

Relation-theoretic contraction principle in metric spaces using multiplicative contraction

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Abstract

Alam and Imdad have presented a novel application of the Banach contraction principle on a complete metric spaces with a binary relation. We have extended the concept of binary relation with the multiplicative contraction in a complete metric spaces. We have also included corollary to demonstrate our results.

Keywords: fixed point, metric spaces, binary relation, multiplicative contraction

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1 Introduction

In many scientific domains, particularly in fixed point theory, the concept of a metric space is extremely useful. This notion has been generalised in numerous directions in recent years, and many notions of a metric-type space have been introduced (b-metric, dislocated space, generalised metric space, quasi-metric space, symmetric space, etc.). The Banach contraction principle's [3] contraction condition has been generalised to numerous forms in the last fifty years. Furthermore, the metric space in the Banach contraction principle has been generalised to a variety of generalised metric spaces. Many authors researched other sorts of fixed point theorems in metric spaces later on, as seen by the and references therein.

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2 Preliminaries

We give the required background material needed to prove our results in this part to make our exposition self-contained. In what follows, N, N_0, Q and R denote the sets of positive integers, non-negative integers, rational numbers and real numbers, respectively.

Aftab and Alam [1] proved new relation-theoretic fixed point theorems on metric spaces in 2015, and then inferred comparable findings in metric spaces.

Metric spaces are sets in which there is a defined a notion of 'distance between pair of points'. The concept of metric spaces was formulated in 1906 by M.Frechet [7], though the definition presently in use given by the German mathematician, Felix Hausdorff.

Definition 2.1. Let M be a non empty arbitrary set and d be a real function from $M \times M$ into R^+ such that for all $u, v, w \in M$ we have

1. $d(u, v) \geq 0$,
2. $d(u, v) = 0 \iff u = v$,
3. $d(u, v) = d(v, u)$ and
4. $d(u, w) \leq d(u, v) + d(v, w)$,

Here (M, d) is called a metric in R and (R, d) is a metric space.

Example 2.1. 1. $d(u, v) = |u - v|$ is a metric space in R .

2. If $d(u, v)$ defined by

$$d(u, v) = \begin{cases} 1 & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

Definition 2.2. [10] Let M be a nonempty set. A subset R of M^2 is called a binary relation on M . Notice that for each pair $u, v \in M$, one of the following conditions holds:

1. $(u, v) \in R$; which amounts to saying that "u is R-related to v" or "u relates to v under R". Sometimes, we write uRv instead of $(u, v) \in R$;
2. $(u, v) \notin R$; which means that "u is not R-related to v" or "u does not relate to v under R".

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Definition 2.3. [10] Let R be a binary relation defined on a nonempty set M and $u, v \in M$. We say that u and v are R -comparative if either $(u, v) \in R$ or $(v, u) \in R$. We denote it by $[u, v] \in R$.

Definition 2.4. [6, 10, 11, 12, 15] A binary relation R defined on a nonempty set M is called

1. reflexive if $(u, u) \in R$ for all $u \in M$,
2. irreflexive if $(u, u) \notin R$ for all $u \in M$,
3. symmetric if $(u, v) \in R$ implies $(v, u) \in R$,
4. antisymmetric if $(u, v) \in R$ implies $(v, u) \notin R$,
5. transitive if $(u, w) \in R$ and $(w, v) \in R$ implies $(u, v) \in R$,
6. complete, connected or dichotomous if $[l, n] \in R$ for all $l, n \in M$,
7. weakly complete, weakly connected or trichotomous if $[u, v] \in R$ or $u = v$ for all $u, v \in M$.
8. strict order or sharp order if R is irreflexive and transitive,
9. near-order if R is antisymmetric and transitive,
10. pseudo-order if R is reflexive and antisymmetric,
11. quasi-order or preorder if R is reflexive and transitive,
12. partial order if R is reflexive, antisymmetric and transitive,
13. simple order if R is weakly complete strict order,
14. weak order if R is complete preorder,
15. total order, linear order or chain if R is complete partial order,
16. tolerance if R is reflexive and symmetric,
17. equivalence if R is reflexive, symmetric and transitive.

Definition 2.5. [4] Let M be a nonempty set and R a binary relation on M . A sequence $\{u_n\} \subset M$ is called R -preserving if

$$(u_n, u_{n+1}) \in R \quad \forall n \in N_0$$

The notion of d - self closeness of a partial order $\preceq$ defined by Turinici [16] is extended to an arbitrary binary relation in the following lines.

Now, we state and prove our main result, which is as follows:

Theorem 2.1. *Let (M, d) be a complete metric space, R a binary relation on M and T a self-mapping on M . Suppose that the following conditions hold: a) $M(f; R)$ is nonempty, b) R is f -closed, c) either f is continuous or R is p -self-closed, d) there exists $\lambda \in [0, 1)$ $d(f(u), f(v)) \leq d(u, v)^\lambda$ for all $u, v \in M$ with $(u, v) \in R$. Then f has a fixed point. Moreover, if e) $\Upsilon(u, v, R^s)$ is nonempty, for each $u, v \in M$, then f has a unique fixed point.*

Proof. Consider a point $u_0 \in M$. Now we define a sequence $\{u_n\}$ of Picard iterates, i.e., $u_n = fu_{n-1}$ for $n = 1, 2, \dots$. From the multiplicative contraction property [13] of f for all $n \in N_0$. As $(u_0, fu_0) \in R$, using condition (b), we get

$$(fu_0, f^2u_0), (f^2u_0, f^3u_0), \dots, (f^nu_0, f^{n+1}u_0), \dots \in R$$

so that

$$(u_n, u_{n+1}) \in R \quad n \in N_0. \quad (1)$$

Thus the sequence $\{u_n\}$ is R -preserving. Applying the contractivity condition (d) to equation (1), we deduce, for all $n \in N_0$, that

$$d(u_{n+1}, u_n) \leq d(u_n, u_{n-1})^\lambda \leq d(u_{n-1}, u_{n-2})^{\lambda^2} \leq \dots \leq d(u_1, u_0)^{\lambda^n}.$$

which by induction yields that

$$d(u_{n+1}, u_{n+2}) \leq d(u_0, fu_0)^{\lambda^{n+1}} \quad n \in N_0. \quad (2)$$

Using equation (2) and triangular inequality, for all $n \in N_0, p \in N, p \geq 2$, we have

$$\begin{aligned} d(u_{n+1}, u_{n+p}) &\leq d(u_{n+1}, u_{n+2}) + d(u_{n+2}, u_{n+3}) + \dots + d(u_{n+p-1}, u_{n+p}) \\ &\leq d(u_1, u_0)^{\lambda^{n+1} + \dots + \lambda^p} \\ &\leq d(u_1, u_0)^{\frac{\lambda^p}{1-\lambda}} \end{aligned}$$

This implies that $d(u_n, u_p) \rightarrow 0$ as $(n, p \rightarrow \infty)$. Hence the sequence $(x_n) = (f^n u_0)$ is multiplicative Cauchy. By the completeness of M , there is $z \in M$ such that $u_n \rightarrow z$ as $n \rightarrow \infty$. Moreover,

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$$d(fz, z) \leq d(fu_n, fz) \cdot d(fu_n, z) \leq d(u_n, z)^\lambda \cdot d(fu_n, z) \rightarrow 1 \text{ as } n \rightarrow \infty,$$

which implies $d(fz, z) = 0$. Therefore this says that z is a fixed point of f ; that is $fz = z$.

Now, if there is another point y such that $fy = y$, then

$$d(z, y) = d(fz, fy) \leq d(z, y)^\lambda.$$

Therefore $d(z, y) = 0$ and $y = z$. This implies that z is the unique fixed point of f .

Alternatively, let us assume that R is d -self-closed. As u_n is an R -preserving sequence and

$$u_n \rightarrow^d u,$$

there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with

$$[u_{n_k}, u] \in R \quad k \in N_0$$

Using (d), $[u_{n_k}, u] \in R$ and $u_n \rightarrow^d u$, we obtain

$$d(u_{n_k+1}, fu) = d(fu_{n_k}, fu) \leq d(u_{n_k}, u)^\lambda \rightarrow 1 \text{ as } k \rightarrow \infty$$

so that $u_{n_k+1} \rightarrow^d f(u)$. Again, owing to the uniqueness of limit, we get $f(u) = u$ so that u is a fixed point of f . To prove uniqueness, take $u, v \in F(f)$, i.e.,

$$f(u) = u \quad \text{and} \quad f(v) = v. \quad (3)$$

By assumption (d), there exists a path (say $\{z_0, z_1, z_2, \dots, z_k\}$) of some finite length k in R^s from u to v so that

$$z_0 = u, z_k = v, [z_i, z_{i+1}] \in R \quad \text{foreach } i(0 \leq i \leq k-1). \quad (4)$$

As R is f -closed, we have

$$[f^n z_i, f^n z_{i+1}] \in R \quad \text{foreach } i(0 \leq i \leq k-1) \quad \text{and} \quad \text{for each } n \in N_0. \quad (5)$$

Making use of equations (3),(4),(5),, triangular inequality and assumption (d), we obtain

$$\begin{aligned} d(u, v) &= d(f^n z_0, f^n z_k) \leq \sum_{i=0}^{k-1} d(f^n z_i, f^n z_{i+1}) \\ &\leq \sum_{i=0}^{k-1} d(f^{n-1} z_i, f^{n-1} z_{i+1})^\lambda \\ &\leq \sum_{i=0}^{k-1} d(f^{n-2} z_i, f^{n-2} z_{i+1})^{\lambda^2} \end{aligned}$$

$$\leq \dots \leq \sum_{k=1}^{i=0} d(z_i, z_{i+1})^{\lambda^n} \\ \rightarrow 0 \text{ as } n \rightarrow \infty$$

so that $u = v$. Hence f has a unique fixed point. $\square$

Corolary 2.1. Let (M, d) be a complete metric space. For ϵ with $\epsilon > 1$ and $u_0 \in M$, consider the multiplicative closed ball, $\overline{B}_\epsilon(u_0)$. Suppose the mapping $f : M \rightarrow M$ satisfies the contraction condition

$$d(f(u), f(v)) \leq d(u, v)^\lambda \quad \text{for all } u, v \in \overline{B}_\epsilon(u_0)$$

where $\lambda \in [0, 1)$ is a constant R is a relation such that $d(fu_0, u_0) \leq \epsilon^{1-\lambda}$. Then f has a unique fixed point in $\overline{B}_\epsilon(u_0)$.

Corolary 2.2. Let (M, d) be a complete metric space. If a mapping $f : M \rightarrow M$ satisfies for some positive integer n ,

$$d(f^n u, f^n v) \leq d(u, v)^\lambda \quad \text{for all } u, v \in M,$$

where $\lambda \in [0, 1)$ is a constant, then f has a unique fixed point in M .

Theorem 2.2. Let (M, d) be a complete metric space, R a binary relation on M and T a self-mapping on M . Suppose that the following conditions hold: a) $M(f; R)$ is nonempty,

b) R is f -closed,

c) either f is continuous or R is p -self-closed,

d) there exists $\lambda \in [0, \frac{1}{2})$ $d(fu, fv) \leq (d(f(u, v).d(fv, u)))^\lambda$ for all $u, v \in M$ with $(u, v) \in R$

Then f has a fixed point. Moreover, if e) $\Upsilon(u, v, R^s)$ is nonempty, for each $u, v \in M$,

then f has a unique fixed point.

Proof. Consider a point $u_0 \in M$. Now we define a sequence $\{u_n\}$ of Picard iterates, i.e., $u_n = fu_{n-1}$ for $n = 1, 2, \dots$ From the multiplicative contraction property of f for all $n \in N_0$. As $(u_0, fu_0) \in R$, using condition 2, we get

$$(fu_0, f^2u_0), (f^2u_0, f^3u_0), \dots, (f^nu_0, f^{n+1}u_0), \dots \in R$$

we have

$$d(u_{n+1}, u_n) = d(fu_n, fu_{n-1}) \leq (d(fu_n, u_n).d(fu_{n-1}, u_{n-1}))^\lambda \\ = (d(u_{n+1}, u_n).d(u_n, u_{n-1}))^\lambda$$

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Thus we have

$$d(u_{n+1}, u_n) \leq (d(u_n, u_{n-1}))^{\frac{\lambda}{1-\lambda}} = d(u_n, u_{n-1})^h,$$

where $h = \frac{\lambda}{1-\lambda}$. For $n > m$, Using triangular inequality, for all $n \in N_0, m \in N, m \geq m$, we have

$$\begin{aligned} d(u_n, u_m) &\leq d(u_n, u_{n-1}).d(u_{n-1}, u_{n-2})...d(u_{m+1}, u_m) \\ &\leq d(u_1, u_0)^{h^{n-1}+h^{n-2}+...+h^m} \leq d(u_1, u_0)^{\frac{h^m}{1-h}} \end{aligned}$$

This implies $d(u_n, u_m) \rightarrow 1$ as $(n, m \rightarrow \infty)$. Hence (u_n) is a Cauchy sequence. By the completeness of M , there is $z \in M$ such that $u_n \rightarrow z$ as $n \rightarrow \infty$. Since

$$\begin{aligned} d(fz, z) &\leq d(fu_n, fz).d(fu_n, z) \\ &\leq (d(fu_n, u_n).d(fz, z))^\lambda.d(u_{n+1}, z), \end{aligned}$$

we have

$$d(u_{n+1}, u_n) \leq (d(u_n, u_{n-1}))^{\frac{\lambda}{1-\lambda}} = d(u_n, u_{n-1})^h,$$

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$$\begin{aligned} d(fz, z) &\leq d(fu_n, fz).d(fu_n, z) \\ &\leq (d(fu_n, u_n).d(fz, z))^\lambda.d(u_{n+1}, z), \end{aligned}$$

we have

$$d(fz, z) \leq (d(fu_n, u_n)^\lambda.d(u_{n+1}, z))^{\frac{1}{1-\lambda}} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Hence $d(fz, z) = 0$. This implies $fz = z$. Finally, it is easy to prove that the fixed point of f is unique.

Alternatively, let us assume that R is d -self-closed. As $\{u_n\}$ is an R -preserving sequence and

$$u_n \xrightarrow{d} u,$$

there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with

$$[u_{n_k}, u] \in R \quad k \in N_0$$

Using (d), $[u_{n_k}, u] \in R$ and $u_{n_k} \rightarrow^d u$, we obtain

$$d(u_{n_k+1}, fu) = d(fu_{n_k}, fu) \leq (d(u_{n_k+1}, u).d(u_{n_k}, u))^\lambda \rightarrow 1 \text{ as } k \rightarrow \infty$$

so that $u_{n_k+1} \rightarrow^d f(u)$. Again, owing to the uniqueness of limit, we obtain $f(u) = u$, so that u is a fixed point of f .

To prove uniqueness, take $u, v \in F(f)$, i.e.,

$$f(u) = u \quad \text{and} \quad f(v) = v. \quad (6)$$

By assumption (e), there exists a path (say $\{z_0, z_1, z_2, \dots, z_k\}$) of some finite length k in R^s from u to v so that

$$z_0 = u, z_k = v, [z_i, z_{i+1}] \in R \quad \text{for each } i(0 \leq i \leq k-1). \quad (7)$$

As R is f -closed, we have

$$[f^n z_i, f^n z_{i+1}] \in R \quad \text{for each } i(0 \leq i \leq k-1) \quad \text{and for each } n \in N_0. \quad (8)$$

Making use of equations (6),(7),(8),, triangular inequality and assumption (d) we obtain

$$\begin{aligned} d(u, v) &= d(f^n z_0, f^n z_k) \leq \sum_{k-1}^{i=0} d(f^n z_i, f^n z_{i+1}) \\ &\leq \sum_{k-1}^{i=0} (d(f^{n-1} z_i, f^{n-1} z_{i+1}).d(f^{n-1} z_{i+1}, f^{n-1} z_i))^\lambda \\ &\leq \sum_{k-1}^{i=0} (d(f^{n-2} z_i, f^{n-2} z_{i+1}).d(f^{n-2} z_{i+1}, f^{n-2} z_i))^{\lambda^2} \\ &\leq \dots \leq \sum_{k-1}^{i=0} (d(z_i, z_{i+1}).d(z_{i+1}, z_i))^{\lambda^n} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

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$$(fu_0, f^2u_0), (f^2u_0, f^3u_0), \dots, (f^n u_0, f^{n+1}u_0), \dots \in R$$

we have

$$\begin{aligned} d(u_{n+1}, u_n) &= d(fu_n, fu_{n-1}) \leq (d(fu_n, u_n) \cdot d(fu_{n-1}, u_{n-1}))^\lambda \\ &= (d(u_{n+1}, u_n) \cdot d(u_n, u_{n-1}))^\lambda \end{aligned}$$

Thus we have

$$d(u_{n+1}, u_n) \leq d(u_n, u_{n-1})^{\frac{\lambda}{1-\lambda}} = d(u_n, u_{n-1})^h,$$

where $h = \frac{\lambda}{1-\lambda}$. For $n > m$, Using triangular inequality, for all $n \in N_0, m \in N, m \geq m$, we have

$$\begin{aligned} d(u_n, u_m) &\leq d(u_n, u_{n-1}) \cdot d(u_{n-1}, u_{n-2}) \dots d(u_{m+1}, u_m) \\ &\leq d(u_1, u_0)^{h^{n-1} + h^{n-2} + \dots + h^m} \leq d(u_1, u_0)^{\frac{h^m}{1-h}} \end{aligned}$$

This implies $d(u_n, u_m) \rightarrow 1$ as $(n, m \rightarrow \infty)$. Hence (u_n) is a Cauchy sequence. By the completeness of M , there is $z \in M$ such that $u_n \rightarrow z$ as $n \rightarrow \infty$. Since

$$\begin{aligned} d(fz, z) &\leq d(fu_n, fz) \cdot d(fu_n, z) \\ &\leq (d(fu_n, u_n) \cdot d(fz, z))^\lambda \cdot d(u_{n+1}, z), \end{aligned}$$

we have

$$d(fz, z) \leq (d(fu_n, u_n)^\lambda \cdot d(u_{n+1}, z))^{\frac{1}{1-\lambda}} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Hence $d(fz, z) = 0$. This implies $fz = z$. Finally, it is easy to prove that the fixed point of f is unique.

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there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with

$$[u_{n_k}, u] \in R \quad k \in N_0$$

Using (d) , $[u_{n_k}, u] \in R$ and $u_{n_k} \rightarrow^d u$, we obtain

$$d(u_{n_k+1}, fu) = d(fu_{n_k}, fu) \leq (d(u_{n_k+1}, u).d(u_{n_k}, u))^\lambda \rightarrow 1 \text{ as } k \rightarrow \infty$$

so that $u_{n_k+1} \xrightarrow{d} f(u)$. Again, owing to the uniqueness of limit, we obtain $f(u) = u$, so that u is a fixed point of f .

To prove uniqueness, take $u, v \in F(f)$, i.e.,

$$f(u) = u \quad \text{and} \quad f(v) = v. \quad (9)$$

By assumption (e), there exists a path (say $\{z_0, z_1, z_2, \dots, z_k\}$) of some finite length k in R^s from u to v so that

$$z_0 = u, z_k = v, [z_i, z_{i+1}] \in R \quad \text{foreach} \quad i(0 \leq i \leq k-1). \quad (10)$$

As R is f -closed, we have

$$[f^n z_i, f^n z_{i+1}] \in R \quad \text{foreach} \quad i(0 \leq i \leq k-1) \quad \text{and} \quad \text{for each } n \in N_0. \quad (11)$$

Making use of equations (9),(10),(11), triangular inequality and assumption (d), we obtain

$$\begin{aligned} d(u, v) &= d(f^n z_0, f^n z_k) \leq \sum_{k-1}^{i=0} d(f^n z_i, f^n z_{i+1}) \\ &\leq \sum_{k-1}^{i=0} (d(f^{n-1} z_i, f^{n-1} z_{i+1}).d(f^{n-1} z_{i+1}, f^{n-1} z_i))^\lambda \\ &\leq \sum_{k-1}^{i=0} (d(f^{n-2} z_i, f^{n-2} z_{i+1}).d(f^{n-2} z_{i+1}, f^{n-2} z_i))^{\lambda^2} \\ &\leq \dots \leq \sum_{k-1}^{i=0} (d(z_i, z_{i+1}).d(z_{i+1}, z_i))^{\lambda^n} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

so that $u = v$. Hence f has a unique fixed point. $\square$

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