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Numerical solution for fuzzy fractional differential equations using fuzzy fractional fourth order Runge-Kutta method based on root mean square and contraharmonic mean

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Abstract

The objective of this research is to determine the approximate solution to Fuzzy Fractional Differential Equations (FFDEs). For Fuzzy Fractional Initial Value Problems (FFIVPs), the methods called Fuzzy Fractional Fourth Order Runge-Kutta method based on Root Mean Square (FFRK4RM) and called Fuzzy Fractional Fourth Order Runge-Kutta method based on Contraharmonic Mean (FFRK4CoM) is developed. In this paper, both linear and nonlinear FFDEs can be solved using triangular and trapezoidal fuzzy numbers. FFRRK4RM and FFRK4CoM can be compared. The tables gives the absolute error between the exact and approximate solutions. From the graphical results, the approximate solution approaches the exact result very closely as the step size gets smaller. The outcomes show that the suggested approach is easy to use, accurate, clear, and convenient for solving both linear and nonlinear FFIVP.

Keywords: FFRK4RM, FFRK4CoM, FFDEs, Triangular fuzzy number, Trapezoidal fuzzy number.

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1 Introduction

Fractional calculus is expanded upon in ordinary calculus. This involves computing a function's derivative in any order. The memory and heredity characteristics of numerous substances and processes in porous media, electrical circuits, control, biology, electromagnetic processes, biomechanics, and electro chemistry have been documented using fractional differential operators Agarwal [2010], Baleanu [2012],Tarasov [2010]. Over the past few decades, fractional calculus and its applications have gained popularity, mostly because it has proven to be a helpful tool for modeling a number of complex processes in a wide range of seemingly unrelated fields of science and engineering Podlubny [1999],Kenneth [1993]. Measurement uncertainty is represented by a fuzzy number. Since its invention by Lotfi Zadeh in 1965, fuzzy sets have found utility in a variety of contexts Diamond [1994],Geotschel [1986],O.Kaleva [1987].

The solution for fractional differential equations in uncertain settings was first introduced by Agarwal [2010]. In applied sciences and the engineering, Fuzzy Fractional Differential Equations (FFDEs) is a crucial topic. Consider the general FFDE with the fuzzy initial conditions of the form:

$$^{c}D^{\delta}\tilde{p}(\zeta,\mu) + L\tilde{p}(\zeta,\mu) + N\tilde{p}(\zeta,\mu) = g(\zeta,\mu), \quad n-1<\delta<n,\ n\in\mathbb{N} \tag{1}$$

with the fuzzy inital condition $D^{k}\tilde{p}(0,\mu) = \tilde{h}^{k}(\zeta), \quad k = 0,1,2,\ldots,n-1$, where $D^{\delta}\tilde{p}(\zeta,\mu)$ represents the Caputo fractional derivative of $\tilde{p}(\zeta,\mu)$ and $\tilde{p}(\zeta,\mu)$ is the fuzzy function. The general linear and nonlinear differential operators are denoted by L and N. The source term is $g(\zeta,\mu)$.

Numerous applications of FFDEs are made in the field of biology and other engineering disciplines O.Kaleva [1987]. There have been several implementations in the FFDE sector to find exact and approximative solutions Chakraverty [2016]. Lydia [2020] proposed the numerical solution of fractional RC and RL electrical circuits by fractional Runge-Kutta method. Nirmala [2011] studied numerical solution of fuzzy differential equation by RK4 approach with higher order derivative approximations. Sharmila [2013a] introduced the numerical solution of Nth order fuzzy initial value problems by RK4 approach based on Centroidal mean. Sharmila [2013b] proposed the numerical solution of fuzzy initial value problems by RK4 approach based on Contraharmonic mean. The aforementioned articles assisted in the development of the FFRK4RM and FFRK4CoM for FFDEs.

The format of the paper is as follows: Some fundamental tools utilising the fuzzy number, Mittag-Leffler function, and Caputo fractional derivative are provided in Section 2. In Section 3, the Fuzzy Fractional Initial Value Problem (FFIVP) is defined. In Section 4, the Runge-Kutta method for FFIVP is illustrated. It is also clearly detailed how to solve the FFIVP using the fourth order Runge- Kutta approach based on root mean square and contraharmonic mean. In

Section 5, numerical illustrations demonstrate the manner in which FFRK4RM and FFRK4CoM work to solve both linear and nonlinear FFDEs. The conclusion offered in Section 6.

2 Basic Tools

Definition 2.1. *The fractional derivative of $f(\zeta)$ in Caputo sense, is defined as follows:*

$$^{c}D_t^{\delta} f(\zeta) = \frac{1}{\Gamma(m-\delta)} \int_0^{\zeta} (\zeta - \Phi)^{m-\delta-1} f^{(m)}(\Phi) \mathrm{d}\Phi, \;\; m-1 < \delta \leq m, \; \zeta > 0, \; m \in \mathbb{N}$$

Definition 2.2. *The Mittag-Leffler function is defined as follows,*

$$E_{\delta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\delta + 1)}, \delta \in \mathbb{C}, Re(\delta) > 0$$

A further generalization of the above equation is defined in the form

$$E_{\delta,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\delta + \beta)}, \delta, \beta \in \mathbb{C}, Re(\delta) > 0, Re(\beta) > 0$$

Definition 2.3. *(Triangular fuzzy number)*

It's a three-pointed fuzzy number represented by $N = (\theta_1, \theta_2, \theta_3)$. N's membership function is as follows:

$$\mu_N(\chi) = \begin{cases} 0, & \chi < \theta_1 \\ \frac{\chi - \theta_1}{\theta_2 - \theta_1}, & \theta_1 \leq \chi \leq \theta_2 \\ \frac{\theta_3 - \chi}{\theta_3 - \theta_2}, & \theta_2 \leq \chi \leq \theta_3 \\ 0, & \chi > \theta_3 \end{cases}$$

Definition 2.4. *(Trapezoidal fuzzy number)*

It's a four-pointed fuzzy number represented by $M = (\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4)$. M's membership function is as follows:

$$\mu_M(\chi) = \begin{cases} 0, & \chi < \Upsilon_1 \\ \frac{\chi - \Upsilon_1}{\Upsilon_2 - \Upsilon_1}, & \Upsilon_1 \leq \chi \leq \Upsilon_2 \\ 1, & \Upsilon_2 \leq \chi \leq \Upsilon_3 \\ \frac{\Upsilon_4 - \chi}{\Upsilon_4 - \Upsilon_3}, & \Upsilon_3 \leq \chi \leq \Upsilon_4 \\ 0, & \chi > \Upsilon_4 \end{cases}$$

3 Fuzzy Fractional Initial Value Problem

Consider the following fuzzy differential equation of fractional order

$$\begin{cases} {}^cD^{\delta}\tilde{p}(\zeta,\mu) = f(\zeta,\tilde{p}) \\ \tilde{p}(\zeta_0) = \tilde{y_0} \in \mathbb{R}_F, \quad \delta \in (0,1] \end{cases} \tag{2}$$

Applying Caputo fractional derivative of order $(1-\delta)$ to equation (2) as in Lydia [2020],

$$D^1\tilde{p}(\zeta,\mu) = {}^cD^{1-\delta}f(\zeta,\tilde{p}) \tag{3}$$

where $f(\zeta,\tilde{p})$ be a fuzzy continuous function and δ times differentiable in the independent variable ζ over the range of $[\zeta_0,\zeta_m]$. Let the interval $[\zeta_0,\zeta_m]$ be subdivided into N sub-intervals $[\zeta_j,\zeta_{j+1}]$ of step size $h = (\zeta_m - \zeta_0)/N$ for $j = 0,1,\ldots,N$.

Fuzzy function $\tilde{p}$ is denoted by $\tilde{p} = [\underline{p},\overline{p}]$. It means that the μ-level set of $\tilde{p}(\zeta)$ for $\zeta \in [\zeta_0,T]$ is $[\tilde{p}(\zeta)]_\mu = [\underline{p}(\zeta,\mu),\overline{p}(\zeta,\mu)]$, $[\tilde{p}(\zeta_0)]_\mu = [\underline{p}(\zeta_0,\mu),\overline{p}(\zeta_0,\mu)]$, $\mu \in (0,1]$. We write

$${}^cD^{1-\delta}f(\zeta,\tilde{p}) = [{}^cD^{1-\delta}\underline{f}(\zeta,p), {}^cD^{1-\delta}\overline{f}(\zeta,p)]$$

$${}^cD^{1-\delta}\underline{f}(\zeta,p) = F[\zeta,\underline{p},\overline{p}], \quad {}^cD^{1-\delta}\overline{f}(\zeta,p) = G[\zeta,\underline{p},\overline{p}]$$

because of $\tilde{p}'(\zeta,\mu) = {}^cD^{1-\delta}f(\zeta,\tilde{p})$.

$${}^cD^{1-\delta}\underline{f}(\zeta,p(\zeta,\mu)) = F[\zeta,\underline{p}(\zeta,\mu),\overline{p}(\zeta,\mu)]$$

$${}^cD^{1-\delta}\overline{f}(\zeta,p(\zeta,\mu)) = G[\zeta,\underline{p}(\zeta,\mu),\overline{p}(\zeta,\mu)]$$

The extension principle of Zadeh leads to the following definition of $f(\zeta,p(\zeta))$ when $p(\zeta)$ is a fuzzy number

$$f(\zeta,p(\zeta))(s) = \sup\{p(\zeta)(\tau)/s = f(\zeta,\tau)\}, s \in \mathbb{R}$$

It follows that

$$[{}^cD^{1-\delta}f(\zeta,p(\zeta))]_r = [{}^cD^{1-\delta}\underline{f}(\zeta,p(\zeta,\mu)), {}^cD^{1-\delta}\overline{f}(\zeta,p(\zeta,\mu))], \ \mu \in (0,1]$$

where

$${}^cD^{1-\delta}\underline{f}(\zeta,p(\zeta,\mu)) = \min\{{}^cD^{1-\delta}f(\zeta,p)|p \in [\tilde{p}(\zeta)]_\mu\}$$

$${}^cD^{1-\delta}\overline{f}(\zeta,p(\zeta,\mu)) = \max\{{}^cD^{1-\delta}f(\zeta,p)|p \in [\tilde{p}(\zeta)]_\mu\}$$

4 Runge-Kutta Method for Fuzzy Fractional Initial Value Problems

Consider the following FFDE

$$\begin{cases} {}^{c}D^{\delta}\tilde{p}(\zeta,\mu) = f(\zeta,\tilde{p}) \\ \tilde{p}(\zeta_0) = \tilde{p_0} \in \mathbb{R}_F, \quad \delta \in (0,1] \end{cases} \tag{4}$$

The difference between the value of y at time ζ_{m+1} and time ζ_m is expressed as the basis for all Runge-Kutta methods.

$$\tilde{p}_{m+1} - \tilde{p}_m = \sum_{i=1}^{q} w_i \tilde{k}_i \tag{5}$$

where for $i = 1, 2, \ldots, m$. w_i's are constants and

$$\tilde{k}_i = hD^{1-\delta} f\left(\zeta_m + c_i h, \tilde{p}_m + \sum_{j=1}^{i-1} a_{ij}\tilde{k}_j\right) \tag{6}$$

where $\tilde{k}_i = [\underline{k}_i, \overline{k}_i]$. Equations (4) is to be exact for powers of h through h^q, because it is match with Taylor series of order q.

4.1 The fourth order Runge- Kutta method based on Root Mean Square for Solving FFIVP

Jansirani [2020] proposed the composite RK4 method for based on variety of means by using intuitionistic fuzzy differential equations. Root mean square formula is as follows:

$$\underline{p}_{m+1} = \underline{p}_m + \frac{1}{3}h\left[\sqrt{\frac{\underline{k}_1^2 + \underline{k}_2^2}{2}} + \sqrt{\frac{\underline{k}_2^2 + \underline{k}_3^2}{2}} + \sqrt{\frac{\underline{k}_3^2 + \underline{k}_4^2}{2}}\right]$$

$$\overline{p}_{m+1} = \overline{p}_m + \frac{1}{3}h\left[\sqrt{\frac{\overline{k}_1^2 + \overline{k}_2^2}{2}} + \sqrt{\frac{\overline{k}_2^2 + \overline{k}_3^2}{2}} + \sqrt{\frac{\overline{k}_3^2 + \overline{k}_4^2}{2}}\right]$$

where

$$\begin{aligned} \underline{k}_1 &= \underline{f}(\zeta_m, \underline{p}_m, \overline{p}_m) \\ \overline{k}_1 &= \overline{f}(\zeta_m, \underline{p}_m, \overline{p}_m) \\ \underline{k}_2 &= \underline{f}(\zeta_m + a_1 h, \underline{p}_m + a_1\underline{k}_1, \overline{p}_m + a_1\overline{k}_1) \\ \overline{k}_2 &= \overline{f}(\zeta_m + a_1 h, \underline{p}_m + a_1\underline{k}_1, \overline{p}_m + a_1\overline{k}_1) \end{aligned}$$

$\underline{k}_3 = \underline{f}(\zeta_m + (a_2 + a_3)h, \underline{p}_m + a_2\underline{k}_1 + a_3\underline{k}_2, \overline{p}_m + a_2\overline{k}_1 + a_3\overline{k}_2)$
$\overline{k}_3 = \overline{f}(\zeta_m + (a_2 + a_3)h, \underline{p}_m + a_2\underline{k}_1 + a_3\underline{k}_2, \overline{p}_m + a_2\overline{k}_1 + a_3\overline{k}_2)$
$\underline{k}_4 = \underline{f}(\zeta_m + (a_4 + a_5 + a_6)h, \underline{p}_m + a_4\underline{k}_1 + a_5\underline{k}_2 + a_6\underline{k}_3, \overline{p}_m + a_4\overline{k}_1 + a_5\overline{k}_2 + a_6\overline{k}_3)$
$\overline{k}_4 = \overline{f}(\zeta_m + (a_4 + a_5 + a_6)h, \underline{p}_m + a_4\underline{k}_1 + a_5\underline{k}_2 + a_6\underline{k}_3, \overline{p}_m + a_4\overline{k}_1 + a_5\overline{k}_2 + a_6\overline{k}_3)$
The parameters are: $a_1 = \frac{1}{2}$,$a_2 = \frac{1}{6}$, $a_3 = \frac{7}{16}$, $a_4 = \frac{1}{8}$, $a_5 = \frac{-17}{56}$, $a_6 = \frac{33}{28}$
The parametric form of (2) is,

$$\begin{cases} {}^cD^\delta \underline{p}(\zeta,\mu) = \underline{f}(\zeta, \underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)), \\ {}^cD^\delta \overline{p}(\zeta,\mu) = \overline{f}(\zeta, \underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)), \\ \underline{p}(\zeta_0) = \underline{p}_0(0,\mu), \quad \overline{p}(\zeta_0) = \overline{p}_0(0,\mu) \\ \delta \in (0,1), \quad \mu \in [0,1] \end{cases} \tag{7}$$

Applying Caputo fractional derivative of order $(1-\delta)$ to equation (12) as in Lydia [2020],

$$\begin{cases} D^1 \underline{p}(\zeta,\mu) = {}^cD^{1-\delta} \underline{f}(\zeta, \underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)), \\ D^1 \overline{p}(\zeta,\mu) = {}^cD^{1-\delta} \overline{f}(\zeta, \underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)) \end{cases} \tag{8}$$

with the same initial condition $\underline{p}(\zeta_0) = \underline{p}_0(0,\mu), \quad \overline{p}(\zeta_0) = \overline{p}_0(0,\mu)$.
Let the exact solution $[\tilde{P}(\zeta)]_\mu = [\underline{P}(\zeta,\mu), \overline{P}(\zeta,\mu)]$ is approximated by some $[\tilde{p}(\zeta)]_\mu = [\underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)]$ From (5), (6),

$$\begin{aligned} \underline{p}(\zeta_{m+1},\mu) - \underline{p}(\zeta_m,\mu) &= \sum_{i=1}^{4} w_i \underline{k}_i(\zeta_m, p(\zeta_m,\mu)) \\ \overline{p}(\zeta_{m+1},\mu) - \overline{p}(\zeta_m,\mu) &= \sum_{i=1}^{4} w_i \overline{k}_i(\zeta_m, p(\zeta_m,\mu)) \end{aligned} \tag{9}$$

where for $i = 1, 2, \ldots, m$. $\tilde{k}_i(\zeta, p(\zeta,\mu)) = [\underline{k}_i(t, p(\zeta,\mu)), \overline{k}_i(t, p(\zeta,\mu))]$ and w_i's are constants.

$$\begin{aligned} \underline{k}_i(\zeta_m, p(\zeta_m,\mu)) &= hD^{1-\delta} f\left(\zeta_m + c_i h, \underline{p}(\zeta_m, p(\zeta_m,\mu)) + \sum_{j=1}^{i-1} a_{ij} \underline{k}_j(\zeta_m, p(\zeta_m,\mu))\right) \\ \overline{k}_i(\zeta_m, p(\zeta_m,\mu)) &= hD^{1-\delta} f\left(\zeta_m + c_i h, \overline{p}(\zeta_m, p(\zeta_m,\mu)) + \sum_{j=1}^{i-1} a_{ij} \overline{k}_j(\zeta_m, p(\zeta_m,\mu))\right) \end{aligned} \tag{10}$$

and

$$\begin{aligned} \underline{k}_1(\zeta_m, p(\zeta_m,\mu)) &= \min\left\{ h\ {}^cD^{1-\delta} f(\zeta_m, p) / p \in (\underline{p}(\zeta_m,\mu), \overline{p}(\zeta_m,\mu)) \right\} \\ \overline{k}_1(\zeta_m, p(\zeta_m,\mu)) &= \max\left\{ h\ {}^cD^{1-\delta} f(\zeta_m, p) / p \in (\underline{p}(\zeta_m,\mu), \overline{p}(\zeta_m,\mu)) \right\} \end{aligned}$$

$$\underline{k}_2(\zeta_m, p(\zeta_m,\mu)) = \min\left\{h\ {}^cD^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_1(\zeta_m, p(\zeta_m,\mu)), \overline{s}_1(\zeta_m, p(\zeta_m,\mu)))\right\}$$
$$\overline{k}_2(\zeta_m, p(\zeta_m,\mu)) = \max\left\{h\ {}^cD^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_1(\zeta_m, p(\zeta_m,\mu)), \overline{s}_1(\zeta_m, p(\zeta_m,\mu)))\right\}$$
$$\underline{k}_3(\zeta_m, p(\zeta_m,\mu)) = \min\left\{h\ {}^cD^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_2(\zeta_m, p(\zeta_m,\mu)), \overline{s}_2(\zeta_m, p(\zeta_m,\mu)))\right\}$$
$$\overline{k}_3(\zeta_m, p(\zeta_m,\mu)) = \max\left\{h\ {}^cD^{1-\delta} f(\zeta_m + \frac{1}{2}h, p)/p \in (\underline{s}_2(\zeta_m, p(\zeta_m,\mu)), \overline{s}_2(\zeta_m, p(\zeta_m,\mu)))\right\}$$
$$\underline{k}_4(\zeta_m, p(\zeta_m,\mu)) = \min\left\{h\ {}^cD^{1-\delta} f(\zeta_m + h, p)/p \in (\underline{s}_3(\zeta_m, p(\zeta_m,\mu)), \overline{s}_3(\zeta_m, p(\zeta_m,\mu)))\right\}$$
$$\overline{k}_4(\zeta_m, p(\zeta_m,\mu)) = \max\left\{h\ {}^cD^{1-\delta} f(\zeta_m + h, p)/p \in (\underline{s}_3(\zeta_m, p(\zeta_m,\mu)), \overline{s}_3(\zeta_m, p(\zeta_m,\mu)))\right\}$$

where

$$\underline{s}_1(\zeta_m, p(\zeta_m,\mu)) = \underline{p}(\zeta_m,\mu) + \frac{1}{2}\underline{k}_1(\zeta_m, p(\zeta_m,\mu))$$
$$\overline{s}_1(\zeta_m, p(\zeta_m,\mu)) = \overline{p}(\zeta_m,\mu) + \frac{1}{2}\overline{k}_1(\zeta_m, p(\zeta_m,\mu))$$
$$\underline{s}_2(\zeta_m, p(\zeta_m,\mu)) = \underline{p}(\zeta_m,\mu) + \frac{1}{16}\underline{k}_1(\zeta_m, p(\zeta_m,\mu)) + \frac{7}{16}\underline{k}_2(\zeta_m, p(\zeta_m,\mu))$$
$$\overline{s}_2(\zeta_m, p(\zeta_m,\mu)) = \overline{p}(\zeta_m,\mu) + \frac{1}{16}\overline{k}_1(\zeta_m, p(\zeta_m,\mu)) + \frac{7}{16}\overline{k}_2(\zeta_m, p(\zeta_m,\mu))$$
$$\underline{s}_3(\zeta_m, p(\zeta_m,\mu)) = \underline{p}(\zeta_m,\mu) + \frac{1}{8}\underline{k}_1(\zeta_m, p(\zeta_m,\mu)) - \frac{17}{56}\underline{k}_2(\zeta_m, p(\zeta_m,\mu)) + \frac{33}{28}\underline{k}_3(\zeta_m, p(\zeta_m,\mu))$$
$$\overline{s}_3(\zeta_m, p(\zeta_m,\mu)) = \overline{p}(\zeta_m,\mu) + \frac{1}{8}\overline{k}_1(\zeta_m, p(\zeta_m,\mu)) - \frac{17}{56}\overline{k}_2(\zeta_m, p(\zeta_m,\mu)) + \frac{33}{28}\overline{k}_3(\zeta_m, p(\zeta_m,\mu))$$

$$\begin{aligned}\underline{F}[\zeta_m, p(\zeta_m,\mu)] = \Bigg[& \sqrt{\frac{\underline{k}_1^2(\zeta_m, p(\zeta_m,\mu)) + \underline{k}_2^2(\zeta_m, p(\zeta_m,\mu))}{2}} \\ & + \sqrt{\frac{\underline{k}_2^2(\zeta_m, p(\zeta_m,\mu)) + \underline{k}_3^2(\zeta_m, p(\zeta_m,\mu))}{2}} \\ & + \sqrt{\frac{\underline{k}_3^2(\zeta_m, p(\zeta_m,\mu)) + \underline{k}_4^2(\zeta_m, y(\zeta_m,\mu))}{2}}\Bigg] \\ \overline{F}[\zeta_m, p(\zeta_m,\mu)] = \Bigg[& \sqrt{\frac{\overline{k}_1^2(\zeta_m, p(\zeta_m,\mu)) + \overline{k}_2^2(\zeta_m, p(\zeta_m,\mu))}{2}} \\ & + \sqrt{\frac{\overline{k}_2^2(\zeta_m, p(\zeta_m,\mu)) + \overline{k}_3^2(\zeta_m, p(\zeta_m,\mu))}{2}}\end{aligned}$$

$$+\sqrt{\frac{\overline{k}_3^2(\zeta_m,p(\zeta_m,\mu))+\overline{k}_4^2(\zeta_m,p(\zeta_m,\mu))}{2}}\Bigg]$$

The exact and approximate solutions at $\zeta_m, 0 \leq m \leq N$ are denoted by $[\tilde{P}(\zeta_m)]_\mu = [\underline{P}(\zeta_m,\mu),\overline{P}(\zeta_m,\mu)]$ and $[\tilde{p}(\zeta_m)]_\mu = [\underline{p}(\zeta_m,\mu),\overline{p}(\zeta_m,\mu)]$ respectively. The solution is calculated by grid points at $a = \zeta_0 < \zeta_1 < \cdots < \zeta_m = b$ and $h = \frac{(b-a)}{N} = \zeta_{i+1} - \zeta_i$.
Then, using the initial condition $\underline{p}(\zeta_0) = \underline{p}_0(0,\mu), \overline{p}(\zeta_0) = \overline{p}_0(0,\mu)$ and the fourth order Runge - Kutta formula based on root mean square for (2) is as follows:
The exact solutions at ζ_{m+1} is given by

$$\underline{P}(\zeta_{m+1},\mu) \approx \underline{P}(\zeta_m,\mu) + \frac{1}{3}\underline{F}[\zeta_m, P(\zeta_m,\mu)]$$

$$\overline{P}(\zeta_{m+1},\mu) \approx \overline{P}(\zeta_m,\mu) + \frac{1}{3}\overline{F}[\zeta_m, P(\zeta_m,\mu)]$$

The approximate solutions at t_{n+1} is given by

$$\underline{p}(\zeta_{m+1},\mu) = \underline{p}(\zeta_m,\mu) + \frac{1}{3}\underline{F}[\zeta_m, p(\zeta_m,\mu)]$$

$$\overline{p}(\zeta_{m+1},\mu) = \overline{p}(\zeta_m,\mu) + \frac{1}{3}\overline{F}[\zeta_m, p(\zeta_m,\mu)]$$

To implement this method, numerical approximation is needed to the fractional derivative operator $D^{1-\delta}$ as in Lydia [2020] fuzzify it, then it becomes

$${}^cD^{1-\delta}\underline{f}(\zeta,\underline{p}(\zeta,\mu),\overline{p}(\zeta,\mu)) \approx \frac{h^{\delta-1}}{\Gamma(1+\delta)}\sum_{j=0}^{m} c_{jm}\underline{f}(\zeta_j,\underline{p}(\zeta_j,\mu),\overline{p}(\zeta_j,\mu))$$

$${}^cD^{1-\delta}\overline{f}(\zeta,\underline{p}(\zeta,\mu),\overline{p}(\zeta,\mu)) \approx \frac{h^{\delta-1}}{\Gamma(1+\delta)}\sum_{j=0}^{m} c_{jm}\overline{f}(\zeta_j,\underline{p}(\zeta_j,\mu),\overline{p}(\zeta_j,\mu))$$

where the coefficients $c_{jm},\ j = 0,1,2,\ldots,m$ are obtained using the quadrature rule Lydia [2020]

$$c_{jm} = \begin{cases} \delta m^{\delta-1} - m^\delta + (m-1)^\delta,\ j=0 \\ (m-j+1)^\delta - 2(m-j)^\delta + (m-j-1)^\delta,\ j = 1,2,\ldots,m-1 \\ 1,\ j=m \end{cases} \tag{11}$$

Clearly $\underline{p}(\zeta,\mu)$ and $\overline{p}(\zeta,\mu)$ converges to $\underline{P}(\zeta,\mu)$ and $\overline{P}(\zeta,\mu)$ whenever $h \to 0$.

4.2 The fourth order Runge- Kutta method based on Contraharmonic Mean for FFIVP

Sharmila [2013a] proposed the fourth order RK method based on Contraharmonic Mean for fuzzy initial value problem as follows:

$$\underline{p}_{m+1} = \underline{p}_m + \frac{1}{3}h\left[\frac{\underline{k}_1^2 + \underline{k}_2^2}{\underline{k}_1 + \underline{k}_2} + \frac{\underline{k}_2^2 + \underline{k}_3^2}{\underline{k}_2 + \underline{k}_3} + \frac{\underline{k}_3^2 + \underline{k}_4^2}{\underline{k}_3 + \underline{k}_4}\right]$$

$$\overline{p}_{m+1} = \overline{p}_m + \frac{1}{3}h\left[\frac{\overline{k}_1^2 + \overline{k}_2^2}{\overline{k}_1 + \overline{k}_2} + \frac{\overline{k}_2^2 + \overline{k}_3^2}{\overline{k}_2 + \overline{k}_3} + \frac{\overline{k}_3^2 + \overline{k}_4^2}{\overline{k}_3 + \overline{k}_4}\right]$$

where

$$\begin{aligned}
\underline{k}_1 &= \underline{f}(\zeta_m, \underline{p}_m, \overline{p}_m) \\
\overline{k}_1 &= \overline{f}(\zeta_m, \underline{p}_m, \overline{p}_m) \\
\underline{k}_2 &= \underline{f}(\zeta_m + a_1 h, \underline{p}_m + a_1\underline{k}_1, \overline{p}_m + a_1\overline{k}_1) \\
\overline{k}_2 &= \overline{f}(\zeta_m + a_1 h, \underline{p}_m + a_1\underline{k}_1, \overline{p}_m + a_1\overline{k}_1) \\
\underline{k}_3 &= \underline{f}(\zeta_m + (a_2 + a_3)h, \underline{p}_m + a_2\underline{k}_1 + a_3\underline{k}_2, \overline{p}_m + a_2\overline{k}_1 + a_3\overline{k}_2) \\
\overline{k}_3 &= \overline{f}(\zeta_m + (a_2 + a_3)h, \underline{p}_m + a_2\underline{k}_1 + a_3\underline{k}_2, \overline{p}_m + a_2\overline{k}_1 + a_3\overline{k}_2) \\
\underline{k}_4 &= \underline{f}(\zeta_m + (a_4 + a_5 + a_6)h, \underline{p}_m + a_4\underline{k}_1 + a_5\underline{k}_2 + a_6\underline{k}_3, \overline{p}_m + a_4\overline{k}_1 + a_5\overline{k}_2 + a_6\overline{k}_3) \\
\overline{k}_4 &= \overline{f}(\zeta_m + (a_4 + a_5 + a_6)h, \underline{p}_m + a_4\underline{k}_1 + a_5\underline{k}_2 + a_6\underline{k}_3, \overline{p}_m + a_4\overline{k}_1 + a_5\overline{k}_2 + a_6\overline{k}_3)
\end{aligned}$$

The parameters are: $a_1 = \frac{1}{2}$,$a_2 = \frac{1}{8}$, $a_3 = \frac{3}{8}$, $a_4 = \frac{1}{4}$, $a_5 = \frac{-3}{4}$, $a_6 = \frac{3}{2}$
The parametric form of (2) is,

$$\begin{cases} {}^cD^\delta \underline{p}(\zeta,\mu) = \underline{f}(\zeta, \underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)), \\ {}^cD^\delta \overline{p}(\zeta,\mu) = \overline{f}(\zeta, \underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)), \\ \underline{p}(\zeta_0) = \underline{p}_0(0,\mu), \quad \overline{p}(\zeta_0) = \overline{p}_0(0,\mu) \\ \delta \in (0,1), \quad \mu \in [0,1] \end{cases} \tag{12}$$

Applying Caputo fractional derivative of order $(1-\delta)$ to equation (12) as in Lydia [2020],

$$\begin{cases} D^1 \underline{p}(\zeta,\mu) = {}^cD^{1-\delta} \underline{f}(\zeta, \underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)), \\ D^1 \overline{p}(\zeta,\mu) = {}^cD^{1-\delta} \overline{f}(\zeta, \underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)) \end{cases} \tag{13}$$

with the same initial condition $\underline{p}(\zeta_0) = \underline{p}_0(0,\mu), \quad \overline{p}(\zeta_0) = \overline{p}_0(0,\mu)$
Let the exact solution $[\tilde{p}(\zeta)]_r = [\underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)]$ is approximated by some $[\tilde{p}(\zeta)]_r =$

$[\underline{p}(\zeta,\mu),\overline{p}(\zeta,\mu)]$. From (5), (6),

$$\begin{aligned}\underline{p}(\zeta_{m+1},\mu)-\underline{p}(\zeta_m,\mu)&=\sum_{i=1}^{4}w_i\underline{k}_i(\zeta_m,p(\zeta_m,\mu))\\ \overline{p}(\zeta_{m+1},\mu)-\overline{p}(\zeta_m,\mu)&=\sum_{i=1}^{4}w_i\overline{k}_i(\zeta_m,p(\zeta_m,\mu))\end{aligned}\tag{14}$$

where for $i=1,2,\ldots,m$. $\tilde{k}_i(\zeta,p(\zeta,\mu))=[\underline{k}_i(\zeta,p(\zeta,\mu)),\overline{k}_i(\zeta,p(\zeta,\mu))]$ and w_i's are constants.

$$\begin{aligned}\underline{k}_i(\zeta_m,p(\zeta_m,\mu))&=hD^{1-\delta}f\left(\zeta_m+c_ih,\underline{p}(\zeta_m,p(\zeta_m,\mu))+\sum_{j=1}^{i-1}a_{ij}\underline{k}_j(\zeta_m,p(\zeta_m,\mu))\right)\\ \overline{k}_i(\zeta_m,p(\zeta_m,\mu))&=hD^{1-\delta}f\left(\zeta_m+c_ih,\overline{p}(\zeta_m,p(\zeta_m,\mu))+\sum_{j=1}^{i-1}a_{ij}\overline{k}_j(\zeta_m,p(\zeta_m,\mu))\right)\end{aligned}\tag{15}$$

and

$$\begin{aligned}
\underline{k}_1(\zeta_m,p(\zeta_m,\mu))&=\min\left\{h\,{}^cD^{1-\delta}f(\zeta_m,p)/p\in(\underline{p}(\zeta_m,\mu),\overline{p}(\zeta_m,\mu))\right\}\\
\overline{k}_1(\zeta_m,p(\zeta_m,\mu))&=\max\left\{h\,{}^cD^{1-\delta}f(\zeta_m,p)/p\in(\underline{p}(\zeta_m,\mu),\overline{p}(\zeta_m,\mu))\right\}\\
\underline{k}_2(\zeta_m,p(\zeta_m,\mu))&=\min\left\{h\,{}^cD^{1-\delta}f(\zeta_m+\frac{1}{2}h,p)/p\in(\underline{s}_1(\zeta_m,p(\zeta_m,\mu)),\overline{s}_1(\zeta_m,p(\zeta_m,\mu)))\right\}\\
\overline{k}_2(\zeta_m,p(\zeta_m,\mu))&=\max\left\{h\,{}^cD^{1-\delta}f(\zeta_m+\frac{1}{2}h,p)/p\in(\underline{s}_1(\zeta_m,p(\zeta_m,\mu)),\overline{s}_1(\zeta_m,p(\zeta_m,\mu)))\right\}\\
\underline{k}_3(\zeta_m,p(\zeta_m,\mu))&=\min\left\{h\,{}^cD^{1-\delta}f(\zeta_m+\frac{1}{2}h,p)/p\in(\underline{s}_2(\zeta_m,p(\zeta_m,\mu)),\overline{s}_2(\zeta_m,p(\zeta_m,\mu)))\right\}\\
\overline{k}_3(\zeta_m,p(\zeta_m,\mu))&=\max\left\{h\,{}^cD^{1-\delta}f(\zeta_m+\frac{1}{2}h,p)/p\in(\underline{s}_2(\zeta_m,p(\zeta_m,\mu)),\overline{s}_2(\zeta_m,p(\zeta_m,\mu)))\right\}\\
\underline{k}_4(\zeta_m,p(\zeta_m,\mu))&=\min\left\{h\,{}^cD^{1-\delta}f(\zeta_m+h,p)/p\in(\underline{s}_3(\zeta_m,p(\zeta_m,\mu)),\overline{s}_3(\zeta_m,p(\zeta_m,\mu)))\right\}\\
\overline{k}_4(\zeta_m,p(\zeta_m,\mu))&=\max\left\{h\,{}^cD^{1-\delta}f(\zeta_m+h,p)/p\in(\underline{s}_3(\zeta_m,p(\zeta_m,\mu)),\overline{s}_3(\zeta_m,p(\zeta_m,\mu)))\right\}
\end{aligned}$$

where

$$\begin{aligned}
\underline{s}_1(\zeta_m,p(\zeta_m,\mu))&=\underline{p}(\zeta_m,\mu)+\frac{1}{2}\underline{k}_1(\zeta_m,p(\zeta_m,\mu))\\
\overline{s}_1(\zeta_m,p(\zeta_m,\mu))&=\overline{p}(\zeta_m,\mu)+\frac{1}{2}\overline{k}_1(\zeta_m,p(\zeta_m,\mu))\\
\underline{s}_2(\zeta_m,p(\zeta_m,\mu))&=\underline{p}(\zeta_m,\mu)+\frac{1}{8}\underline{k}_1(\zeta_m,p(\zeta_m,\mu))+\frac{3}{8}\underline{k}_2(\zeta_m,p(\zeta_m,\mu))\\
\overline{s}_2(\zeta_m,p(\zeta_m,\mu))&=\overline{p}(\zeta_m,\mu)+\frac{1}{8}\overline{k}_1(\zeta_m,p(\zeta_m,\mu))+\frac{3}{8}\overline{k}_2(\zeta_m,p(\zeta_m,\mu))
\end{aligned}$$

$$\underline{s}_3(\zeta_m, p(\zeta_m,\mu)) = \underline{p}(\zeta_m,\mu) + \frac{1}{4}\underline{k}_1(\zeta_m, p(\zeta_m,\mu)) - \frac{3}{4}\underline{k}_2(\zeta_m, p(\zeta_m,\mu)) + \frac{3}{2}\underline{k}_3(\zeta_m, p(\zeta_m,\mu))$$
$$\overline{s}_3(\zeta_m, p(\zeta_m,\mu)) = \overline{p}(\zeta_m,\mu) + \frac{1}{4}\overline{k}_1(\zeta_m, p(\zeta_m,\mu)) - \frac{3}{4}\overline{k}_2(\zeta_m, p(\zeta_m,\mu)) + \frac{3}{2}\overline{k}_3(\zeta_m, p(\zeta_m,\mu))$$
$$\begin{aligned}\underline{F}[\zeta_m, p(\zeta_m,\mu)] = \Bigg[& \frac{\underline{k}_1^2(\zeta_m, p(\zeta_m,\mu)) + \underline{k}_2^2(\zeta_m, p(\zeta_m,\mu))}{\underline{k}_1(\zeta_m, p(\zeta_m,\mu)) + \underline{k}_2(\zeta_m, p(\zeta_m,\mu))} \\ & + \frac{\underline{k}_2^2(\zeta_m, p(\zeta_m,\mu)) + \underline{k}_3^2(\zeta_m, p(\zeta_m,\mu))}{\underline{k}_2(\zeta_m, p(\zeta_m,\mu)) + \underline{k}_3(\zeta_m, p(\zeta_m,\mu))} \\ & + \frac{\underline{k}_3^2(\zeta_m, p(\zeta_m,\mu)) + \underline{k}_4^2(\zeta_m, p(\zeta_m,\mu))}{\underline{k}_3(\zeta_m, p(\zeta_m,\mu)) + \underline{k}_4(\zeta_m, p(\zeta_m,\mu))}\Bigg]\end{aligned}$$
$$\begin{aligned}\overline{F}[\zeta_m, p(\zeta_m,\mu)] = \Bigg[& \frac{\overline{k}_1^2(\zeta_m, p(\zeta_m,\mu)) + \overline{k}_2^2(\zeta_m, p(\zeta_m,\mu))}{\overline{k}_1(\zeta_m, p(\zeta_m,\mu)) + \overline{k}_2(\zeta_m, p(\zeta_m,\mu))} \\ & + \frac{\overline{k}_2^2(\zeta_m, p(\zeta_m,\mu)) + \overline{k}_3^2(\zeta_m, p(\zeta_m,\mu))}{\overline{k}_2(\zeta_m, p(\zeta_m,\mu)) + \overline{k}_3(\zeta_m, p(\zeta_m,\mu))} \\ & + \frac{\overline{k}_3^2(\zeta_m, p(\zeta_m,\mu)) + \overline{k}_4^2(\zeta_m, p(\zeta_m,\mu))}{\overline{k}_3(\zeta_m, p(\zeta_m,\mu)) + \overline{k}_4(\zeta_m, p(\zeta_m,\mu))}\Bigg]\end{aligned}$$

The exact and approximate solutions at $\zeta_m, 0 \le m \le N$ are denoted by $[\tilde{P}(\zeta_m)]_\mu = [\underline{P}(\zeta_m,\mu), \overline{P}(\zeta_m,\mu)]$ and $[\tilde{p}(\zeta_m)]_\mu = [\underline{p}(\zeta_m,\mu), \overline{p}(\zeta_m,\mu)]$ respectively. The solution is calculated by grid points at $a = \zeta_0 < \zeta_1 < \cdots < \zeta_m = b$ and $h = \frac{(b-a)}{N} = \zeta_{i+1} - \zeta_i$.
Then, using the initial condition $\underline{p}(\zeta_0) = \underline{p}_0(0,\mu), \overline{p}(\zeta_0) = \overline{p}_0(0,\mu)$ and the fourth order Runge - Kutta formula based on Contraharmonic Mean for (2)is as follows:
The exact solutions at ζ_{m+1} is given by

$$\underline{P}(\zeta_{m+1}) \approx \underline{P}(\zeta_m) + \frac{1}{3}\underline{F}[\zeta_m, P(\zeta_m,\mu)]$$
$$\overline{P}(\zeta_{m+1}) \approx \overline{P}(\zeta_m) + \frac{1}{3}\overline{F}[\zeta_m, P(\zeta_m,\mu)]$$

The approximate solutions at ζ_{m+1} is given by

$$\underline{p}(\zeta_{m+1}) = \underline{p}(\zeta_m) + \frac{1}{3}\underline{F}[\zeta_m, p(\zeta_m,\mu)]$$
$$\overline{p}(\zeta_{m+1}) = \overline{p}(\zeta_m) + \frac{1}{3}\overline{F}[\zeta_m, p(\zeta_m,\mu)]$$

To implement this method, numerical approximation is needed to the fractional derivative operator $D^{1-\delta}$ as in Lydia [2020] fuzzify it, then it becomes

$${}^cD^{1-\delta}\underline{f}(\zeta, \underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)) \approx \frac{h^{\delta-1}}{\Gamma(1+\delta)} \sum_{j=0}^{m} c_{jm}\underline{f}(\zeta_j, \underline{p}(\zeta_j,\mu), \overline{p}(\zeta_j,\mu))$$

$$^{c}D^{1-\delta}\overline{f}(\zeta, \underline{p}(\zeta,\mu), \overline{p}(\zeta,\mu)) \approx \frac{h^{\delta-1}}{\Gamma(1+\delta)} \sum_{j=0}^{m} c_{jm} \overline{f}(\zeta_j, \underline{p}(\zeta_j,\mu), \overline{p}(\zeta_j,\mu))$$

where the coefficients $c_{jm},\ j = 0, 1, 2, \ldots, m$ are obtained using the quadrature rule

$$c_{jm} = \begin{cases} \delta m^{\delta-1} - m^{\delta} + (m-1)^{\delta},\ j = 0 \\ (m-j+1)^{\delta} - 2(m-j)^{\delta} + (m-j-1)^{\delta},\ j = 1, 2, \ldots, m-1 \\ 1,\ j = m \end{cases}$$

Clearly $\underline{p}(\zeta,\mu)$ and $\overline{p}(\zeta,\mu)$ converges to $\underline{P}(\zeta,\mu)$ and $\overline{P}(\zeta,\mu)$ whenever $h \to 0$.

5 Numerical Examples

In this section, examples for linear and nonlinear FFDE is discussed using the methodology of FFRK4RM and FFRK4CoM.

Example 5.1. *Consider the linear triangular FIVP for fractional order*

$$D^{\delta}\tilde{p}(\zeta,\mu) = \tilde{p}(\zeta,\mu) + 1,\ \ 0 < \delta \leq 1,\ \mu \in [0, 1] \tag{16}$$

subject to the initial condition, $\tilde{p}(0,\mu) = [0.2\mu - 0.2, 0.2 - 0.2\mu]$

The Exact solution of (16) is

$$\underline{p}(\zeta,\mu) = (0.2\mu - 0.2)E_{\delta}(\zeta^{\delta}) + \zeta^{\delta}E_{\delta,\delta+1}(\zeta^{\delta})$$

$$\overline{p}(\zeta,\mu) = (0.2 - 0.2\mu)E_{\delta}(\zeta^{\delta}) + \zeta^{\delta}E_{\delta,\delta+1}(\zeta^{\delta})$$

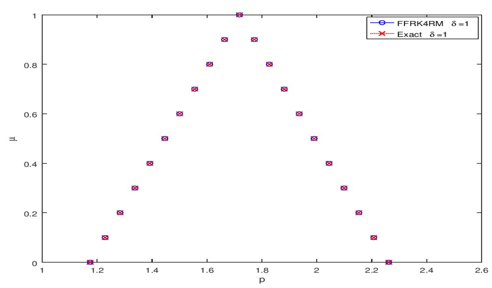

Figure 1: Exact and FFRK4RM Solution for Example 1 for $h = 0.1$, $\zeta = 1$ and $\delta = 1$

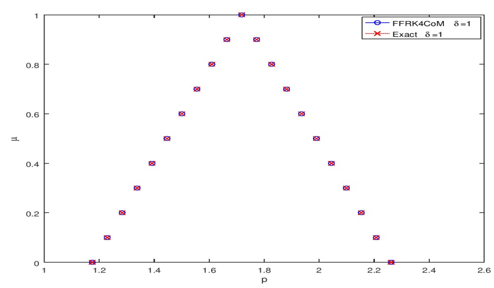

Figure 2: Exact and FFRK4CoM Solution for Example 1 for $h = 0.1$, $\zeta = 1$ and $\delta = 1$

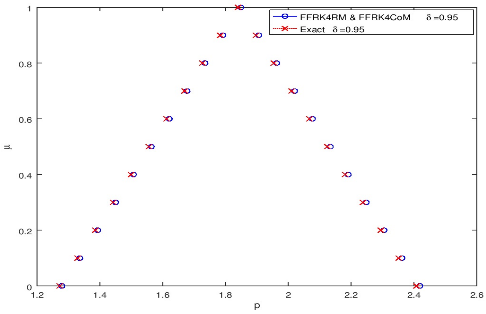

Figure 3: Exact and FFRK4RM & FFRK4CoM Solution for Example 1 for $h = 0.1$, $\zeta = 1$ and $\delta = 0.95$

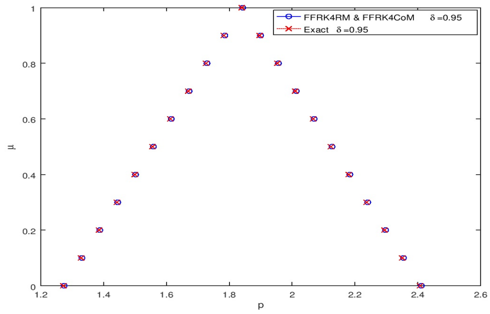

Figure 4: Exact and FFRK4RM & FFRK4CoM Solution for Example 1 for $h = 0.01$, $\zeta = 1$ and $\delta = 0.95$

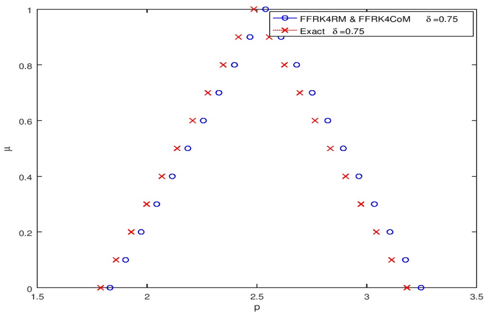

Figure 5: Exact and FFRK4RM & FFRK4CoM Solution for Example 1 for $h = 0.1$, $\zeta = 1$ and $\delta = 0.75$

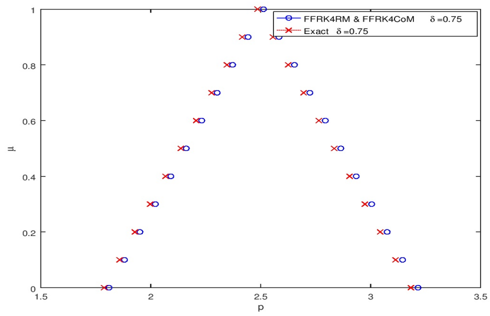

Figure 6: Exact and FFRK4RM & FFRK4CoM Solution for Example 1 for $h = 0.01$, $\zeta = 1$ and $\delta = 0.75$

	FFRK4RM		FFRK4CoM	
μ	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$
0	3.503367e-007	5.255050e-007	2.993491e-006	4.490236e-006
0.1	3.590951e-007	5.167466e-007	3.068328e-006	4.415399e-006
0.2	3.678535e-007	5.079882e-007	3.143165e-006	4.340561e-006
0.3	3.766119e-007	4.992298e-007	3.218002e-006	4.265724e-006
0.4	3.853703e-007	4.904713e-007	3.292840e-006	4.190887e-006
0.5	3.941288e-007	4.817129e-007	3.367677e-006	4.116050e-006
0.6	4.028872e-007	4.729545e-007	3.442514e-006	4.041212e-006
0.7	4.116456e-007	4.641961e-007	3.517351e-006	3.966375e-006
0.8	4.204040e-007	4.554377e-007	3.592189e-006	3.891538e-006
0.9	4.291624e-007	4.466793e-007	3.667026e-006	3.816701e-006
1	4.379208e-007	4.379208e-007	3.741863e-006	3.741863e-006

Table 1: Absolute Error between FFRK4RM and FFRK4CoM of Example 1 for $h = 0.1$, $\zeta = 1$ & $\delta = 1$

	FFRK4RM		FFRK4CoM	
μ	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$
0	7.016838e-003	1.052526e-002	7.020171e-003	1.053026e-002
0.1	7.192259e-003	1.034984e-002	7.195676e-003	1.035475e-002
0.2	7.367679e-003	1.017441e-002	7.371180e-003	1.017925e-002
0.3	7.543100e-003	9.998994e-003	7.546684e-003	1.000374e-002
0.4	7.718521e-003	9.823573e-003	7.722189e-003	9.828240e-003
0.5	7.893942e-003	9.648152e-003	7.897693e-003	9.652736e-003
0.6	8.069363e-003	9.472731e-003	8.073197e-003	9.477231e-003
0.7	8.244784e-003	9.297310e-003	8.248701e-003	9.301727e-003
0.8	8.420205e-003	9.121889e-003	8.424206e-003	9.126223e-003
0.9	8.595626e-003	8.946468e-003	8.599710e-003	8.950719e-003
1	8.771047e-003	8.771047e-003	8.775214e-003	8.775214e-003

Table 2: Absolute Error between FFRK4RM and FFRK4CoM of Example 1 for $h = 0.1$, $\zeta = 1$ & $\delta = 0.95$

	FFRK4RM		FFRK4CoM	
μ	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$
0	4.280237e-002	6.420356e-002	4.281712e-002	6.422568e-002
0.1	4.387243e-002	6.313350e-002	4.388755e-002	6.315525e-002
0.2	4.494249e-002	6.206344e-002	4.495798e-002	6.208483e-002
0.3	4.601255e-002	6.099338e-002	4.602841e-002	6.101440e-002
0.4	4.708261e-002	5.992332e-002	4.709883e-002	5.994397e-002
0.5	4.815267e-002	5.885326e-002	4.816926e-002	5.887354e-002
0.6	4.922273e-002	5.778320e-002	4.923969e-002	5.780311e-002
0.7	5.029279e-002	5.671314e-002	5.031012e-002	5.673269e-002
0.8	5.136285e-002	5.564308e-002	5.138055e-002	5.566226e-002
0.9	5.243291e-002	5.457302e-002	5.245097e-002	5.459183e-002
1	5.350297e-002	5.350297e-002	5.352140e-002	5.352140e-002

Table 3: Absolute Error between FFRK4RM and FFRK4CoM of Example 1 for $h = 0.1$, $\zeta = 1$ & $\delta = 0.75$

Example 5.2. *Consider a nonlinear trapezoidal FIVP for fractional order*

$$D_\zeta^\delta \tilde{p}(\zeta) = 2\tilde{p}(\zeta) - \tilde{p}^2(\zeta) + 1, \tag{17}$$

subject to the initial condition $\tilde{p}(0) = [0.75 + 0.1\mu, 1.125 - 0.125\mu]$

The exact solution for (17) is $\tilde{p}(t) = \frac{\tilde{p}(0)((1+\sqrt{2})e^{2\sqrt{2}\zeta}+\sqrt{2}-1)+e^{2\sqrt{2}\zeta}-1}{\tilde{p}(0)(e^{2\sqrt{2}\zeta}-1)+(\sqrt{2}-1)e^{2\sqrt{2}\zeta}+\sqrt{2}+1}$ at $\delta = 1$

	FFRK4RM		FFRK4CoM	
μ	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$
0	1.376584e-006	2.304300e-007	1.007102e-005	1.212671e-005
0.1	1.341864e-006	2.372080e-007	1.015498e-005	1.212002e-005
0.2	1.307424e-006	2.439957e-007	1.023728e-005	1.211324e-005
0.3	1.273258e-006	2.507932e-007	1.031792e-005	1.210638e-005
0.4	1.239362e-006	2.576005e-007	1.039691e-005	1.209943e-005
0.5	1.205730e-006	2.644176e-007	1.047423e-005	1.209238e-005
0.6	1.172358e-006	2.712445e-007	1.054991e-005	1.208525e-005
0.7	1.139241e-006	2.780813e-007	1.062394e-005	1.207803e-005
0.8	1.106374e-006	2.849278e-007	1.069632e-005	1.207073e-005
0.9	1.073753e-006	2.917843e-007	1.076707e-005	1.206333e-005
1	1.041373e-006	2.986507e-007	1.083618e-005	1.205584e-005

Table 4: Absolute Error between FFRK4RM and FFRK4CoM of Example 2 for $h = 0.1, \zeta = 1$ & $\delta = 1$

	$\delta = 0.95$				$\delta = 0.75$			
	FFRK4RM		FFRK4CoM		FFRK4RM		FFRK4CoM	
μ	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$	$\underline{p}(\zeta,\mu)$	$\overline{p}(\zeta,\mu)$
0	2.184525	2.267603	2.184537	2.267615	2.134656	2.214349	2.134673	2.214366
0.1	2.187265	2.267155	2.187277	2.267167	2.137047	2.213872	2.137064	2.213889
0.2	2.189969	2.266706	2.189981	2.266718	2.139421	2.213395	2.139438	2.213412
0.3	2.192637	2.266255	2.192649	2.266267	2.141777	2.212917	2.141794	2.212934
0.4	2.195271	2.265803	2.195282	2.265815	2.144115	2.212439	2.144132	2.212456
0.5	2.19787	2.26535	2.197882	2.265363	2.146436	2.211959	2.146453	2.211977
0.6	2.200436	2.264897	2.200449	2.264909	2.148741	2.21148	2.148758	2.211497
0.7	2.20297	2.264441	2.202982	2.264454	2.151029	2.21100	2.151046	2.211017
0.8	2.205472	2.263985	2.205485	2.263997	2.153301	2.210519	2.153318	2.210536
0.9	2.207943	2.263528	2.207955	2.26354	2.155558	2.210037	2.155575	2.210054
1	2.210384	2.263069	2.210396	2.263081	2.157799	2.209555	2.157816	2.209572

Table 5: Approximate solution for FFRK4RM and FFRK4CoM of Example 1 for $\delta = 0.95$, $\delta = 0.75$ at $\zeta = 1$.

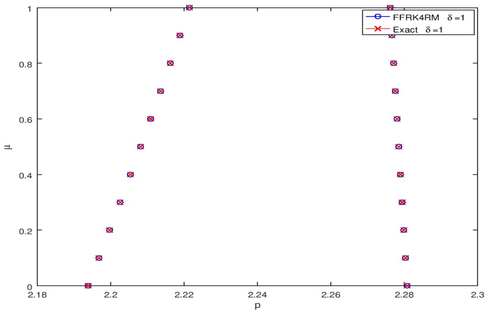

Figure 7: Exact and FFRK4RM & FFRK4CoM Solution for Example 5.2 for $h = 0.1$, $\zeta = 1$ and $\delta = 1$

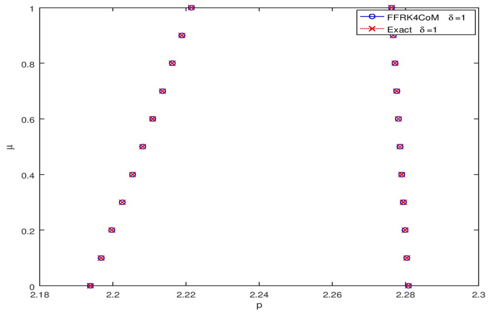

Figure 8: Exact and FFRK4RM & FFRK4CoM Solution for Example 5.2 for $h = 0.1$, $\zeta = 1$ and $\delta = 1$

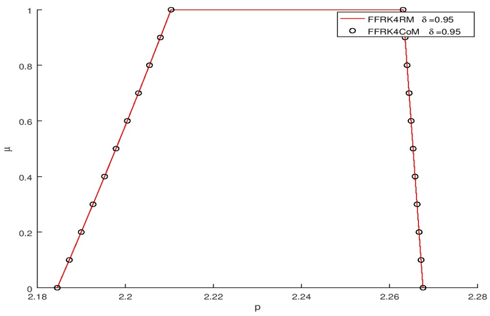

Figure 9: FFRK4RM & FFRK4CoM Solution for Example 5.2 for $h = 0.1$, $\zeta = 1$ and $\delta = 0.95$

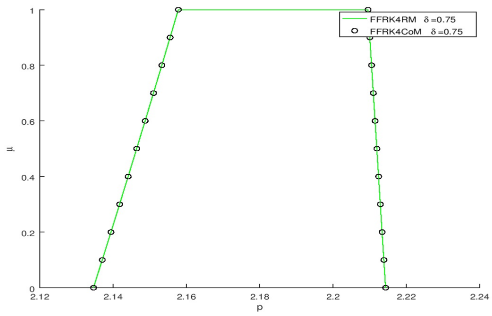

Figure 10: FFRK4RM & FFRK4CoM Solution for Example 5.2 for $h = 0.1$, $\zeta = 1$ and $\delta = 0.75$

6 Conclusion

FFRK4RM and FFRK4CoM have been effectively utilized in this paper for both linear and nonlinear FFIVPs. When the derivative is in fractional order, FFRK4RM and FFRK4CoM almost offer the same result. However, when the derivative order is an integer, FFRK4RM performs better than FFRK4CoM in terms of result. FFRK4RM and FFRK4CoM both methods are given the best result compare with the max-min improved Euler methods. The computational load will be reduced by using the Octave program. The tables and figures show the proposed method is how effectively approaches the exact result very closely as the step size gets smaller. The absolute errors of linear and nonlinear FFIVPs are investigated graphically and numerically for various values of δ in order to

evaluate the dependability and efficiency of the suggested approach. The analysis of the 2D plots and tables for various values of δ confirms that the approximate solution converges quickly to the exact solution.
The benefits of using these techniques are:

- The problem does not need any small or big parametric assumptions.
- Both weak and highly nonlinear problems can be solved using these techniques.
- The suggested approaches generate solutions without perturbation, linearization, or discretization, as opposed to other analytical approximation methods.
- He's and Adomian polynomials are not required for solving nonlinear problems.

It may be inferred that the suggested approaches are rapid, exact, and simple to apply and produce excellent outcomes. As fractional order differential systems in uncertain environments, these methods have been used in the future for many significant applications in engineering and science.

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