995] tried to improve Theorem C by taking higher order derivatives. By the following example he concluded that in Theorem C the second derivative cannot be straight way replaced by any higher order derivatives.

**Example 1.1.** [Zhong, 1995] Let \( k(\geq 3) \) be a positive integer and \( \omega(\neq 1) \) be a \((k - 1)\text{th root of unity. If } f = e^{\omega z} + \omega - 1, \text{ then } f, f^{(1)}, \text{ and } f^{(k)} \text{ share the value } \omega \text{ CM, but } f \neq f^{(1)}. \)
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Considering two consecutive higher order derivatives H. Zhong [Zhong, 1995] improved Theorem C in another direction. The following is the improved result.

**Theorem D.** [Zhong, 1995] Let \( f \) be a non-constant entire function and \( a \) be a non-zero finite value. If \( E(a; f) = E(a; f^{(1)}) \) and \( \overline{E}(a; f) \subset \overline{E}(a; f^{(n)}) \cap E(a; f^{(n+1)}) \) for \( n(\geq 1) \), then \( f \equiv f^{(n)} \).

For further discussion we need the following notation. Let \( f \) be a non-constant meromorphic function, \( a = a(z) \) be a polynomial and \( A \) be a set of complex numbers. We denote by \( n_A(t, a; f) \), the number of zeros of \( f - a \), counted according to their multiplicities which lie in \( A \). The integrated counting function \( N_A(r, a; f) \) of the zeros of \( f - a \) which lie in \( A \cap \{ z : |z| \leq r \} \) is defined as

\[
N_A(r, a; f) = \int_0^r \frac{n_A(t, a; f) - n_A(0, a; f)}{t} dt + n_A(0, a; f) \log r,
\]

where \( n_A(0, a; f) \) denotes the multiplicity of zeros of \( f - a \) at origin. \( N_A(r, a; f) \) be the reduced counting function of zeros of \( f - a \) in \( A \cap \{ z : |z| \leq r \} \). Clearly if \( A = \mathbb{C} \) then \( N_A(r, a; f) = N(r, a; f) \) and \( \overline{N}_A(r, a; f) = \overline{N}(r, a; f) \).

For standard definitions and notations of the value distribution theory we refer the reader to [Hayman, 1964] and [Yang and Yi, 2003].

Recently I. Lahiri and I. Kaish [Lahiri and Kaish, 2017] improved Theorem D by considering a shared polynomial. They proved the following result.

**Theorem E.** [Lahiri and Kaish, 2017] Let \( f \) be a non-constant entire function and \( a = a(z)(\neq 0) \) be a polynomial with \( \deg(a) \neq \deg(f) \). Suppose that \( A = \overline{E}(a; f) \Delta \overline{E}(a; f^{(1)}) \) and \( B = \overline{E}(a, f^{(1)}) \setminus \{ \overline{E}(a, f^{(n)}) \cap \overline{E}(a, f^{(n+1)}) \} \), where \( \Delta \) denotes the symmetric difference of sets and \( n(\geq 1) \) is an integer. If

(i) \( N_A(r, a; f) + N_A(r, a; f^{(1)}) = O(\log T(r, f)) \);

(ii) \( N_B(r, a; f^{(1)}) = S(r, f) \) and

(iii) each common zero of \( f - a \) and \( f^{(1)} - a \) has the same multiplicity,

then \( f = \lambda e^z \), where \( \lambda(\neq 0) \) is a constant.

Throughout the paper we denote by \( L = L(f) \) a nonconstant linear differential polynomial generated by \( f \) of the form

\[
L = L(f) = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)},
\]

where \( a_1, a_2, \ldots, a_n(\neq 0) \) are constants.

Considering Linear differential polynomial P.Li [Li, 1999] improved Theorem D in the following way.
Theorem F. [Li, 1999]. Let $f$ be a non-constant entire function and $L$ be defined in (1) and $a$ be a non-zero finite complex number. If $E(a; f) = E(a; f^{(1)}) \subset E(a; L) \cap E(a; L^{(1)})$ then $f = f^{(1)} = L$.

In this paper we extend Theorem D and Theorem F in the following way.

Theorem 1.1. Let $f$ be a non-constant entire function, $L$ be defined in (1) and $a = a(z) \not\equiv 0$ be a polynomial with $\deg(a) \neq \deg(f)$. Suppose that $A = E(a; f) \Delta E(a; f^{(1)})$ and $B = E(a, f^{(1)}) \setminus \{E(a, L^{(p)}) \cap E(a, L^{(q)})\}$ where $p, q$ are integers satisfying $q > p \geq \deg(a)$.

If

(i) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$,

(ii) $N_B(r, a; f^{(1)}) = S(r, f)$ and

(iii) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,

then $f = L = \lambda e^z$, where $\lambda (\neq 0)$ is a constant.

Putting $A = B = \emptyset$ we get the following corollary.

Corollary 1.1. Let $f$ be a non-constant entire function, $L$ be defined in (1) and $a = a(z) \not\equiv 0$ be a polynomial with $\deg(a) \neq \deg(f)$. If $E(a; f) = E(a; f^{(1)})$ and $E(a, f^{(1)}) \subset E(a, L^{(p)}) \cap E(a, L^{(q)})$ where $p, q$ are integers satisfying $q > p \geq \deg(a)$, then $f = L = \lambda e^z$, where $\lambda (\neq 0)$ is a constant.

Remark 1.1. If in Corollary 1.1, $a$ is a non-zero constant and $p = \deg(a) = 0, q = p + 1$ then it is a particular form of Theorem F.

Remark 1.2. If in (1), $a_1 = a_2 = \ldots a_{n-1} = 0$ and $a_n = 1$ then $L = f^{(n)}$ and if in Corollary 1.1, $a$ is a non-zero constant and $p = \deg(a), q = p + 1$, then Corollary 1.1 is the Theorem D.

Remark 1.3. It is an open problem whether the Theorem 1.1 is valid or not if we omit the condition $p \geq \deg(a)$.

2 Lemmas

In this section we present some necessary lemmas.

Lemma 2.1. [Lahiri and Kaish, 2017]. Let $f$ be a transcendental entire function of finite order and $a = a(z) \not\equiv 0$ be a polynomial and $A = E(a; f) \Delta E(a; f^{(1)})$.

If
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(i) \(N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}\),

(ii) each common zero of \(f - a\) and \(f^{(1)} - a\) has the same multiplicity, 
then \(m(r, a; f) = S(r, f)\).

**Lemma 2.2.** [Lain, 1993]. Suppose \(f\) be an entire function, \(a_0, a_1, \ldots, a_n\) are polynomials and \(a_0, a_n\) are not identically zero. Then each solution of the linear differential equation \(a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0 f = 0\) is of finite order.

**Lemma 2.3.** [Hayman, 1964]. Let \(f\) be a non-constant meromorphic function and \(a_1, a_2, a_3\) be three distinct meromorphic functions satisfying \(T(r, a_\nu) = S(r, f)\) for \(\nu = 1, 2, 3\) then
\[
T(r, f) \leq \overline{N}(r, 0; f - a_1) + \overline{N}(r, 0; f - a_2) + \overline{N}(r, 0; f - a_3) + S(r, f).
\]

**Lemma 2.4.** Let \(f\) be a transcendental entire function and \(a = a(z)(\neq 0)\) be a polynomial. Also let \(L(f), L(a)\) be the linear differential polynomials generated by \(f\) and \(a\) respectively. Suppose
\[
h = \frac{(a - a^{(1)})(L^{(p)}(f) - L^{(p)}(a)) - (a - L^{(p)}(a))(f^{(1)} - a^{(1)})}{f - a},
\]
\[
A = \overline{E}(a; f) \setminus \overline{E}(a; f^{(1)}) \quad \text{and} \quad B = \overline{E}(a, f^{(1)}) \setminus \{\overline{E}(a, L^{(p)}) \cap \overline{E}(a, L^{(q)})\}, \quad \text{where} \quad p, q \text{ are integers satisfying } 0 \leq p < q.
\]
If

(i) \(N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)\),

(ii) each common zero of \(f - a\) and \(f^{(1)} - a\) has the same multiplicity,

(iii) \(h\) is a transcendental entire or meromorphic, 
then \(m(r, a, f^{(1)}) = S(r, f)\).

**Proof.** Since \(a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)\), if \(z_0\) be a common zero of \(f - a\) and \(f^{(1)} - a\) with multiplicity \(r(\geq 2)\), then \(z_0\) is a zero of \(a - a^{(1)}\) with multiplicity \(r - 1\). So
\[
N_{(2)}(r, a; f) \leq 2N(r, 0; a - a^{(1)}) + N_A(r, a; f) = S(r, f), \quad \text{(2)}
\]
where \(N_{(2)}(r, a; f)\) be the counting function of multiple zeros of \(f - a\).

Using (2) and from the hypothesis we get
\[
N(r, h) \leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N_{(2)}(r, a; f) + S(r, f) = S(r, f)
\]
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Since \( m(r, h) = S(r, f) \), we have \( T(r, h) = S(r, f) \).

From \( h = \frac{(a - a^{(1)})(L^{(p)}(f) - L^{(p)}(a)) - (a - L^{(p)}(a))(f^{(1)} - a^{(1)})}{f - a} \), we get

\[
f = a + \frac{1}{h}((a - a^{(1)})(L^{(p)}(f) - L^{(p)}(a)) - (a - L^{(p)}(a))(f^{(1)} - a^{(1)}))
= a + \frac{1}{h}((a - a^{(1)})(L^{(p)}(f) - a) - (a - L^{(p)}(a))(f^{(1)} - a)) \tag{3}
\]

**Case I.** Let \( p > 0 \). Differentiating (3) we get

\[
f^{(1)} = a^{(1)} + \frac{1}{h}(a^{(1)} - (a - a^{(1)})(L^{(p)}(f) - a) - (a - L^{(p)}(a))(f^{(1)} - a)) + \frac{1}{h}((a^{(1)} - a^{(2)})(L^{(p)}(f) - a) + (a - a^{(1)})(L^{(p+1)}(f - a)) + \frac{1}{h}(a - a^{(1)})(L^{(p+1)}(f - a^{(1)} - \frac{1}{h}(a - L^{(p)}(a))(f^{(2)} - a^{(1)}),
\]

This implies

\[
(f^{(1)} - a)(1 + (\frac{a}{h}))(a - L^{(p)}(a)) + \frac{1}{h}(a^{(1)} - L^{(p+1)}(a))
= a^{(1)} - a + (\frac{a}{h})^{(1)}(a - a^{(1)})(L^{(p)}(f) - a) + \frac{1}{h}(a^{(1)} - a^{(2)})(L^{(p)}(f) - a) + \frac{1}{h}(a - a^{(1)})(L^{(p+1)}(f - a^{(1)} - \frac{1}{h}(a - L^{(p)}(a))(f^{(2)} - a^{(1)}),
\]

or,

\[
(f^{(1)} - a)(1 + (\frac{a - L^{(p)}(a)}{h}))^{(1)} = (a^{(1)} - a) + ((\frac{a - L^{(p)}(a)}{h})(L^{(p+1)}(f - a^{(1)}) + (\frac{a - L^{(p)}(a)}{h})(L^{(p)}(f - L^{(p+1)}(a)) + \frac{a - L^{(p)}(a)}{h}((L^{(p)}(f) - L^{(p)}(a)) - \frac{1}{h}(a - L^{(p)}(a))(f^{(2)} - a^{(1)}),
\]

or

\[
\frac{1}{f^{(1)} - a} = \frac{h_{1}}{h_{2}} - \frac{1}{h_{2}}(a - a^{(1)})^{(1)}(L^{(p)}(f) - L^{(p+1)}(a))
= \frac{h_{1}}{h_{2}} - \frac{1}{h_{2}}(a - a^{(1)})^{(1)}(L^{(p+1)}(f) - L^{(p)}(a))
= \frac{h_{1}}{h_{2}} - \frac{1}{h_{2}}(a - L^{(p)}(a))(f^{(2)} - a^{(1)}) \tag{4}
\]

where \( h_{1} = 1 + (\frac{a - L^{(p)}(a)}{h})^{(1)}, \)

\( h_{2} = a^{(1)} - a + ((\frac{a - L^{(p)}(a)}{h})(L^{(p-1)}(a)) - a)^{(1)} \).

We now verify that \( h_{1} \neq 0, h_{2} \neq 0. \)
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If \( h_1 \equiv 0 \), then \( 1 + \left( \frac{a - L_p(a)}{h} \right)^{(1)} \equiv 0 \). Integrating we get \( \frac{1}{h} = \frac{a - L_p(a)}{h^2} \), where \( c_j \) is a constant. This is a contradiction, because \( h \) is transcendent.

If \( h_2 \equiv 0 \), then \( a^{(1)} - a + \{(a - a^{(1)})(L^{(p-1)}(a) - a)\}^{(1)} \equiv 0 \). Integrating we get \( h = \frac{(a - a^{(1)})(L^{(p-1)}(a) - a)}{P(z)} \), where \( P(z) \) is a polynomial. This is again a contradiction. Therefore \( h_1 \neq 0, h_2 \neq 0 \). Again \( T(r, h_1) + T(r, h_1) = S(r, f) \), since \( T(r, h) = S(r, f) \).

Now from (4) and using Lemma of logarithmic derivative we get \( m(r, a; \ f^{(1)} = m(r, \frac{1}{f^{(1)} - a}) = S(r, f) \).

Case 2. Let \( p = 0 \). Then \( L_p(f) = L(f) \).

Suppose \( L(f) = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)} \)

and \( \ f(a) = a_1 a^{(1)} + a_2 a^{(2)} + \ldots + a_n a^{(n)} \), where \( a_1, a_2, \ldots, a_n \neq 0 \) are constant, \( n \geq 1 \) be an integer.

From the definition of \( h \) we get

\[ f = a + \frac{1}{h} \{(a - a^{(1)})(L(f) - a) - (a - L(a))(f^{(1)} - a)\} \]

Differentiating we get

\[ f^{(1)} = a^{(1)} + \left( \frac{1}{h} \right)^{(1)} \{(a - a^{(1)})(L(f) - a) - (a - L(a))(f^{(1)} - a)\} \]

\[ + \frac{1}{h} \{(a^{(1)} - a^{(2)})(L(f) - a) + (a - a^{(1)})(L^{(1)}(f) - a^{(1)})\} \]

\[ - \frac{1}{h} \{(a^{(1)} - L^{(1)}(a))(f^{(1)} - a) - (a - L(a))(f^{(2)} - a^{(1)})\} \]

This implies

\[ (f^{(1)} - a) \{1 + \left( \frac{a - L(a)}{h} \right)^{(1)}\} = \{a^{(1)} - a\} + \left( \frac{a - a^{(1)}}{h} \right)^{(1)} \{(L(f) - a) + \frac{a - a^{(1)}}{h} (L^{(1)}(f) - a^{(1)}) - \frac{a - L(a)}{h} (f^{(2)} - a^{(1)})\} \]

\[ + \left( \frac{a - a^{(1)}}{h} \right) (L^{(1)}(f) - L(a)) + \left( \frac{a - a^{(1)}}{h} \right) (L(a) - a^{(1)}) - \frac{a - L(a)}{h} (f^{(2)} - a^{(1)})\} = \{a^{(1)} - a\} \]

\[ + \left\{(a^{(1)} - L^{(1)}(a) - a)\right\} \]

\[ + \left\{(a^{(1)})(L^{(1)}(f) - L(a)) + \left( \frac{a - a^{(1)}}{h} \right) (L^{(1)}(f) - L(a)) \right\} \]

Or

\[ \frac{1}{f^{(1)} - a} = \frac{h_3}{h} - \frac{1}{h_4} \left( \frac{a - a^{(1)}}{h} \right)^{(1)} \left( L(f) - L_1(a) \right) \]

\[ + \left( \frac{a - a^{(1)}}{hh_4} \right) \left( \frac{L^{(1)}(f) - L(a)}{f^{(1)} - a} \right) - \left( \frac{a - L(a)}{hh_4} \right) \left( \frac{f^{(2)} - a^{(1)}}{f^{(1)} - a} \right) \]

where

\[ L_1(a) = a_1 a + a_2 a^{(1)} + \ldots + a_n a^{(n-1)} \]

\[ h_3 = 1 + \left( \frac{a - L(a)}{h} \right)^{(1)} \] and

\[ h_4 = a^{(1)} - a + \left\{(a - a^{(1)}) (L_1(a) - a)\right\}^{(1)} \]

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Similarly as in Case 1, \( h_3 \not\equiv 0, h_4 \not\equiv 0 \). Also \( T(r, h_3) + T(r, h_4) = S(r, f) \).

Therefore from (5) and using Lemma of logarithmic derivative we get
\[
m(r, a; f^{(1)}) = m(r, \frac{1}{f^{(1)} - a}) = S(r, f).
\]
This completes the proof of the lemma.

\[\square\]

**Lemma 2.5.** Let \( f \) be a transcendental entire function, \( a = a(z)(\not\equiv 0) \) be a polynomial and \( L = L(f) \) be define in (1). Suppose

(i) \( N_A(r, a; f) + N_A(r, a; f^{(1)}) = S(r, f) \), where \( A = E(a; f)\Delta E(a; f^{(1)}) \)

(ii) \( N_B(r, a; f^{(1)}) = S(r, f) \), where \( B = E(a, f^{(1)}) \setminus \{ E(a, L^{(p)}) \cap E(a, L^{(q)}) \} \)

\( p, q \) are integers satisfying \( q > p \geq \deg(a) \).

(iii) each common zero of \( f - a \) and \( f^{(1)} - a \) has the same multiplicity,

(iv) \( m(r, a; f) = S(r, f) \), then \( f = L = \lambda e^z \), where \( \lambda(\not\equiv 0) \) is a constant.

**Proof.** Let
\[
\alpha = \frac{f^{(1)} - a}{f - a}.
\]

From the hypothesis we get,
\[
N(r, \alpha) \leq N_A(r, a; f) + S(r, f) = S(r, f)
\]
and
\[
m(r, \alpha) = m(r, \frac{f^{(1)} - a}{f - a})
\leq m(r, \frac{f^{(1)} - a^{(1)} + a^{(1)} - a}{f - a})
\leq m(r, a; f) + S(r, f)
= S(r, f).
\]

Therefore \( T(r, \alpha) = S(r, f) \).

From (6) we get
\[
f^{(1)} = \alpha f + a(1 - \alpha) = \alpha_1 f + \beta_1,
\]
where \( \alpha_1 = \alpha \) and \( \beta_1 = a(1 - \alpha) \)

Differentiating we get,
\[
f^{(2)} = \alpha_2 f + \beta_2,
\]

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where \( \alpha_2 = \alpha_1^{(1)} + \alpha_1 \alpha_1 \) and \( \beta_2 = \beta_1^{(1)} + \alpha_1 \beta_1 \).

Similarly,

\[ f^{(k)} = \alpha_k f + \beta_k, \]

where \( \alpha_{k+1} = \alpha_k^{(1)} + \alpha_1 \alpha_k \) and \( \beta_{k+1} = \beta_k^{(1)} + \alpha_k \beta_1 \).

Clearly \( T(r, \alpha_k) + T(r, \beta_k) = S(r, f) \), because \( T(r, \alpha) = S(r, f) \).

Now

\[
L^{(p)} = \sum_{k=1}^{n} a_k f^{(p+k)} = \left( \sum_{k=1}^{n} a_k \alpha_{p+k} \right) f + \left( \sum_{k=1}^{n} a_k \beta_{p+k} \right) = \mu_1 f + \nu_1, \tag{7}
\]

where \( \mu_1 = \sum_{k=1}^{n} a_k \alpha_{p+k} \), \( \nu_1 = \sum_{k=1}^{n} a_k \beta_{p+k} \)

\[
L^{(q)} = \sum_{k=1}^{n} a_k f^{(q+k)} = \left( \sum_{k=1}^{n} a_k \alpha_{q+k} \right) f + \left( \sum_{k=1}^{n} a_k \beta_{q+k} \right) = \mu_2 f + \nu_2, \tag{8}
\]

where \( \mu_2 = \sum_{k=1}^{n} a_k \alpha_{q+k} \), \( \nu_2 = \sum_{k=1}^{n} a_k \beta_{q+k} \).

Clearly \( T(r, \mu_i) + T(r, \nu_i) = S(r, f) \), \( i = 1, 2. \)

Let \( D = \overline{E}(a; f) \cap \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L^{(p)}) \cap \overline{E}(a; L^{(q)}). \)

Note that \( D \neq \emptyset \), because otherwise, \( N(r, a; f) = S(r, f) \). Then from the hypothesis \( T(r, f) = S(r, f) \), a contradiction.

Let \( z_1 \in D \) then \( f(z_1) = f^{(1)}(z_1) = L^{(p)}(z_1) = L^{(q)}(z_1) = a(z_1) \).

Now from (7) and (8) we get \( a(z_1) = \mu_1(z_1) a(z_1) + \nu_1(z_1) \) and \( a(z_1) = \mu_2(z_1) a(z_1) + \nu_2(z_1) \)

If \( \mu_1 a + \nu_1 - a \not\equiv 0 \), then

\[
N(r, a; f) \leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N_D(r, a; f) + S(r, f) \\
\leq N_A(r, 0; \mu_1 a + \nu_1 - a) + S(r, f) \\
= S(r, f),
\]

a contradiction. Therefore

\[
\mu_1 a + \nu_1 - a \equiv 0. \tag{9}
\]
Similarly

$$\mu_2 a + \nu_2 - a \equiv 0. \quad (10)$$

From (9) and (10) we get $\mu_1 \equiv \mu_2 \equiv 1$ and $\nu_1 \equiv 0 \equiv \nu_2$.

Then from (7)

$$L^{(p)} \equiv f. \quad (11)$$

Also $\mu_1 \equiv 1$ implies

$$\sum_{k=1}^{n} a_k \alpha^{p+k} \equiv 1. \quad (12)$$

From (12) we see that $\alpha$ has no pole. Because if $\alpha$ has a pole of order $d(\geq 1)$
then the left hand side of (12) has a pole of order $(p+k)d$ but the right hand side
is a constant.

Again by simple calculation from (12) we get

$$a_n \alpha^n + P[\alpha] \equiv 0. \quad (13)$$

where $P[\alpha]$ is a differential polynomial in $\alpha$ with degree not exceeding $(n + p - 1)$.

If $\alpha$ is transcendental entire, then by Clunie’s Lemma we have $m(r, \alpha) = S(r, \alpha)$, a contradiction.

If $\alpha$ is a nonconstant polynomial then left hand side of (13) is also a noncon-
stant polynomial, which is again a contradiction.

Therefore $\alpha$ is a constant.

Now from $f^{(1)} = a$, we get $f^{(1)} - \alpha f = a(1 - \alpha)$.

Integrating we get

$$e^{-\alpha z} f = (1 - \alpha) \int ae^{-\alpha z} dz$$

$$= (1 - \alpha) P(z)e^{-\alpha z} + \lambda,$$

where $\lambda(\neq 0)$ is a constant and $P(z)$ is a polynomial of degree at most $\text{deg}(a)$,
or, $f = (1 - \alpha) P(z) + \lambda e^{\alpha z}$.

Now $f^{(r+1)} = \lambda \alpha^{r+1} e^{\alpha z}$, if $r = \text{deg}(a)$. 

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polynomial

Therefore

\[ L^{(p)} = \sum_{k=1}^{n} a_k f^{(p+k)} \]
\[ = \left( \sum_{k=1}^{n} a_k \alpha^{p+k} \right) \lambda e^{\alpha z} \]
\[ = \lambda e^{\alpha z} \]
\[ = \frac{f^{(1)}}{\alpha} - \frac{1 - \alpha}{\alpha} P^{(1)}(z), \]  
(14)

Suppose \( \alpha \neq 1 \).
Since \( D = \overline{E}(a; f) \cap \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L^{(p)}) \cap \overline{E}(a; L^{(q)}) \neq \emptyset \),
we have \( f(z_2) = f^{(1)}(z_2) = L^{(p)}(z_2) = L^{(q)}(z_2) = a(z_2) \), for some \( z_2 \in D \).
From (14) we get

\[ a(z_2) = \frac{a(z_2)}{\alpha} - \frac{1 - \alpha}{\alpha} P^{(1)}(z_2) \]
or,
\[ a(z_2)(1 - \frac{1}{\alpha}) + \frac{1 - \alpha}{\alpha} P^{(1)}(z_2) = 0 \]
or,
\[ (\alpha - 1)\{a(z_2) - P^{(1)}(z_2)\} = 0 \]
or,
\[ a(z_2) - P^{(1)}(z_2) = 0. \]
Clearly \( a(z) - P^{(1)}(z) \neq 0 \), because \( \text{deg}(P^{(1)}(z)) \) is less than \( \text{deg}(a) \).

\[ N(r, a; f) \leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N_D(r, a; f) + S(r, f) \]
\[ \leq N(r, 0; a - P^{(1)}) + S(r, f) \]
\[ = S(r, f). \]

Then from the hypothesis \( T(r, f) = S(r, f) \), a contradiction.
Therefore \( \alpha = 1 \), so \( f = \lambda e^z \).
Again

\[ L = \sum_{k=1}^{n} a_k f^{(k)} \]
\[ = \left( \sum_{k=1}^{n} a_k \alpha^k \right) \lambda e^{\alpha z} \]
\[ = \lambda e^z. \]

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Therefore $f = L = \lambda e^z$.
This completes the lemma.

\box

3 Proof of the Main Theorem

Proof. First we claim that $f$ is a transcendental entire function.

If $f$ is a polynomial, then

$T(r, f) = O(\log r)$ and $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O(\log r)$.

Then from the hypothesis we get $O(\log r) = O(\log T(r, f)) = S(r, f)$, which implies $T(r, f) = S(r, f)$, a contradiction. Therefore $A = \emptyset$.

Similarly $N_B(r, a; f^{(1)}) = S(r, f)$ implies $B = \emptyset$.

Therefore $E(a; f) = E(a; f^{(1)})$ and $E(a; f^{(1)}) \subset E(a; L^{(p)}) \cap E(a; L^{(q)})$.

Let $\deg(f) = m$ and $\deg(a) = r$. If $m \geq r + 1$ then $\deg(f - a) = m$ and $\deg(f^{(1)} - a) \leq m - 1$ which contradicts that $E(a, f) = E(a, f^{(1)})$.

If $m \leq r - 1$, then $\deg(f - a) = \deg(f^{(1)} - a) = r$. Since $E(a, f) = E(a, f^{(1)}), (f - a) = t(f^{(1)} - a)$, where $t(\neq 0)$ is a constant.

If $t = 1$, then $f = f^{(1)}$, which is a contradiction because $f$ is a polynomial.

If $t \neq 1$ then $tf^{(1)} - f \equiv (t - 1)a$, which is impossible because $\deg((t - 1)a) = r$ and $\deg(tf^{(1)} - f) = m$ and $m < r$. Therefore our claim "$f$ is transcendental entire function" is established. Now we prove the result into two cases.

Case 1. Let $f \equiv L^{(p)}$. Then

$$m(r, a; f) = m(r, a; \frac{1}{f - a})$$

$$\leq m(r, a; \frac{1}{f - a}) + S(r, f)$$

$$= m(r, a; \frac{1}{f - a}) + 1 - 1 + S(r, f)$$

$$\leq m(r, a; \frac{1}{f - a}) + 1 + S(r, f)$$

$$\leq m(r, f; \frac{1}{f - a}) + S(r, f)$$

$$= m(r, f; \frac{L^{(p)}}{f - a}) + S(r, f), \quad (15)$$

since $p \geq \deg(a)$, by Lemma of logarithmic derivative, $m(r, \frac{L^{(p)}}{f - a}) = S(r, f)$. So from (15) $m(r, a; f) = S(r, f)$. Therefore by Lemma 5, $f = L = \lambda e^z, \lambda(\neq 0)$ is a constant.

Case 2. Let $f \neq L^{(p)}$. This case can be divided into two subcases.

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Subcase 2.1. Let \( f^{(1)} \not= L^{(p)} \).
Since \( a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a) \), a common zero of \( f - a \) and \( f^{(1)} - a \) of multiplicity \( s(\geq 2) \) is a zero of \( a - a^{(1)} \) with multiplicity \( s - 1(\geq 1) \).
Therefore \( N_{(2)}(r; a; f^{(1)}) \mid f = a \) \( \leq 2N(r; 0; a - a^{(1)}) = S(r, f) \),
where \( N_{(2)}(r; a; f^{(1)} \mid f = a) \) denotes the counting function (counted with multiplicities) of those multiple zeros of \( f^{(1)} - a \) which are also zeros of \( f - a \).

Now
\[
N_{(2)}(r; a; f^{(1)}) \leq N_A(r, a; f^{(1)}) + N_B(r, a; f^{(1)}) + N_{(2)}(r, a; f^{(1)} \mid f = a) + S(r, f) \\
= S(r, f). \tag{16}
\]

Using (16) and from the hypothesis we get
\[
N(r, a; f^{(1)}) \leq N_B(r, a; f^{(1)}) + N(r, \frac{a - L^{(p)}(a)}{a - a^{(1)}}; \frac{L^{(p)}(f) - L^{(p)}(a)}{f^{(1)} - a^{(1)}}) + S(r, f) \\
\leq T(r, \frac{a - L^{(p)}(a)}{a - a^{(1)}}; \frac{L^{(p)}(f) - L^{(p)}(a)}{f^{(1)} - a^{(1)}}) + S(r, f) \\
= N(r, \frac{L^{(p)}(f) - L^{(p)}(a)}{f^{(1)} - a^{(1)}}) + S(r, f) \\
\leq N(r, a^{(1)}; f^{(1)}) + S(r, f). \tag{17}
\]

Again
\[
m(r, a; f) = m(r, \frac{f^{(1)} - a^{(1)}}{f^{(1)} - a}) \frac{1}{f^{(1)} - a^{(1)}} \\
\leq m(r, a^{(1)}; f^{(1)}) + S(r, f) \\
= T(r, f^{(1)}) - N(r, a^{(1)}; f^{(1)}) + S(r, f) \\
= m(r, f^{(1)}) - N(r, a^{(1)}; f^{(1)}) + S(r, f) \\
\leq m(r, f) - N(r, a^{(1)}; f^{(1)}) + S(r, f) \\
= T(r, f) - N(r, a^{(1)}; f^{(1)}) + S(r, f),
\]
i.e
\[
N(r, a^{(1)}; f^{(1)}) \leq N(r, a; f) + S(r, f).
\]

So from (17) we get
\[
N(r, a; f^{(1)}) \leq N(r, a; f) + S(r, f). \tag{18}
\]

Also
\[
N(r, a; f) \leq N_A(r, a; f) + N(r, a; f \mid f^{(1)} = a) \\
\leq N(r, a; f^{(1)}) + S(r, f). \tag{19}
\]
From (18) and (19) we get

$$N(r, a; f^{(1)}) = N(r, a; f) + S(r, f). \quad (20)$$

Let

$$h = \frac{(a - a^{(1)})(L^{(p)}(f) - L^{(p)}(a)) - (a - L^{(p)}(a))(f^{(1)} - a^{(1)})}{f - a},$$

which is defined in Lemma 2.4.

Clearly $T(r, h) = S(r, h)$.

Now

$$T(r, f) = m(r, f)$$

$$= m(r, a + \frac{1}{h}\{(a - a^{(1)})(L^{(p)}(f) - L^{(p)}(a)) - (a - L^{(p)}(a))(f^{(1)} - a^{(1)})\})$$

$$\leq m(r, (a - a^{(1)})L^{(p)}(f) - (a - L^{(p)})f^{(1)}) + S(r, f)$$

$$\leq m(r, f^{(1)}) + T(r, f^{(1)}) + S(r, f)$$

$$= m(r, f^{(1)}) + S(r, f)$$

$$\leq m(r, f^{(1)}) + S(r, f)$$

$$= T(r, f) + S(r, f).$$

Therefore

$$T(r, f^{(1)}) = T(r, f) + S(r, f). \quad (21)$$

If $h$ is transcendental, then by Lemma 2.4, $m(r, a; f^{(1)}) = S(r, f)$ and from (20) and (21) $m(r, a; f) = S(r, f)$. So from Lemma 2.5, $f = L = \lambda e^z$, $\lambda(\neq 0)$, is a constant.

If $h$ is rational, then by Lemma 2.2 we see that $f$ is of finite order. So by Lemma 2.1 we get $m(r, a; f) = S(r, f)$.

Therefore from Lemma 2.5, $f = L = \lambda e^z$, $\lambda(\neq 0)$ is a constant.
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Subcase 2.2. Let \( f^{(1)} \equiv L^{(p)} \). Now

\[
\begin{align*}
m(r, a; f) &= m(r, \frac{a^{(1)}}{f - a}) \\
&\leq m(r, \frac{a^{(1)}}{f - a}) + S(r, f) \\
&= m(r, \frac{f^{(1)} - (f^{(1)} - a^{(1)})}{f - a}) + S(r, f) \\
&\leq m(r, \frac{f^{(1)}}{f - a}) + S(r, f) \\
&= m(r, \frac{L^{(p)}}{f - a}) + S(r, f). \tag{22}
\end{align*}
\]

Since \( p \geq \text{deg}(a) \), by Lemma of logarithmic derivative, \( m(r, \frac{L^{(p)}}{f - a}) = S(r, f) \), so from (22) \( m(r, a; f) = S(r, f) \).

Therefore from Lemma 2.5, we get \( f = L = \lambda e^z \), \( \lambda(\neq 0) \), is a constant.

This completes the proof of the Main Theorem.

4 Conclusions

Finally we arrive at the conclusion that a non-constant entire function sharing a polynomial with its linear differential polynomial with some conditions defined in Theorem (1.1) belongs to the class of functions \( \mathfrak{F} = \{ \lambda e^z : \lambda \in \mathbb{C} \setminus \{0\} \} \).

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