Radio Heronian Mean k-Graceful Labeling on Degree Splitting of Graphs

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Abstract

A mapping $g: V(G) \to \{k, k+1, \dots, k+N-1\}$ is a radio heronian mean k-labeling such that if for any two distinct vertices s and t of G, $d(s,t) + \left\lceil \frac{g(s)+g(t)+\sqrt{g(s)g(t)}}{3} \right\rceil \geq 1 + D$,for every $s, t \in$ V(G), where D is the diameter of G. The radio heronian mean k-number of g, $rrhmn_k(g)$, is the maximum number assigned to any vertex of G. The radio heronian mean number of G, $rhmn_k(g)$, is the minimum value of $rhmn_k(g)$ taken overall radio heronian mean labelings g of G. If $rhmn_k(g) = |V(G)| + \mathrm{k} - 1$, we call such graphs as radio heronian mean k-graceful graphs. In this paper, we investigate the radio heronian mean k-graceful labeling on degree splitting of graphs such as comb graph $P_n \odot K_1$, rooted tree graph $RT_{n,n}$ hurdle graph Hd_n and twig graph TW_n.

Keywords: Radio heronian mean labeling, k-graceful labeling, degree splitting graphs, comb graph, rooted tree graph, hurdle graphand twig graph.

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1. Introduction

Throughout this paper we consider the graphs are finite, simple, undirected and connected. Let $V(G)$ and $E(G)$ respectively denote the vertex set and edge set of G.A labeling of a graph $G$is an assignment of integers to the vertices or edges or both subject to certain conditions. Radio labeling or multilevel distance labeling is motivated by the channel assignment problem for radio transmitters [5]. Ponraj et al [8] introduced the notion of radio mean labeling of graphs and investigated radio number of some graphs. The radio mean number of,$rmn(G)$, is the lowest span taken overall radio mean labelings of the graph G. The span of a labelling is the maximum integer that g maps to a vertex of G. S. S. Sandya, et al [10] introduced heronian mean labeling of graphs. Y. Lavanya et al [6], raised the ideas about radio mean graceful graphs. In this sequel, we introduced radio heronian mean k-graceful labelling on degree splitting of graphs. For standard terminology and notations, we follow Harary [4] and Gallian [3].

2. Preliminaries

Definition 2.1. [1] The distance $d(s,t)$from a vertex $s$to a vertex $t$in a connected graph $G$is the minimum length of the $s-t$paths in$G$.

Definition 2.2. [1] The eccentricity $e(t)$ of a vertex t in a connected graph $G$is the distance between $t$and a vertex farthest from $t$in$G$.

Definition 2.3. [1] The diameter $D$is the greatest eccentricity among the vertices of G.

Definition 2.4. [8][9] A radio mean labeling of a connected graph G is a one-to-one map $g : V(G) \to N$such that for any two distinct vertices $s$and $t$of$G$, $d(s,t) + \left\lceil \frac{g(s)+g(t)}{2} \right\rceil \geq 1 + D$ for every $s,t \in V(G)$, where $D$is the diameter of$G$. The radio mean number of $G$is denoted by$rmn(G)$.

Definition 2.5. [10] A graph $G = (V,E)$ with $p$vertices and $q$edges is said to be a heronian mean graph if it is possible to label the vertices $s \in V$with distinct labels $g(s)$ from $1,2,..,q+1$ in such way that when each edge $e = st$is labelled with, $g(e = s,t) = \left\lceil \frac{g(s)+g(t)+\sqrt{g(s)g(t)}}{3} \right\rceil$ (or) $\left\lfloor \frac{g(s)+g(t)+\sqrt{g(s)g(t)}}{3} \right\rfloor$ then the edge labels are distinct. In this case $g$is called a heronian mean labelling of$G$.

Definition 2.6. [1] Let $G = (V,E)$ be a graph with $V = S_1 \cup S_2 \cup \ldots S_i \cup T$, where each $S_i$is a set of vertices having atleast two vertices and having the same degree and $T = V - \cup S_i$. The degree splitting graph of Gdenoted by $DS(G)$ is obtained from $G$by adding vertices$w_1, w_2, \ldots, w_t$and joining $w_i$to each vertex of $S_i (1 \leq i \leq t)$.

Definition 2.7. A radio heronian mean k −labeling of a connected graph G is a one to one map $g : V(G) \to \{k, k+1, k+2, \ldots, k+N-1\}$ such that for any two distinct vertices $s$and $t$of$G$,$d(s,t)+\left\lceil\frac{g(s)+g(t)+\sqrt{g(s)g(t)}}{3}\right\rceil \geq 1+D$, for every $s,t \in V(G)$ where $D$is the diameter of G. The Radio heronian mean k-number of $G$is denoted by$rhmn_k(g)$. If$rhmn_k(g) = |V(G)| + k - 1$, we call such graphs as radio heronian mean k-graceful graphs.

Definition 2.8. [1] The comb graph $P_n \odot K_1$is obtained by joining a single pendent edge to each vertex of a pathP_n.

Definition 2.9. [2] A tree in which one vertex is distinguished from all the other is called a rooted tree and the vertex is called the root of the tree. A rooted tree with $n$vertices is denoted by $RT_{n,n}$

Definition 2.10. [7] A graph obtained from a path P_n by attaching a pendant edge to every internal vertex of the path is called the hurdle graph with $n-2$ hurdles and is denoted by Hd_n.

Definition 2.11. [7] A twig graph $TW_n$is obtained from a path $P_n$by attaching exactly two pendant edges to each internal vertices of the pathP_n.

3. Main Results

Theorem 3.1. The degree splitting of comb graph $DS(P_n \odot K_1)$ is radio heronian mean $k-$graceful for $n \geq 4$ and$k \geq 1$.

Proof. Let $s_i, 1 \leq i \leq n$ be the vertices of the pathK_1. Join the pendent vertex t_i to the vertex $s_i$of the path$P_n, 1 \leq i \leq n$. The resultant graph is$P_n \odot K_1$. Introduce three new vertices $s, s'$and$t$. Join $s'$to s_1,$s_n$and s to $s_i, 2 \leq i \leq n-1$ and t to $t_i, 1 \leq i \leq n$ respectively. The resulting graph is$DS(P_n \odot K_1)$ whose edge set is$E(DS(P_n \odot K_1)) = \{\{s_i s_{i+1}/1 \leq i \leq n-1\} \cup \{s_i t_i, tt_i/1 \leq i \leq n\} \cup \{ss_i/2 \leq i \leq n-1\} \cup \{s's_1, s's_n\}\}$ and diameter of $DS(P_n \odot K_1)$ is $D = 3$.

Define a function $g : V\left(DS(P_n \odot K_1)\right) \rightarrow \{k, k+1, \ldots, k+N-1\}$ by

$g(s') = k, k \geq 1$;

$g(s) = k+1, k \geq 1$;

$g(t) = k+2, k \geq 1$;

$g(s_i) = n+k+i+2, k \geq 1, 1 \leq i \leq n$;

$g(t_i) = k+i+2, k \geq 1, 1 \leq i \leq n$;

Now we verify the radio heronian mean condition for g

Case a: Consider the pair (s,s')

$$d(s,s')+\left\lceil\frac{g(s)+g(s')+\sqrt{g(s)g(s')}}{3}\right\rceil \geq 3+\left\lceil\frac{2k+1+\sqrt{k(k+1)}}{3}\right\rceil \geq 4 = 1+D$$

Case b: Consider the pair (t,s')

$$d(t,s') + \left\lceil \frac{g(t) + g(s') + \sqrt{g(t)g(s')}}{3} \right\rceil \geq 3 + \left\lceil \frac{2k + 2 + \sqrt{(k+2)(k+1)}}{3} \right\rceil \geq 4$$

Case c: Consider the pair (s,t)

$$d(s,t) + \left\lceil \frac{g(s) + g(t) + \sqrt{g(s)g(t)}}{3} \right\rceil \geq 3 + \left\lceil \frac{2k + 3 + \sqrt{(k+2)(k+1)}}{3} \right\rceil \geq 4$$

Case d: Consider the pair (s', s_i), $1 \leq i \leq n$

$$d(s', s_i) + \left\lceil \frac{g(s') + g(s_i) + \sqrt{g(s')g(s_i)}}{3} \right\rceil$$
$$\geq 1 + \left\lceil \frac{n + 2k + i + 2 + \sqrt{k(n+k+i+1)}}{3} \right\rceil \geq 4$$

Case e: Consider the pair (s', t_i), $1 \leq i \leq n$

$$d(s', t_i) + \left\lceil \frac{g(s') + g(t_i) + \sqrt{g(s')g(t_i)}}{3} \right\rceil \geq 2 + \left\lceil \frac{2k + i + 2 + \sqrt{k(k+i+1)}}{3} \right\rceil \geq 4$$

Case f: Consider the pair (s, s_i), $1 \leq i \leq n$

$$d(s, s_i) + \left\lceil \frac{g(s) + g(s_i) + \sqrt{g(s)g(s_i)}}{3} \right\rceil$$
$$\geq 1 + \left\lceil \frac{n + 2k + i + 3 + \sqrt{(k+1)(n+k+i+2)}}{3} \right\rceil \geq 4$$

Case g: Consider the pair (s, t_i), $1 \leq i \leq n$

$$d(s, t_i) + \left\lceil \frac{g(s) + g(t_i) + \sqrt{g(s)g(t_i)}}{3} \right\rceil \geq 2 + \left\lceil \frac{2k + i + 3 + \sqrt{(k+1)(k+i+2)}}{3} \right\rceil$$
$$\geq 4$$

Case h: Consider the pair (t, s_i), $1 \leq i \leq n$

$$d(t, s_i) + \left\lceil \frac{g(t) + g(s_i) + \sqrt{g(t)g(s_i)}}{3} \right\rceil$$
$$\geq 2 + \left\lceil \frac{n + 2k + i + 4 + \sqrt{(k+2)(n+k+i+2)}}{3} \right\rceil \geq 4$$

Case i: Consider the pair (t, t_i), $1 \leq i \leq n$

$$d(t, t_i) + \left\lceil \frac{g(t) + g(t_i) + \sqrt{g(t)g(t_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{2k + i + 4 + \sqrt{(k+2)(k+i+2)}}{3} \right\rceil$$
$$\geq 4$$

Case j: Consider the pair (s_i, s_j), $i \neq j, 1 \leq i \leq n$

$$d(s_i, s_j) + \left\lceil \frac{g(s_i) + g(s_j) + \sqrt{g(s_i)g(s_j)}}{3} \right\rceil$$
$$\geq 1 + \left\lceil \frac{2n + 2k + i + j + 4 + \sqrt{(n + k + i + 2)(n + k + j + 2)}}{3} \right\rceil \geq 4$$

Case k: Consider the pair (t_i, t_j), $i \neq j, 1 \leq i \leq n$

$$d(t_i, t_j) + \left\lceil \frac{g(t_i) + g(t_j) + \sqrt{g(t_i)g(t_j)}}{3} \right\rceil$$
$$\geq 2 + \left\lceil \frac{2k + i + j + 4 + \sqrt{(k + i + 2)(k + j + 2)}}{3} \right\rceil \geq 4$$

Case j: Consider the pair (s_i, t_j), $1 \leq i \leq n$

$$d(s_i, t_j) + \left\lceil \frac{g(s_i) + g(t_j) + \sqrt{g(s_i)g(t_j)}}{3} \right\rceil$$
$$\geq 1 + \left\lceil \frac{n + 2k + i + j + 4 + \sqrt{(n + k + i + 2)(k + j + 2)}}{3} \right\rceil \geq 4$$

Thus, the radio heronian mean condition is satisfied for all pairs of vertices. Hence $g$is a valid radio heronian mean k-labeling of$\mathrm{DS}(\mathrm{P_n} \odot \mathrm{K_1})$
Therefore, $rhmn_k(\mathrm{DS}(\mathrm{P_n} \odot \mathrm{K_1})) = 2n + k + 2, n \geq 4, k \geq 1$.
Clearly $|V\ (DS(P_n \odot K_1))| = 2n + 3$.
Thus $|V\ (DS(P_n \odot K_1)))| + k - 1 = 2n + k + 2, n \geq 4, k \geq 1$.
Therefore, $rhmn_k(\mathrm{DS}(\mathrm{P_n} \odot \mathrm{K_1})) = |V\,(\mathrm{DS}(\mathrm{P_n} \odot \mathrm{K_1}))| + k - 1$.
Hence the degree splitting of comb graph$\mathrm{DS}(\mathrm{P_n} \odot \mathrm{K_1})$ is radio heronian mean k-graceful for n ≥ 4 and k ≥ 1. ■

Theorem 3.2. The degree splitting of rooted tree graph $DS(RT_{n,n})$ is radio heronian mean k-graceful for $n \geq 2$and $k \geq 1$.
Proof. Let $s$be the root of the tree and $s_i, 1 \leq i \leq n$ be the vertices which are joined to the vertex $s$of the tree. Let $t_i, 1 \leq i \leq n$be the vertices which are joined to the vertex$s_i, 1 \leq i \leq n$. Introduce two new vertices s'andt'. Join $s'$to s_i and $t'$to $t_i$the resultant graph is $DS(RT_{n,n})$ whose edge set is $E(DS(RT_{n,n})) = \{ss_i, s_it_i, s's_i, t't_i / 1 \leq i \leq n\}$and diameter of $DS(RT_{n,n})$ is $D = 3$.
Define a $function\ g : V\ (DS(RT_{n,n})) \rightarrow \{k, k + 1, \ldots, k + N - 1\}\ by$
$g(s) = k + 2, k \geq 1$;
$g(s') = k + 1, k \geq 1$;
$g(s_i) = k + i + 2, k \geq 1, 1 \leq i \leq n$

$g(si) = k + i + 2, k \geq 1, 1 \leq;$
$g(t') = k, k \geq 1$
$g(t_i) = n + k + i + 2, k \geq 1, 1 \leq i \leq n;$
Now we verify the radio heronian mean condition for$g$.

Case a: Consider the pair (s, s')

$$d(s,s') + \left\lceil \frac{g(s) + g(s') + \sqrt{g(s)g(s')}}{3} \right\rceil \geq 2 + \left\lceil \frac{2k + 3 + \sqrt{(k+2)(k+1)}}{3} \right\rceil \geq 4$$
$$= 1 + D$$

Case b: Consider the pair (s, s_i), $1 \leq i \leq n$

$$d(s,s_i) + \left\lceil \frac{g(s) + g(s_i) + \sqrt{g(s)g(s_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{2k + i + 4 + \sqrt{(k+2)(k+i+2)}}{3} \right\rceil$$
$$\geq 4$$

Case c: Consider the pair (s, t')

$$d(s,t') + \left\lceil \frac{g(s) + g(t') + \sqrt{g(s)g(t')}}{3} \right\rceil \geq 3 + \left\lceil \frac{2k + 2 + \sqrt{(k+2)(k)}}{3} \right\rceil \geq 4$$

Case d: Consider the pair (s, t_i), $1 \leq i \leq n$

$$d(s,t_i) + \left\lceil \frac{g(s) + g(t_i) + \sqrt{g(s)g(t_i)}}{3} \right\rceil$$
$$\geq 2 + \left\lceil \frac{n + 2k + i + 4 + \sqrt{(k+2)(n+k+i+2)}}{3} \right\rceil \geq 4$$

Case e

Consider the pair (s_i, s_j), $i \neq j, 1 \leq i \leq n$

$$d(s_i,s_j) + \left\lceil \frac{g(s_i) + g(s_j) + \sqrt{g(s_i)g(s_j)}}{3} \right\rceil$$
$$\geq 2 + \left\lceil \frac{2k + i + j + 4 + \sqrt{(k+i+2)(k+j+2)}}{3} \right\rceil \geq 4$$

Case f: Consider the pair (t_i, t_j), $i \neq j, 1 \leq i \leq n$

$$d(t_i,t_j) + \left\lceil \frac{g(t_i) + g(t_j) + \sqrt{g(t_i)g(t_j)}}{3} \right\rceil$$
$$\geq 2 + \left\lceil \frac{2n + 2k + i + j + 4 + \sqrt{(n+k+i+2)(n+k+j+2)}}{3} \right\rceil \geq 4$$

Case g: Consider the pair (s_i, t_j), $1 \leq i, j \leq n$

$$d(s_i,t_j)+\left\lceil\frac{g(s_i)+g(t_j)+\sqrt{g(s_i)g(t_j)}}{3}\right\rceil$$

$$\geq 1+\left\lceil\frac{n+2k+i+j+4+\sqrt{(k+i+2)(n+k+j+2)}}{3}\right\rceil\geq 4$$

Case h: Consider the pair (s_i,t'),$1\leq i\leq n$)

$$d(s_i,t')+\left\lceil\frac{g(s_i)+g(t')+\sqrt{g(s_i)g(t')}}{3}\right\rceil\geq 2+\left\lceil\frac{2k+i+2+\sqrt{(k+i+2)(k)}}{3}\right\rceil\geq 4$$

Case i: Consider the pair (s',t_i), $1\leq i\leq n$

$$d(s',t_i)+\left\lceil\frac{g(s')+g(t_i)+\sqrt{g(s')g(t_i)}}{3}\right\rceil$$

$$\geq 2+\left\lceil\frac{n+2k+i+3+\sqrt{(k+1)(n+k+i+2)}}{3}\right\rceil\geq 4$$

Case j: Consider the pair (s',s_i), $1\leq i\leq n$

$$d(s',s_i)+\left\lceil\frac{g(s')+g(s_i)+\sqrt{g(s')g(s_i)}}{3}\right\rceil$$

$$\geq 1+\left\lceil\frac{2k+i+3+\sqrt{(k+1)(k+i+2)}}{3}\right\rceil\geq 4$$

Case k: Consider the pair (s',t')

$$d(s',t')+\left\lceil\frac{g(s')+g(t')+\sqrt{g(s')g(t')}}{3}\right\rceil\geq 3+\left\lceil\frac{2k+1+\sqrt{k(k+1)}}{3}\right\rceil\geq 4$$

Case l: Consider the pair (s,t')

$$d(s,t')+\left\lceil\frac{g(s)+g(t')+\sqrt{g(s)g(t')}}{3}\right\rceil\geq 3+\left\lceil\frac{2k+2+\sqrt{(k+2)(k+1)}}{3}\right\rceil\geq 4$$

Thus, the radio heronian mean condition is satisfied for all pairs of vertices. Hence $g$is a valid radio heronian mean k-labeling of$DS(RT_{n,n})$.
Therefore,$rhmn_k(DS(RT_{n,n}))=2n+k+2, n\geq 2, k\geq 1$.
Clearly,$|V(DS(RT_{n,n}))|=2n+3$.
Thus, $rhmn_k(DS(RT_{n,n}))=|V(DS(RT_{n,n}))|+k-1$.
Hence the degree splitting of rooted tree graph $DS(RT_{n,n})$ is radio heronian mean $k-$ graceful for $\geq 2, k\geq 1$. ■

Theorem 3.3. The degree splitting of hurdle graph $DS(Hd_n)$is radio heronian mean k-graceful for n ≥ 4 and k ≥ 1.

Proof. Let$s_i, 1 \leq i \leq n$be the vertices of the path P_n by attaching a pendant edge to each internal vertices $s_i$of the pathP_n, $2 \leq i \leq n-1$. The resultant graph isHd_n. Introduce two new vertices $s$and$t$. Join $s$tos_i, $2 \leq i \leq n-1$ and$t$ to$s_1, s_n, t_i, 2 \leq i \leq n-1$. The resultant graph is $DS(Hd_n)$ whose edge set is$E(DS(Hd_n)) = \{\{s_i s_{i+1}, 1 \leq i \leq n-1\} \cup \{s_i t_{i-1}, tt_{i-1}, ss_i, 2 \leq i \leq n-1\} \cup \{ts_1, ts_n\}\}$and diameter of DS (Hd_n) is$D = 3$. Define a function $g : V(DS(Hd_n)) \to \{k, k+1, \dots, k+N-1\} by$

$g(s) = k, k \geq 1$;

$g(t) = k+1, k \geq 1$;

$g(s_i) = n+k+i-1, k \geq 1, 1 \leq i \leq n$;

$g(t_i) = k+i+1, k \geq 1, 1 \leq i \leq n-2$;

Now we verify the radio heronian mean condition for g

Case a: Consider the pair (s, t)

$$d(s,t) + \left\lceil \frac{g(s) + g(t) + \sqrt{g(s)g(t)}}{3} \right\rceil \geq 3 + \left\lceil \frac{2k+1+\sqrt{k(k+1)}}{3} \right\rceil \geq 4 = 1 + D$$

Case b: Consider the pair (s, s_i), $1 \leq i \leq n$

$$d(s,s_i) + \left\lceil \frac{g(s) + g(s_i) + \sqrt{g(s)g(s_i)}}{3} \right\rceil$$

$$\geq 1 + \left\lceil \frac{n+2k+i-1+\sqrt{k(n+k+i-1)}}{3} \right\rceil \geq 4$$

Case c: Consider the pair (s, t_i), $1 \leq i \leq n-2$

$$d(s,t_i) + \left\lceil \frac{g(s) + g(t_i) + \sqrt{g(s)g(t_i)}}{3} \right\rceil \geq 2 + \left\lceil \frac{2k+i+1+\sqrt{k(k+i+1)}}{3} \right\rceil \geq 4$$

Case d: Consider the pair (t, s_i), $1 \leq i \leq n$

$$d(t,s_i) + \left\lceil \frac{g(t) + g(s_i) + \sqrt{g(t)g(s_i)}}{3} \right\rceil$$

$$\geq 1 + \left\lceil \frac{n+2k+i+\sqrt{(k+1)(n+k+i-1)}}{3} \right\rceil \geq 4$$

Case e: Consider the pair (t, t_i), $1 \leq i \leq n-2$

$$d(t,t_i) + \left\lceil \frac{g(t) + g(t_i) + \sqrt{g(t)g(t_i)}}{3} \right\rceil \geq 1 + \left\lceil \frac{2k+i+2+\sqrt{(k+1)(k+i+1)}}{3} \right\rceil$$

$$\geq 4$$

Case f: Consider the pair (s_i, s_j), $i \neq j, 1 \leq i \leq n$

$$d(s_i,s_j) + \left\lceil \frac{g(s_i) + g(s_j) + \sqrt{g(s_i)g(s_j)}}{3} \right\rceil$$

$$\geq 1 + \left\lceil \frac{2n + 2k + i + j - 2 + \sqrt{(n + k + i - 1)(n + k + j - 1)}}{3} \right\rceil \geq 4$$

Case g: Consider the pair (t_i, t_j), $i \neq j, 1 \leq i \leq n - 2$

$$d(t_i, t_j) + \left\lceil \frac{g(t_i) + g(t_j) + \sqrt{g(t_i)g(t_j)}}{3} \right\rceil$$

$$\geq 2 + \left\lceil \frac{2k + i + j + 2 + \sqrt{(k + i + 1)(k + j + 1)}}{3} \right\rceil \geq 4$$

Case h: Consider the pair (s_i, t_j), $1 \leq i \leq n, 1 \leq i \leq n - 2$

$$d(s_i, t_j) + \left\lceil \frac{g(s_i) + g(t_j) + \sqrt{g(s_i)g(t_j)}}{3} \right\rceil$$

$$\geq 1 + \left\lceil \frac{n + 2k + i + j + 4 + \sqrt{(n + k + i + 2)(k + j + 2)}}{3} \right\rceil \geq 4$$

Thus, the radio heronian mean condition is satisfied for all pairs of vertices. Hence g is a valid radio heronian mean k-labeling of $DS(Hd_n)$.

Therefore, $rhmn_k(DS(Hd_n)) = 2n + k - 1, n \geq 4, k \geq 1$. Clearly $|V(DS(Hd_n))|$ = $2n$. Thus $|V(DS(Hd_n))| + k - 1 = 2n + k - 1, n \geq 4, k \geq 1$.

Therefore, $(rhmn_k(DS(Hd_n)) = |V(DS(Hd_n))| + \mathrm{k} - 1$. Hence the degree splitting of hurdle graph $DS(Hd_n)$ is radio heronian mean k-graceful for $n \geq 4$ and $k \geq 1$ ■

Theorem 3.4. The degree splitting of twig graph DS (TW_n) is radio heronian mean k-graceful for n ≥ 4 and k ≥ 1.

Proof. Let $s_i, 1 \leq i \leq n$ be the vertices of the path P_n by attaching exactly two pendent edges to each internal vertices s_i of the path $P_n, 2 \leq i \leq n - 1$. The resultant graph is TW_n. Introduce two new vertices s and t. Join s to $s_i, 2 \leq i \leq n - 1$ and t to $s_1, s_n, t_i, t_i', 2 \leq i \leq n - 2$. the resultant graph is $DS(TW_n)$ whose edge set is $E\big(DS(TW_n)\big) = \big\{\{s_i s_{i+1}/1 \leq i \leq n - 1\} \cup \{s_i t_{i-1}, s_i t'_{i-1}, tt_{i-1}, tt'_{i-1}, ss_i/2 \leq i \leq n - 1\}\big\}$ and diameter of $DS(TW_n)$ is $D = 3$.

Define a function $g : V\big(DS(TWn)\big) \rightarrow \{k, k + 1, \dots, k + N - 1\} by$

$g(s) = k, k \geq 1$;

$g(t) = k + 1, k \geq 1$;

$g(s_i) = 2n + k + i - 3, k \geq 1, 1 \leq i \leq n$;

$g(t_i) = k + i + 1, k \geq 1, 1 \leq i \leq n - 2$;

$g(t'_i) = n + k + i - 1, k \geq 1, 1 \leq i \leq n - 2$;

Now we verify the radio heronian mean condition for g

Case a: Consider the pair (s, t)

$$d(s,t)+\left\lceil\frac{g(s)+g(t)+\sqrt{g(s)g(t)}}{3}\right\rceil \geq 3+\left\lceil\frac{2k+1+\sqrt{k(k+1)}}{3}\right\rceil \geq 4=1+D$$

Case b: Consider the pair (s,s_i), $1\leq i\leq n$

$$d(s,s_i)+\left\lceil\frac{g(s)+g(s_i)+\sqrt{g(s)g(s_i)}}{3}\right\rceil$$
$$\geq 1+\left\lceil\frac{2n+2k+i-3+\sqrt{k(2n+k+i-3)}}{3}\right\rceil \geq 4$$

Case c: Consider the pair (s,t_i), $1\leq i\leq n-2$

$$d(s,t_i)+\left\lceil\frac{g(s)+g(t_i)+\sqrt{g(s)g(t_i)}}{3}\right\rceil \geq 2+\left\lceil\frac{2k+i+1+\sqrt{k(k+i+1)}}{3}\right\rceil \geq 4$$

Case d: Consider the pair (s,t'_i), $1\leq i\leq n-2$

$$d(s,t_i')+\left\lceil\frac{g(s)+g(t'_i)+\sqrt{g(s)g(t'_i)}}{3}\right\rceil \geq 2+\left\lceil\frac{n+2k+i-1+\sqrt{k(k+i-1)}}{3}\right\rceil$$
$$\geq 4$$

Case e: Consider the pair (t,s_i), $1\leq i\leq n$

$$d(t,s_i)+\left\lceil\frac{g(t)+g(s_i)+\sqrt{g(t)g(s_i)}}{3}\right\rceil$$
$$\geq 1+\left\lceil\frac{2n+2k+i-2+\sqrt{(k+1)(2n+k+i-3)}}{3}\right\rceil \geq 4$$

Case f: Consider the pair (t,t_i), $1\leq i\leq n-2$

$$d(t,t_i)+\left\lceil\frac{g(t)+g(t_i)+\sqrt{g(t)g(t_i)}}{3}\right\rceil \geq 1+\left\lceil\frac{2k+i+2+\sqrt{(k+1)(k+i+1)}}{3}\right\rceil$$
$$\geq 4$$

Case g: Consider the pair (t,t'_i), $1\leq i\leq n-2$

$$d(t,t'_i)+\left\lceil\frac{g(t)+g(t'_i)+\sqrt{g(t)g(t'_i)}}{3}\right\rceil$$
$$\geq 1+\left\lceil\frac{n+2k+i+\sqrt{(k+1)(n+k+i+1)}}{3}\right\rceil \geq 4$$

Case h: Consider the pair (s_i,s_j), $i\neq j, 1\leq i\leq n$

$$d(s_i,s_j)+\left\lceil\frac{g(s_i)+g(s_j)+\sqrt{g(s_i)g(s_j)}}{3}\right\rceil$$
$$\geq 1+\left\lceil\frac{4n+2k+i+j-6+\sqrt{(2n+k+i-3)(2n+k+j-3)}}{3}\right\rceil$$
$$\geq 4$$

Case i: Consider the pair (t_i,t_j), $i\neq j, 1\leq i\leq n-2$

$$d(t_i,t_j)+\left\lceil\frac{g(t_i)+g(t_j)+\sqrt{g(t_i)g(t_j)}}{3}\right\rceil$$
$$\geq 2+\left\lceil\frac{2k+i+j+2+\sqrt{(k+i+1)(k+j+1)}}{3}\right\rceil\geq 4$$

Case j: Consider the pair (t'_i,t'_j), $i\neq j, 1\leq i\leq n-2$

$$d(t'_i,t'_j)+\left\lceil\frac{g(t'_i)+g(t'_j)+\sqrt{g(t'_i)g(t'_j)}}{3}\right\rceil$$
$$\geq 2+\left\lceil\frac{2n+2k+i+j-2+\sqrt{(n+k+i1)(n+k+j-1)}}{3}\right\rceil\geq 4$$

Case k: Consider the pair (s_i,t_j), $1\leq i\leq n, 1\leq i\leq n-2$

$$d(s_i,t_j)+\left\lceil\frac{g(s_i)+g(t_j)+\sqrt{g(s_i)g(t_j)}}{3}\right\rceil$$
$$\geq 1+\left\lceil\frac{2n+2k+i+j-2+\sqrt{(2n+k+i-3)(k+j+1)}}{3}\right\rceil\geq 4$$

Case l: Consider the pair (s_i,t'_j), $1\leq i\leq n, 1\leq i\leq n-2$

$$d(s_i,t'_j)+\left\lceil\frac{g(s_i)+g(t'_j)+\sqrt{g(s_i)g(t'_j)}}{3}\right\rceil$$
$$\geq 1+\left\lceil\frac{3n+2k+i+j-4+\sqrt{(2n+k+i-3)(n+k+j-1)}}{3}\right\rceil\geq 4$$

Case m: Consider the pair (t_i,t'_j), $1\leq i,j\leq n-2$

$$d(t_i, t'_j) + \left\lceil \frac{g(t_i) + g(t'_j) + \sqrt{g(t_i)g(t'_j)}}{3} \right\rceil$$
$$\geq 1 + \left\lceil \frac{n + 2k + i + j + \sqrt{(k + i1)(n + k + j - 1)}}{3} \right\rceil \geq 4$$

Thus, the radio heronian mean condition is satisfied for all pairs of vertices. Hence,$g$is a valid radio heronian mean k-labeling of $DS(TW_n)$.
Therefore,$rhmn_k(DS(TW_n)) = 3n + k - 3, n \geq 4, k \geq 1$.
Clearly, $|V\ (DS(TWn))| = 3n - 2$.
Thus, $|V\ (DS(TW_n))| + k - 1 = 3n + k - 3, n \geq 4, k \geq 1$.
Therefore,$rhmn_k(DS(TW_n)) = |V\ (DS(TW_n))| + k - 1$.
Hence the degree splitting of twig graph $DS(TW_n)$ is radio heronian mean k-graceful for$n \geq 4, k \geq 1$.

4. Conclusion

In this paper, we investigate degree splitting of comb graph, rooted tree graph, hurdle graph and twig graph are radio heronian mean k- graceful. In future, we investigate degree splitting on m-graceful labeling of graphs.

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