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Solving Fully Fuzzy Linear Systems with Triangular and Pentagonal Fuzzy Numbers

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Abstract

In this paper algorithms are proposed to solve fully fuzzy linear system (FFLS) using triangular fuzzy number and pentagonal fuzzy number by converting the system into block system of linear equations known as associated fuzzy system. The method is illustrated by numerical examples.

Keywords: Fully fuzzy linear systems, Triangular fuzzy number, Pentagonal fuzzy number, fuzzy systems.

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1 Introduction

In many real-life applications, fuzzy sets (Zadeh 1965) play an important role instead of crisp approach. Many applications have made use of the fuzzy concept including computer applications, communication such as. solving fuzzy linear system of equations (FLS) is one field of applied mathematics that shows in various applications when the coefficients of system can be taken by fuzzy numbers. The solutions of FLS were first introduced by Friedman et al. (1998) by proposing a method for solving n x n system which includes crisp coefficients matrix with an arbitrary fuzzy number vector as right-hand side. A fuzzy linear system where all the elements of the system are fuzzy numbers is called a fully fuzzy linear system (FFLS). In this paper we use the multiplication operation on matrices to solve fully fuzzy linear system. In section 2, we define some of definitions and basics of triangular and pentagonal fuzzy numbers. In section 3, an algorithm to solve triangular fuzzy system of equations and pentagonal fuzzy system of equations is discussed in section 4. In section 5, numerical examples are solved. In section 6, conclusion is given.

2 Preliminaries

Definition 2.1. The characteristic function $\mu_{\tilde{A}}$ of a crisp set A of X assigns a value either 0 or 1 to each member in X. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e., $\mu_{\tilde{A}}$: X → [0,1] . The assigned value indicates the membership grade of the element in the set A. The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A}=\{(x, \mu_{\tilde{A}}(\mathrm{x})); x \in X\}$ is called fuzzy set.

Definition 2.2. A triangular fuzzy number $\tilde{A}$ is fuzzy number whose membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 + \dfrac{x-a}{\epsilon}, & if\ a - \epsilon \leq x \leq a \\ 1 - \dfrac{x-a}{\epsilon}, & if\ a \leq x \leq a + \delta \\ 0, & otherwise \end{cases}$$

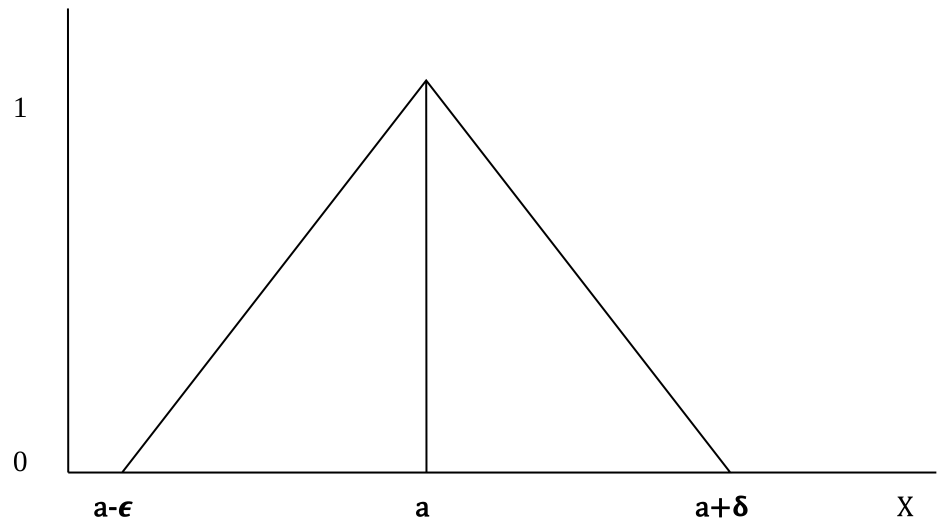

Figure 1.1 Triangular Fuzzy Number

and simply denoted by $\tilde{A} = (a, \epsilon, \delta)$

Definition 2.3. A fuzzy number $\tilde{A}$ is positive $\mu_{\tilde{A}}(x) = 0$ for every $x \leq 0$.

Definition 2.4. Let $\tilde{A} = (a, \varepsilon_1, \delta_1)$ and $\tilde{B} = (b, \varepsilon_2, \delta_2)$ are two triangular fuzzy numbers. Then addition of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \oplus \tilde{B} = (a, \varepsilon_1, \delta_1) \oplus (b, \varepsilon_2, \delta_2)$$
$$= (a + b, \varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2)$$

Definition 2.5. Let $\tilde{A} = (a, \varepsilon_1, \delta_1)$ and $\tilde{B} = (b, \varepsilon_2, \delta_2)$ are two triangular fuzzy numbers. Then product of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \otimes \tilde{B} = (a, \varepsilon_1, \delta_1) \otimes (b, \varepsilon_2, \delta_2) = (\mathcal{A}, \mathcal{B}, \mathcal{C})$$

Where, $\mathcal{A} = ab$, $\mathcal{B} = a\varepsilon_2 + b\varepsilon_1 - \varepsilon_1\varepsilon_2$ $\mathcal{C} =, a\delta_2 + b\delta_1 - \delta_1\delta_2$

Definition 2.6. A fuzzy number $\tilde{A} = (a, b, c, d, e)$ is said to be a pentagonal fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0\,, & if x < \ a - \varepsilon_2 \\ \frac{1}{2}\left(\frac{x - a + \varepsilon_1}{\varepsilon_1 - \varepsilon_2}\right), & if a - \varepsilon_1 \leq x \leq a - \varepsilon_2 \\ 1 + \frac{1}{2}\left(\frac{x - a}{\varepsilon_2}\right), & if\ a - \varepsilon_2 \leq x \leq a \\ 1\,, & if\ x = a \\ 1 - \frac{1}{2}\left(\frac{x - a}{\delta_1}\right), & if\ a \leq x \leq a + \delta_1 \\ \frac{1}{2}\left(\frac{x - a\delta_2}{\varepsilon_1 - \delta_2}\right), & if a + \delta_1 \leq x \leq a + \delta_2 \\ 0\,, & if\ x > a + \delta_2 \end{cases}$$

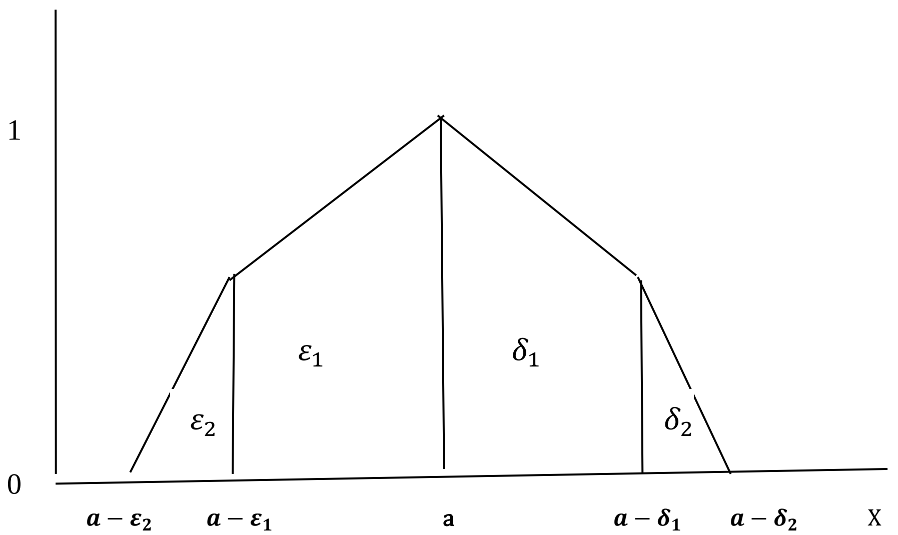

Figure 2.2 Pentagonal Fuzzy Number

Definition 2.7. Let $\tilde{A} = (a, \varepsilon_1, \delta_1, \gamma_1, \theta_1)$ and $\tilde{B} = (b, \varepsilon_2, \delta_2, \gamma_2, \theta_2)$ are two pentagonal fuzzy numbers. Then addition of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A} \oplus \tilde{B} = (a, \varepsilon_1, \delta_1, \gamma_1, \theta_1) \oplus (b, \varepsilon_2, \delta_2, \gamma_2, \theta_2)$$
$$= (a + b, \varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2, \gamma_1 + \gamma_2, \theta_1 + \theta_2)$$

Definition 2.8. Let $\widetilde{A} = (a, \varepsilon_1, \delta_1, \gamma_1, \theta_1)$ and $\tilde{B} = (b, \varepsilon_2, \delta_2, \gamma_2, \theta_2)$ are two pentagonal fuzzy numbers. Then product of $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{A}\otimes\tilde{B} = (a,\varepsilon_1,\delta_1,\gamma_1,\theta_1)\otimes(b,\varepsilon_2,\delta_2,\gamma_2,\theta_2) = (\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{E})$$

where $\mathcal{A} = ab$, $\mathcal{B} = a\varepsilon_2 + b\varepsilon_1 - \varepsilon_1\varepsilon_2$, $\mathcal{C} = a\delta_2 + b\delta_1 - \delta_1\delta_2$
$\mathcal{D} = a\gamma_2 + b\gamma_1 - \gamma_1\gamma_2$, $\mathcal{E} = a\theta_2 + b\theta_1 - \theta_1\theta_2$

3 Triangular fuzzy linear systems

Consider the fully fuzzy triangular linear system $AX = B$. where $A = (a_{ij})$ is the coefficient matrix, $X = (x_1, x_2, x_3, \dots x_j)$ is the unknown variable, $B = (b_1, b_2, b_3, \dots b_j)$ is the constant vector. Let $a_{ij} = (a_{ij}, \varepsilon_{ij}, \delta_{ij})$, $b_j = (b_j, g_j, h_j)$ and $x_j = (x_j, y_j, z_j)$ be triangular fuzzy numbers by $i = 1,2, \dots, n$ and j $= 1,2, \dots, n$.

Then $AX = B$ ------ (1)

is rewritten in the form

$$(a_{ij}, \varepsilon_{ij}, \delta_{ij})\otimes(x_j, y_j, z_j) = (b_j, g_j, h_j)$$

$$\left(\sum a_{ij}x_j, \sum a_{ij}y_j + \sum x_j\varepsilon_{ij} - \sum y_j\varepsilon_{ij}, \sum a_{ij}z_j + \sum x_j\delta_{ij} - \sum z_j\delta_{ij}\right) = \left(b_j, g_j, h_j\right) \text{---- (2)}$$

$$\left(\sum a_{ij}x_j, \sum a_{ij}y_j + \sum \varepsilon_{ij}(x_j - y_j), \sum a_{ij}z_j + \sum \delta_{ij}(x_j - z_j)\right) = \left(b_j, g_j, h_j\right) \text{------- (3)}$$

Comparing on both sides, the system is rewritten as

$Sx = b$ -------- (4)

$Sy + T(x - y) = g$ -------- (5)

$Sz + Q(x - z) = h$ -------- (6)

Where, $S = \sum a_{ij}$, $T = \sum \varepsilon_{ij}$, $Q = \sum \delta_{ij}$

Equation (4) to (6) can be rewritten as a $3n \times 3n$ block system linear equation. This system is known as the associated system of fuzzy system and is written in matrix representation as follows,

$$\begin{pmatrix} S & 0 & 0 \\ T & (S-T) & 0 \\ Q & 0 & (S-Q) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ g \\ h \end{pmatrix} \qquad \text{-------- (7)}$$

The solution of the above matrix is

$$\left.\begin{aligned} x &= S^{-1}b \\ \mathrm{y} &= (\mathrm{T}^{-1}\mathrm{S} - \mathrm{I})^{-1}(\mathrm{T}^{-1}\mathrm{h} - \mathrm{Ix}) \\ \mathrm{z} &= (\mathrm{Q}^{-1}\mathrm{s} - \mathrm{I})^{-1}(\mathrm{Q}^{-1}\mathrm{h} - \mathrm{Ix}) \end{aligned}\right\} \qquad \text{-------- (8)}$$

4 Pentagonal fuzzy linear systems

Consider the fully fuzzy pentagonal linear system $AX = B$. where $A = (a_{ij})$ is the coefficient matrix, $X = (x_1, x_2, x_3, \dots x_j)$ is the unknown variable, $B = (b_1, b_2, b_3, \dots b_j)$ is the constant vector. $a_{ij} = (a_{ij}, \varepsilon_{ij}, \delta_{ij}, \gamma_{ij}, \theta_{ij})$, $b_j = \left(b_j, g_j, h_j, k_j, l_j\right)$ and

$x_j = (x_j, y_j, z_j, w_j, v_j)$ are pentagonal fuzzy numbers by $i = 1,2,\dots,n$ and $j = 1,2,\dots,n$.

Then $AX = B$------- (9)

is rewritten in the form

$\left(a_{ij}, \varepsilon_{ij}, \delta_{ij}, \gamma_{ij}, \theta_{ij}\right) \otimes \left(b_j, g_j, h_j, k_j, l_j\right) = (b_j, g_j, h_j, k_j, l_j)$-------- (10)

$\left(\sum a_{ij}x_j, \sum a_{ij}y_j + \sum x_j\varepsilon_{ij} - \sum y_j\varepsilon_{ij}, \sum a_{ij}z_j + \sum x_j\delta_{ij} - \sum z_j\delta_{ij}, \sum a_{ij}w_j + \sum x_j\gamma_{ij} - \sum w_j\gamma_{ij}, \sum a_{ij}v_j + \sum x_j\theta_{ij} - \sum v_j\theta_{ij}\right) = \left(b_j, g_j, h_j, k_j, l_j\right)$ ------- (11)

$\left(\sum a_{ij}x_j, \sum a_{ij}y_j + \sum \varepsilon_{ij}(x_j - y_j), \sum a_{ij}z_j + \sum \delta_{ij}(x_j - z_j), \sum a_{ij}w_j + \sum \gamma_{ij}(x_j - w_j), \sum a_{ij}v_j + \sum \theta_{ij}(x_j - v_j)\right) = \left(b_j, g_j, h_j, k_j, l_j\right)$ -------- (12)

Comparing on both sides, the system is rewritten as

$$\left.\begin{aligned} Sx &= b \\ Sy + T(x-y) &= g \\ Sz + Q(x-z) &= h \\ Sw + R(x-w) &= k \\ Sv + H(x-v) &= l \end{aligned}\right\} \text{-------- (13)}$$

where,

$$\left.\begin{aligned} S &= \sum a_{ij} \\ T &= \sum \varepsilon_{ij} \\ Q &= \sum \delta_{ij} \\ R &= \sum \gamma_{ij} \\ H &= \sum \theta_{ij} \end{aligned}\right\} \text{-------- (14)}$$

Equation (13) can be rewritten as a $5n \times 5n$ block system linear equation. This system is known as the associated system of fuzzy system and is written in matrix representation as follows,

$$\begin{pmatrix} S & 0 & 0 & 0 & 0 \\ T & (S-T) & 0 & 0 & 0 \\ Q & 0 & (S-Q) & 0 & 0 \\ R & 0 & 0 & (S-R) & 0 \\ H & 0 & 0 & 0 & (S-H) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \\ v \end{pmatrix} = \begin{pmatrix} b \\ h \\ g \\ k \\ l \end{pmatrix} \text{------- (15)}$$

$$\begin{pmatrix} \mathrm{I} & 0 & 0 & 0 & 0 \\ \mathrm{I} & (\mathrm{T}^{-1}\mathrm{S} - \mathrm{I}) & 0 & 0 & 0 \\ \mathrm{I} & 0 & (\mathrm{Q}^{-1}\mathrm{S} - \mathrm{I}) & 0 & 0 \\ \mathrm{I} & 0 & 0 & (\mathrm{R}^{-1}\mathrm{S} - \mathrm{I}) & 0 \\ \mathrm{I} & 0 & 0 & 0 & (\mathrm{H}^{-1}\mathrm{S} - \mathrm{I}) \end{pmatrix} \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \\ \mathrm{w} \\ \mathrm{v} \end{pmatrix} = \begin{pmatrix} \mathrm{S}^{-1}\mathrm{b} \\ \mathrm{T}^{-1}\mathrm{h} \\ \mathrm{Q}^{-1}\mathrm{g} \\ \mathrm{R}^{-1}\mathrm{k} \\ \mathrm{H}^{-1}\mathrm{l} \end{pmatrix} \text{-------- (16)}$$

The solution of the above matrix is

$$\left.\begin{aligned} x = S^{-1}b\mathrm{y} &= (\mathrm{T}^{-1}\mathrm{S} - \mathrm{I})^{-1}(\mathrm{T}^{-1}\mathrm{h} - \mathrm{Ix}) \\ \mathrm{z} &= (\mathrm{Q}^{-1}\mathrm{s} - \mathrm{I})^{-1}(\mathrm{Q}^{-1}\mathrm{h} - \mathrm{Ix}) \\ \mathrm{w} &= (\mathrm{R}^{-1}\mathrm{S} - \mathrm{I})^{-1}(\mathrm{R}^{-1}\mathrm{K} - \mathrm{Ix}) \\ \mathrm{v} &= (\mathrm{H}^{-1}\mathrm{S} - \mathrm{I})^{-1}(\mathrm{H}^{-1}\mathrm{l} - \mathrm{Ix}) \end{aligned}\right\} \text{------- (17)}$$

5 Numerical Examples

1. Consider the following triangular fully fuzzy linear system

$$\begin{bmatrix}(3,1,2) & (5,2,3)\\(4,2,1) & (6,1,2)\end{bmatrix}\begin{bmatrix}(x_1,x_2,x_3)\\(y_1,y_2,y_3)\end{bmatrix}=\begin{bmatrix}(11,15,12)\\(14,18,20)\end{bmatrix}$$

Let us find the 2 x 2 coefficient matrices defined in Section 3.

$$S=\begin{bmatrix}3 & 5\\4 & 6\end{bmatrix},\ T=\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}, Q=\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}.$$

The corresponding right-hand side vectors are

$$b=\begin{bmatrix}11\\14\end{bmatrix},\quad g=\begin{bmatrix}15\\18\end{bmatrix},\quad h=\begin{bmatrix}12\\20\end{bmatrix}.$$

The solution of the matrix is presented in section 3.

$$x=S^{-1}b.$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}.$$

$$\mathrm{y}=(\mathrm{T}^{-1}\mathrm{S}-\mathrm{I})^{-1}(\mathrm{T}^{-1}\mathrm{h}-\mathrm{Ix}).$$

$$\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}7\\4\end{bmatrix}.$$

$$\mathrm{z}=(\mathrm{Q}^{-1}\mathrm{s}-\mathrm{I})^{-1}(\mathrm{Q}^{-1}\mathrm{h}-\mathrm{Ix}).$$

$$\begin{bmatrix}z_1\\z_2\end{bmatrix}=\begin{bmatrix}16\\-9\end{bmatrix}.$$

2. Consider the following pentagonal fully fuzzy linear system

$$\begin{bmatrix}(3,1,1,2,2) & (1,2,2,3,3)\\(1,3,3,4,4) & (4,2,2,5,5)\end{bmatrix}\begin{bmatrix}(x_1,x_2,x_3)\\(y_1,y_2,y_3)\end{bmatrix}=\begin{bmatrix}(22,10,10,12,12)\\(24,12,12,14,14)\end{bmatrix}$$

As per first step let us find the 2 x 2 matrices defined section 4.

$$S=\begin{bmatrix}3 & 1\\1 & 4\end{bmatrix},\ T=\begin{bmatrix}1 & 2\\3 & 2\end{bmatrix},\ Q=\begin{bmatrix}1 & 2\\3 & 2\end{bmatrix},$$

$$R=\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}, H=\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$$

The corresponding right-hand side vectors:

$$b=\begin{bmatrix}22\\24\end{bmatrix}, g=\begin{bmatrix}10\\12\end{bmatrix}, h=\begin{bmatrix}10\\12\end{bmatrix}, k=\begin{bmatrix}12\\14\end{bmatrix}, l=\begin{bmatrix}12\\14\end{bmatrix}$$

The solution presented in section 4.

$$x=S^{-1}b.$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}5.82\\4.55\end{bmatrix}.$$

$$\mathrm{y}=(\mathrm{T}^{-1}\mathrm{S}-\mathrm{I})^{-1}(\mathrm{T}^{-1}\mathrm{h}-\mathrm{Ix}).$$

$$\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}-15.34\\2.27\end{bmatrix}.$$

$$\mathrm{z}=(\mathrm{Q}^{-1}\mathrm{s}-\mathrm{I})^{-1}(\mathrm{Q}^{-1}\mathrm{h}-\mathrm{Ix}).$$

$$\begin{bmatrix}z_1\\z_2\end{bmatrix}=\begin{bmatrix}22.45\\5.32\end{bmatrix}.$$

$$\mathrm{w}=(\mathrm{R}^{-1}\mathrm{s}-\mathrm{I})^{-1}(\mathrm{R}^{-1}\mathrm{k}-\mathrm{Ix}).$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 10.25 \\ 7.67 \end{bmatrix}.$$

$$v = (H^{-1}s - I)^{-1}(H^{-1}l - Ix).$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 14.45 \\ -2.38 \end{bmatrix}.$$

6 Conclusion

In this paper we consider fully fuzzy triangular linear system of equations and fully fuzzy pentagonal system of equations, the solution of FFLS is obtained by constructing associated system of equations by multiplication operation on matrices. This method yields solution faster than other existing ones.

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