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Detour Self-Decomposition of Corona Product of Graphs

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Abstract

Decomposition of a graph G is the collection of edge-disjoint subgraphs of G. The longest distance between any two vertices of G is its detour distance. A subset S of $V(G)$ is a detour set if every vertex of G lie on some $u - v$ detour path, where $u, v \in S$. If a graph G can be decomposed into subgraphs $G_1, G_2, ..., G_n$ with same detour number as G then the decomposition $\Pi = (G_1, G_2, ..., G_n)$ is called detour self-decomposition. The cardinality of maximum such possibility of detour self-decomposition in G is the detour self-decomposition number of G and is denoted by $\pi_{sdn}(G)$. The bounds of detour self-decomposition number of corona product of graphs based on few properties have been discussed here.

Keywords: Detour number; Detour self-decomposition; Detour self-decomposition number; Corona product;
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1 Introduction

The graphs $G = (V, E)$ that we have used in this work are all finite, simple, connected and undirected. We refer to [4] for important graph theory terms.

G. Chartrand, P. Zhang, and G.L. Johns [1] introduced the notion of the detour number. In the graph G for any two vertices x and y, the notation $D(x, y)$ refers to the detour distance which is the longest $x - y$ path of length in G. The $x - y$ detour means a $x - y$ path of $D(x, y)$ length. Vertices lying at any $x - y$ detour of G are represented by $I_D[x, y]$ and for any of the subset S of $V(G)$, $I_D[S]$ implies $\cup_{x,y \in S} I_D[x, y]$. If $I_D[S] = V$, S is considered as a detour set and also the detour set with the least number of vertices in G is the minimum detour set and also the cardinality of this set is a detour number. It has applications in game theory, telephone switching centres, facility location, distributed computing, information retrieval and communication networks.

Graph decompositions is one of the interesting areas of research in Graph Theory. The edge disjoint subgraphs collection $G_1, G_2, ..., G_n$ of G represents the decomposition of G if $G's$ each edge is in exactly one $G_i, 1 \leq i \leq n$ [6].

The idea behind H-decomposition was introduced by L.Posa, P. Erdos, and A. W. Goodman [3] and various problems related to H-decomposition has been studied in recent years. Recent researchers found interest in graph decompositions based on some graph properties. The concept ”steiner decomposition number of graphs” [7] introduced by M. Mahiba & E.E.R. Merly, in which the graph and its subgraphs had the same steiner number motivated us to introduce the concept of detour self-decomposition of graphs. In [5], we introduced the concept of ”detour self-decomposition of graphs”, some general properties and bounds of detourself-decomposition of some graphs has been studied.

Many research articles have been published based on graph operations. Researchers are particularly drawn to the corona product among the several graph operations now in use because of its complicated yet distinctive structure. The corona $G \odot H$ of the given graphs G and H is well-defined as the graph acquired with taking one copy of G and $|V(G)|$ number of copies of H. Here, the i^{th} vertex of G has been adjacent to every vertex at the i^{th} copy of H. Due to the structural properties of the corona product of graphs, it is interesting to study its detour self-decomposition.

In this paper, keeping the first graph as any general graph and giving certain restrictions to the second graph, the detour number and the bounds of the detour self-decomposition number of the corona product of such graphs have been studied.

2 Preliminaries

The subsequent work makes use of the following definitions and theorems.

Definition 2.1. *[5] The decomposition $\Pi = (G_1, G_2, ..., G_n)$ of G is stated to be detour self-decomposition if $dn(G) = dn(G_i), 1 \leq i \leq n$ and detour self-decomposition number of G is the greatest cardinality of such decomposition and is represented as $\pi_{sdn}(G)$.*

Theorem 2.1. *[2] Every detour-set for the non-trivial connected graph G consists every end vertices of that graph.*

Theorem 2.2. *[2] A tree T having k end-vertices has detour number k.*

Theorem 2.3. *[9] Any G of an order $p \geq 2, 2 \leq dn(G) \leq p$*

3 Main Results

Lemma 3.1. *Consider v as the cut vertex for G. If $G - \{v\}$ has m components then $dn(G) \geq m$ i.e., the detour set at the least consists one of the vertex from every component of $G - \{v\}$.*

Proof. Let v represent a cut vertex of the graph, G.

Let $G - \{v\}$ has m components $H_1, H_2, ..., H_m$.

To prove: $dn(G) \geq m$ i.e., to prove the minimum number of detour set of G contains atleast m vertices.

Suppose $dn(G) = m_1$,where $m_1 < m$.

Since $m_1 < m$, some components of $G - \{v\}$ does not have any vertices in S, let H_i be one of the components of $G - \{v\}$ that does not have any vertices at the S. Meanwhile $I_D[S] = V(G)$ means there is a vertex pair $v_j, v_k \in S$ such that the vertices of H_i lies in some $v_j - v_k$ detour path in G. Also since v is a cut vertex and $\{v_j, v_k\} \notin V(H_i)$ the $v_j - v_k$ detour paths contains a detour path $v_j - v$ followed by a $v - v$ detour path covering all the vertices of H_i again followed by another $v - v_k$ detour path. But this does not form a path since the cut vertex v is repeated twice in this sequence which is a contradiction to the definition of path. Hence for G the detour set contains at least one vertex from every $G - \{v\}$ components. $\square$

Lemma 3.2. *Consider G and H as any two graphs and detour number of H is 2. Then $dn(G \odot H) = |V(G)|$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_{|V(G)|}\}$ and the vertex set of i^{th} copy of H be $\{u_{i_1}, u_{i_2}, ..., u_{i_{|V(H)|}}\}, 1 \leq i \leq |V(G)|$. Meanwhile $dn(H) = 2$ so rename the vertices of i^th copy of H in such a way that $\{u_{i_1}, u_{i_2}\}$ is the detour set of i^th copy of $H, 1 \leq i \leq |V(G)|$.

In $G \odot H$, every vertex of G is a cutvertex and one of the components of $G \odot H - \{v_i\}$ is the i^{th} copy of H, from lemma 3.1 the minimum detour set of $G \odot H$ contains atleast one vertex from the i^th copy of H .

Also for any i^{th} and j^{th} copy of H, there exists two vertices u_{i_1}, u_{j_1} such that all the vertices of i^{th} and j^{th} copy of H and all/some vertices of G lie in some $u_{i_1} - u_{j_1}$ detour paths.

In this way the set $S = \{u_{1_1}, u_{2_1}, ..., u_{|V(G)|_1}\}$ satisfies the condition for detour set and this set is minimum since removal of a vertex from S contradicts lemma 3.1.

Hence $dn(G \odot H) = |S| = |V(G)|$. □

Theorem 3.1. *Let G and H be any of the two graphs and detour number of H is 2. If $|V(H)| > |V(G)|$ then $G \odot H$ is detour self-decomposable and $\pi_{sdn}(G \odot H) \geq (\frac{|V(H)|}{|V(G)|} - 1)|V(G)| + 1$*

Proof. Let G and H be the graphs and $dn(H) = 2$.

By lemma 3.2, $dn(G \odot H) = |V(G)|$.

Let $V(G) = \{v_1, v_2, ..., v_{|V(G)|}\}$.

Let the i^{th} copy of H have the vertex set $\{u_{i_1}, u_{i_2}, \ldots, u_{i_{|V(H)|}}\}$.

Since $dn(H) = 2$, rename the vertices of i^{th} copy of H in such a way that $\{u_{i_1}, u_{i_2}\}$ as the detour set for the graph H.

Assume $|V(H)| > |V(G)|$

Then $|V(H)| = a|V(G)| + b$,where $a, b \in \mathbb{Z}^+, 0 \leq b < |V(G)|$

In $G \odot H, deg(v_i) \geq |V(H)| + 1 = a|V(G)| + b + 1$.

Consider the case when $b = 0$.

Take $G_{j|V(G)|+k}$ as the star graph with center v_k and pendant vertices $u_{k_{j|V(G)|+2}}, u_{k_{j|V(G)|+3}}, ..., u_{k_{(j+1)|V(G)|}}, u_{k_{(j+1)|V(G)|+1}}$, for all $0 \leq j \leq a - 2$ and $1 \leq k \leq |V(G)|$.

Removal of edges of $G_i, 1 \leq i \leq (a - 1)|V(G)|$ from $G \odot H$ results into a connected graph G' since i^{th} copy of H is connected to G through edges $v_iu_{i_1}, v_iu_{i_{(a-1)|V(G)|+2}}, ..., v_iu_{i_{a|V(G)|+b}}$.

Again there exists $|V(G)|$ stars with $|V(G)|$ pendant vertices in G' but removal of these subgraphs from G' results into a disconnected graph and the components may have different detour number which may or may not be further decomposed into subgraphs with same detour number.

In G', each $v_i, 1 \leq i \leq |V(G)|$ is a cutvertex of G' and one of the components of $G' - \{v_i\}$ is the i^{th} copy of H, from lemma 3.1 the minimum detour-set of G' consists atleast a vertex from the i^{th} copy of the graph H.

Also for any i^{th} and j^{th} copy of H in G', there exist two vertices u_{i_2}, u_{j_2} such that all the vertices of i^{th} and j^{th} copy of H and all/some vertices of G' lie in some $u_{i_2} - u_{j_2}$ detour paths.

In this way the set $S = \{u_{1_2}, u_{2_2}, ..., u_{|V(G)|_2}\}$ satisfies the condition for detour set and this set is minimum since removal of a vertex from S contradicts lemma 3.1 Hence $dn(G') = |S| = |V(G)|$.

Thus G' itself is a subgraph of $G \odot H$ with same detour number. Depending upon G and H it can further be decomposed into subgraphs with same detour number.

Hence $\pi_{sdn}(G \odot H) \geq (\frac{|V(H)|}{|V(G)|} - 1)|V(G)| + 1$.

□

Theorem 3.2. *Let G and H be any of the two graphs and detour number of H is 2. If $|V(H)| > |V(G)|$ and $|V(G) \nmid |V(H)|$ then $\pi_{sdn}(G \odot H) \geq \left\lfloor \frac{|V(H)|}{|V(G)|} \right\rfloor |V(G)| + 1$*

Proof. Let G and H be the two graphs and $dn(H) = 2$.

By lemma 3.2, $dn(G \odot H) = |V(G)|$.

Let $V(G) = \{v_1, v_2, ..., v_|V(G)|\}$.

Let the i^{th} copy of H have the vertex set $\{u_{i_1}, u_{i_2}, ..., u_{i_{|V(H)|}}\}$.

Since $dn(H) = 2$, rename the vertices of i^{th} copy of H in such a way that $\{u_{i_1}, u_{i_2}\}$ as the detour set for the graph H.

Assume $|V(H)| > |V(G)|$ and $|V(G)| \nmid |V(H)|$

Then $|V(H)| = a|V(G)| + b$,where $a, b \in \mathbb{Z}+, 0 < b < |V(G)|$

In $G \odot H, deg(v_i) \geq |V(H)| + 1 = a|V(G)| + b + 1$.

Take $G_{j|V(G)|+k}$ as the star graph with center v_k and pendant vertices $u_{k_{j|V(G)|+2}}, u_{k_{j|V(G)|+3}}, ..., u_{k_{(j+1)|V(G)|}}, u_{k_{(j+1)|V(G)|+1}}$, for all $0 \leq j \leq a - 1$ and $1 \leq k \leq |V(G)|$.

Removal of edges of $G_i, 1 \leq i \leq a|V(G)|$ from $G \odot H$ results into a connected graph G' since $H's$ i^{th} copy is connected to G atleast through an edge $v_i u_{i_1}$.

In G', each $v_i, 1 \leq i \leq |V(G)|$ is the cut vertex. Also one of the components of $G' - \{v_i\}$ is the i^{th} copy of H, from lemma 3.1 for G' the minimum detour set consists atleast a vertex from each copy of H.

Also for any i^{th} and j^{th} copy of H in G', there exist two vertices u_{i_2}, u_{j_2} such that all the vertices of i^{th} and j^{th} copy of H and all/some vertices of G lie in some $u_{i_2} - u_{j_2}$ detour paths.

In this way the set $S = \{u_{1_2}, u_{2_2}, ..., u_{|V(G)|_2}\}$ satisfies the condition for detour set and this set is minimum since removal of a vertex from S contradicts lemma 3.1. Hence $dn(G') = |S| = |V(G)|$.

Thus G' itself is the $G \odot H$ subgraph with same detour number. Depending upon G and H it can further be decomposed into subgraphs with same detour number. Hence $\pi_{sdn}(G \odot H) \geq \left\lfloor \frac{|V(H)|}{|V(G)|} \right\rfloor |V(G)| + 1$. □

Lemma 3.3. *Let G be any graph and H be any tree with exactly one center then $dn(G \odot H) = V(G)$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_{|V(G)|}\}$.

Let the i^{th} copy of H have the vertex set $\{u_{i_1}, u_{i_2}, ..., u_{i_{|V(H)|}}\}$. Let the center of i^{th} copy of H be u_{i_1}.

In $G \odot H$, each vertex of G is a cutvertex and one of the components of $G \odot H - \{v_i\}$ is the i^{th} copy of H, from lemma 3.1 the $G \odot H$ minimum detour set consists at least a vertex from the i^{th} copy of H .

Again, considering the subgraph $H + \{v_i\}$ in $G \odot H$, the set $\{u_{i_1}, v_i\}$ is the minimum detour set for this subgraph.

Also for any i^{th} and j^{th} copy of H, there exists two vertices u_{i_1}, u_{j_1} such that all the vertices of i^{th} and j^{th} copy of H and all/some vertices of G lie in some $u_{i_1} - u_{j_1}$ detour paths. In this way the set $S = \{u_{1_1}, u_{2_1}, ..., u_{|V(G)|_1}\}$ satisfies the condition for detour set and this set is minimum since removal of a vertex from S contradicts lemma 3.1. Hence $dn(G \odot H) = |S| = |V(G)|$. $\square$

Theorem 3.3. *Let G be any of the graph along with the H be any tree with exactly one center. If $|V(H)| > |V(G)|$ then $G \odot H$ is detour self-decomposable and $\pi_{sdn}(G \odot H) \geq (\frac{|V(H)|}{|V(G)|} - 1)|V(G)| + 1$*

Proof. From lemma 3.3 $dn(G \odot H) = |V(G)|$.

The proof is same as the theorem 3.1. $\square$

Theorem 3.4. *Let G be the any of the graph and the H be any tree with exactly one center. If $|V(H)| > |V(G)|$ and $|V(G) \nmid |V(H)|$ then $\pi_{sdn}(G \odot H) \geq \left\lfloor \frac{|V(H)|}{|V(G)|} \right\rfloor |V(G)| + 1$*

Proof. From lemma 3.3 $dn(G \odot H) = |V(G)|$.

The proof is same as the theorem 3.2. $\square$

4 Conclusions

In this paper, a decomposition parameter on the basis of the detour number is studied. The detour number and few lower bounds for the detour self-decomposition number on corona product of some graphs that satisfy such decomposition is discussed. Although the first graph in the corona product is general, the lower bound for general graphs remains unsolved. Further research can be obtained by analyzing the graphs satisfying this decomposition with other graph parameters and graph operations.

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