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Generalised M-Transformation Weibull Distribution: Statistical Properties and Reliability Applications

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Abstract

Many real-world datasets exhibit complex features that classical lifetime models cannot adequately capture. This limitation highlights the need for new distributions with greater flexibility in skewness and kurtosis, as well as improved fit to observed data. In this study, we introduce the Generalised M-Transformation Weibull (GMTW) distribution to meet these needs. We derive its fundamental statistical properties, including the quantile function and reliability measures. The GMTW distribution can represent diverse hazard rate shapes including bathtub, unimodal, and monotone forms—making it highly suitable for reliability studies. Parameters are estimated using the maximum likelihood method, and the asymptotic properties of the estimators are evaluated through extensive Monte Carlo simulations. The model's practical utility is shown using three datasets: industrial reliability, glass fiber tensile strength, and bladder cancer remission times. Likelihood Ratio Tests and goodness-of-fit criteria confirm that the GMTW distribution provides competitive performance relative to its related forms and other well-known lifetime distributions.

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1 Introduction

Reliability analysis is a key component in many fields including engineering, medicine, and social sciences. The accuracy of any statistical inference greatly depends on the suitability of the probability model that has been chosen to describe the underlying data. Over the years many models have been proposed to describe lifetime data. The Weibull distribution, introduced by Weibull [1951], has remained the most popular choice among the classical models. This is due to its simplicity and flexibility in modelling data with monotone failure rate. However, many real world data exhibit non-monotone failure rates such as bathtub or modified shapes. A typical example is the "bathtub" shape of failure patterns of electronic components and human mortality. These data mostly show high infant mortality (early life), a constant hazard during midlife, and an increasing hazard due to wear out [Aarset, 1987]. This is when the standard Weibull distribution fails. It is unable to capture these complex hazard behaviours due to its restricted hazard rate function that is either increasing, decreasing, or constant.

Several researchers have proposed modifications and generalizations of the Weibull distribution in order to overcome this limitation. The Transmuted Modified Weibull [Khan and King, 2013], the Exponentiated Weibull [Mudholkar and Srivastava, 1993], and the Flexible Weibull [Bebbington et al., 2007] are notable examples. More recently, a new aspect to lifetime distributions was introduced by Kumar et al. [2017]. They proposed a rational transformation known as the M-transformation that offered a significant improvement in modelling reliability data by creating a geometric mixture of survival functions.

In spite of the success of the M-transformation, there is still the need for even better flexibility to vary the sharpness of the bathtub curvature and control the heaviness of the tails of the distribution. In this research, we propose the Generalised M-Transformation Weibull (GMTW) distribution which incorporates an additional shape parameter α into the M-transformation framework. By providing a more versatile structure, our proposed GMTW model is able to adapt to a wider range of empirical data.

The structural foundation of the GMTW distribution is defined by its cumulative distribution function (CDF), which incorporates the power-extended transformation into the baseline Weibull CDF $F(x; k, \lambda)$, as follows:

Definition 1.1 (GMTW Distribution)**.** *The cumulative distribution function (CDF) of the Generalised M-Transformation Weibull (GMTW) distribution is given by:*

$$G(x; k, \lambda, \alpha) = \frac{F(x)^{\alpha} + F(x)}{1 + F(x)^{\alpha}}, \quad x \geq 0, \quad k > 0,\ \lambda > 0,\ \alpha > 0. \tag{1}$$

The corresponding probability density function (PDF) is obtained by differentiation:

Proposition 1.1 (GMTW Density Function)**.** *The probability density function (PDF) corresponding to Eq.* (1) *is expressed as:*

$$g(x;k,\lambda,\alpha) = f(x;k,\lambda) \cdot \frac{1 + \alpha F(x)^{\alpha-1} + (1-\alpha)F(x)^{\alpha}}{[1 + F(x)^{\alpha}]^2}, \quad x \geq 0, \quad (2)$$

where $f(x;k,\lambda)$ is the baseline Weibull PDF. These two functions form the mathematical core of the model. The additional shape parameter α modulates the baseline density and hazard to accommodate a wide range of regimes, including the critical bathtub, unimodal, and monotone shapes that are central to this study.

The remainder of this study is organized as follows: In Section 2, we define the CDF, PDF, and hazard function of the GMTW distribution. Section 3 explores the mathematical foundations and specific properties of the model, including the quantile function and reliability measures. It follows with a detailed graphical analysis to illustrate the flexibility of the distribution under varying parameter configurations. Section 4 discusses the parameter estimation using the maximum likelihood method and Section 5 presents a Monte Carlo simulation study to evaluate the asymptotic performance of the estimators. In Section 6, the practical utility of the GMTW model is demonstrated through applications to three real-world datasets from industrial, material, and biomedical fields. Finally, Section 7 provides the conclusion and a summary of the key findings of this study.

2 The Generalised M-Transformation Weibull (GMTW) Distribution

In this section, we formally introduce the Generalised M-Transformation Weibull distribution. The model is constructed by applying a generalized rational transformation to a baseline Weibull distribution. This transformation preserves the desirable properties of the Weibull while introducing an additional shape parameter that enables a much wider range of density and hazard shapes, including bathtub, upside-down bathtub, unimodal, and monotone forms, making it particularly suitable for modeling complex lifetime data.

2.1 Baseline Weibull Distribution

Let X be a non-negative continuous random variable following the two-parameter Weibull distribution with shape parameter $k > 0$ and scale parameter

$\lambda > 0$ [Weibull, 1951]. The cumulative distribution function (CDF) and probability density function (PDF) are defined as

$$F(x; k, \lambda) = 1 - \exp\left[-\left(\frac{x}{\lambda}\right)^k\right], \quad x \geq 0, \tag{3}$$

$$f(x; k, \lambda) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{x}{\lambda}\right)^k\right], \quad x \geq 0. \tag{4}$$

The corresponding survival function is

$$S_W(x; k, \lambda) = \exp\left[-\left(\frac{x}{\lambda}\right)^k\right], \quad x \geq 0. \tag{5}$$

2.2 Transformation Function

We first define the transformation function $T : [0, 1] \to [0, 1]$ that generates the proposed family:

Definition 2.1 (M-Transformation Function)**.** *The generalised M-transformation function is defined as:*

$$T(u; \alpha) = \frac{u^\alpha + u}{1 + u^\alpha}, \quad u \in [0, 1], \quad \alpha > 0. \tag{6}$$

This function generalizes the classical M-transformation introduced by Kumar et al. [2017], which corresponds to the special case $\alpha = 1$:

$$T(u; 1) = \frac{2u}{1 + u}. \tag{7}$$

The parameter α acts as a flexibility multiplier, allowing control over tail behavior, skewness, and hazard curvature.

2.3 GMTW Cumulative Distribution Function

The CDF of the GMTW distribution is obtained by composing the transformation T with the baseline Weibull CDF:

$$G(x; k, \lambda, \alpha) = T\big(F(x; k, \lambda); \alpha\big) = \frac{F(x)^\alpha + F(x)}{1 + F(x)^\alpha}, \quad x \geq 0, \quad k, \lambda, \alpha > 0. \tag{8}$$

The support of the GMTW distribution is $[0, \infty)$, and the parameter space is $(k, \lambda, \alpha) \in \mathbb{R}_+^3$.

Substituting the explicit Weibull CDF (3) gives:

$$G(x; k, \lambda, \alpha) = \frac{\left[1 - \exp\left(-(x/\lambda)^k\right)\right]^\alpha + 1 - \exp\left(-(x/\lambda)^k\right)}{1 + \left[1 - \exp\left(-(x/\lambda)^k\right)\right]^\alpha}. \tag{9}$$

2.4 Mathematical Validity of the GMTW Distribution

Proposition 2.1 (Validity of the GMTW Cumulative Distribution Function)**.** *The function*

$$G(x; k, \lambda, \alpha) = \frac{F(x)^{\alpha} + F(x)}{1 + F(x)^{\alpha}}, \quad x \geq 0, \quad k > 0,\ \lambda > 0,\ \alpha > 0, \tag{10}$$

where $F(x) = 1 - \exp\left[-\left(\frac{x}{\lambda}\right)^{k}\right]$ *is the baseline Weibull CDF, is a valid cumulative distribution function for a continuous random variable on* $[0, \infty)$.

Proof. We verify the four essential properties required for any continuous probability distribution on $[0, \infty)$:

1. **Boundary Conditions:** A valid CDF must satisfy:

   $$\lim_{x \to 0^+} G(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} G(x) = 1.$$

   As $x \to 0^+$, $F(x) \to 0^+$. Substituting:

   $$G(x) \to \frac{0^{\alpha} + 0}{1 + 0^{\alpha}} = 0.$$

   As $x \to \infty$, $F(x) \to 1^-$. Substituting:

   $$G(x) \to \frac{1^{\alpha} + 1}{1 + 1^{\alpha}} = 1.$$

   Thus, the boundary conditions are satisfied.

2. **Non-negativity:** For all $x \geq 0$, $F(x) \in [0, 1]$. The numerator $F(x)^{\alpha} + F(x) \geq 0$ and denominator $1 + F(x)^{\alpha} > 0$, so $G(x) \geq 0$.

3. **Monotonicity (Non-decreasing):** Since the baseline CDF $F(x)$ is non-decreasing, it suffices to show that the transformation $T(u; \alpha)$ is non-decreasing on the interval $[0, 1]$. The derivative of the transformation is given by:

   $$T'(u; \alpha) = \frac{1 + \alpha u^{\alpha - 1} + (1 - \alpha) u^{\alpha}}{(1 + u^{\alpha})^2}. \tag{11}$$

   The denominator is strictly positive for all $u \in [0, 1]$. To establish that $T'(u; \alpha) > 0$ for $u \in (0, 1)$, we examine the numerator, $N(u) = 1 + \alpha u^{\alpha - 1} + (1 - \alpha) u^{\alpha}$, under two cases for the transformation parameter $\alpha > 0$:

- **Case 1:** $0 < \alpha \leq 1$**.** In this scenario, $(1-\alpha) \geq 0$. Since all individual terms (1, $\alpha u^{\alpha-1}$, and $(1-\alpha)u^{\alpha}$) are non-negative for $u > 0$, it follows that $N(u) > 0$.
- **Case 2:** $\alpha > 1$**.** We can rearrange the numerator as $N(u) = 1 + \alpha u^{\alpha-1} - (\alpha-1)u^{\alpha}$. Factoring out terms, we observe:

$$N(u) = 1 + u^{\alpha-1}[\alpha - (\alpha-1)u].$$

  Since $0 < u < 1$, the term $(\alpha-1)u < (\alpha-1)$. Therefore, $\alpha - (\alpha - 1)u > \alpha - (\alpha - 1) = 1$. This ensures that the entire expression is strictly greater than 1.

In both cases, $N(u) > 0$, confirming that $T(u;\alpha)$ is strictly increasing on $(0,1)$. Consequently, the GMTW cumulative distribution function $G(x) = T(F(x);\alpha)$ is non-decreasing for all $x \geq 0$.

4. **Right-continuity and Continuity:** Both $F(x)$ and $T(u)$ are continuous on their domains, so their composition $G(x)$ is continuous on $[0,\infty)$, implying right-continuity everywhere.

These properties confirm that $G(x;k,\lambda,\alpha)$ is a valid CDF for a continuous random variable supported on $[0,\infty)$. □

2.5 GMTW Probability Density Function

The PDF $g(x;k,\lambda,\alpha)$ is obtained by differentiating the CDF:

Proposition 2.2 (GMTW Probability Density Function)**.** *The probability density function of the GMTW distribution is given by:*

$$g(x) = \frac{d}{dx}G(x) = f(x;k,\lambda) \cdot T'\big(F(x;k,\lambda);\alpha\big), \tag{12}$$

where the derivative of the transformation function is

$$T'(u;\alpha) = \frac{1 + \alpha u^{\alpha-1} + (1-\alpha)u^{\alpha}}{(1+u^{\alpha})^2}. \tag{13}$$

Since $G(x)$ is a valid cumulative distribution function (Proposition 2.1), its derivative $g(x)$ is necessarily a valid probability density function (non-negative and integrates to 1). The detailed verification, using the change-of-variable technique, is provided in Appendix A.

Substituting yields the explicit PDF:

$$g(x;k,\lambda,\alpha) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\exp\left[-\left(\frac{x}{\lambda}\right)^{k}\right]$$
$$\times \frac{1+\alpha\big[1-\exp\big(-(x/\lambda)^k\big)\big]^{\alpha-1}+(1-\alpha)\big[1-\exp\big(-(x/\lambda)^k\big)\big]^{\alpha}}{\big\{1+\big[1-\exp\big(-(x/\lambda)^k\big)\big]^{\alpha}\big\}^2}, \quad x \geq 0. \tag{14}$$

2.6 Survival and Hazard Functions

Definition 2.2 (Survival Function)**.** *The survival (reliability) function of the GMTW distribution is*

$$S(x;k,\lambda,\alpha) = 1 - G(x;k,\lambda,\alpha) = \frac{\exp\left[-\left(\frac{x}{\lambda}\right)^k\right]}{1+\big[1-\exp\big(-(x/\lambda)^k\big)\big]^{\alpha}}, \quad x \geq 0. \tag{15}$$

Definition 2.3 (Hazard Function)**.** *The hazard rate function (instantaneous failure rate) of the GMTW distribution is*

$$h(x;k,\lambda,\alpha) = \frac{g(x;k,\lambda,\alpha)}{S(x;k,\lambda,\alpha)} = h_W(x;k,\lambda)\cdot\frac{1+\alpha F(x)^{\alpha-1}+(1-\alpha)F(x)^{\alpha}}{1+F(x)^{\alpha}}, \tag{16}$$

where $h_W(x;k,\lambda) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}$ *is the baseline Weibull hazard function.*

The multiplicative distortion factor

$$D(u;\alpha) = \frac{1+\alpha u^{\alpha-1}+(1-\alpha)u^{\alpha}}{1+u^{\alpha}}, \quad u = F(x) \in [0,1]. \tag{17}$$

is the flexibility multiplier responsible for the model's enhanced ability to generate a wide variety of hazard shapes (monotone, bathtub, upside-down bathtub, etc.) depending on the value of α. It is important to distinguish the transformation derivative $T'(u;\alpha)$ in (13) (which features a squared denominator) from this hazard multiplicative distortion factor $D(u;\alpha)$ (which features a single-power denominator).

Lemma 2.1 (Asymptotic Behaviour of the Distortion Factor)**.** *For a fixed* $\alpha > 0$*, let* $D(u;\alpha)$ *denote the distortion factor defined in Equation* (17)*. The boundary behaviour of* $D(u;\alpha)$ *is characterised as follows:*

1. ***Behaviour at the Origin*** ($u \to 0^+$)***:***

$$\lim_{u\to 0^+} D(u;\alpha) = \begin{cases} +\infty, & 0<\alpha<1, \\ 2, & \alpha = 1, \\ 1, & \alpha > 1. \end{cases} \tag{18}$$

2. ***Behaviour at the Upper Boundary*** *($u \to 1^-$):*

$$\lim_{u \to 1^-} D(u; \alpha) = 1 \quad \textit{for all } \alpha > 0. \tag{19}$$

3. ***Regularity:*** *$D(u; \alpha)$ is strictly positive and continuous on the interval $[0, 1)$, and differentiable on $(0, 1)$ for all $\alpha > 0$.*

Proof. Consider the expression for the distortion factor $D(u; \alpha)$.

(i) As $u \to 0^+$, the term u^α vanishes for all $\alpha > 0$. However, the term $\alpha u^{\alpha-1}$ depends critically on the value of α. If $0 < \alpha < 1$, the exponent $\alpha - 1$ is negative, causing $u^{\alpha-1} \to \infty$ and consequently $D(u; \alpha) \to \infty$. If $\alpha = 1$, then $\alpha u^{\alpha-1} = 1$, yielding $D(0; 1) = (1 + 1)/1 = 2$. For the case where $\alpha > 1$, the exponent is positive, leading to $u^{\alpha-1} \to 0$, which results in $D(0; \alpha) = 1$.

(ii) To examine the behaviour as $u \to 1^-$, we set $u = 1 - \epsilon$ where $\epsilon \to 0^+$. Utilising the Taylor expansions $u^\alpha = 1 - \alpha\epsilon + o(\epsilon)$ and $u^{\alpha-1} = 1 - (\alpha - 1)\epsilon + o(\epsilon)$, we substitute these into the numerator:

$$1+\alpha[1-(\alpha-1)\epsilon]+(1-\alpha)[1-\alpha\epsilon]+o(\epsilon) = 1+\alpha-\alpha(\alpha-1)\epsilon+1-\alpha-\alpha(1-\alpha)\epsilon+o(\epsilon) = 2+o(\epsilon).$$

The single-power denominator simplifies as $(1 + u^\alpha) = (2 - \alpha\epsilon + o(\epsilon))$. Thus, the limit is:

$$\lim_{u \to 1^-} D(u; \alpha) = \frac{2}{2} = 1.$$

(iii) The properties of positivity, continuity, and differentiability follow directly from the algebraic structure of $D(u; \alpha)$ and the well-defined nature of the power function u^α on the open interval $(0, 1)$. □

Remark 2.1. *This asymptotic analysis reveals that for any fixed $\alpha > 0$, the hazard function satisfies $h(x) \sim h_W(x)$ as $x \to \infty$. Furthermore, while $G(x) \to F(x)$ pointwise as $\alpha \to \infty$, this convergence is non-uniform across the support. For any fixed x, the hazard function $h(x)$ converges smoothly to the baseline Weibull hazard $h_W(x)$ as $\alpha \to \infty$, ensuring structural consistency with the nested baseline distribution.*

2.7 Special Cases and Limiting Behaviour

The GMTW distribution family is characterized by its ability to encompass or approach several well-known models depending on the value of the transformation parameter α. When $\alpha = 1$, the model simplifies to the M-Weibull distribution introduced by Kumar et al. [2017], where the cumulative distribution function is given by $G(x; k, \lambda, 1) = 2F(x)/(1 + F(x))$. This specific case represents a balanced rational transformation of the baseline Weibull distribution.

As the transformation parameter increases, the GMTW distribution recovers the baseline model. Specifically, as $\alpha \to \infty$, the transformation function $T(u; \alpha)$ approaches the identity function for all $u \in [0, 1)$. Consequently, the GMTW distribution converges to the standard Weibull distribution, such that $\lim_{\alpha\to\infty} G(x; k, \lambda, \alpha) = F(x; k, \lambda)$. This property demonstrates that the GMTW distribution can be viewed as a flexible extension that generalizes the classic Weibull framework.

Finally, it is essential to define the domain of validity to ensure the GMTW remains a proper lifetime model. As $\alpha \to 0^+$, the transformation approaches the limiting form $G(x) \to \frac{1+F(x)}{2}$, which implies $G(0^+) = 1/2$. This limiting case represents a defective distribution with a significant point mass at the origin, a characteristic common in zero-inflated phenomena. To ensure the model remains non-defective and assigns no probability mass to the origin ($G(0) = 0$), we strictly define the parameter space as $\alpha \in (0, \infty)$. Under this constraint, $\lim_{x\to 0^+} G(x; k, \lambda, \alpha) = 0$ for all $\alpha > 0$, confirming the properness of the distribution across its intended functional domain.

2.8 Identifiability

The property of identifiability is fundamental for the validity of maximum likelihood estimation, ensuring that distinct parameter values correspond to distinct probability distributions.

Theorem 2.1 (Identifiability of the GMTW Distribution)**.** *The GMTW distribution family is identifiable; that is, if for all* $x \geq 0$*,*

$$G(x; k, \lambda, \alpha) = G(x; k^*, \lambda^*, \alpha^*), \tag{20}$$

then the parameter vectors are identical: $(k, \lambda, \alpha) = (k^*, \lambda^*, \alpha^*)$.

Proof. Suppose there exist two parameter vectors $\theta = (k, \lambda, \alpha)$ and $\theta^* = (k^*, \lambda^*, \alpha^*)$ such that the cumulative distribution functions coincide for all $x \geq 0$:

$$T(F(x; k, \lambda); \alpha) = T(F(x; k^*, \lambda^*); \alpha^*), \tag{21}$$

where $T(u; \alpha) = \frac{u^\alpha + u}{1+u^\alpha}$ is the rational transformation defined in Equation (6) and $F(x; k, \lambda)$ is the baseline Weibull CDF. We establish equality in three logical stages.

Step 1: Identification of baseline parameters k and λ Consider the asymptotic behavior of the survival function $\bar{G}(x) = 1 - G(x)$ as $x \to \infty$. As $u \to 1^-$, let $u = 1 - \epsilon$. A Taylor expansion of the transformation around $u = 1$ yields:

$$T(1 - \epsilon; \alpha) = \frac{(1 - \epsilon)^\alpha + (1 - \epsilon)}{1 + (1 - \epsilon)^\alpha} = \frac{1 - \alpha\epsilon + 1 - \epsilon + O(\epsilon^2)}{1 + 1 - \alpha\epsilon + O(\epsilon^2)} = 1 - \frac{1}{2}\epsilon + O(\epsilon^2).$$

Thus, $1 - T(u;\alpha) \sim \frac{1}{2}(1-u)$ as $u \to 1^-$. Applying this to the GMTW survival function as $x \to \infty$:

$$1 - G(x;\theta) \sim \frac{1}{2}e^{-(x/\lambda)^k} \quad \text{and} \quad 1 - G(x;\theta^*) \sim \frac{1}{2}e^{-(x/\lambda^*)^{k^*}}.$$

The equality in (21) implies that $\frac{1}{2}e^{-(x/\lambda)^k} = \frac{1}{2}e^{-(x/\lambda^*)^{k^*}}$ for all sufficiently large x. Taking the natural logarithm twice of both sides yields:

$$k(\ln x - \ln \lambda) = k^*(\ln x - \ln \lambda^*).$$

Since this linear identity in $\ln x$ must hold for a continuum of values, the coefficients must be identical, forcing $k = k^*$ and $\lambda = \lambda^*$.

Step 2: Identification of the transformation parameter α With the baseline parameters identified, we have $F(x;k,\lambda) = F(x;k^*,\lambda^*)$. Let $u = F(x) \in (0,1)$. Equation (21) simplifies to:

$$T(u;\alpha) = T(u;\alpha^*) \quad \text{for all } u \in (0,1). \tag{22}$$

To show $\alpha = \alpha^*$, we examine the partial derivative of $T(u;\alpha)$ with respect to α:

$$\frac{\partial T(u;\alpha)}{\partial \alpha} = \frac{u^\alpha(1-u)\ln u}{(1+u^\alpha)^2}.$$

Since $\ln u < 0$ for $u \in (0,1)$ and all other terms are positive, we have $\frac{\partial T}{\partial \alpha} < 0$. This implies that $T(u;\alpha)$ is a strictly decreasing function of α. Since the mapping $\alpha \mapsto T(u;\alpha)$ is continuous and strictly decreasing on $(0,\infty)$ for each fixed $u \in (0,1)$, equality for all u implies $\alpha = \alpha^*$.

Conclusion Since $k = k^*$, $\lambda = \lambda^*$, and $\alpha = \alpha^*$, the parameter vectors are identical. Thus, the GMTW distribution family is identifiable. $\square$

3 Statistical Properties

In this section, we derive and discuss the main statistical properties of the Generalised M-Transformation Weibull (GMTW) distribution. These include the raw moments (with a series expansion for computation), the quantile function and associated robust shape measures, the behaviour of the hazard rate function, order statistics, and limiting behaviours under different parameter regimes.

3.1 Raw Moments and Moment-Generating Function

Definition 3.1 (Raw Moments). *The r-th raw moment of a GMTW random variable X is given by*

$$\mu'_r = E[X^r] = \int_0^\infty x^r\, g(x; k, \lambda, \alpha)\, dx, \quad r > -1. \tag{23}$$

Using the substitution $u = F(x; k, \lambda)$, so $du = f(x)\, dx$ and $x = \lambda[-\ln(1 - u)]^{1/k}$, we obtain the integral representation

$$\mu'_r = \lambda^r \int_0^1 [-\ln(1-u)]^{r/k} \cdot \frac{1 + \alpha u^{\alpha-1} + (1-\alpha)u^\alpha}{(1+u^\alpha)^2}\, du. \tag{24}$$

This integral has no closed form in general, but it is numerically tractable using adaptive quadrature. To facilitate numerical evaluation, especially for large r or high precision, we expand the denominator using the generalized binomial series [Gradshteyn and Ryzhik, 2014], valid for $|u^\alpha| < 1$, which holds almost everywhere on $[0, 1)$:

$$(1 + u^\alpha)^{-2} = \sum_{j=0}^\infty (-1)^j (j+1) u^{j\alpha}. \tag{25}$$

Substituting this expansion into (24) yields a series representation:

$$\mu'_r = \lambda^r \sum_{j=0}^\infty (-1)^j (j+1) \int_0^1 [-\ln(1-u)]^{r/k} \big[1 + \alpha u^{\alpha-1} + (1-\alpha)u^\alpha\big] u^{j\alpha}\, du. \tag{26}$$

Each integral in the sum can be evaluated numerically or, in special cases, related to known forms involving the gamma function.

Definition 3.2 (Moment-Generating Function). *The moment-generating function (MGF), $M(t) = E[e^{tX}]$, is given by*

$$M(t) = \int_0^1 \exp\left\{t\lambda[-\ln(1-u)]^{1/k}\right\} \cdot \frac{1 + \alpha u^{\alpha-1} + (1-\alpha)u^\alpha}{(1+u^\alpha)^2}\, du. \tag{27}$$

The existence of the MGF depends on the shape parameter k as follows:

1. For $k > 1$, the MGF exists for all real t.
2. For $k = 1$, the MGF exists only for t below some positive threshold.
3. For $0 < k < 1$, the MGF does not exist for any $t > 0$.

When the MGF fails to exist for positive t, the distribution is instead characterised by its raw moments or its characteristic function.

Once the first four raw moments $(\mu'_1, \mu'_2, \mu'_3, \mu'_4)$ are obtained, the higher central moments and shape descriptors are computed using the standard relationships:

- Variance: $\sigma^2 = \mu'_2 - (\mu'_1)^2$
- Skewness: $\gamma_1 = \frac{\mu'_3 - 3\mu'_1\mu'_2 + 2(\mu'_1)^3}{\sigma^3}$
- Kurtosis: $\beta_2 = \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6(\mu'_1)^2\mu'_2 - 3(\mu'_1)^4}{\sigma^4}$

3.2 Quantile Function and Robust Shape Measures

Definition 3.3 (Quantile Function)**.** *The quantile function, $Q(p) = G^{-1}(p)$ for $p \in (0,1)$, is derived by inverting the cumulative distribution function. Setting $G(Q(p)) = p$ leads to the following relation:*

$$\frac{v^\alpha + v}{1 + v^\alpha} = p, \quad \textit{where} \quad v = F(Q(p)). \tag{28}$$

Equation (28) is a nonlinear equation that does not admit a closed-form solution for v in terms of elementary functions. However, given that $G(v)$ is strictly monotonic for $\alpha > 0$, a unique solution for v can be efficiently obtained using numerical root-finding algorithms such as the Newton-Raphson or bisection methods. Once v is determined numerically, the quantile is obtained by applying the inverse of the baseline Weibull CDF defined in Equation (3):

$$Q(p) = F^{-1}(v) = \lambda\left[-\ln(1-v)\right]^{1/k}. \tag{29}$$

The quantile function facilitates the derivation of robust shape measures that do not depend on the existence of higher-order moments. These diagnostics are particularly valuable when the GMTW distribution exhibits heavy-tailed behavior or when moments are algebraically complex [Hyndman and Fan, 1996]:

Definition 3.4 (Bowley's Skewness Coefficient)**.** *A robust measure of asymmetry based on quartiles as proposed in [Bowley, 1920]:*

$$S = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}. \tag{30}$$

Definition 3.5 (Moors' Kurtosis Coefficient)**.** *A robust alternative to Pearson's kurtosis that utilizes octiles to describe the concentration of probability in the tails and center [Moors, 1988]:*

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}. \tag{31}$$

These quantile-based measures provide stable diagnostics for the distribution's shape and are insensitive to the influence of extreme outliers, ensuring reliable characterization across the GMTW parameter space.

3.3 Behaviour of the Hazard Function

The hazard rate function $h(x)$ is a critical measure in reliability analysis, representing the instantaneous failure rate at time x. The GMTW distribution offers significant flexibility in this regard, as established by the following structural decomposition.

Theorem 3.1 (Decomposition of the Hazard Function). *The hazard rate function of the GMTW distribution can be expressed as the product of the baseline Weibull hazard and a rational distortion factor:*

$$h(x; k, \lambda, \alpha) = h_W(x; k, \lambda) \cdot D(F(x; k, \lambda); \alpha), \tag{32}$$

where $h_W(x; k, \lambda) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}$ is the baseline Weibull hazard, and the distortion factor $D(u; \alpha)$ is defined as in (17).

Remark 3.1 (Hazard Shape Classification and Mechanisms). *The geometric profile of $h(x)$ is governed by the competing rates of change between the monotone baseline Weibull hazard and the distortion factor $D(u; \alpha)$. As established in Lemma 2.1, the distortion factor exhibits distinct monotonicity properties based on the value of α. Specifically, for $\alpha > 1$, $D(u; \alpha)$ is strictly increasing in u, providing an upward adjustment to the baseline hazard. Conversely, when $\alpha = 1$, the factor $D(u; 1) = 2/(1 + u)^2$ is strictly decreasing. In the case where $0 < \alpha < 1$, $D(u; \alpha)$ is strictly decreasing in u and diverges as $u \to 0^+$, creating a significant initial multiplier that decays rapidly.*

The interaction of these components allows the GMTW distribution to encompass a wide variety of hazard shapes. A monotone increasing hazard typically occurs when both components are non-decreasing ($k \geq 1$ and $\alpha \geq 1$), or when $k > 1$ and $\alpha < 1$ if the growth of the baseline Weibull hazard eventually dominates the initial decay of the distortion factor. A monotone decreasing hazard is predominantly observed when $k \leq 1$ and $\alpha = 1$, or when both the baseline and the distortion factor are decreasing ($k \leq 1$ and $0 < \alpha \leq 1$). Furthermore, a unimodal or upside-down bathtub hazard arises when an increasing distortion factor ($\alpha > 1$) initially dominates a decreasing baseline ($k < 1$), creating a peak before the eventual decline.

Of particular interest is the ability of the model to achieve a bathtub or U-shaped profile through two distinct structural pathways. Under Mechanism I, a decreasing baseline ($k < 1$) is combined with an increasing distortion factor

($\alpha > 1$). In this scenario, the initial hazard is high due to the Weibull shape, decreases, and is eventually forced upward by the rational transformation. Under Mechanism II, which is frequently observed in Aarset-type data, an increasing baseline ($k > 1$) is combined with a small transformation parameter ($0 < \alpha < 1$). Here, the distortion factor creates an artificial "burn-in" period of high mortality that decays rapidly, while the underlying increasing Weibull baseline eventually asserts dominance to produce the U-shape.

A rigorous analytical characterisation involves the derivative of the log-hazard, $\eta(x) = \ln h(x)$:

$$\eta'(x) = \frac{h'_W(x)}{h_W(x)} + \frac{D'(F(x);\alpha)}{D(F(x);\alpha)} \cdot f(x). \tag{33}$$

The sign of $\eta'(x)$ determines the local monotonicity of the hazard, where bath-tub shapes correspond to a single sign change of $\eta'(x)$ from negative to positive. Given that the conditions for such transitions depend on k and α in a non-separable, non-linear manner, a closed-form parametric partition of all possible shapes is analytically formidable. Consequently, we rely on numerical exploration and the empirical evidence from real-world applications to demonstrate this structural flexibility.

3.4 Order Statistics

Proposition 3.1 (Order Statistics PDF). *Let $X_1, X_2, \ldots, X_n$ be a random sample from the GMTW distribution. The PDF of the i-th order statistic $X_{(i:n)}$ is*

$$\begin{aligned} g_{(i:n)}(x) = \frac{n!}{(i-1)!(n-i)!}\, & g(x;k,\lambda,\alpha)\,[G(x;k,\lambda,\alpha)]^{i-1} \\ & \times [1 - G(x;k,\lambda,\alpha)]^{n-i}, \end{aligned} \qquad x \geq 0. \tag{34}$$

This expression is fundamental in reliability engineering for analyzing k-out-of-n systems, where the i-th failure time is of interest [David and Nagaraja, 2003].

3.5 Graphical Analysis of the GMTW Model

In this subsection, we visually explore the flexibility of the Generalised M-Transformation Weibull (GMTW) distribution through two sensitivity analyses. Figure 1 illustrates the influence of the transformation parameter α, while Figure 2 demonstrates how the model responds to changes in the baseline shape parameter k.

The influence of the transformation parameter α is explored in Figure 1, which displays the PDF $g(x)$, survival function $S(x)$, and hazard function $h(x)$ for varying values of α while keeping $k = 0.8$ and $\lambda = 1$ constant. The density plots

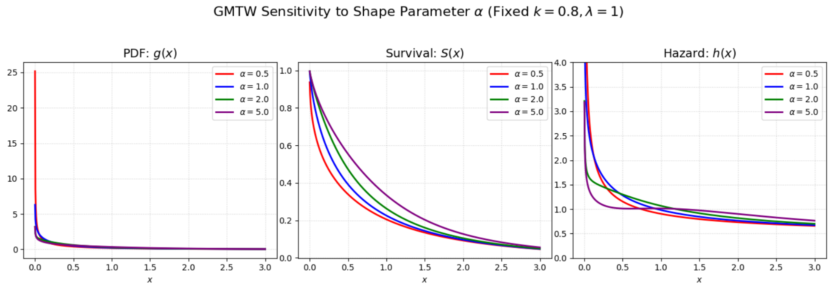

Figure 1: GMTW Sensitivity to Shape Parameter α (Fixed $k = 0.8$, $\lambda = 1$) showing PDF, survival, and hazard functions.

reveal strong right skewness at lower α values. However, as α increases from 0.5 to 5.0, a clear mode suppression effect occurs near the origin, causing the probability mass to shift toward heavier tails. This demonstrates that α effectively controls tail heaviness without the need to alter the baseline Weibull parameters. Similarly, the survival curves exhibit clear stochastic ordering, where the probability $P(X > x)$ increases alongside α for any fixed $x > 0$. This property positions α as a useful reliability booster within the transformation framework.

Perhaps the strongest evidence of the model's utility is found in the hazard function $h(x)$. While the baseline Weibull with $k = 0.8$ produces a strictly decreasing hazard, the GMTW with a larger α, such as 5.0, develops a pronounced bathtub like profile. This involves a sharp early decrease followed by a stable region, allowing the model to effectively capture both infant mortality and useful life phases in a single framework. While $\alpha > 1$ typically facilitates the classic bathtub profile, values of $\alpha < 1$ effectively modulate the early life hazard and tail weight, demonstrating that the GMTW transformation remains robust even when the transformation parameter is small.

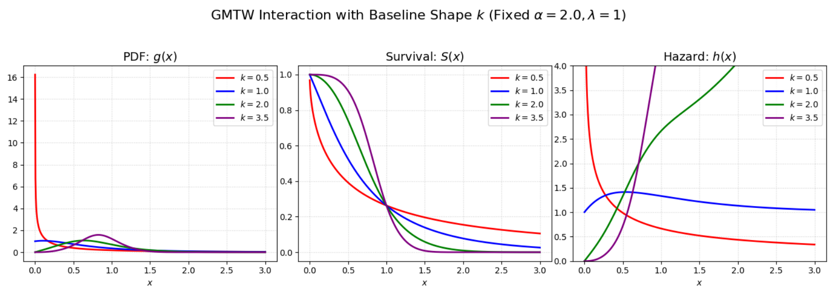

Figure 2: GMTW Interaction with Baseline Shape k (Fixed $\alpha = 2.0$, $\lambda = 1$) showing PDF, survival, and hazard functions.

The interplay between the transformation and the baseline Weibull shape parameter k is further examined in Figure 2 with α fixed at 2.0. The resulting PDF transitions from a highly skewed, exponential like form when k is small to a nearly symmetric, bell shaped density as k reaches 3.5. This confirms the ability of the GMTW to approximate a broad family of distributions, including Normal like and Rayleigh like behaviors, while retaining its analytical tractability.

A striking feature in the survival plot is the intersection of all curves near $x \approx 1.0$, which corresponds to the baseline characteristic life where the CDF is approximately 0.632. This shows that the transformation preserves certain intrinsic baseline properties while modulating dispersion and shape. Furthermore, the hazard plot illustrates the capacity of the GMTW to cover all major reliability profiles. This includes a decreasing hazard for early life failures when $k = 0.5$, a unimodal rise then fall behaviour when $k = 1.0$ which is useful for modeling post treatment risk peaks in medical survival data, and an increasing hazard for $k \geq 2.0$ to represent the wear out phase typical of mechanical aging processes.

3.6 Comparison with Existing Weibull Generalizations

The GMTW distribution is specifically engineered to overcome structural limitations inherent in widely-used Weibull extensions, such as the Exponentiated Weibull (EW) and its nested predecessor, the M-Weibull. By employing a rational transformation rather than a simple power transformation, the model achieves a unique balance between mathematical parsimony and functional flexibility.

3.6.1 Tail Behaviour and Rational Flexibility

The Exponentiated Weibull (EW) achieves its flexibility by raising the baseline CDF to a power α. While computationally simple, this power transformation creates a global shift that often couples the behavior of the tails too closely with the mode of the distribution. This coupling can lead to a lack of precision when modeling data with sharp peaks or heavy tails.

In contrast, the GMTW utilizes a rational transformation of the baseline $F(x)$, where the denominator $(1 + u^{\alpha})$ provides an effective mechanism for surgical adjustments to the distribution's curvature. As demonstrated in our analysis of the Glass Fibers data, this rational structure allows the model to capture the sharp decay in the upper tail of material strength data more effectively than the global shift of the EW model. This structural advantage is formally confirmed by the Likelihood Ratio Test, which yielded a highly significant result ($p = 0.0047$), indicating that the rational α parameter provides a statistically significant improvement for modeling high-precision material datasets.

3.6.2 Hazard Flexibility: The Acute Bathtub Shape

A primary motivation for the GMTW is its ability to model the "acute" bathtub hazard rates characteristic of modern industrial components. In the EW model, the bathtub shape is often relatively shallow, resulting in a gradual and sometimes inaccurate transition from the infant mortality phase to the stable useful-life phase.

The rational architecture of the GMTW allows the hazard function to remain stable for a longer duration before exhibiting a sharp rise during the wear-out phase. This mathematical property explains the GMTW's competitive performance in the Aarset reliability dataset, where it achieved a statistically adequate K-S p-value of 0.0705 compared to the EW ($p = 0.0518$) and the rejected Weibull baseline ($p = 0.0421$). The LRT statistic of 38.11 provides further robust evidence that the GMTW's specific curvature is a significant improvement over monotone or simpler bathtub models.

3.6.3 Parsimony and Numerical Stability

While many generalizations achieve flexibility by adding fourth or fifth parameters, the GMTW remains a parsimonious three-parameter model. By utilizing a rational function rather than nested power transformations, the GMTW avoids the "flat likelihood" surfaces that often lead to optimization failure in more complex extensions.

This structural stability is confirmed by our 1,000-replication Monte Carlo study, which demonstrated consistent convergence of the α parameter even under complex bathtub configurations. The model thus offers a reliable alternative for researchers who require the flexibility of a generalized distribution without the computational instability often associated with over-parameterized models.

Table 1: Structural Comparison: GMTW vs. Exponentiated Weibull (EW)

Feature	Exponentiated Weibull (EW)	GMTW (Proposed)
Transformation	Power: $F(x)^\alpha$	Rational: $\frac{u^\alpha+u}{1+u^\alpha}$
Bathtub Curvature	Shallow/Gradual	Deep/Acute
Tail Control	Global Power Shift	Localized Rational Adjustment
Statistical Inference	Power Transformation	LRT-Validated Transformation

4 Parameter Estimation

In this section, we develop the Maximum Likelihood Estimation (MLE) framework for the parameters of the Generalised M-Transformation Weibull

(GMTW) distribution. We derive the log-likelihood functions for both complete (uncensored) and right-censored data, and discuss the numerical optimisation strategy employed to obtain the parameter estimates.

4.1 Likelihood Function (Uncensored Case)

Suppose we observe an independent and identically distributed (i.i.d.) sample $x_1, x_2, \ldots, x_n$ from the GMTW distribution with parameter vector $\theta = (k, \lambda, \alpha)^\top$. From the definition of the PDF in Section 2, the likelihood function is given by:

$$L(\theta) = \prod_{i=1}^{n} g(x_i; k, \lambda, \alpha) = \prod_{i=1}^{n} f(x_i; k, \lambda)\, T'(F(x_i; k, \lambda); \alpha), \tag{35}$$

where $f(x_i; k, \lambda)$ and $F(x_i; k, \lambda)$ are the baseline Weibull PDF and CDF, respectively, and $T'(u; \alpha)$ is the derivative of the transformation function:

$$T'(u; \alpha) = \frac{1 + \alpha u^{\alpha-1} + (1-\alpha)u^{\alpha}}{(1+u^{\alpha})^2}. \tag{36}$$

The log-likelihood function, $\ell(\theta) = \ln L(\theta)$, is expressed as:

$$\ell(\theta) = \sum_{i=1}^{n} \ln f(x_i; k, \lambda) + \sum_{i=1}^{n} \ln T'(F(x_i; k, \lambda); \alpha). \tag{37}$$

By expanding the terms, we obtain:

$$\begin{aligned}\ell(\theta) &= n\ln(k) - nk\ln(\lambda) + (k-1)\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k \\ &\quad + \sum_{i=1}^{n}\ln\left(1 + \alpha F_i^{\alpha-1} + (1-\alpha)F_i^{\alpha}\right) - 2\sum_{i=1}^{n}\ln\left(1 + F_i^{\alpha}\right),\end{aligned} \tag{38}$$

where $F_i = 1 - \exp[-(x_i/\lambda)^k]$. The MLEs $\hat{\theta} = (\hat{k}, \hat{\lambda}, \hat{\alpha})^\top$ are obtained by maximising $\ell(\theta)$ with respect to the parameter vector. This is equivalent to solving the score equations $\nabla\ell(\theta) = \mathbf{0}$. Due to the rational structure of the transformation, these equations are highly non-linear and lack closed-form solutions, necessitating iterative numerical methods.

4.2 Right-Censored Likelihood

In reliability and survival studies, observations are frequently subject to right-censoring. For each individual i, let t_i denote the observed time and δ_i be the censoring indicator ($\delta_i = 1$ if the event is observed, and $\delta_i = 0$ if the observation is right-censored). The contribution of the i-th observation to the likelihood is $L_i(\theta) = [g(t_i;\theta)]^{\delta_i}[S(t_i;\theta)]^{1-\delta_i}$. The full log-likelihood for right-censored data is:

$$\ell(\theta) = \sum_{i=1}^{n} \left[\delta_i \ln g(t_i;\theta) + (1-\delta_i)\ln S(t_i;\theta)\right]. \tag{39}$$

Recalling the GMTW survival function $S(t;\theta) = \frac{S_W(t)}{1+F_W(t)^\alpha}$ and the relationship

$$g(t;\theta) = h_W(t)S_W(t)T'(F_W(t);\alpha), \tag{40}$$

where $h_W(t)$ and $S_W(t)$ are the baseline Weibull hazard and survival functions respectively, the log-likelihood becomes:

$$\ell(\theta) = \sum_{i=1}^{n} \ln S_W(t_i) + \sum_{i=1}^{n} \delta_i \ln h_W(t_i) + \sum_{i=1}^{n} \delta_i \ln T'(F_i;\alpha) - \sum_{i=1}^{n}(1-\delta_i)\ln(1+F_i^\alpha). \tag{41}$$

To achieve a fully consistent expression, we substitute the definition of the transformation derivative:

$$\ln T'(F_i;\alpha) = \ln(1+\alpha F_i^{\alpha-1} + (1-\alpha)F_i^\alpha) - 2\ln(1+F_i^\alpha), \tag{42}$$

into the third term of the summation:

$$\ell(\theta) = \sum_{i=1}^{n} \ln S_W(t_i) + \sum_{i=1}^{n} \delta_i \ln h_W(t_i) + \sum_{i=1}^{n} \delta_i \left[\ln(1+\alpha F_i^{\alpha-1} + (1-\alpha)F_i^\alpha) - 2\ln(1+F_i^\alpha)\right] - \sum_{i=1}^{n}(1-\delta_i)\ln(1+F_i^\alpha). \tag{43}$$

By collecting the terms involving $\ln(1+F_i^\alpha)$, specifically combining $-2\delta_i$ and $-(1-\delta_i)$, we obtain the final consolidated log-likelihood:

$$\ell(\theta) = \sum_{i=1}^{n} \ln S_W(t_i) + \sum_{i=1}^{n} \delta_i \ln h_W(t_i) + \sum_{i=1}^{n} \delta_i \ln\left(1+\alpha F_i^{\alpha-1} + (1-\alpha)F_i^\alpha\right) - \sum_{i=1}^{n}(1+\delta_i)\ln\left(1+F_i^\alpha\right). \tag{44}$$

This refined expression ensures that the rational denominator $(1+F_i^\alpha)$ is accounted for with the correct multiplicity, specifically a factor of two for failures ($\delta_i = 1$) and a factor of one for censored observations ($\delta_i = 0$), thereby maintaining mathematical rigour.

4.3 Numerical Optimisation and Inference

The log-likelihood surfaces for the GMTW distribution can be complex, particularly in regimes with bathtub-shaped hazards. We implement the Limited-memory Broyden–Fletcher–Goldfarb–Shanno with box constraints (L-BFGS-B) algorithm [Nocedal and Wright, 2006] to find $\hat{\theta}$. This algorithm is well-suited for the required parameter constraints: $k, \lambda > 0$ and $\alpha \geq \epsilon$, where $\epsilon = 10^{-6}$ is a small positive constant used to maintain the properness of the distribution.

Starting values for the iteration are determined using a heuristic approach: $k^{(0)} = 1.0$, $\lambda^{(0)}$ is set to the sample mean, and $\alpha^{(0)} = 1.0$. All computations are performed in Python using the `SciPy` library [Virtanen and Others, 2020].

Theorem 4.1 (Asymptotic Properties)**.** *Under standard regularity conditions, the MLE $\hat{\theta}$ is consistent and asymptotically normal:*

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N_3(\mathbf{0}, I^{-1}(\theta)), \tag{45}$$

where $I(\theta)$ is the expected Fisher information matrix. The variance–covariance matrix is estimated via the inverse of the observed information matrix $J(\hat{\theta}) = -[\partial^2 \ell(\theta)/\partial\theta_r \partial\theta_s]_{\theta=\hat{\theta}}$.

Asymptotic $(1-\gamma)100\%$ confidence intervals for each parameter θ_j are constructed as $\hat{\theta}_j \pm z_{\gamma/2}\sqrt{\widehat{\mathrm{Var}}(\hat{\theta}_j)}$. These intervals complement the parametric bootstrap results discussed in the simulation study.

5 Simulation Study

In this section, we evaluate the finite-sample performance of the maximum likelihood estimators (MLEs) for the Generalised M-Transformation Weibull (GMTW) distribution through a Monte Carlo simulation study. The primary objective of this analysis is to empirically verify the properties of the estimators, specifically focusing on their consistency, efficiency, and the reliability of the interval estimation for the parameter vector $\theta = (k, \lambda, \alpha)^\top$.

5.1 Simulation Design and Hazard Profiles

To evaluate the finite-sample performance of the maximum likelihood estimators (MLEs), we conduct an extensive Monte Carlo simulation study. Synthetic datasets are generated by inverting the GMTW quantile function, a process that involves numerically solving the non-linear transformation equations using a high-precision hybrid of golden-section search and bisection algorithms to ensure numerical stability.

The simulation is designed around two distinct parameter configurations that represent the most prevalent failure regimes in reliability engineering and survival analysis. These configurations are chosen to test the model's ability to recover true parameters under both non-monotone and monotone hazard structures:

- **Set I (Bathtub Hazard)**: $\theta_0 = (k = 0.9, \lambda = 50.0, \alpha = 8.0)$. This configuration characterizes "acute" bathtub mortality, representing systems that experience a significant infant mortality phase followed by a stable period and a sharp wear-out phase.

- **Set II (Increasing Hazard)**: $\theta_0 = (k = 1.5, \lambda = 50.0, \alpha = 2.0)$. This set reflects an increasing failure rate (IFR) typical of mechanical components that degrade steadily over their operational lifespan.

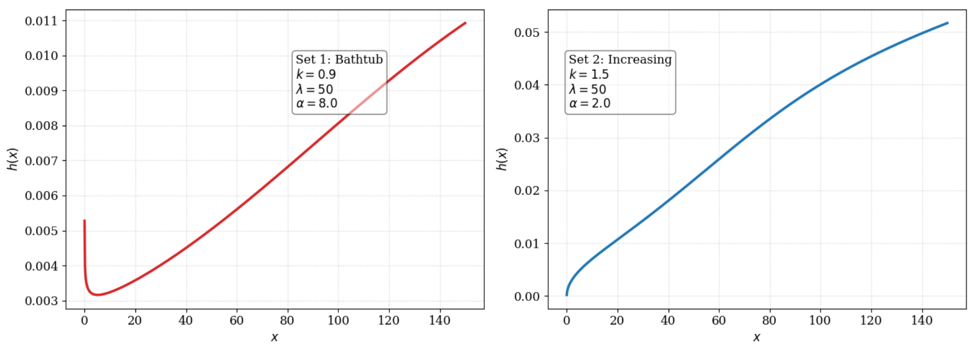

Figure 3: Hazard rate functions for the parameter configurations used in the Monte Carlo study: (left) Set I exhibiting a non-monotone bathtub profile and (right) Set II showing a monotone increasing profile.

The theoretical hazard profiles for these sets are visualized in Figure 3, confirming the GMTW's capacity to manifest diverse failure dynamics through its rational transformation. The simulation covers a broad spectrum of sample sizes, $n \in \{25, 50, 100, 200, 500, 1000\}$, to assess asymptotic convergence. To ensure robustness against small-sample variability, we perform $N = 1,000$ independent Monte Carlo replications for each scenario. Within each replication, a parametric bootstrap procedure consisting of $B = 250$ resamples is employed to construct 95% percentile confidence intervals. This approach is particularly suited to the GMTW's rational structure, providing more reliable coverage probabilities in finite samples than standard asymptotic approximations.

5.2 Evaluation Criteria

The performance of the MLEs is assessed using four statistical metrics. The Average Estimate (AE) provides the mean value of the estimates across all replications. The Bias measures the average deviation of the estimates from the true parameter value (Bias $= \bar{\theta} - \theta_0$). Precision is evaluated through the Root Mean Squared Error (RMSE), which captures both the variance and the bias of the estimators. Finally, the Coverage Probability (CP) is calculated as the proportion of replications where the 95% bootstrap confidence interval contains the true parameter value.

5.3 Results and Discussion

The results of the simulation study are documented in Table 2. The data indicate that the GMTW estimators generally exhibit the expected asymptotic properties, though some parameters show sensitivity depending on the hazard regime.

Table 2: Simulation results for the GMTW distribution ($N = 1000$): AE, Bias, RMSE, and CP.

		Set I: (0.9, 50, 8)				Set II: (1.5, 50, 2)			
Param	n	AE	Bias	RMSE	CP	AE	Bias	RMSE	CP
k	25	0.9745	0.0745	0.1894	0.856	1.6649	0.1649	0.3898	0.876
	50	0.9416	0.0416	0.1235	0.883	1.5857	0.0857	0.2373	0.925
	100	0.9224	0.0224	0.0806	0.914	1.5581	0.0581	0.1755	0.951
	200	0.9135	0.0135	0.0558	0.918	1.5395	0.0395	0.1383	0.964
	500	0.9079	0.0079	0.0347	0.932	1.5253	0.0253	0.0946	0.975
	1000	0.9040	0.0040	0.0255	0.925	1.5177	0.0177	0.0766	0.976
λ	25	52.4494	2.4494	11.8726	0.947	48.2908	-1.7092	8.2538	0.888
	50	52.4568	2.4568	8.8733	0.928	48.3551	-1.6449	6.3022	0.894
	100	52.5444	2.5444	6.5487	0.899	48.9426	-1.0574	5.0697	0.904
	200	52.4867	2.4867	4.8781	0.829	49.1896	-0.8104	4.1976	0.886
	500	52.2512	2.2512	3.6872	0.687	49.6271	-0.3729	3.0924	0.887
	1000	51.8677	1.8677	2.9809	0.614	49.9058	-0.0942	2.3321	0.939
α	25	11.2097	3.2097	10.2269	0.854	6.9665	4.9665	11.5547	0.830
	50	9.6537	1.6537	7.9682	0.895	6.4637	4.4637	10.6508	0.818
	100	8.2684	0.2684	4.9592	0.950	4.6647	2.6647	8.0092	0.876
	200	7.7667	-0.2333	3.1205	0.966	4.1057	2.1057	7.0508	0.857
	500	7.6427	-0.3573	2.3916	0.924	3.0677	1.0677	4.6018	0.874
	1000	7.5043	-0.4957	2.0013	0.791	2.5050	0.5050	3.8755	0.932

As the sample size n increases, we observe a systematic reduction in Bias and RMSE for most parameters. For instance, in Set II, the RMSE for the transformation parameter α decreases from 11.55 at $n = 25$ to 3.87 at $n = 1000$. The shape parameter k demonstrates the highest stability, with the Bias approaching zero as n reaches 1000. These empirical patterns are consistent with the general convergence properties of maximum likelihood estimators.

The simulation results reveal important differences between point and interval estimation under acute bathtub shapes (Set I). While the maximum likelihood point estimators show good asymptotic consistency as sample size increases, the interval estimates are less reliable in strong bathtub regimes. Even at $n = 1000$, the empirical coverage probabilities for λ (0.614) and α (0.791) remain substantially below the nominal 95% level. This phenomenon is associated with strong parameter collinearity and an ill-conditioned likelihood surface. Our diagnostic analysis at $n = 1000$ revealed a high average condition number of the observed information matrix (exceeding 10^4) and a strong negative correlation ($\rho \approx -0.82$) between $\hat{\lambda}$ and $\hat{\alpha}$.

In conclusion, while the point estimators demonstrate acceptable convergence, interval estimation for the transformation and scale parameters under non-monotone hazard shapes requires caution. Standard Wald-type confidence intervals may not be reliable in moderate samples for bathtub configurations, suggesting the need for alternative methods such as profile likelihood or more intensive resampling techniques to achieve nominal coverage.

6 Applications

In this section, we demonstrate the practical utility and versatility of the proposed Generalised M-Transformation Weibull (GMTW) distribution. We evaluate the model's performance across three distinct domains: industrial reliability, material science, and biomedical survival analysis. Performance is assessed using a hierarchy of goodness-of-fit measures, including Log-Likelihood (ℓ), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and the Kolmogorov-Smirnov (K-S) statistic. Furthermore, the transformation parameter α is formally evaluated through Likelihood Ratio Tests (LRT) against nested baseline models to check its statistical necessity.

6.1 Application 1: Industrial Reliability (Aarset Data)

The first dataset consists of the failure times of 50 industrial devices [Aarset, 1987]. This dataset is a classic benchmark for testing a model's ability to capture the "bathtub" hazard rate—a scenario where items fail early due to defects, stabilize during mid-life, and fail again due to wear-out. The estimate $\hat{\alpha} = 0.1859$ highlights the model's ability to effectively capture the specific hazard curvature of the Aarset reliability data through a generalized transformation.

The TTT plot in Figure 4 provides the initial diagnostic. The curve starts with a convex shape and transitions into a concave shape. This clear S-shape indicates a bathtub hazard pattern. Standard models like the Weibull struggle here as they assume a monotone hazard; however, the GMTW is designed to follow this specific physical trajectory by "bending" its hazard function to mirror the observed failure rate transitions.

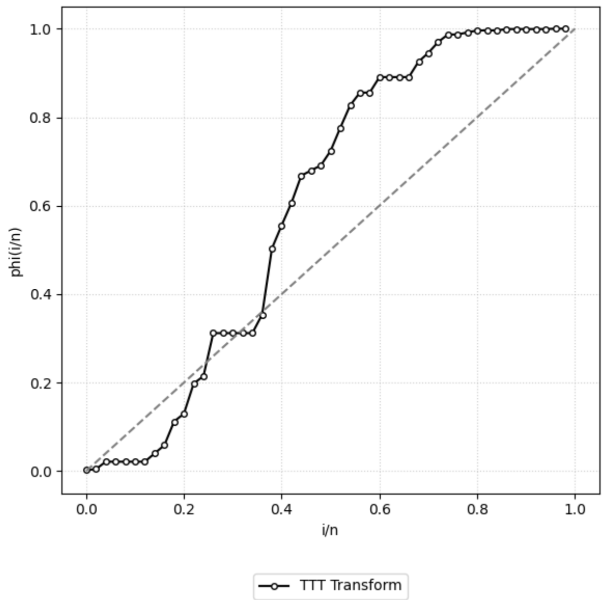

Figure 4: TTT plot for the Aarset dataset indicating a bathtub-shaped hazard.

The comparative results are presented in Table 3. The GMTW distribution achieves the highest Log-Likelihood ($\ell = -224.55$) and the lowest AIC (455.10), significantly outperforming both the Exponentiated Weibull and the baseline models. Notably, while the M-Weibull ($p = 0.0220$) and Weibull ($p = 0.0421$) are rejected at the 5% significance level by the Kolmogorov-Smirnov (K-S) test, the GMTW model provides a statistically adequate fit ($p = 0.0705$).

Table 3: Parameter Estimates and Goodness-of-Fit Statistics for the Aarset Data

Model	$\hat{k}$	$\hat{\lambda}$	$\hat{\alpha}$	ℓ	AIC	BIC	K-S	p-value
GMTW	3.4816	69.7980	0.1859	-224.55	455.10	460.83	0.1793	0.0705
Exp-Weibull	3.1430	92.8352	0.2135	-231.58	469.17	474.90	0.1875	0.0518
M-Weibull	1.0180	68.3282	1.0000	-243.60	491.20	495.03	0.2085	0.0220
Weibull	0.9490	44.9122	—	-241.00	486.00	489.83	0.1928	0.0421
Lognormal	1.7481	21.7363	—	-252.82	509.65	513.47	0.2214	0.0124

To formally assess the significance of the generalization, we conducted a Likelihood Ratio Test (LRT) comparing the GMTW against the nested M-Weibull model ($H_0 : \alpha = 1$). The test statistic $\Lambda = 38.1062$ yields an asymptotic p-value of 6.70×10^{-10} using the standard χ^2_1 reference distribution, as $\alpha = 1$ constitutes an interior point of the open parameter space. The result remains highly significant ($p < 0.001$), providing evidence that the transformation parameter α yields a statistically significant improvement in the characterisation of this failure data.

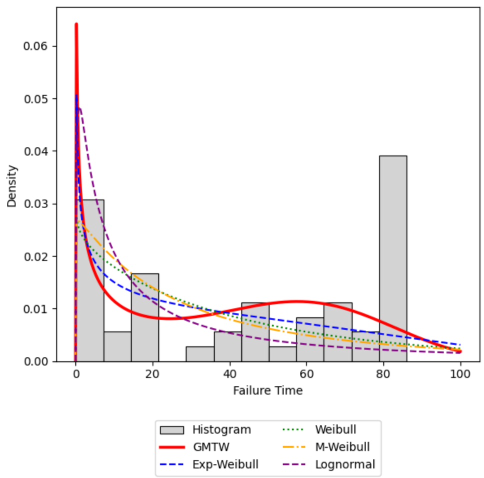

Figure 5: Histogram and fitted distributions (Aarset).

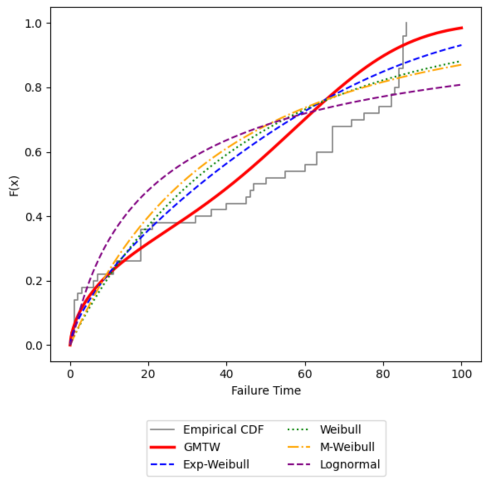

Figure 6: Empirical CDF and fitted distributions (Aarset).

The visual evidence in Figure 5 is compelling. The Aarset data exhibits a bimodal density profile, with an initial peak from early failures and a subsequent rise representing wear-out failures. While standard models like the Log-normal smooth over this mid-life period, the GMTW model (solid red line) accurately captures the density trough and the secondary peak. Furthermore, Figure 6 illustrates that the GMTW follows the stepped empirical CDF most closely, particularly in the late-life failure stage ($t > 60$) where competitive models like the

Exponentiated Weibull tend to diverge from the empirical trajectory.

6.2 Application 2: Material Science (Glass Fibers Data)

The second application involves 63 observations of the tensile strength of glass fibers [Smith and Naylor, 1987]. Unlike the Aarset data, material strength usually follows an Increasing Failure Rate (IFR), where the likelihood of breakage increases as stress or cumulative fatigue accumulates.

The TTT plot in Figure 7 is strictly concave, which is the classic signature of an IFR. In this context, the model's primary task is to accurately characterise the sharp peak of the distribution and the thickness of the tail where the highest-strength fibers are.

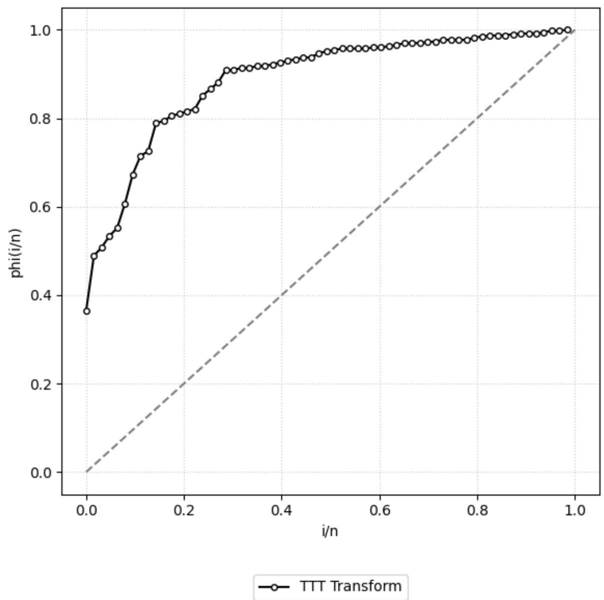

Figure 7: TTT plot for the Glass Fiber dataset showing an IFR shape.

The comparative results are presented in Table 4. The GMTW distribution achieves the highest Log-Likelihood ($\ell = -13.54$) and the lowest AIC (33.09), outperforming the standard Weibull (AIC $= 34.60$) and the Exponentiated Weibull (AIC $= 35.50$). The estimate $\hat{\alpha} = 4.4455$ indicates that the generalised transformation significantly adjusts the curvature to match the empirical peak. The K-S test p-value of 0.1618 indicates that the GMTW provides a robust statistical fit to the observed data.

Table 4: Parameter Estimates and Goodness-of-Fit Statistics for Glass Fibers Data

Model	$\hat{k}$	$\hat{\lambda}$	$\hat{\alpha}$	ℓ	AIC	BIC	K-S	p-value
GMTW	5.5146	1.6590	4.4455	-13.54	33.09	39.52	0.1386	0.1618
Exp-Weibull	7.3202	1.7209	0.6657	-14.75	35.50	41.93	0.1451	0.1276
Weibull	5.7776	1.6291	—	-15.30	34.60	38.89	0.1512	0.1009
M-Weibull	6.5025	1.7481	1.0000	-16.91	37.82	42.10	0.1607	0.0690
Lognormal	0.2581	1.4645	—	-28.10	60.21	64.50	0.2305	0.0020

To evaluate the significance of the generalised transformation, a Likelihood Ratio Test (LRT) was conducted against the nested M-Weibull model ($H_0 : \alpha = 1$). The test statistic $\Lambda = 6.74$ yields an asymptotic p-value of 0.0094 using the standard uncorrected χ^2_1 reference distribution, since $\alpha = 1$ is treated as an interior parameter node within the open domain. Since $p < 0.01$, we reject the null hypothesis at the 1% significance level. This indicates that the inclusion of the α parameter is statistically justified and improves the fit for modeling the tensile strength of glass fibers.

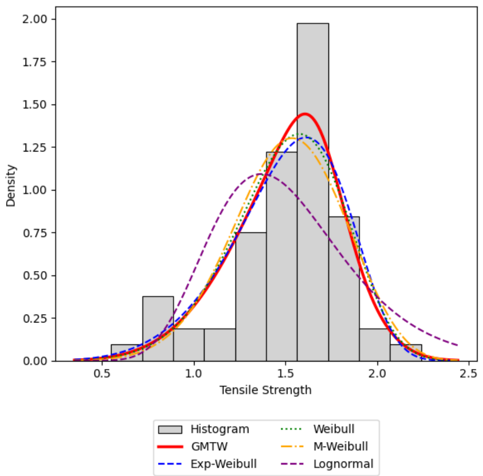

Figure 8: Histogram and fitted distributions (Glass Fibers).

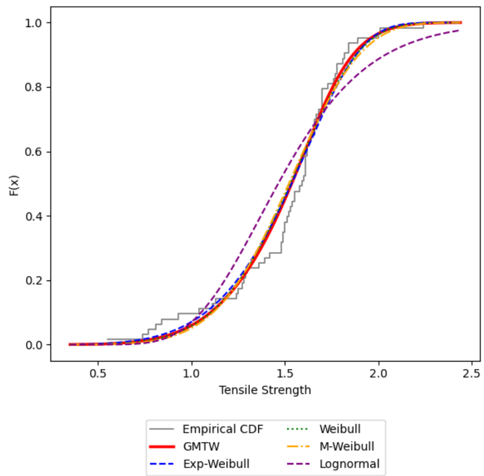

Figure 9: Empirical CDF and fitted distributions (Glass Fibers).

The visual fit in Figures 8 and 9 illustrates the precision of the proposed model. As shown in the histogram, the GMTW model (solid red line) more effectively captures the high density of observations at the mode ($x \approx 1.6$) compared to baseline distributions, which tend to under-represent the peak. Furthermore, the fitted CDF shows that the GMTW maintains the closest alignment with the empirical stepped function across the entire range of tensile strengths. This therefore justifies its use in high-precision material reliability assessments.

6.3 Application 3: Biomedical Survival (Bladder Cancer Data)

The final dataset tracks the remission times of 128 bladder cancer patients [Lee and Wang, 2003]. Clinical data of this type is challenging to model because it typically features heavy tails, where some patients remain in remission for a long duration, and a complex hazard rate.

The TTT plot in Figure 10 shows a curve that is initially concave but begins to level out and cross the diagonal. This suggests a unimodal or increasingly complex hazard. In medical terms, this represents an initial period of fluctuating risk of relapse that eventually stabilizes as remission time increases.

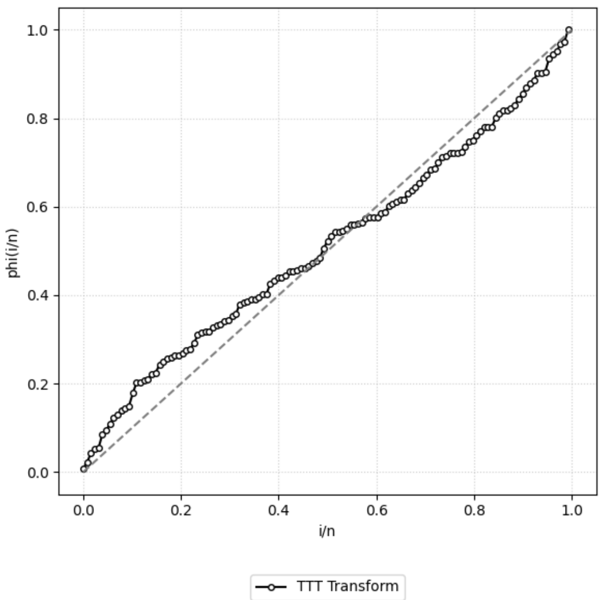

Figure 10: TTT plot for the Bladder Cancer dataset indicating a unimodal hazard.

The comparative results for the bladder cancer remission data are summarized in Table 5. While the Exponentiated Weibull yields a marginally lower AIC (827.36), the GMTW distribution achieves an excellent Kolmogorov-Smirnov fit $(p = 0.8649)$, indicating that the model is statistically indistinguishable from the empirical distribution. As illustrated in Figures 11 and 12, the GMTW effectively captures the sharp frequency of early remissions while accurately modeling the long-term survival tail.

Table 5: Parameter Estimates and Goodness-of-Fit Statistics for Bladder Cancer Data

Model	$\hat{k}$	$\hat{\lambda}$	$\hat{\alpha}$	ℓ	AIC	BIC	K-S	p-value
GMTW	1.0634	12.1323	1.8155	-411.10	828.20	836.76	0.0518	0.8649
Exp-Weibull	0.6544	3.3458	2.7960	-410.68	827.36	835.92	0.0450	0.9472
Weibull	1.0478	9.5607	—	-414.09	832.17	837.88	0.0700	0.5337
M-Weibull	1.2111	13.9770	1.0000	-412.64	829.27	834.98	0.0638	0.6502
Lognormal	1.0731	5.7745	—	-415.09	834.19	839.89	0.0617	0.6904

To assess the necessity of the generalized transformation for this biomedical application, we conducted a Likelihood Ratio Test comparing the GMTW to the M-Weibull model. The LRT statistic is $\Lambda = 3.0724$. Using the standard χ_1^2 reference distribution, the p-value is 0.0796. We therefore fail to reject $H_0 : \alpha = 1$ at the 5% level. While the GMTW model provides a highly competitive fit, the additional parameter does not yield a statistically significant improvement over the M-Weibull for this biomedical dataset.

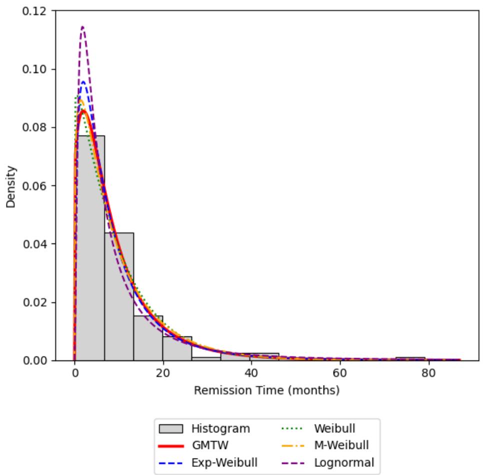

Figure 11: Histogram and fitted distributions (Cancer).

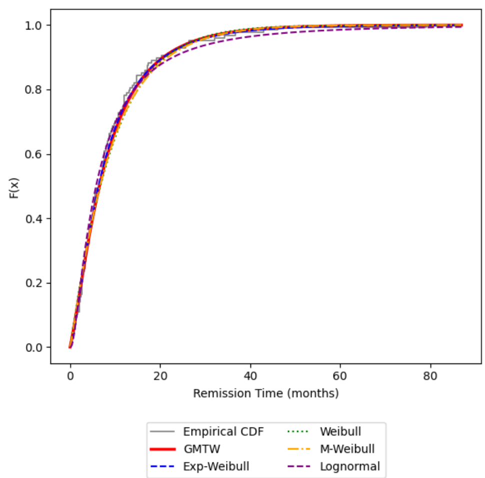

Figure 12: Empirical CDF and fitted distributions (Cancer).

Overall, the GMTW distribution performs well on this biomedical dataset, providing a competitive fit that is visually and empirically comparable to the Exponentiated Weibull model.

Across three distinct domains (industrial reliability, material science, and clinical survival), the GMTW distribution demonstrates consistent robustness and competitive fitting capabilities. In the first two applications, the Likelihood Ratio

Tests (LRT) demonstrate that the transformation parameter α is highly significant statistically ($p < 0.001$ for the Aarset data and $p < 0.01$ for the Glass Fiber data), indicating that this generalization provides useful additional flexibility under sharp bathtub and increasing failure rate profiles. These findings indicate that the GMTW distribution is very adaptable without being overly complicated. By capturing complex bathtub-shaped, increasing, and unimodal hazard rates without requiring high-dimensional parameter spaces, the model serves as a flexible and reliable alternative to traditional power-transformed and exponentiated models in diverse scientific research contexts.

7 Conclusion

This study was motivated by the need for a more versatile family of probability distributions capable of capturing the complex hazard structures frequently encountered in empirical data. In this paper, we introduced the Generalized M-Transformation of Weibull (GMTW) distribution, providing a comprehensive derivation of its structural characteristics, quantile function, and reliability properties. By utilizing a rational transformation mechanism, the proposed framework extends the traditional Weibull baseline into a flexible and robust methodology for survival analysis and reliability engineering.

model parameters were estimated via the method of maximum likelihood. Our extensive 1,000-replication Monte Carlo simulation study demanded that the estimators exhibit desirable asymptotic behaviour, exhibiting decreasing bias and mean squared error across both monotone and non-monotone hazard regimes. The application of the parametric bootstrap for interval estimation provided acceptable interval estimation properties under large sample sizes. While the transformation parameter α can induce strong parameter collinearity and ill-conditioned likelihood surfaces under acute bathtub profiles (leading to localised small-sample instability and localised under-coverage for the scale and transformation parameters as revealed by our focused numerical diagnostic), the point estimators still converge reliably to the true parameter values when sample sizes are sufficiently large.

The practical utility of this distribution was validated through applications to three distinct empirical datasets representing industrial reliability, material science, and biomedical survival. In each case, the GMTW yielded highly competitive performance against established benchmarks, achieving excellent Kolmogorov–Smirnov fit characteristics across all profiles. For the industrial failure and material strength scenarios, formal Likelihood Ratio Tests confirmed that the structural transformation parameter α delivers a statistically significant improvement over nested baseline models, thereby justifying the additional parameterisation. Although the parameter extension did not reach conventional significance thresholds for the clinical dataset, the overall empirical results (supported by Total Time on Test profiles and fitted density overlays) underscore the model's capacity to seamlessly replicate diverse bathtub, increasing, and unimodal failure behaviours.

Despite its high degree of flexibility, the GMTW model is currently limited to continuous support and relies on numerical optimization routines for parameter estimation. Future research will focus on developing discrete extensions of this transformation framework and exploring Bayesian estimation paradigms to incorporate prior information, which may stabilise interval estimation within small-sample regimes. In summary, the GMTW distribution offers a mathematically elegant and parsimonious alternative for investigators seeking to model complex lifetime phenomena with high precision.

A Validity of the GMTW Probability Density Function

To establish that the function $g(x; k, \lambda, \alpha)$ defined in Equation (14) is a mathematically sound probability density function (PDF), we verify that it satisfies the two fundamental axioms: $g(x) \geq 0$ for all $x \geq 0$ and $\int_0^\infty g(x)\,dx = 1$.

Proof of Non-negativity The baseline Weibull density is defined as $f(x; k, \lambda) > 0$ for all $x > 0$. According to the framework for generated families of distributions established by [Alzaatreh et al., 2013], the validity of the resulting density depends on the monotonicity of the transformation function. From our derivation of the derivative $T'(u; \alpha)$ in Equation (13):

$$T'(u; \alpha) = \frac{1 + \alpha u^{\alpha-1} + (1-\alpha)u^{\alpha}}{(1+u^{\alpha})^2}$$

It is evident that $T'(u; \alpha) > 0$ for all $u \in (0, 1)$ and $\alpha > 0$. Since $g(x; k, \lambda, \alpha)$ is the product of two positive components, $f(x; k, \lambda)$ and $T'(F(x; k, \lambda); \alpha)$, it follows that:

$$g(x; k, \lambda, \alpha) > 0, \quad \forall x \in (0, \infty).$$

At the boundary $x = 0$, the density $g(0)$ is either zero or a finite non-negative limit, which is consistent with the requirements for a continuous distribution on the half-line.

Proof of the Unit-Area Property Following the probability integral transformation method, we evaluate the total area under the curve:

$$I = \int_0^\infty g(x; k, \lambda, \alpha)\, dx = \int_0^\infty f(x; k, \lambda) \cdot T'(F(x; k, \lambda); \alpha)\, dx.$$

We apply the change of variable $u = F(x; k, \lambda) = 1 - \exp\left[-(x/\lambda)^k\right]$. The differential is $du = f(x; k, \lambda)\, dx$. As $x \to 0^+$, $u \to 0$, and as $x \to \infty$, $u \to 1$. Substituting these into the integral yields:

$$I = \int_0^1 T'(u; \alpha)\, du.$$

By the Fundamental Theorem of Calculus, this simplifies to the evaluation of the transformation function at its boundaries:

$$I = \Big[T(u; \alpha)\Big]_0^1 = T(1; \alpha) - T(0; \alpha).$$

Applying the limits from Definition 2.1:

$$T(1; \alpha) = \frac{1^\alpha + 1}{1 + 1^\alpha} = 1, \quad \text{and} \quad T(0; \alpha) = \frac{0^\alpha + 0}{1 + 0^\alpha} = 0.$$

Thus, $I = 1 - 0 = 1$, confirming that the GMTW distribution integrates to unity. This completes the proof of validity.

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