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Common Fixed Point Theorems For Ćirić-Reich-Rus Type Contractions In F-Metric Spaces

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Abstract

In this work, we introduce a Ćirić-Reich-Rus type contraction for four mappings and prove a common fixed point theorem for mappings satisfying this contractive condition in F-metric spaces. Additionally, we present several related results. To illustrate the validity and applicability of our theoretical findings, we provide an application that bridges abstract theory and practical implementation.

Keywords: F-metric, common fixed point, weakly compatible mappings.
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1 Introduction

The notion of F-metric space was introduced as a generalization of metric spaces. F-metric spaces satisfy the properties of non-negativity, identity, symmetry and a generalized triangular inequality by means of an F-function defined by Wardowski. The authors also established the Banach contraction principle in F-metric spaces in 2018 [Jleli and Samet, 2018]. Later Bera et al. were interested in the topological properties of F-metric spaces[Bera et al., 2019].
Let Ω denote the family of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following properties;

F_1) F is increasing,

F_2) For every sequence $\{x_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\ x_n) = -\infty$ [Wardowski, 2012].

Definition 1.1. *[Jleli and Samet, 2018] Let X be a nonempty set and let $(F, \alpha) \in \Omega \times [0, +\infty)$. A mapping $d : X \times X \to [0, \infty)$ is called an F-metric if it satisfies:*

D_1) $d\,(x, y) = 0 \Leftrightarrow\ x = y,$

D_2) $d\,(x, y) = d\,(y, x)$

D_3) *for $\forall n \geq 2$ ($n \in \mathbb{N}$) and $\forall \{u_i\} \subset X$ with $(u_1, u_n) = (x, y)$ such that*

$$d\,(x, y) > 0 \Longrightarrow F\,(d\,(x, y)) \leq F\left(\Sigma_{i=1}^{n-1} d\,(u_i, u_{i+1})\right) + \alpha$$

for all $x, y \in X$. In this case (X, d) is called an F-metric space.

Definition 1.2. *[Jleli and Samet, 2018] Let $\{x_n\} \subset X$ be a sequence.*

i. *$\{x_n\}$ is said to be F-convergent if there exists a $x \in X$ such that $d\,(x_n, x) \to 0$ as $n \to \infty$.*

ii. *$\{x_n\}$ is said to be an F-Cauchy sequence if $d\,(x_n, x_m) \to 0$ as $n, m \to \infty$.*

iii. *(X, d) is said to be F-complete if every F-Cauchy sequence is F-convergent.*

Theorem 1.1. *[Bera et al., 2019] Every $F-$metric space is Hausdorff.*

Theorem 1.2. *[Jleli and Samet, 2018] Let (X, d) be an $F-$complete $F-$metric space and let $T : X \to X$ be a self-mapping. Suppose that*

$$d(Tx, Ty) \leq ad(x, y) \tag{1}$$

for all $x, y \in X$, where $a \in (0, 1)$. Then T has a unique fixed point.

Many researchers have developed various contraction principles and proved fixed point results for a self-mapping in F-metric spaces. In F-metric spaces, Lateefa [Lateefa and Ahmad, 2019] and [Mitrovic et al., 2019] generalized (1) as the following contractions:

i. Das Gupta contraction:

$$d(Tx,Ty) \leq a_1 d(x,y) + a_2 \frac{[1+d(x,Tx)]\, d(y,Ty)}{1+d(x,y)}$$

where $a_1 \in (0,1)$ with $a_1 + a_2 < 1$.

ii. Reich contraction:

$$d(Tx,Ty) \leq a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty)$$

where $a_1, a_2, a_3 \in (0,1)$ with $a_1 + a_2 + a_3 < 1$.

Al-Mezel et al. [Mezel et al., 2020], Faraji et al. [Faraji et al., 2022] defined $\alpha-\beta-$admissible-type contraction. Acar and Ozturk [Acar and Ozturk, 2024] defined $\alpha-$admissible Berinde type contraction and Hussain and Kanwal [Hussain and Kanwal, 2018] defined $\alpha-$admissible Ćirić type contraction. Ozturk [Ozturk, 2023] defined Ćirić-Presic type contraction. Mitrovic et al. [Mitrovic et al., 2019] introduced following contraction principle and proved common fixed point theorems in F-metric spaces.

Theorem 1.3. *[Mitrovic et al., 2019] Let (X,d) be an F-complete F-metric space and let $T,S : X \to X$ be self-mappings satisfying*

$$d(Tx,Ty) \leq a_1 d(Sx,Sy) + a_2 d(Sx,Tx) + a_3 d(Sy,Ty)$$

for all $x,y \in X$, where $a_1 \in (0,1)$ and $a_2, a_3 \in [0,1)$ with $a_1 + a_2 + a_3 < 1$.

In this work, we generalize the concept of Ćirić-Reich-Rus type contractions in F-metric spaces for four mappings and we prove a fixed point theorem for weakly compatible mappings.

Definition 1.3. *[Jungck, 1996] Let X be a nonempty set. Two mappings $T,S : X \to X$ are said to be weakly compatible if $TSx = STx$ whenever $Tx = Sx$.*

2 Main Results

Definition 2.1. *Let (X,d) be an F-complete F-metric space. The self-mappings $T,S,A,B : X \to X$ are called a Ćirić-Reich-Rus type contraction if there exist $a_1 \in (0,1)$ and $a_2, a_3 \in [0,1)$ with $a_1 + a_2 + a_3 < 1$ such that*

$$d(Ax,By) \leq a_1 d(Sx,Ty) + a_2 d(Ax,Sx) + a_3 d(By,Ty) \tag{2}$$

for all $x,y \in X$.

Theorem 2.1. *Let (X,d) be an F-complete F-metric space. Let $T,S,A,B : X \to X$ be Ćirić-Reich-Rus type contraction mappings (2). Assume that*

$i.$ $A(X) \subseteq T(X), B(X) \subseteq S(X)$,

$ii.$ $T(X)$ *or* $S(X)$ *is a closed subset of* X,

$iii.$ *the pairs* (A,S) *and* (B,T) *are weakly compatible.*

Then A,B,S and T have a unique common fixed point.

Proof

Let $x_0 \in X$ be arbitrary. Since $Ax_0 \in T(X)$, there exists $x_1 \in X$ such that $Tx_1 = Ax_0$. Similarly, since $Bx_1 \in S(X)$, there exists $x_2 \in X$ such that $Sx_2 = Bx_1$. Continuing inductively, we obtain a sequence $\{x_n\}$ satisfying $Tx_{2k+1} = Ax_{2k}$ and $Sx_{2k+2} = Bx_{2k+1}$, $k = 0,1,2,\dots$.
Define

$$y_{2k} = Tx_{2k+1} = Ax_{2k}$$

$$y_{2k+1} = Sx_{2k+2} = Bx_{2k+1}.$$

Now we show that $\{y_n\}$ is an F-Cauchy sequence. Let $(F,\alpha) \in \Omega \times [0,\infty)$ be the pair satisfying condition (D_3). By (F_2), there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow F(t) < F(\epsilon) - \alpha. \tag{3}$$

Using the contractive condition (2),

$$\begin{aligned} d(y_{2k}, y_{2k+1}) &= d(Ax_{2k}, Bx_{2k+1}) \\ &\leq a_1 d(Sx_{2k}, Tx_{2k+1}) + a_2 d(Ax_{2k}, Sx_{2k}) + a_3 d(Bx_{2k+1}, Tx_{2k+1}) \\ &= a_1 d(y_{2k-1}, y_{2k}) + a_2 d(y_{2k}, y_{2k-1}) + a_3 d(y_{2k+1}, y_{2k}). \end{aligned}$$

Thus, we get

$$(1-a_3) d(y_{2k}, y_{2k+1}) \leq (a_1 + a_2) d(y_{2k}, y_{2k-1}).$$

So, we have

$$d(y_{2k}, y_{2k+1}) \leq \left(\frac{a_1 + a_2}{1 - a_3}\right) d(y_{2k}, y_{2k-1}).$$

Set $d_{2k} = d(y_{2k}, y_{2k+1})$

$$d_{2k} \leq \left(\frac{a_1 + a_2}{1 - a_3}\right) d_{2k-1}.$$

Since $a_1 + a_2 + a_3 < 1$, we get $d_{2k} \leq d_{2k-1}$ for $a = \frac{a_1+a_2}{1-a_3} < 1$. Similarly, it is easy to see that $d_{2k+1} \leq d_{2k}$. Repeating this process n times we obtain

$$\begin{aligned} d(y_{2k}, y_{2k+1}) &\leq a(y_{2k-1}, y_{2k}) \\ &\leq a^2 d(y_{2k-2}, y_{2k-1}) \\ &. \\ &. \\ &. \\ &\leq a^n d(y_0, y_1). \end{aligned}$$

Hence, for $m > n$

$$\sum_{i=n}^{m-1} d(y_i, y_{i+1}) \leq \left(\frac{a^n}{1-a}\right) d(y_0, y_1).$$

Since $a < 1$,

$$\lim_{n\to\infty} \left(\frac{a^n}{1-a}\right) d(y_0, y_1) = 0.$$

There exists some $N \in \mathbb{N}$ such that

$$0 < \left(\frac{a^n}{1-a}\right) d(y_0, y_1) < \delta$$

for $n \geq N$. By (3) and $(F1)$, for $m > n > N$, we have

$$F\left(\sum_{i=n}^{m-1} d(y_i, y_{i+1})\right) \leq F\left(\left(\frac{a^n}{1-a}\right) d(y_0, y_1)\right) \leq F(\epsilon) - \alpha. \tag{4}$$

Using (D_3) and (4),

$$d(y_n, y_m) > 0 \Rightarrow F\left(d(y_n, y_m)\right) \leq F\left(\sum_{i=n}^{m-1} d(y_i, y_{i+1})\right) + \alpha < F(\epsilon).$$

By $(F1)$, we get $d(y_n, y_m) < \epsilon$.
Thus $\{y_n\}$ is an F-Cauchy sequence. By F-completeness of (X, d) there exists $y \in X$ such that

$$\lim_{k\to\infty} Tx_{2k+1} = \lim_{k\to\infty} Ax_{2k} = \lim_{k\to\infty} Bx_{2k+1} = \lim_{k\to\infty} Sx_{2k+2} = y.$$

Assume without loss of generality that $T(X)$ is a closed subset of X. Then there exists $v \in X$ such that $Tv = y$.
Suppose $Bv \neq y$. Applying the contractive condition (2) we have

$$d(Ax_{2k}, Bv) \leq a_1 d(Sx_{2k}, Tv) + a_2 d(Ax_{2k}, Sx_{2k}) + a_3 d(Bv, Tv).$$

Letting $k \to \infty$ yields

$$d(y, Bv) \leq a_1 d(y, Tv) + a_2 d(y, y) + a_3 d(Bv, Tv)$$

and

$$d(y, Bv) \leq a_1 d(y, y) + a_2 d(y, y) + a_3 d(Bv, y).$$

So

$$d(y, Bv) \leq a_3 d(Bv, y)$$

which is contradiction since $a_3 < 1$. Hence, $Bv = y$. Since (B, T) is weakly compatible, we have $Ty = TBv = BTv = By$.
Now we show that $By = y$. Apply the contractive condition (2)

$$d(Ax_{2k}, By) \leq a_1 d(Sx_{2k}, Ty) + a_2 d(Ax_{2k}, Sx_{2k}) + a_3 d(By, Ty)$$

Letting $k \to \infty$, we get

$$d(y, By) \leq a_1 d(y, Ty) + a_2 d(y, y) + a_3 d(By, Ty).$$

Thus we get

$$d(y, By) \leq 0.$$

Hence $y = By$. Similarly, we show that y is also fixed point of A and S. Therefore $Ay = Ty = Sy = By = y$.
Now, we show that y is unique fixed point. Suppose z is another common fixed point of A, B, S, T. Then

$$d(z, y) = d(Az, By) \leq a_1 d(Sz, Ty) + a_2 d(Az, Sz) + a_3 d(By, Ty) = 0.$$

So $d(z, y) = 0$. Hence $z = y$.

Example 2.1. *Let $X = [0, 1]$ be an F-metric space equipped with the F-metric $d(x, y) = \begin{cases} 2^{|x-y|}, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases}$ and let $F(t) = \frac{-1}{t}$ with $\alpha = 1$. Define the mappings $A, B, S, T : X \to X$ by*

$$A(x) = \left(\frac{x}{2}\right)^8, \quad B(x) = \left(\frac{x}{2}\right)^2, \quad T(x) = x^2, \text{ and } S(x) = \left(\frac{x}{2}\right)^2.$$

The pairs (A, S) and (B, T) are weakly compatible, and both $T(X)$ and $S(X)$ are closed subsets of X. Furthermore, for $x \neq y$, the contractive condition holds as follows for suitable $a_1, a_2, a_3 > 0$ with $a_1 + a_2 + a_3 < 1$:

$$\begin{aligned} d(Ax, By) &= 2^{\left|\left(\frac{x}{2}\right)^8 - \left(\frac{y}{2}\right)^2\right|} = a_1 2^{\left|\left(\frac{x}{2}\right)^2 - y^2\right|} + a_2 2^{\left|\left(\frac{x}{2}\right)^8 - \left(\frac{x}{2}\right)^2\right|} + a_3 2^{\left|\left(\frac{y}{2}\right)^2 - y^2\right|} \\ &\leq a_1 d(Sx, Ty) + a_2 d(Ax, Sx) + a_3 d(By, Ty). \end{aligned}$$

$x = 0$ is the unique common fixed point of A, S, T, B.

If we take $S = T = I$ identity map in Theorem 2.1, we obtain following corollary.

Corollary 2.1. *Let (X, d) be an F-complete F-metric space and let $A, B : X \to X$ satisfy*

$$d(Ax, By) \leq a_1 d(x, y) + a_2 d(Ax, x) + a_3 d(By, y) \tag{5}$$

where $a_1, a_2, a_3 > 0$ with $a_1 + a_2 + a_3 < 1$. Then A and B have a unique common fixed point.

If we take $A = B$ in Corollary 2.1, we obtain following corollary.

Corollary 2.2. *Let (X, d) be an F-complete F-metric space and let $A : X \to X$ satisfies*

$$d(Ax, Ay) \leq a_1 d(x, y) + a_2 d(Ax, x) + a_3 d(Ay, y) \tag{6}$$

where $a_1, a_2, a_3 > 0$ with $a_1 + a_2 + a_3 < 1$. Then A has a unique fixed point.

Corollary 2.3. *Let (X, d) be an F-complete F-metric space and let $A, B, S, T : X \to X$ satisfy*

$$d(Ax, By) \leq a d(Sx, Ty) \tag{7}$$

where $0 < a < 1$ with the same conditions (i) and (ii) as in Theorem 2.1. Then A, B, S and T have a unique common fixed point.

3 Application

Let us consider the integral equation

$$x(t) = \alpha \int_0^1 T_i(t, s, x(s))\, ds + \beta \int_0^1 W_j(t, s, x(s))\, ds, i, j = 1, 2 \tag{8}$$

where α and β are real numbers and $T_i, W_j : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is continuous function. Equip $C[0,1]$ with the F-metric $d(x, y) = |x - y|^2$.

Theorem 3.1. *Suppose there exists $\gamma \in [0,1)$ such that for all $t, s \in [0,1]$ and $x, y \in C[0,1]$,*

$$|T_1(t, s, x(s)) - T_2(t, s, y(s))| \leq \sqrt{\gamma}\, |x(t) - y(t)|,$$

$$|W_1(t, s, x(s)) - W_2(t, s, y(s))| \leq \sqrt{\gamma}\, |x(t) - y(t)|.$$

Then the integral equation (8) has a unique solution with $\left(\frac{\alpha\sqrt{\gamma}}{1-\beta\sqrt{\gamma}}\right)^2 < 1$.

Proof. Define the operators

$$\begin{aligned} Ax(t) &= \alpha \int_0^1 T_1(t,s,x(s))\,ds \\ Bx(t) &= \alpha \int_0^1 T_2(t,s,x(s))\,ds \\ Sx(t) &= Ix(t) - \beta \int_0^1 W_1(t,s,x(s))\,ds \\ Tx(t) &= Ix(t) - \beta \int_0^1 W_2(t,s,x(s))\,ds. \end{aligned}$$

For all $x, y \in X$ and $t \in [0,1]$, we have

$$\begin{aligned} |Ax(t) - By(t)|^2 &= \left| \begin{array}{c} \alpha \int_0^1 T_1(t,s,x(s))\,ds \\ -\alpha \int_0^1 T_2(t,s,y(s))\,ds \end{array} \right|^2 \\ &\leq \alpha^2 \int_0^1 |T_1(t,s,x(s)) - T_2(t,s,y(s))|^2\,ds \\ &\leq \alpha^2\gamma \int_0^1 |x(t)-y(t)|^2\,ds \\ &\leq \alpha^2\gamma\,|x(t)-y(t)|^2 \int_0^1 ds. \end{aligned}$$

Thus, we get

$$d(Ax, By) \leq \alpha^2\gamma d(x,y). \tag{9}$$

Similarly,

$$\begin{aligned} |Sx(t) - Ty(t)|^2 &= \left| \begin{array}{c} Ix(t) - \beta \int_0^1 W_1(t,s,x(s))\,ds \\ -Iy(t) + \beta \int_0^1 W_2(t,s,y(s))\,ds \end{array} \right|^2 \\ &= \left| x(t) - y(t) - \beta\left(\int_0^1 [W_1(t,s,x(s)) - W_2(t,s,y(s))\,ds] \right) \right|^2 \\ &\geq \left[|x(t)-y(t)| - \left| \beta \left(\int_0^1 [W_1(t,s,x(s)) - W_2(t,s,y(s))\,ds] \right) \right| \right]^2 \\ &\geq \left[(1-\beta\sqrt{\gamma})\,|x(t)-y(t)| \right]^2. \end{aligned}$$

Hence we have

$$d(Sx, Ty) \geq (1-\beta\sqrt{\gamma})^2\, d(x,y).$$

From (9),

$$d(Ax, By) \leq \alpha^2\gamma \frac{d(Sx,Ty)}{\left(1-\beta\sqrt{\gamma}\right)^2} = \left(\frac{\alpha\sqrt{\gamma}}{1-\beta\sqrt{\gamma}} \right)^2 d(Sx,Ty)$$

All conditions of Corollary 2.3 are satisfied. Therefore integral equation (8) has a unique solution.

4 Discussion and Conclusions

Banach's fixed point theorem is a fundamental result guaranteeing the existence and uniqueness of fixed points. Over the last century, this result has been generalized to two, three, and four mappings, yielding various common fixed point theorems. The present study is the first to extend the Ćirić-Reich-Rus type contraction from two mappings to four mappings in the setting of F-metric spaces. An application to an integral equation is provided to demonstrate the practical utility of the theoretical results. The findings obtained here will serve as a guide for future research on fixed point theory in generalized metric spaces and will contribute to the solution of more complex problems in nonlinear analysis and its applications.

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