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Some results on pseudo MV-algebras with n-roots

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Abstract

The paper provides a study of pseudo MV-algebras with n-roots. We outline their main properties and establish that the class of MV-algebras with n-roots forms a variety. Then, we introduce the concept of a strict n-root to classify the class of pseudo MV-algebras with n-root and demonstrate an equivalence between this variety and the class of n-divisible unital ℓ-groups. Finally, we find a relationship between strongly atomless pseudo MV-algebra and strict pseudo MV-algebras.
Keywords: Pseudo MV-algebra; Square root; n-root; n-strict MV-algebra; Boolean algebra; Strongly atomless
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1 Introduction

MV-algebras, as algebraic models for infinite-valued logic, have been pivotal in the study of root theories such as square roots and n-roots. Building on the foundational work of Chang [Chang, 1958, 1959] and Mundici [Mundici, 1986], who established their equivalence with ℓ-groups, the research has extended to pseudo MV-algebras, non-commutative generalizations that enrich the algebraic framework for non-classical logics. The study of n-roots in pseudo MV-algebras addresses unique challenges arising from non-commutative operations, offering deeper insights into algebraic structures and their connections to divisibility and atomicity.

In MV-algebras, n-roots generalize square roots, revealing decompositions into Boolean algebras and strict MV-algebras, and establishing links with n-divisible ℓ-groups. Pseudo MV-algebras, however, introduce non-commutative operations ($\oplus$ and $\odot$), necessitating a redefinition of roots to accommodate asymmetry. For instance, weak square roots in pseudo MV-algebras satisfy existence and order-compatibility conditions, while strict square roots impose additional idempotent constraints on the root of zero, linking to strongly atomless properties .

Due to the non-commutativity of operations in pseudo MV-algebras, the research methods for the nth roots of MV-algebras cannot be directly applied. As a result, issues such as the rationality of the definition of n-roots in pseudo MV-algebras, the conditions for their existence, the determination of their uniqueness, and their interaction with filters, ideals, and homomorphic mappings have not been effectively solved.

In fact, the research on the nth roots of pseudo MV-algebras holds significant theoretical value and practical significance. From a theoretical perspective, clarifying the properties of n-roots is conducive to further revealing the non-commutative structural characteristics of pseudo MV-algebras, improving the element classification theory of pseudo MV-algebras, and providing a key link for constructing a more complete non-commutative many valued logical algebra system. From an application perspective, in non-commutative fuzzy reasoning systems based on pseudo MV-algebras, n-roots can be used to describe the ”detailed splitting” and ”inverse operation” of proposition truth values, providing a new algebraic tool for handling the iterative calculation of truth values in complex reasoning scenarios.

These roots interact deeply with the algebraic structure:

- **Decomposition**: Similar to MV-algebras, pseudo MV-algebras with n-roots can decompose into a Boolean algebra and a strict pseudo MV-algebra, a result rooted in the categorical equivalence with ℓ-groups and the properties of normal prime ideals.

- **Divisibility**: The existence of strict n-roots corresponds to the n-divisibility of the underlying ℓ-group, ensuring unique root extraction and structural simplicity.

The main goals of this paper are to provide a study on pseudo MV-algebra with n-roots and present their main properties:

- Define n-roots and weak n-roots that in the case of MV-algebras coincide.
- Present strict n-roots on the pseudo MV-algebras.
- Decompose n-roots into two parts, one as the identity on the Boolean part and one as a strict pseudo MV-algebras.
- Study strongly atomless and find their relationship with strict pseudo MV-algebras.
- Represent n-roots by addition in unital ℓ-groups on representable symmetric pseudo MV-algebras.

2 Preliminaries

In this section, we will review several definitions and fundamental statements that will serve as the foundation for the subsequent content of this paper. For more details about pseudo MV-algebras, we refer to [Dvurečenskij, 2001a,b, 2002, Georgescu and Iorgulescu, 2001].

Definition 2.1. *[Georgescu and Iorgulescu, 2001] A pseudo MV-algebra is an algebra $(M; \oplus, -, \sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$,*
(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(A2) $x \oplus 0 = 0 \oplus x = x$,
(A3) $x \oplus 1 = 1 \oplus x = 1$,
(A4) $1^- = 1^\sim = 0$,
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,
(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$,
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$,

(A8) $(x^-)^\sim = x$,
where

$$x \odot y = (y^- \oplus x^-)^\sim.$$

If the operation $\oplus$ is commutative, equivalently, $\odot$ is commutative, then M is an MV-algebra, and of course $x^- = x^\sim$ for each $x \in M$. We denote it by $x' := x^-$. For more about MV-algebras, see [Cignoli et al., 2013].
It is well-known that every pseudo MV-algebra is a bounded distributive lattice [Georgescu and Iorgulescu, 2001], where axiom (A6) defines $x \vee y$ and (A7) describes $x \wedge y$.

An algebra $(G; \vee, \wedge, +, -, 0)$ is said to be a lattice-ordered group or ℓ-group if :(1) $(G; \vee, \wedge)$ forms a lattice, (2) $(G; +, -, 0)$ is a group, and (3) the group operation $+$ is order-preserving. In an ℓ-group G, we use G^+ to denote $\{w \in G : 0 \leq w\}$. Additionally, for any $z \in G$, we define the positive part $z^+ := z \vee 0$ and the negative part $z^- := -z \vee 0$. A strong unit of G is an ℓ-group, and it is equipped with a fixed strong unit denoted as $u \in G$.

If $(G; \vee, \wedge, +, -, 0)$ is an ℓ-group, given $0 \leq u$, we define an interval $[0, u] := \{x \in G | 0 < x < u\}$. The interval $[0, u]$ can be converted into a pseudo MV-algebra $\Gamma(G, u) = ([0, u]; \oplus, -, \sim, 0, u)$ as follows: $x \oplus y = (x + y) \wedge u,\ x \odot y = (x - u + y) \vee 0$, $x^- = u - x$ and $x^\sim = -x + u$, $x, y \in [0, u]$.

Conversely, due to the principal result [Dvurečenskij, 2002], every pseudo MV-algebra is isomorphic to $\Gamma(G, u)$ for some unital ℓ-group (G, u). The unital ℓ-group (G, u) is unique up to isomorphism of unital ℓ-groups. Moreover, the category of unital ℓ-groups is categorical equivalent to the category of pseudo MV-algebras.

Let $\mathcal{UG}$ be the category of unital ℓ-groups whose objects are unital ℓ-groups (G, u) and morphisms between objects are ℓ-group homomorphisms preserving fixed strong units. We denote by $\mathcal{PMV}$ the category of pseudo MV-algebras whose objects are pseudo MV-algebras and morphisms are homomorphisms of pseudo MV-algebras. Consider the functor $\Gamma : \mathcal{UG} \to \mathcal{PMV}$ which is defined as follows

$$\Gamma(G, u) = (\Gamma(G, u); \oplus, -, \sim, 0, u),$$

and if $h : (G, u) \to (H, v)$ is a morphism, then $\Gamma(h)$ is the restriction of h onto the interval $[0, u]$. On the other side, there is a functor from the category of $\mathcal{PMV}$ to $\mathcal{UG}$ sending a pseudo MV-algebra M to a unital ℓ-group (G, u) such that $M \cong \Gamma(G, u)$ which is denoted by $\Psi : \mathcal{PMV} \to \mathcal{UG}$. If $\varphi : M_1 \to M_2$ is a morphism of pseudo MV-algebras, then $\Psi(\phi)$ is unique extension of ϕ onto a morphism of unital ℓ-groups. It is possible to show that if $\phi : M_1 \to M_2$ is an injective (surjective) homomorphism, so is $\Psi(\phi)$.

Theorem 2.1. *[Dvurečenskij, 2002] The composite functors $\Gamma \circ \Psi$ and $\Psi \circ \Gamma$ are naturally equivalent to the identity functors of $\mathcal{PMV}$ and $\mathcal{UG}$, respectively. Therefore, the categories $\mathcal{PMV}$ and $\mathcal{UG}$ are categorically equivalent.*

A pseudo MV-algebra M is said to be $symmetric$ if $x^\sim = x^-$ for each $x \in M$. We underline that if M is symmetric, then $\oplus$ is not necessarily commutative. In every pseudo MV-algebra M, we can define a partial operation $+$ in the following form: $x + y$ is defined in M if $x \leq y^-$, equivalently $y \leq x^\sim$, equivalently $y \odot x = 0$, and in either case, we set $x + y := x \oplus y$, see [Dvurečenskij, 2002]. We note that if $M = \Gamma(G, u)$, then $+$ is the restriction of the group addition of the group G to the interval $[0, u]$.

For any integer $n \geq 0$ and any $x \in M$, we can define

$$0.x = 0,\ and\ n.x = (n-1).x \oplus x,\ n \neq 1,$$

$$0x = 0,\ and\ nx = (n-1)x + x,\ n \neq 1,$$

assuming $(n-1)x$ and $(n-1)x+x$ are defined in M. An element $x \in M$ is called a Boolean element if $x \oplus x = x$. The set $B(M)$ denotes the set of all Boolean elements of M, which is a Boolean algebra and a subalgebra of M.

In what follows, we will assume usually that M is such that $0 \neq 1$.

If that $(M; \oplus, -, \sim, 0, 1)$ is pseudo MV-algebra, for each $a \in B(M)$, the interval $([0, a]; \oplus, ^{-a}, ^{\sim a}, 0, a)$ is a pseudo MV-algebra, where $x^{-a} = x^- \wedge a$ and $x^{\sim a} = x^\sim \wedge a$, and the mapping $\varphi_a : M \to M_a := [0, a]$ defined by $\phi_a(x) = a \wedge x,\ x \in M$, is a surjective homomorphism of pseudo MV-algebras.

In each pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$, we define two additional binary operations $\to$ and $\rightsquigarrow$ by

$$x \to y = x^- \oplus y,\ x \rightsquigarrow y := y \oplus x^\sim.$$

Proposition 2.1. *[Dvurečenskij, 2001a, Georgescu and Iorgulescu, 2001] In any pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$, the following properties hold for all $x, y, z \in M$:*
(1) $x \odot (x \to y) = x \wedge y \leq y,\ and\ (x \rightsquigarrow y) \odot x = x \wedge y \leq y.$
(2) $x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y.$

(3) $(x \odot y) \to z = y \to (x \to z)$ *and* $(x \odot y) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z)$.
(4) $x \to (x \odot y) = (x \to 0) \vee y$ *and* $x \rightsquigarrow (y \odot x) = (x \rightsquigarrow 0) \vee y$.
(5) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$.
(6)$x \odot y \leq z \Longleftrightarrow y \leq x \to z \Longleftrightarrow x \leq y \rightsquigarrow z$.
(7) $x \wedge y = x \odot (x \to y) = y \odot (y \to x) = (y \rightsquigarrow x) \odot y = (x \rightsquigarrow y) \odot x$.
(8) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$ *and* $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \wedge (x \rightsquigarrow z)$.
(9) $x \to (y \wedge z) = (x \to y) \odot ((x \wedge y) \to z)$ *and* $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \vee (x \rightsquigarrow z)$.
(10) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$ *and* $(x \wedge y) \rightsquigarrow z = ((x \wedge y) \rightsquigarrow z) \odot (x \rightsquigarrow y)$.
(11) $If\ x \leq y,\ then\ y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$.

A non-empty subset I of a pseudo MV-algebra M is called an ideal if (1) for each $y \in M,\ y \leq x \in I$ implies that $y \in I$; (2) I is closed under $\oplus$. An ideal I of M is said to be (i) prime if $x \wedge y \in I$ implies $x \in I$ or $y \in I$; (ii) normal if $x \oplus I = I \oplus x$ for any $x \in M$, where $x \oplus I := \{x \oplus i | i \in I\}$ and $I \oplus x = \{i \oplus x | i \in I\}$. We recall that an ideal I is normal if and only if given $x, y \in M$, $x \odot y^- \in I$ if and only if $y^\sim \odot x \in I$. We note that the kernel of any homomorphism of pseudo MV-algebras is a normal ideal.

A one-to-one relationship exists between congruences and normal ideals of a pseudo MV-algbera, [Georgescu and Iorgulescu, 2001]. If I is a normal of pseudo MV-algebra, then the relation $\sim_I$, defined by $x \sim_I y$ if and only if $x \odot y^-, y \odot x^- \in I$, is a congruence, and $M/I = \{x/I : x \in M\}$ and x/I is the equivalence class containing x.

$$x/I \oplus y/I = (x \oplus y)/I,\ (x/I)^- = x^-/I,\ (x/I)^\sim = x^\sim/I,\ (0/I)^\sim = 1/I.$$

Conversely, if θ is a congruence on M, then $I_\theta = \{x \in M | x \sim 0\}$ is a normal ideal such that $\sim_{I_\theta} = \sim$.

A pseudo MV-algebra M is said to be representable if M is a subdirect product of a system of linearly ordered pseudo MV-algebras. By ([Dvurečenskij, 2001b], Prop, 6.9), M is representable if and only if $a^\perp = \{x \in M | x \wedge a = 0\}$ is a normal ideal of M for each $a \in M$. Moreover, $M = \Gamma(G, u)$ is representable iff G is a representable ℓ-group, see ([Dvurečenskij, 2001b], Prop, 6.9).

Proposition 2.2. *[Cignoli R L, 2013] For each element x, y of representable pseudo MV-algebra* $(M; \oplus, -, \sim, 0, 1)$ *and each* $i, j \in \mathbb{N}$ *with the property* $i + j = n$*, we have* $x^i \odot y^j \leq x^n \vee y^n$.

Proof. Since each representable pseudo MV-algebra is a subdirect product of a family of linearly ordered MV-algebra, it suffices to prove it only for the linearly ordered case. Assume that M is a chain and $x, y \in M$. If $x \leq y$, then $x^i \odot y^j \leq y^{i+j} = y^n \leq x^n \vee y^n$ and similarly, if $y \leq x$, then $x^i \odot y^j \leq x^{i+j} = x^n \leq x^n \vee y^n$. $\square$

3 n-roots on pseudo MV-algebras

Definition 3.1. *[Höhle, 1995] Let M be an MV-algebra and $n \geq 1$ be an integer. A unary operation $r : M \to M$ is called an n^{th} root or simply an n-root, if it fulfills (R1) and (R2) for all $z \in M$:*
(R1) $r(z)^n = \underbrace{r(z) \odot \ldots \odot r(z)}_{n} = z$,
(R2) $\omega^n \leq z$ *implies that* $\omega \leq r(z)$ *for each* $\omega \in M$.
If $n = 2$, a 2-root is simply a square root.

Definition 3.2. *Let $(M; \oplus, ^{-}, ^{\sim}, 0, 1)$ be a pseudo MV-algebra. A mapping $r : M \to M$ is said to be an n-root if it satisfies the following conditions:*
(Sq1) for all $x \in M$, $\underbrace{r(x) \odot \ldots \odot r(x)}_{n} = x$;
(Sq2) for each $x, y \in M$, $y^n \leq x$ implies $y \leq r(x)$;
(Sq3) for each $x \in M$, $r(x^{-}) = r(x) \to r(0)$ and $r(x\ ) = r(x) \rightsquigarrow r(0)$;
and a weak n-root if it satisfies only (Sq1) and (Sq2). If $n = 2$, a 2-root is simply a square root.

A pseudo MV-algebra $(M; \oplus, ^{-}, ^{\sim}, 0, 1)$ has n-roots (weak n-roots) if there exists an n-root (weak n-root) r on M. Sometimes, one writes $x^{1/n}$ for n-roots $r(x)$.

A standard n-root is an n-root (weak n-root) r satisfying the following additional condition:
(Sq4) for all $x \in M$, $r(x) \odot r(0) = r(0) \odot r(x)$.

For each $m \in N$ and each $x \in M$, we define $r^0(x) = x$, $r^1(x) = r(x)$ and $r^{m+1}(x) = r \circ r^m(x) = r(r^m(x))$, $m \geq 1$.

Remark 3.1. *(i) Any weak n-root $r : M \to M$ is a one-to-one map. If $r(x) = r(y)$, then $x = r(x)^n = r(y)^n = y$.*
(ii) If r_1 and r_2 are two weak n-roots on a pseudo MV - algebra M, then $r_1 = r_2$. By (Sq2), for each $x \in M$, $r_1(x)^n \leq x$ implies $r_1(x) \leq r_2(x)$. In a similar way, $r_2(x) \leq r_1(x)$. That is, $r_1 = r_2$.
(iii) If r is a weak n-root on a pseudo MV-algebra M, then $r(x) \odot r(y) = r(y) \odot$

$r(x)$ implies that $x\odot y = y\odot x$. Indeed, $x\odot y = (r(x))^n \odot (r(y))^n = (r(x))^{n-1}\odot (r(x)\odot r(y))\odot(r(y))^{n-1} = (r(x))^{n-1}\odot(r(y)\odot r(x))\odot(r(y))^{n-1} = (r(x))^{n-2}\odot r(x)\odot(r(y)\odot r(x))\odot r(y)\odot(r(y))^{n-2} = (r(x))^{n-2}\odot(r(y)\odot r(x))\odot(r(y)\odot r(x))\odot(r(y))^{n-2} = \ldots = (r(y))^n\odot(r(x))^n = y\odot x$.

Proposition 3.1. *Let r be an n-root on a pseudo MV-algebra $(M;\oplus,-,\ ,0,1)$. For each $x,y\in M$, we have:*
(1) $x\leq x\vee r(0)\leq r(x)$ and $r(x)\odot x = x\odot r(x)$.
(2) $x\leq y$ implies that $r(x)\leq r(y)$.
(3) $x\wedge y\leq r(x)\odot r(y)$, $r(y)\odot r(x)$. Furthermore ,if b is a Boolean element of M such that $r(0)\geq b$, then $b=0$.
(4) $x\leq r(x^n)\leq r(x)$ and $r(x\odot x)^n = r(x)^n\odot r(x)^n = x\odot x$.
(5) $(x\wedge x^-)^n\vee(x\wedge x^\sim)^n\leq r(0)$.
(6) $r(x)\wedge r(y) = r(x\wedge y)$.
(7) $r(x)\rightarrow r(y)\leq r(x\rightarrow y)$ and $r(x)\rightsquigarrow r(y)\leq r(x\rightsquigarrow y)$. Moreover, $r(x)\odot r(y)\leq r(x\odot y)$ for all $x,y\in M$ if and only if $r(x)\rightarrow r(y) = r(x\rightarrow y)$ and $r(x)\rightsquigarrow r(y) = r(x\rightsquigarrow y)$.
(8) $r(x\vee y) = r(x)\vee r(y)$.
(9) If M is representable, $r(x)^i\odot r(y)^j\leq x\vee y$ for all $i,j\in\mathbb{N}$ such that $i+j=n$ and $y'\wedge y\leq r(0)$.
(10) $r(x\odot y)\leq(r(x)\odot r(y))\vee r(0)$, and $r(x^m) = r(x)^m\vee r(0)$ for all $m\in\mathbb{N}$, and $r(y^n) = y\vee r(0)$. Consequently, $y = r(y^n)$ for all $r(0)\leq y\in M$.
(11) $r(x\oplus y)\geq(r(x)\odot r(0)^-)\oplus r(y)$.
(12) For each $m,n\in\mathbb{N}$ with $n\leq m$, $r^m(x)^{n^k} = r^{m-k}(x)$.
(13) $r(y) = \max\{x\wedge(x^{n-1}\rightarrow y)\}$.
(14) $y = r(y)$ if and only if $r(y)\in B(M)$.
(15) For each idempotent element $a\in B(M)$, the map $r_a : [0,a]\rightarrow[0,a]$ sending x to $r(x)\odot a$ is an n-root on the pseudo MV-algebra $[0,a]$.

Proof. (1) $x\leq r(x)$ follows from (Sq1). On the other hand, $r(0)^n = 0\leq x,$ by (Sq2), $r(0)\leq r(x)$. Hence,$x = r(x)^n\leq r(x)$. For each $x\in M$, $r(x)\odot x = r(x)\odot r(x)^n = r(x)^n\odot r(x) = x\odot r(x)$.
(2) From (Sq2) and $r(x)^n = x\leq y$, we have $r(x)\leq r(y)$.
(3) When $n=2$, it holds.(see [Dvurecenskij and Zahiri, 2022]) By (2), $x\wedge y = r(x\wedge y)^n\leq r(x\wedge y)\odot r(x\wedge y)\leq r(x)\odot r(y)$. Choose $b\in B(M)$ such that $r(0)\geq b$. Then, by (Sq1), $b = b^n\leq r(0)^n = 0$.
(4) $x^n\leq x^n$ and (Sq2) imply that $x\leq r(x^n)$. The second part follows from (Sq1): Indeed, $r(x\odot x)^n = x\odot x = r(x)^n\odot r(x)^n$.
(5) We have $(x\wedge x^-)^n\leq(x\wedge x^-)^2 = (x\wedge x^-)\odot(x\wedge x^-)\leq x\odot x^- = 0$. Similarly, $(x\wedge x^\sim)^n\leq(x\wedge x^\sim)^2\leq x\odot x^\sim = 0,\ n\geq 2$. Thus by (Sq2), $x\wedge x^-,\ x\wedge x^\sim\leq r(0)$.
(6) From (2) and $x\wedge y\leq x,y$, we have that $r(x\wedge y)\leq r(x)\wedge r(y)$. If $n=2$,

$r(x \wedge y)^2 = (r(x) \odot r(x)) \wedge (r(x) \odot r(y)) \wedge (r(y) \odot r(x)) \wedge (r(y) \odot r(y)) = r(x)^2 \wedge (r(x) \odot r(y)) \wedge (r(y) \odot r(x)) \wedge r(y)^2 \leq r(x)^2 \wedge r(y)^2 = x \wedge y$, assume the formula is true for some positive integer $k-1$: $(r(x) \wedge r(y))^{k-1} \leq r(x)^{k-1} \wedge r(y)^{k-1} = x \wedge y$. Show the formula is true for $n = k$: $(r(x) \wedge r(y))^n = (r(x) \wedge r(y))^{k-1} \odot (r(x) \wedge r(y)) \leq (r(x)^{k-1} \wedge r(y)^{k-1}) \odot (r(x) \wedge r(y)) = r(y)^k \wedge (r(x)^{k-1} \odot r(y)) \wedge (r(y)^{k-1} \odot r(x)) \wedge r(x)^k \leq x \wedge y$. By mathematical induction, the formula $(r(x) \wedge r(y))^n \leq x \wedge y$ is true for all positive integers n. So by (Sq2), $r(x) \wedge r(y) \leq r(x \wedge y)$. Thus, $r(x) \wedge r(y) = r(x \wedge y)$.
(7) Let $x, y \in M$. Then

$$\begin{aligned}
(x \wedge y) \odot (r(x) \to r(y))^n &= r(x \wedge y)^n \odot (r(x) \to r(y))^n \\
&= r(x \wedge y)^{n-1} \odot r(x \wedge y) \odot (r(x) \to r(y)) \\
&\quad \odot (r(x) \to r(y))^{n-1} \\
&\leq r(x \wedge y)^{n-1} \odot r(x) \odot (r(x) \to r(y)) \\
&\quad \odot (r(x) \to r(y))^{n-1} \\
&= r(x \wedge y)^{n-1} \odot (r(x) \wedge r(y)) \\
&\quad \odot (r(x) \to r(y))^{n-1} \quad \text{(by Prop 2.1(1))} \\
&\leq r(x \wedge y)^{n-1} \odot r(x) \odot (r(x) \to r(y))^{n-1} \\
&= r(x \wedge y)^{n-1} \odot r(x) \odot (r(x) \to r(y)) \\
&\quad \odot (r(x) \to r(y))^{n-2} \\
&\leq \cdots \\
&\leq r(x \wedge y)^{n-1} \odot r(x) \odot (r(x) \to r(y)) \\
&\leq r(x \wedge y)^{n-1} \odot r(y) \\
&\leq r(y)^{n-1} \odot r(y) \\
&= y.
\end{aligned}$$

From Proposition 2.1 (10), it follows that $(r(x) \to r(y))^n \leq (x \wedge y) \to y \leq x \to y$ and so by (Sq1), $r(x) \to r(y) \leq r(x \to y)$.
Now, let $r(x) \odot r(y) \leq r(x \odot y)$ for all $x, y \in M$. Then $r(x) \odot r(x \to y) \leq r(x \odot (x \to y)) \leq r(y)$ which entails $r(x \to y) \leq r(x) \to r(y)$. Therefore, $r(x \to y) = r(x) \to r(y)$.
Conversely, let $r(x \to y) = r(x) \to r(y)$ for all $x, y \in M$. Then from ([Georgescu and Iorgulescu, 2001], Prop 1.13), it follows that $r(x) \to r(x \odot y) = r(x \to (x \odot y)) = r(x^- \vee y) \geq r(y)$ and so $r(x) \odot r(y) \leq r(x \odot y)$.
Similarly, we can prove the relations with $\rightsquigarrow$.

(8) By part (6) and (Sq3), we have

$$\begin{aligned} r(x \vee y) &= r(((x \to 0) \wedge (y \to 0)) \rightsquigarrow 0) = r(x \to 0) \wedge (y \to 0)) \rightsquigarrow r(0) \\ &= (r(x \to 0) \rightsquigarrow r(0)) \vee (r(y \to 0) \rightsquigarrow r(0)) \qquad \text{(by Prop 2.1(10))} \\ &= r((x \to 0) \rightsquigarrow 0) \vee r((y \to 0) \rightsquigarrow 0) \qquad \text{(by Prop 2.1(2))} \\ &= r(x) \vee r(y). \end{aligned}$$

(9) We know that $r(x)^i \odot r(y)^j \leq r(x)^n \vee r(y)^n = x \vee y$. In addition, $(y \wedge y')^n \leq (y \wedge y')^2 \leq y \wedge y' = 0$ and so by (Sq2), $y \wedge y' \leq r(0)$.
(10) By (6), (7) and (Sq3), we have

$$\begin{aligned} r(x \odot y) &= r(((x \odot y)^-)^\sim) = r(((x \odot y) \to 0) \rightsquigarrow 0) \\ &= r((y \to (x \to 0)) \rightsquigarrow 0) = r(y \to (x \to 0)) \rightsquigarrow r(0) \\ &\leq (r(y) \to r(x \to 0)) \rightsquigarrow r(0) \qquad \text{(by (7) and Prop 2.1(11))} \\ &= (r(y) \to (r(x) \to r(0))) \rightsquigarrow r(0) \\ &= ((r(x) \odot r(y)) \to r(0)) \rightsquigarrow r(0) \\ &= (r(x) \odot r(y)) \vee r(0). \qquad \text{(by Prop 2.1(2))} \end{aligned}$$

Also,

$$\begin{aligned} r(x^m) &\leq (r(x) \odot r(x^{m-1})) \vee r(0) \\ &\leq (r(x) \odot ((r(x) \odot r(x^{m-2})) \vee r(0))) \vee r(0) \\ &= (r(x) \odot (r(x) \odot r(x^{m-2})) \vee (r(x) \odot r(0))) \vee r(0) \\ &= (r(x) \odot (r(x) \odot r(x^{m-2})) \vee r(0) \leq \ldots = r(x)^m \vee r(0). \end{aligned}$$

For $m = n$, $r(x^n) \leq r(x)^n \vee r(0) = x \vee r(0)$. By part (4), $x,\ r(0) \leq r(x^n)$, so $x \vee r(0) \leq r(x^n)$. Therefore, $r(x^n) = x \vee r(0)$.
(11) By (7), $r(x \oplus y) = r((x^\sim)^- \oplus y) = r(x^\sim \to y) \geq r(x^\sim) \to r(y) = (r(x) \rightsquigarrow r(0)) \to r(y) = (r(0) \oplus r(x)^\sim) \oplus r(y) = (r(x) \odot r(0)^-) \oplus r(y)$.
(12) Choose $m \in \mathbb{N}$. We have $r^m(x)^n = r(r^{m-1}(x))^n = r^{m-1}(x)$. If $n = 2$, then $r^m(x)^{n^2} = (r(r^{m-1}(x))^n = (r^{m-1}(x))^n = r^{m-2}(x)$,
$r^m(x)^{n^k} = (r^{m-1}(x))^{n^{k-1}} = \ldots = r^{m-k}(x)$, for all $1 \leq k \leq m$.
(13) Since $r(y)^n = y$, we get $r(y) \leq r(y)^{n-1} \to y$. Hence

$$r(y) = r(y) \wedge (r(y)^{n-1} \to y) \in \{x \wedge (x^{n-1} \to y) : x \in M\}.$$

For each $x \in M$,

$$(x \wedge (x^{n-1} \to y))^n = (x \wedge (x^{n-1} \to y))^{n-1} \odot x \wedge (x^{n-1} \to y) \leq x^{n-1} \odot (x^{n-1} \to y) \leq y$$

so by (Sq2), $(x^{n-1} \to y) \leq r(y)$. Therefore, $r(y) = \max\{x \wedge (x^{n-1} \to y) : x \in M\}$.

(14) If $r(y) \in B(M)$, then by (Sq1), $y = r(y)^n = r(y) \in B(M)$. Conversely, suppose that $r(y) = y$. Then $y = r(y)^n \leq r(y) = y$. $r(y) = r(y)^n$, and so $r(y) \in B(M)$.
(15) Due to ([Georgescu and Iorgulescu, 2001], Pro 4.3), we have $r_a(x) = r(x) \wedge a = a \odot r(x) = r(x) \odot a$, $x \in [0, a]$. Since $r_a(x)^n = (a \odot r(x))^n = a^n \odot r(x)^n = a \odot r(x)$, conditions (Sq1) and (Sq2) are clear. Since the mapping $\psi : M \to [0, a]$ defined by $\psi_a(x) = x \wedge a$, $x \in M$, is a homomorphism of pseudo MV-algebras, (Sq3) holds. $\square$

Proposition 3.2. *Let r be a weak n-root on a pseudo MV-algebra. The following statements hold:*
(1) x is a Boolean element if and only if $r(x) = x \oplus r(0)$ if and only if $r(x) = r(0) \oplus x$.
(2) If $a, b \in M$, $a \leq b$, then $r([a, b]) = [r(a), r(b)]$. In particular $r(M) = [r(0), 1]$.
(3) $(r(0) \to 0)^n = (r(0) \rightsquigarrow 0)^n \in B(M)$.

Proof. First, we claim that

$$r(n.y) = y \oplus r(0) = r(0) \oplus y, \ \forall y \in M.$$

Check

$$\begin{aligned}
r(n.y) &= r(((y^{\sim})^n)^{-}) = r((y^{\sim})^n) \to r(0) \\
&= ((r(y^{\sim})^n) \vee r(0)) \to r(0) = (y^{\sim} \vee r(0)) \to r(0) \qquad \text{(by (10))} \\
&= (y^{\sim} \to r(0)) \wedge (r(0) \to r(0)) = y^{\sim} \to r(0) = y \oplus r(0), \\
r(n.y) &= r(((y^{-})^n)^{\sim}) = r((y^{-})^n) \rightsquigarrow r(0) \\
&= ((r(y^{-})^n) \vee r(0)) \rightsquigarrow r(0) = (y^{-} \vee r(0)) \rightsquigarrow r(0) \qquad \text{(by (10))} \\
&= (y^{-} \rightsquigarrow r(0)) \wedge (r(0) \rightsquigarrow r(0)) = y^{-} \rightsquigarrow r(0) = r(0) \oplus y.
\end{aligned}$$

(1) If $x \in B(M)$, we can yield $r(x) = x \oplus r(0) = r(0) \oplus x$. Conversely, let $r(x) = x \oplus r(0)$, we can $r(n.x) = x \oplus r(0) = r(x)$, since r is one-to-one, $n.x = x$ and $x \in B(M)$.
(2) Let $a, b \in M$ with $a \leq b$ be given. By Proposition 3.1 (2), $r([a, b]) \subset [r(a), r(b)]$. Let $r(a) \leq y \leq r(b)$. We have $y \geq r(a) \geq r(0)$, $y = y \vee r(0)$ and by (10), $y = r(y^n)$. From $a = r(a)^n \leq y^n \leq r(b)^n = b$ it follows that $y = r(y^n) \in r([a, b])$. Consequently, $r([a, b]) = [r(a), r(b)]$. As a special case, we have $r(M) = r([0, 1]) = [r(0), r(1)] = [r(0), 1]$.
(3) We show that $(r(0) \to 0)^n \in B(M)$. By (1), it suffices to prove that $r((r(0) \to 0)^n) = (r(0) \to 0)^n \oplus r(0)$. By (10), $r((r(0) \to 0)^n) = (r(0) \to 0) \vee r(0)$. On the other hand, $(r(0) \to 0)^n \oplus r(0) = (r(0)^{-})^n \oplus r(0) = r(0)^{-} \vee r(0)$ (see [Georgescu and Iorgulescu, 2001]). It follows that $(r(0) \to 0)^n \in B(M)$. In a similar way, we can show that $(r(0) \rightsquigarrow 0)^n \in B(M)$. $\square$

Proposition 3.3. *Let r be a weak n-root on a pseudo MV-algebra. The following statements are equivalent:*
(1) r is an n-root on M.
(2) $r(x) \odot r(x^-) \leq r(0)$.
(3) $r(x^\sim) \odot r(x) \leq r(0)$.
(4) $r(x^-) = r(x) \to r(0)$.
(5) $r(x^\sim) = r(x) \rightsquigarrow r(0)$.

Proof. Simlarly to Proposition 3.1 (7), for a weak n-root r, we always have $r(x) \to r(0) \leq r(x \to 0) = r(x^-)$ and $r(x) \rightsquigarrow r(0) \leq r(x \rightsquigarrow 0) = r(x^\sim)$.
Let $r(x) \odot r(x^-) \leq r(0)$. It implies $r(x^-) \leq r(x) \to r(0)$. Hence $r(x) \to r(0) = r(x^-)$. Conversely $r(x^-) \leq r(x) \to r(0)$ implies that $r(x) \odot r(x^-) \leq r(0)$.
This proves that (1)-(5) are equivalent. □

Proposition 3.4. *Let r be an n-root on a pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$. The following properties hold for $x \in M$:*
(1) $x \leq x \vee r(0) = r(x^n) \leq r(x)$,
$r(x^-)^\sim \vee r(x^\sim)^- \leq n.(r(x^-)^\sim) = x \leq x \vee r(0) = r(x^n) \leq r(x)$. Moreover, $x \in B(M)$ iff $r(x) = x \vee r(0)$.
(2) $r(x) \leq (x \oplus r(0)) \wedge (r(0) \oplus x)$.
(3) $r(x^\sim)^- \oplus r(0) = r(x) = r(0) \oplus r(x^-)^\sim$.
(4) $x \oplus r(x) = r(x) \oplus x$, $x \oplus r(0) = r(0) \oplus x$.

Proof. (1) By Proposition 3.1 (1) and (4), we have $x \leq x \vee r(0) \leq r(x)$ and $x \leq r(x^n)$. Also,by part (10) from this proposition, $r(x^n) = r(x)^n \vee r(0) = x \vee r(0)$, so we have equation (1). Hence, if $x \in M \setminus B(M)$, then $x^n \neq x$, thus $r(0) \vee x < r(x)$. Otherwise, if $r(0) \vee x = r(x)$, then $r(x) = r(x^n) = x \vee r(0)$, that is $x^n = x$. Moreover, $x \in B(M)$ if and only if $r(x) = r(0) \vee x$ (since r is a one-to-one map).
By Proposition 3.1 (1), replacing x with x^- and $x^\sim$, we have $x^- \leq r(x^-)$ and $x^\sim \leq r(x^\sim)$ which imply that $r(x^-)^\sim \vee r(x^\sim)^- \leq x$. Also, $n.(r(x^-)^\sim) = (r(x^-)^n)^\sim = (x^-)^\sim = x$. In a similar way, $x = n.(r(x^\sim)^-)$. Therefore, we have (2).
(2) According to the definition of an n-root, $r(x) = max\{z \in M | z^n \leq x\}$. We claim that $x \oplus r(0)$ is an upper bound for $r(x)$. Indeed, set $S := \{z \in M | z^n \leq 0\}$. If $z \in M$ such that $z^n \leq x$, then $z^n \odot x^- = 0$ and $x^\sim \odot z^n = 0$ which means $(z \odot x^-)^n \leq z^{n-1} \odot (z \odot x^-) = z^n \odot x^- = 0$ and $(x^\sim \odot z)^n \leq (x^\sim \odot z) \odot z^{n-1} = x^\sim \odot z^n = 0$, thus $x^\sim \odot z$, $z \odot x^- \in S$. It follows that $x^\sim \odot z$, $z \odot x^- \leq r(0)$ and hence,

$$z \leq x \vee z = x \oplus (x^\sim \odot z) \leq x \oplus r(0),$$

$$z \leq x \vee z = (z \odot x^-) \oplus x \leq r(0) \oplus x.$$

Since $r(x)$ is the greatest element of S, we conclude that (2) holds.
(3) Due to (1) and (2), $r(x^-)^\sim \leq x$ and $r(x) \leq r(0) \oplus x$. Then

$$\begin{aligned} r(0) \oplus r(x^-)^\sim &= r(0) \oplus (r(x) \to r(0))^\sim \\ &= r(0) \oplus (r(x)^- \oplus r(0))^\sim \\ &= r(0) \oplus (r(0)^\sim \odot r(x)) \\ &= r(0) \vee r(x) = r(x). \qquad \text{(by Prop 3.1(2))} \end{aligned}$$

In a similar way, $r(x) = r(x^\sim)^- \oplus r(0)$. Hence, (3) is established.
(4) (i) By (1) and (3), $x = n.(r(x)^-)^\sim = n.(r(x^\sim))^-$, $r(x) = r(x^\sim)^- \oplus r(0) = r(0) \oplus r(x^-)^\sim$. It follows that

$$\begin{aligned} x \oplus r(x) &= n.r(x^\sim)^- \oplus r(0) \oplus r(x^-)^\sim \\ &= (n-1).r(x^\sim)^- \oplus (r(x^\sim)^- \oplus r(0)) \oplus r(x^-)^\sim \\ &= (n-1).r(x^\sim)^- \oplus (r(0) \oplus r(x^-)^\sim) \oplus r(x^-)^\sim \\ &= (n-1).r(x^\sim)^- \oplus r(0) \oplus 2.r(x^-)^\sim \\ &= \dots \\ &= r(x^\sim)^- \oplus r(0) \oplus n.r(x^-)^\sim \\ &= r(x) \oplus x \end{aligned}$$

(ii) For each $x \in M$, $x \oplus r(0) = r(0) \oplus x$. □

Similarly to the definition of the n-root of a pseudo MV-algebra, we can define an n-root of an individual element of a pseudo MV-algebra. Let $m \in \mathbb{N}$. We called an element $x \in M$ has an m-root if there exists $a \in M$ satisfies (Sq1) and (Sq2), that is $a^m = x$ and $y^m \leq x$ implies that $y \leq a$. The condition (Sq2) implies that if x has an m-root, then it is unique, so that, we can denote it by $\sqrt[m]{x}$ for simplicity. Clearly, if r is an m-root on M, then each element of M has an m-root, indeed, $\sqrt[m]{x} = r(x)$.

Theorem 3.1. *Suppose that M is a pseudo MV-algebra with an n-root r, and m and p are natural numbers such that $n = mp$. If $x \in M$ satisfies the condition $r(0) \leq r(x)^p$, then x has an m-root, $\sqrt[m]{x} = r(x)^p$.*

Proof. Let $mp = n$. Consider the map $v : M \to M$ defined by $v(x) = r(x)^p$ for all $x \in M$. So, $v(x)^m = r(x)^{mp} = r(x)^n = x$. In addition, if $y \in M$ such that $y^m \leq x$, then $y^n \leq x^p$, whence $y \leq r(x^p)$. By Proposition 3.1(10),

$$r(x^p) = r(x)^p \vee r(0) = v(x) \vee r(0).$$

Suppose that $r(0) \leq r(x)^p$ for some $x \in M$. Then $y \leq v(x) \vee r(0) = v(x)$, that is, v satisfies (Sq2), so $\sqrt[m]{x} = v(x)$. □

Proposition 3.5. *Let $f : M_1 \to M_2$ be a homomorphism of pseudo MV-algebras and r be an n-root on M_1. Then $\tau : Im(f) \to Im(f)$, defined by $\tau(f(x)) = f(\tau(x))$ for all $x \in M_1$, is an n-root on $Im(f)$.*

Proof. (i) For each $r \in M_1$, we have $f(r(x))^n = f(r(x)^n) = f(x)$.
(ii) Let $x, y \in M_1$, be such that $f(y)^n \leq f(x)$. By Proposition 3.1(8) and (10), $r(y^n \vee x) = r(y^n) \vee r(x) = (y \vee r(0)) \vee r(x)$, whence

$$\begin{aligned}
\tau(f(x)) &= \tau(f(y)^n \vee f(x)) = \tau(f(y^n \vee x)) \\
&= f(r(y^n \vee x)) = f(r(y^n) \vee r(x)) \qquad \text{by Proposition 3.1(8)} \\
&= f((y \vee r(0)) \vee r(x)) \qquad \text{by Proposition 3.1(10)} \\
&= (f(y) \vee f(r(0))) \vee f(r(x)) \\
&= (f(y) \vee f(r(0))) \vee \tau(f(x)).
\end{aligned}$$

That is, $f(y) \leq \tau(f(x))$. Hence (Sq2) holds.
(iii) We have $\tau(f(x)) = f(r(x))^- = f(r(x)^-) = f(r(x) \to r(0)) = f(r(x)) \to f(r(0)) = \tau(f(x)) \to \tau(f(0))$. Similarly for the second equality, that is,(Sq3) holds.
(iv) We show $\tau : Im(f) \to Im(f)$ is well-defined. Indeed, if $f(x) = f(y)$, then $f(r(x))$ and $f(r(y))$ are n-roots of $f(x) = f(y)$. Parts (i) and (ii) imply that $f(r(x))^n = f(x) = f(y)$ and so $f(r(x)) \leq f(r(y))$. In similar way, $f(r(y)) \leq f(r(x))$. Thus, $\tau : Im(f) \to Im(f)$ sending $f(x)$ to $f(r(x))$ is a correctly defined n-root. ☐

Corolary 3.1. *Let I be a normal ideal of a pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$ with an n-root r. Then:*
(1) The mapping $r_I : M/I \to M/I$ defined by $r_I(x/I) = r(x)/I$ is an n-root on M/I.
(2) For each $x, y \in M$, if $x \odot y^-$, $y \odot x^- \in I$, then $r(x) \odot r(y)^-$, $r(y) \odot r(x)^- \in I$.

Proof. The proof of part (1) follows from Proposition 3.5.
(2) Let $x, y \in M$ such that $x \odot y^-, y \odot x^- \in I$. Then $x/I = y/I$ and so by (i), $r(x)/I = r(y)/I$ which means $r(x) \odot r(y)^-,\ r(y) \odot r(x)^- \in I$. ☐

Theorem 3.2. *Let r be an n-root on a pseudo MV-algebra $(M; \vee, \wedge, \oplus, 0)$. Then M is a Boolean algebra if and only if $r(0) = 0$.*

Proof. Let $r(0) = 0$ and $x \in M$. By Proposition 3.3(2), we have $x \leq r(x^n) = r(0) \vee x \leq r(x) \leq r(0) \oplus x = x$ which implies that $r(x) = r(x^n)$ and so $x = x^n$ (since r is one-to-one). That is $M = B(M)$ is a Boolean algebra.
Conversely, let M be a Boolean algebra. Then $r(0)^n = r(0)$ and $r(0)^n = 0$. Therefore, $r(0) = 0$. ☐

Let $(M;\oplus,-,\sim,0,1)$ be a pseudo MV-algebra with an n-root r and I be a normal ideal of M. An ideal I of M is called r-invariant if $r(I)\subseteq I$.

Proposition 3.6. *Let $(M;\oplus,-,\sim,0,1)$ be a pseudo MV-algebra with an n-root r and I be a normal ideal of M. Then I is r-invariant if and only if I is a Boolean ideal of M.*

Proof. Suppose that I is an r-invariant normal ideal of M. By Corollary 3.1, $r_I : M/I \to M/I$ is an n-root on M/I. Since $r_I(0/I) = r(0)/I = 0/I$, Theorem 3.2 implies that M/I is a Boolean algebra and so I is a Boolean ideal of M. Conversely, let I be a Boolean ideal of M. Then M/I is a Boolean algebra and $r_I : M/I \to M/I$ is the identity map on M/I. Hence for each $x \in I$, we have $r(x)/I = r_I(x/I) = x/I = 0/I$, that is $r(x)\in I$. Therefore, I is r-invariant. $\square$

Remark 3.2. *Let $(M;\oplus,-,\sim,0,1)$ be a pseudo MV-algebra and $a \in B(M)\setminus\{0,1\}$. Set $b := a^-$. Then $[0,a]$ and $[0,b]$ are proper subsets of M. Consider the pseudo MV-algebras $([0,a];\oplus,^{-a},^{\sim a},0,a)$ and $([0,b];\oplus,^{-b},^{\sim b},0,b)$. The map $\varphi : M \to [0,a]\times[0,b]$ sending $x \in M$ to $(x\wedge a, x\wedge b)$ is an isomorphism of pseudo MV-algebras [see [Georgescu and Iorgulescu, 2001], page 23]. Hence, each pseudo MV-algebra with the non-empty set $B(M)\setminus\{0,1\}$ is not directly indecomposable. Now, let r be an n-root on M. By Proposition 3.1(15), r_a and r_b are n-roots on the pseudo MV-algebras $[0,a]$ and $[0,b]$, respectively. Therefore, in this case, M has a decomposition of pseudo MV-algebras with n-roots.*

$B\check{e}lohl\acute{a}vek$ in [Bělohlávek, 2003] showed that the class of all commutative residuated lattices with square roots is a variety. Consequently, the class of MV-algebra with square roots is also a variety. The following theorem shows that the class of pseudo MV-algebra with n-roots is a variety, too.

Theorem 3.3. *The class of all pseudo MV-algebras with n-roots is a variety. The same is true for the class of pseudo MV-algebras with weak n-roots.*

Proof. Let V be the class of all pseudo MV-algebras $(M;\oplus,-,\sim,0,1)$ such that there is an n-root on M. Consider the class W of algebras of type $(M;\oplus,-,\sim,r,0,1)$ satisfying the identities (A1)-(A8) and the following additional three identities:
(1) $r(x)^n = x$;
(2) $r(y^n\vee x)\wedge y = y$;
(3) $r(x^-) = r(x)\to r(0)$ and $r(x^\sim) = r(x)\rightsquigarrow r(0)$.
First, suppose that $M \in V$. There exists an n-root r on M. We show that $(M;\oplus,-,\sim,r,0,1)$ belongs to W. We only need to show (2). For each $x,y\in M$ by Proposition 3.1(8) and 3.1(1), $r(y^n\vee x) = r(y^n)\vee r(x) \geq y\vee r(x)$ and so $r(y^n\vee x)\wedge y = y$. Hence $(M;\oplus,-,\sim,r,0,1)\in W$.

Conversely, let $(M; \oplus, -, \sim, r, 0, 1)$ belong to W. Clearly by (A1)-(A8), $(M; \oplus, -, \sim, 0, 1)$ is a pseudo MV-algebra. Let $y^n \leq x$. Then by (2), $y = r(y^n \vee x) \wedge y = r(x) \wedge y$ which means $y \leq r(x)$. From (3), we get (Sq3). Hence, $(M; \oplus, -, \sim, 0, 1)$ is a pseudo MV-algebra with the n-root r. That is, $(M; \oplus, -, \sim, 0, 1) \in V$. Therefore , V is a variety.
Using equations (1)-(2), we can prove that the class of pseudo MV-algebras with weak n-roots is a variety, too. □

Theorem 3.4. *(1) Let M_1 and M_2 be two pseudo MV-algebras and $r_1 : M_1 \to M_1$ and $r_2 : M_2 \to M_2$ be two n-roots. If $f : M_1 \to M_2$ is a homomorphism of pseudo MV-algebras, then f preserves n-roots if and only if $Im(f)$ is closed under r_2.*
(2) If $f : M_1 \to M_2$ is an isomorphism and r_1 is a (weak) n-root on M_1, then $r_2 = f \circ r_1 \circ f^{-1}$ is a (weak) n-root on M_2.

Proof. (1) ($\Leftarrow$) Suppose that $Im(f)$ is closed under r_2. Then $r_2|_{Im(f)} : Im(f) \to Im(f)$ is an n-root on the pseudo MV-algebra $Im(f)$. Since by Proposition 3.5, the map $\tau : Im(f) \to Im(f)$ defined by $\tau(f(x)) = f(r_1(x)), x \in M_1$, is an n-root, then $\tau = r_2$ which implies that $r_2(f(x)) = \tau(f(x)) = f(r_1(x))$ for all $x \in M$.
($\Rightarrow$) Let f preserve n-roots. For each $f(x) \in Im(f)$, we have $r_2(f(x)) = f(r_1(x)) \in Im(f)$. Therefore, $Im(f)$ is closed under r_2.
(2) (Sq1): Let $z \in M_2$. Then $r_2(z)^n = f(r_1(f^{-1}(z)))^n = f(r_1(f^{-1}(z)^n)) = f(f^{-1}(z)) = z$.
(Sq2): Let $x, y \in M_2$ and $y^n \leq x$. Then $f^{-1}(x)^n \leq f^{-1}(x)$, so that, $f^{-1}(x) \leq r_1(f^{-1}(x))$ and $y \leq f(r_1(f^{-1}(x))) = r_2(x)$.
(Sq3): Let $x \in M_2$, so $r_2(0) = f(r_1(0))$, and $r_2(x^-) = f(r_1(f^{-1}(x))) = f(r_1(f^{-1}(x)) \to f(r_1(0)) = r_2(x) \to r_2(0)$. □

Theorem 3.5. *Let $(M; \oplus, -, \sim, 0, 1)$ be a pseudo MV-algebra with an n-root r. Then the structure $([r(0), 1]; \uplus, ', *, r(0), 1)$ is a pseudo MV-algebra, where for given $r(x), r(y) \in r(M)$,*

$$r(x) \uplus r(y) = r(x \oplus y),\ r(x)' = r(x)^- \oplus r(0),\ r(x)^* = r(x)^\sim \oplus r(0).$$

Furthermore, the pseudo MV-algebras $(M; \oplus, -, \sim, 0, 1)$ and $([r(0), 1]; \uplus, ', *, r(0), 1)$ are isomorphic, and the restriction of r onto $r(M)$ is an n-root on $r(M)$.

Proof. (i) First, we note that r is one-to-one, so by Proposition 3.2(2), the operations $\uplus, '$ and $*$ are well-defined, and $[r(0), 1]$ is closed under these operations. We verify conditions (A1)-(A8). Let $x, y, z \in M$ be given.
(1)

$$\begin{aligned} r(x) \uplus (r(y) \uplus r(z)) &= r(x) \uplus r(y \oplus z) = r(x \oplus (y \oplus z)) \\ &= r((x \oplus y)) \uplus r(z) = (r(x) \uplus r(y)) \uplus r(z). \end{aligned}$$

(2) $r(x) \uplus r(0) = r(x \oplus 0) = r(x) = r(0 \oplus x) = r(0) \uplus r(x)$.
(3) $r(x) \uplus r(1) = r(x \oplus 1) = r(1) = r(1 \oplus x) = r(1) \uplus r(x)$.
(4) $1' = 1' \oplus r(0) = 0 \oplus r(0) = r(0)$. Similarly, $1^* = r(0)$.
(5) For each $x \in M$, by Proposition 3.4 (3) the following identities hold:

$$r(x)^* = r(x)^\sim \oplus r(0) = r(0) \oplus r(x)^\sim = r(x) \rightsquigarrow r(0) = r(x \rightsquigarrow 0) = r(x^\sim), \tag{a}$$

$$r(x)' = r(x)^- \oplus r(0) = r(x) \to r(0) = r(x \to 0) = r(x^-). \tag{b}$$

It follows that $(r(x)')^* = r(x^-)^* = r(x^{-\sim}) = r(x)$ and $(r(x)^*)' = r(x^\sim)' = r(x^{\sim -}) = r(x)$.
(6) From (a) and (b) deduce that

$$(r(x)' \uplus r(y)')^* = (r(x^-) \uplus r(y^-))^* = r((x^- \oplus y^-)^\sim) = r((x^\sim \oplus y^\sim)^-).$$

A similar calculus shows that $(r(x)^* \uplus r(y)^*)' = r((x^\sim \oplus y^\sim)^-)$. Hence $(r(x)^* \uplus r(y)^*)' = (r(x)' \uplus r(y)')^*$.
(7) Let $u \oslash v := (v' \uplus u')^*$ for all, $u, v \in [r(0), 1]$. Then from (a) and (b) it follows that

$$\begin{aligned} r(x) \oslash r(y) &= (r(y)' \uplus r(x)')^* = (r(y^-) \uplus r(x^-))^* \\ &= r(y^- \uplus x^-)^* = r((y^- \uplus x^-)^\sim) = r(x \odot y). \end{aligned}$$

Similarly to (6), we can prove (A6) and (A7) hold. Therefore, $([r(0), 1]; \uplus, ', *, r(0), 1)$ is a pseudo MV-algebra.
(ii) Consider the mapping $f : M \to [r(0), 1]$ defined by $f(x) = r(x)$. Clearly, $f(x) \in Im(f) = [r(0), 1]$. By (a) and (b), for each $x \in M$, $f(x)^* = r(x)^* = r(x^\sim) = f(x^\sim)$ and $f(x)' = r(x)' = r(x^-) = f(x^-)$. In addition, for each $x, y \in M$, $f(x) \uplus f(y) = r(x) \uplus r(y) = r(x \oplus y) = f(x \uplus y)$, and $f(0) = r(0)$, $f(1) = r(1) = 1$. Therefore, f is an isomorphism. Hence, M is isomorphic to $Im(r)$, and using Theorem 3.2 (2) for each $x \in M$, we have $f(r(f^{-1}(f(x)))) = f(r(x)) = r(r(x))$, is an n-root on $r(M) = [r(0), 1]$. $\square$

4 Strict n-roots and n-strict MV-algebras

In this section, we present an important class of pseudo MV-algebra with strict n-root.

Definition 4.1. *An n-root r: $M \to M$ on a pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$ is said to be strict if $r(0)^{n-1} = r(0)^- = r(0)^\sim$.*
A pseudo MV-algebra possessing a strict n-root is said to be an n-strict pseudo MV-algebra.

Theorem 4.1. *Let* $(M; \oplus, -, \sim, 0, 1)$ *be a totally ordered symmetric pseudo MV-algebra with an* n*-root* $r : M \to M$ *and let* (G, u) *be its related unital* ℓ*-group. If* $r(0) = 0$*, then* $r = Id_M$*. If* $r(0) \geq 0$*, for each element* $x \in M$*, the element* $(x + (n-1)u)/n$ *exists in* M*, and* $r(x) = (x + (n-1)u)/n$ *for* $x \in M$*, where* $+$ *denotes the group addition in* G*.*

Proof. Since $M = \Gamma(G, u)$, we have G is also totally ordered, so it enjoys unique extraction of roots.
(1) If $r(0) = 0$, by Theorem 3.2, M is the Boolean algebra $\{0, 1\}$, and $r = Id_M$.
(2) Assume that $r(0) \neq 0$. For each $x \in M$, we have two cases:
(i) If $x \neq 0$, then $x = r(x)^n = (nr(x) - (n-1)u) \vee 0 = (nr(x) - (n-1)u)$ because M is totally ordered. Then $x = nr(x) - (n-1)u$ and $x + (n-1)u = nr(x)$. The unique extraction of roots property implies that the element $(x + (n-1)u)/n$ exists in M. Hence, for each $x \in M \setminus \{0\}$, $r(x) = (x + (n-1)u)/n$.
(ii) If $x = 0$, then $0 = r(0)^n = (nr(0) - (n-1)u) \vee 0$ entails that $(nr(0) - (n-1)u) \leq 0$. That is $nr(0) \leq (n-1)u$, On the other hand, if $y \in M$, we have $y^n \leq 0$ if and only if $ny \leq (n-1)u$. By (Sq2), $ny \leq (n-1)u$ implies that $y \leq r(0)$. That is $r(0) = max\{z \in M : nz \leq (n-1)u\}$. Also, if $g \in G$ be such that $ng \leq (n-1)u$, then $g \in G^-$ implies that $g \leq r(0)$ and from $g \in G^+$ we get $g \leq ng \leq (n-1)u$, that is $g = [0, u] = M$ which means $g \leq r(0)$. Hence

$$r(0) = \max\{z \in G : nz \leq (n-1)u\}. \tag{c}$$

By Proposition 3.2, $(r(0)^-)^n \in B(M)$, so $(r(0)^-)^n = 0$ or $(r(0)^-)^n = 1$ (since M is totally ordered). If $(r(0)^-)^n = 1$, then $r(0)^- = 1$ entails that $r(0) = 0$, but this is excluded. If $(r(0)^-)^n = 0$,then $nr(x) = (n-1)u$ entails $nr(0) \geq (n-1)u$, so by (c), $nr(0) = u$. In this case, $r(0) = (0 + (n-1)u)/n$. □

Theorem 4.2. *(Representation of* n*-strict MV-algebras) Let* $M = \Gamma(G, u)$ *be a representation symmetric pseudo MV-algebra with a strict* n*-root* r*, where* (G, u) *is a unital* ℓ*-group. Then* $(x + (n-1)u)/n$ *exists for each* $x \in M$*, and*

$$r(x) = \frac{x + (n-1)u}{n}, \; x \in M,$$

where $+$ *denotes the group addition in* G*.*

Proof. Assume that X is the set of all normal prime ideals $P \neq M$ of M. The set X is non-empty because M is representable. We note that also G is representable. Consider the embedding $\varphi : M \to M_0 := \Pi_{P \in X} M/P$ defined by $\varphi(x) = (x/P)_{P \in X}$. We have $r(0) > 0$.
(1) For each $P \in X$ we have $r(0) \notin P$(since r is strict, and $u = n.r(0)$), consequently, $r(0)/P \neq 0/P$, and $r_P(x/P) = r(x)/P$. Hence, we have

$$r_P(x/P)^{n-1} = r(x)^{n-1}/P = r(x)^-/P = (r(x)/P)^- = r_P(x/P)^-.$$

Hence, r_P is an n-root on the totally ordered pseudo MV-algebra M/P and r_P is strict.
(2) By Proposition 3.4, the homomorphism $\pi_P \circ \varphi : M \to M/P$ includes a strict n-root $t_P : M/P \to M/P$ defined by $t_P(x/P) = r(x)/P$. Let $\hat{P}$ be the ℓ-ideal (i.e. a normal convex ℓ-subgroup of G) of G generated by P. Then $\Gamma(G/\hat{P}, u/\hat{P})$ is totally ordered, symmetric pseudo MV-algebra such that $M/P \cong \Gamma(G/\hat{P}, u/\hat{P})$. Since M/P is totally ordered,(1) implies that M/P is not a Boolean algebra and so by Theorem 4.1,

$$r(x)/P = t_p(x/P) = (x/P +_P (n-1)u/P)/n,$$

where $+_P$ denotes the group addition in the unital ℓ-group $(G/\hat{P}, u/\hat{P}) = \Psi(M/P)$. If we set $t((x/P)_{P\in X}) = (t_P(x/P))_{P\in X}$, t is an n-root on M_0 and r is the restriction of t on M.
(3) Consider the unital ℓ-group $\Psi(\Pi_{P\in X} M/P) = \Pi_{P\in X}(\Psi(M/P))$. Due to Theorem 2.1, the functor $\Psi : \mathcal{PMV} \to \mathcal{UG}$ includes an ℓ-group homomorphism $\Psi(\varphi) : (G, u) \to \Psi(\Pi_{P\in X} M/P)$ preserving the strong unit that is injective. Recall that $\varphi(g) = \Psi(\varphi)(g)$ for all $g \in M$. We have $\varphi(r(x)) = (r(x)/P)_P \in X = (t_P(x/P))_{P\in X} = ((x/P +_P (n-1)u/P)/n)_{P\in X}$. Also,

$$(x/P +_P (n-1)u/P)_{P\in X} = ((x +_P (n-1)u)/P)_{P\in X}.$$

(4) By(3),

$$\varphi(nr(x)) = n\varphi(r(x)) = n\Psi\varphi(r(x)) = ((x+(n-1)u)/P)_{P\in X} = \varphi(x+(n-1)u).$$

entails that $nr(x) = (x+(n-1)u)$ and consequently, $r(x) = (x+(n-1)u)/n$. $\square$

We introduce the following notions. Let X be a non-empty subset of partially ordered set $(P, \leq)$. We defined $\uparrow X := \{a \in P | x \leq a \ for\ some\ x \in P\}$ and $\downarrow X := \{a \in P | a \leq x \ for\ some\ x \in P\}$.

Theorem 4.3. *Let $s : M \to M$ be a strict n-root on a pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$. The following statements hold:*
(1) For each $x \in M$, $s(x)^- \vee s(x)^\sim \leq s(0)$ and $s(0) \oplus s(x) = s(x) \oplus s(0) = 1$.
(2) For each $b \in B(M)$, $s(0) \leq b \leq 1$ implies that $b = 1$, consequently, $(\downarrow s(0) \cup \uparrow s(0)) \cap B(M) = \{0, 1\}$ and if M is not proper, then $B(M) \cap s(M) = \{1\}$.
(3) For all $x, y \in M$, $s(x) \to (s(x) \odot s(y)) = s(y) = s(x) \rightsquigarrow (s(y) \odot s(x))$.

Proof. (1) For each $x \in M$, we have $s(0) \leq s(x)$ and so $s(x)^- \leq s(0)^- = s(0)^{n-1}$ and $s(x)^\sim \leq s(0)^\sim = s(0)^{n-1}$. Due to [[Georgescu and Iorgulescu, 2001]. Prop.1.9], thus $s(x) \oplus s(0)^{n-1} = s(0)^{n-1} \oplus s(x) = 1$. Then $1 = s(x) \oplus s(0)^{n-1} \leq s(x) \oplus s(0)$, and so $s(x) \oplus s(0) = 1$.

(2) Let $b \in B(M)$ such that $s(0) \leq b \leq 1$. First we note that if $0 \neq 1$, then $s(0) < b$ (otherwise, by Proposition 3.1 (14), $0 = s(0)^n = b^n = 0$). Then $b^- < s(0)^- = s(0)^{n-1}$ and so $b^- = (b^-)^2 \leq s(0)^{n-1} \odot s(0)^{n-1} = s(0)^n \odot s(0)^{n-2} = 0$ which implies that $b = 1$. Therefore, by Proposition 3.1 (3), $(\downarrow s(0) \cup \uparrow s(0)) \cap B(M) = \{0, 1\}$. Clearly, $1 = s(1) \in B(M) \cap s(M)$. Now, let $b \in B(M) \cap s(M)$. Then by (1), $b^- \leq s(0)$, so $b = 1$.
(3) By Proposition 2.1 (4), we have

$$s(x) \to (s(x) \odot s(y)) = s(x)^- \vee s(y), s(x) \rightsquigarrow (s(y) \odot s(x)) = s(x)^\sim \vee s(y).$$

By (1), $s(x)^- \vee s(x)^\sim \leq s(0) \leq s(y)$. So we have $s(x) \to (s(x) \odot s(y)) = s(y) = s(x) \rightsquigarrow (s(y) \odot s(x))$ for all $x, y \in M$. □

Theorem 4.4. *For each n-root of a representable symmetric pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1) = \Gamma(G, u)$, the following statements hold:*
(1) $n.r(0) = r(0) \oplus r(0)$ and it is a Boolean element of M.
(2) $r(0) = 0$ or $r(0) \oplus r(0) = u$.
(3) $r(0) \geq 0$, then $r(0)^2 \neq 0$.
(4) $r(0) \geq 0$, then $r(0)^{n-1} \oplus r(0) = u$. In addition, $r(0)^{n-1} \geq 0$.
(5) $M = \{0, 1\}$ or M is n-strict.

Proof. (1) By Proposition 3.1 (2), we have

$$n.r(0) \leq r(n.r(0)) = r(0) \oplus r(0) \leq n.r(0) \; and \; r(0) \leq r(n.r(0)).$$

It follows that $r(n.r(0)) = n.r(0) = n.r(0) \vee r(0)$. By Proposition 3.4 (1), $n.r(0) \in B(M)$. Therefore, $r(0) \oplus r(0) = 0$ if and only if $r(0) = 0$.
(2) By (1), $r(0) \oplus r(0) = 0$ or $r(0) \oplus r(0) = u$. Moreover, $r(0) \oplus r(0) = 0$ if and only if $r(0) = 0$.
(3) For $n = 2$, the proof is straightforward (refer to [Dvurečenskij and Zahiri, 2023]). Let $n \geq 3$. Since $r(0) \neq 0$, by(2), $r(0) \oplus r(0) = 1$ and so $r(0)^- \leq r(0)$.
(i) For each $y \in M \setminus \{0\}$, $0 < y = r(y)^n = (nr(y) - (n-1)u) \vee 0$ and so $y = nr(y) - (n-1)u$ entails that $nr(y) = y + (n-1)u$, whence $r(y) = (y + (n-1)u)/n$.
(ii) We claim that $r(0)^2 > 0$, otherwise, $r(0)^2 = 0$ consequently, $r(0) \leq r(0)^-$. Thus, $r(0) = r(0)^-$ or equivalently, $2r(0) = u$, that is $r(0) = u/2 \in G$. It follows that $r(u/2) = (u/2 + (n-1)u)/n = ((2n-1)u)/2n$. So, $-u/2n = ((2n-1)u)/2n - u \in G$ which implies $u/2n$, u/n, $(n-1)u/n \in G$. From $((n-1)u/n)^n = (n(n-1)u/n - (n-1)u)/n = 0$ and (Sq2) it follows that $(n-1)u/n \leq r(0) = u/2$ which is a contradiction. Therefore, $r(0)^n \neq 0$ which implies that $r(0)^2 = 2r(0) - u$.
(4) Set $y = r(0)^{n-1} \oplus r(0)$. First, we will show that $r(y) = 1$. By (2), $y \oplus r(0) =$

$r(0)^{n-1} \oplus r(0) \oplus r(0) = 1$, so $r(y) = y \oplus r(0) = 1$, and Proposition 3.2 implies that $y \in B(M)$. On the other hand M is a chain and $0 < r(0) \leq y \in B(M)$, thus $y = 1$. Hence, it suffices to prove that $r(y) = 1$ (note that $u = 1$).

$$\begin{aligned} r(y) &= r\big(r(0)^{n-1} \oplus r(0)\big) \\ &= r\big(r(0)^{n-1}\big) \oplus \big(r\big(r(0)\big) \odot r(0)^{-}\big), \quad \text{(by Theorem 3.12)} \\ &= \big(r\big(r(0)\big)^{n-1} \vee r(0)\big) \oplus \big(r\big(r(0)\big) \odot r(0)^{-}\big). \text{(by Prop3.4 (10))} \end{aligned}$$

Since $r(0) \neq 0$, by part (1) in the proof of (iii), we have

$$\begin{aligned} r\big(r(0)\big) \odot r(0)^{-} &= \left(\frac{r(0) + (n-1)u}{n} - u + \big(u - r(0)\big)\right) \vee 0 \\ &= \frac{(n-1)u - (n-1)r(0)}{n} \vee 0 \\ &= \frac{(n-1)u - (n-1)r(0)}{n}, \quad \text{(since } r(0) \leq u\text{)}, \\ r\big(r(0)\big)^{n-1} \vee r(0) &= \left(\frac{r(0) + (n-1)u}{n}\right)^{n-1} \vee r(0) \\ &= \left((n-1)\frac{r(0) + (n-1)u}{n} - (n-2)u\right) \vee 0 \vee r(0) \\ &= \left(\frac{(n-1)r(0) + (n-1)^2 u - n(n-2)u}{n}\right) \vee 0 \vee r(0) \\ &= \left(\frac{(n-1)r(0) + (n^2 - 2n + 1)u - (n^2 - 2n)u}{n}\right) \vee 0 \vee r(0) \\ &= \frac{(n-1)r(0) + u}{n} \vee r(0) \\ &= \frac{(n-1)r(0) + u}{n}, \quad \text{(since } r(0) \leq u\text{)}. \end{aligned}$$

It follows that

$$\begin{aligned} r(y) &= \frac{(n-1)r(0) + u}{n} \oplus \frac{(n-1)u - (n-1)r(0)}{n} \\ &= \left(\frac{(n-1)r(0) + u}{n} + \frac{(n-1)u - (n-1)r(0)}{n}\right) \wedge u \\ &= (nu/n) \wedge u = u. \end{aligned}$$

$r(0)^{n-1} = 0$ implies that $r(0) = r(0)^{n-1} \oplus r(0) = u = r(u)$, so $0 = u$ which is absurd. It follows that, $r(0)^{n-1} > 0$.
(5) Consider $a = r(0)^{n-1} \oplus r(0)$. According to (4), $a \in M$, so $a = 0$ or $a = 1$.

From $r(0) \leq a = 0$ and Theorem we get that $r = Id_M$ and M is a Boolean algebra, so $M = \{0, 1\}$. Moreover, if $a = 1$, then $r(0)^{n-1} \geq r(0)^-$. Whence $r(0)^{n-1} = r(0)^-$ which means r is a strict n-root. Note that, in each n-root, we always have $r(0)^{n-1} \leq r(0)^-$. □

Theorem 4.5. *For each n-root of a representable symmetric pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1) = \Gamma(G, u)$ and for all $i \in \{1, 2, \ldots, n-1\}$, $r(0)^i \oplus r(0) \in B(M)$.*

Proof. If $r(0) = 0$, then by Theorem 3.2, $r = Id_M$ and $M = B(M)$. So the proof is evident. Now, let $r(0) \neq 0$. Consider the symbols and notations in the proof of Theorem 4.2. For each $P \in Y$, $r_P : M/(P \cap M) \to M/(P \cap M)$ is an n-root on a representation symmetric pseudo MV-algebra $M/(P \cap M)$, so by Theorem (4), $r_P(0/P)^{n-1} \oplus r_p(0/P) = 0$ or $r_P(0/P)^{n-1} \oplus r_P(0/P) = u/P$. In each case, $r_P(0/P)^i \oplus r_P(0/P) \in B(M)$ for all $i \in \{1, 2, \ldots, n-1\}$. Let $i \in \{1, 2, \ldots, n-1\}$ be given.

$$\Gamma(f)(r(0)^i \oplus r(0)) = (\frac{r(0)^i \oplus r(0)}{P \cap M})_{P \in Y} = (\frac{r(0)^i}{P \cap M})_{P \in Y} \oplus (\frac{r(0)}{P \cap M})_{P \in Y} \quad (1)$$
$$= (\frac{r(0)}{P \cap M})^i_{P \in Y} \oplus (\frac{r(0)}{P \cap M})_{P \in Y} \quad (2)$$
$$= (r_P(\frac{0}{P \cap M}))^i_{P \in Y} \oplus (r_P(\frac{0}{P \cap M}))_{P \in Y} \quad (3)$$
$$= (r_P(\frac{0}{P \cap M})^i \oplus r_P(\frac{0}{P \cap M}))_{P \in Y} \in B(\prod_{P \in Y} \Gamma(G/P, u/P)). \quad (4)$$

Hence $r(0)^i \oplus r(0) \in B(M)$. □

Lemma 4.1. *Let r be an n-root on a represention symmetric pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$. Then $r(0)^- = r(0)^\sim$, and there exists a unique idempotent element $u \in B(M)$ such that $u \vee r(0)^{n-1} = r(0)^- = r(0)^\sim$.*

Proof. Let $u := (n-1).r(0)^- \odot r(0)^-$ be an n-root. By Proposition 4.5, $u \in B(M)$ and we have

$$u \vee r(0)^{n-1} = ((r(0)^{n-1})^- \odot r(0)^-) \oplus r(0) \qquad \text{by Proposition 3.1(10)}$$
$$= r(0)^- \vee r(0)^{n-1} = r(0)^-,$$

since $r(0)^n = 0$ which yields $r(0)^{n-1} \leq r(0)^-$. In a similar way, using Proposition 4.5, we can show that $u \vee r(0) = r(0)^\sim$.
Now, let $v \in B(M)$ such that $v \vee r(0)^{n-1} = r(0) \to 0$. Then we have

$$(v \vee r(0))^{n-1} = v^{n-1} \vee (v^{n-2} \odot r(0)) \vee \ldots \vee (v \odot r(0)^{n-2}) \vee r(0)^{n-1}$$
$$= v^{n-1} \vee r(0)^{n-1} = v \vee r(0)^{n-1} = r(0)^-.$$

On the other hand, by Proposition 3.1 (10), $v \vee r(0) = r(v^n) = r(v)$, so $r(v)^{n-1} = r(0)^-$. In a similar way, $r(u)^{n-1} = r(0)^-$. It follows that $u = v$. $\square$

Theorem 4.6. *Let $r : M \to M$ be an n-root on a pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$. Then M satisfies only one of the following statements:*
(1) The pseudo MV-algebra M is a Boolean algebra.
(2) The pseudo MV-algebra M is a strict pseudo MV-algebra.
(3) There exist MV-algebra M_1 and M_2 such that M_1 is a Boolean algebra, M_2 is an n-strict MV-algebra, and $M \cong M_1 \times M_2$. In addition, this representation for M is unique up to isomorphic.

Proof. Due to Proposition 3.2, the element $a = r(0)^{n-1} \oplus r(0)$ and $b = (n-1).r(0)^- \odot r(0)^-$ are Boolean elements, $b = a^- = a^\sim$. We have three cases.
Case 1. If $a = 0$, then $0 = r(0)$, so by Theorem 3.2, M is a Boolean algebra.
Case 2. If $a = 1$, then $r(0)^- \leq r(0)^{n-1}$. Hence, $r(0)^- = r(0)^{n-1}$, r is a strict n-root.
Case 3. If $0 < a < 1$, then consider the pseudo MV-algebras $([0, b]; \oplus, ^{-b}, ^{\sim b}, 0, b)$ and $([0, a]; \oplus, ^{-a}, ^{\sim a}, 0, a)$. By Proposition 3.1 (15), they have n-root $r_a : [0, a] \to [0, a]$ and $r_b : [0, b] \to [0, b]$, defined by $r_a(x) = r(x) \wedge a$ and $r_b(y) = r(y) \wedge b$ for all $x \in [0, a]$ and $y \in [0, b]$.

$$r_b(0) = b \wedge r(0) = b \odot r(0) = ((n-1).r(0)^-) \odot r(0)^- \odot r(0) = 0.$$

$$r_a(0) = r(0) \wedge a = r(0) \wedge (r(0)^{n-1} \oplus r(0)) = r(0).$$

So by Theorem 3.2 $[0, b]$ is a Boolean algebra, and from Lemma 4.5, we have$[0, a]$ is an n-strict pseudo MV-algebra.
In addition, by Remark 3.2, $M \cong [0, a] \times [0, b]$.
Now, we prove the second part of (3), which is the uniqueness of the direct product. Let $M \cong_\varphi M_1 \times M_2$, where M_1 is a Boolean algebra and M_2 is a strict pseudo MV-algebra. Let 0_1 and 0_2 (1_1 and 1_2) be the least (greatest) elements of M_1 and M_2, respectively. Consider the n-square root $s_i : M_i \to M_i$ $(i - 1, 2)$ (by the assumption, $s_1 : Id_{M_1}$ and s_2 is strict). Clearly, $s := (s_1, s_2) : M_1 \times M_2 \to M_1 \times M_2$, defined by $s(x_1, x_2) = (s_1(x_1), s_2(x_2))$, is an n-root on $M_1 \times M_2$. Also,

$$\begin{aligned} s(0_1, 0_2) \to (0_1, 0_2) &= (s_1(0_1), s_2(0_2)) \to (0_1, 0_2) = (0_1 \to 0_1, s_2(0) \to 0_2) \\ &= (1_1, s_2(0_2) \to 0_2) = (1_1, 0_2) \vee (0_1, s_2(0_2) \to 0_2) \\ &= (1_1, 0_2) \vee (0_1, s_2(0)^{n-1}) = (1_1, 0_2) \vee (0_1, s_2(0_2))^{n-1} \\ &= (1_1, 0_2) \vee (s_1(0_1), s_2(0_2))^{n-1} = (1_1, 0_2) \vee s(0_1, 0_2)^{n-1}. \end{aligned}$$

Hence, $(1_1, 0_2)$ satisfies the conditions of Lemma 4.5 which means $\varphi^{-1}(1_1, 0_2)$ satisfies the conditions, too. Lemma 4.5 implies that $\varphi^{-1}(1_1, 0_2) = b$ and, consequently, $[0, b] \cong [(0_1, 0_2), (1_1, 0_2)] \cong M_1$. Also,$b^\sim = \varphi^{-1}(1_1, 0_2)^\sim = \varphi^{-1}(0_1, 1_2)$, so $[0, a] \cong [(0_1, 0_2), (0_1, 1_2)] \cong M_2$. $\square$

Theorem 4.7. *Each n-root r of a representation symmetric pseudo MV-algebra $M = \Gamma(G, u)$ satisfies the following equality:*

$$r(x) = (x \wedge b) \vee \frac{(x \wedge a) + (n-1)a}{n}, x \in M,$$

where $b = r(0)^- \odot (n-1)r(0)^-$, $a = r(0) \oplus r(0)^{n-1}$, and + is the group addition in G. Moreover, $a \neq 1$, the interal $[0, a]$ is n-divisible.

Proof. According to the proof of Theorem 4.5, $M_1 = [0, b]$ is a Boolean algebra with n-root r_1, $M_2 = [0, a]$ is an n-strict pseudo MV-algebra with an n-root r_2, where $r_1(x) = r(x) \odot b$ and $r_2(y) = r(y) \odot a$ for all $x \in M_1$ and $y \in M_2$. The mapping $f : M_1 \times M_2 \to M$, defined by $f(x, y) = x \vee y$. is an isomorphism. By Proposition 3.4, $\tau(z) = f((r_1, r_2)(x, y))$ is an n-root on M, where $f(x, y) = z$. Clearly, $x = z \wedge b$ and $y = z \wedge a$. This implies that $\tau = r$. It follows that for each $z \in M$, we have

$$\begin{aligned} r(z) = \tau(z) &= f((r_1, r_2)(z \wedge b, z \wedge a)) = f(r_1(z \wedge b), r_2(z \wedge a)) \\ &= r_1(z \wedge b) \vee r_2(z \wedge a) \\ &= (z \wedge b) \vee r_2(z \wedge a) \\ &= (z \wedge b) \vee \frac{(z \wedge a) + (n-1)a}{n}. \end{aligned}$$

Note that, $M_2 = \{z \in M : z \leq a\} = \{z \in G : 0 \leq z \leq a\} = \Gamma(G, a)$. Therefore, the formula for r can be obtained if we know $r(0)$. □

Corolary 4.1. *Suppose that r is an n-root on MV-algebra $M = (\Gamma, u)$. Then for each $m \in \mathbb{N}$ with $m|n$, the MV-algebra M has an m-root.*

Proof. Set $p = n/m$. Define v: $M \to M$ by $v(z) = r(z)^p$. According to the proof of Theorem 4.7, we have $r(z) = r_1(z \wedge b) \vee r_2(z \wedge a)^p$. Due to $a \wedge b = 0$, we have $(r_1(z \wedge b) \vee r_2(z \wedge a))^p = r_1(z \wedge b)^p \vee r_2(z \wedge a)^p$. In view of the fact r_2 is a strict n-root and Corollary 3.1, $v_2 : M_2 \to M_2$ defined by $v_2(z) = r_2(z)^p$, $z \in M_2$, is a strict m-root and $v_2(z) = (z - (m-1)u)/m$. Hence, $r_2(z \wedge a)^p = v_2(x \wedge a) = ((z \wedge a) + (m-1)a)/m$ for all $z \in M$. We have

$$\begin{aligned} r(z)^p &= (r_1(z \wedge b) \vee r_2(z \wedge a))^p = r_1(z \wedge b)^p \vee r_2(z \wedge a)^p \\ &= (z \wedge b)^p \vee r_2(z \wedge a)^p = (z \wedge b) \vee r_2(z \wedge a)^p \\ &= (z \wedge b) \vee \frac{(z \wedge a) + (m-1)a}{m}. \end{aligned}$$

The identity map is an m-root on M_1, since M_1 is a Boolean algebra. On the other hand, since M_2 is n-strict and $m|n$, by Corollary 3.1, the mapping $(x +$

$(m-1)a)/m$, $x \in M_2$, is an m-root on M_2. Similar to the proof of Theorem 4.7, $\tau(x) = (x \wedge b) \vee ((x \vee a) + (m-1)a)/m$ is an m-root on M. Clearly, $\tau = v$. Therefore, M has an m-root. $\square$

Similar to [[Belluce, 1992], Page 16], we defined a strongly atomless pseudo MV-algebra. It is a pseudo MV-algebra M such that for each non-zero element $x \in M \setminus \{0\}$, there exists a normal prime ideal P such that $x \notin P$ and x/P is not an atom of M/P. Clearly, each strongly atomless pseudo MV-algebra is representable.

Lemma 4.2. *[Dvurecenskij and Zahiri, 2022] In each representable pseudo MV-algebra $(M; \oplus, -, \sim, 0, 1)$, the following conditions are equivalent:*
(1) M is strongly atomless.
(2) For each $x \in M \setminus \{0\}$ there exists $y \in M$ such that $0 \le y \le x$ and $y \wedge (x \odot y^-) \neq 0$.
(3) For each $x \in M \setminus \{0\}$ there exists $y \in M$ such that $0 \le y \le x$ and $y \wedge (y^\sim \odot x) \neq 0$.

Proposition 4.1. *Let $(M; \oplus, -, \sim, 0, 1)$ be a representable pseudo MV-algebra. If M is n-strict, then it is strongly atomless.*

Proof. Assume $M = \Gamma(G, u)$, where (G, u) is a unital ℓ-group. Let $r : M \to M$ be a strict n-root. Due to Theorem 4.1, $r(x) = (x + (n-1)u)/n$ for each $x \in M$. According to Lemma 4.7, we need to prove that for each $x \in M \setminus \{0\}$, there is $y \in M$ such that $0 < y < x$ and $y \wedge (x \odot y^-) \neq 0$. If $|M| = 1$, then $M = \{0\}$ and the proof is clear. Let $|M| \ge 2$. Choose $x \in M \setminus \{0\}$.
(1) If $x = 1$, then choose $y \in M \setminus B(M)$ such that $0 < y < x$ (since M is strict, by Theorem 4.5, M is not a Boolean algebra and so we can find such an element). Then $y \wedge (x \odot y^-) = y \wedge y^- > 0$, since y is not a Boolean element (see [Georgescu and Iorgulescu, 2001]).
(2) Let $x \in M \setminus \{0, 1\}$ such that $x \in B(M)$. By Proposition 3.4, $r(x) = x \vee r(0)$ and $r(x^-) = x^- \vee r(0)$. Set $y := r(x^-)^\sim$. From relation 3.6(i), we know that $0 \le y \le x$. If $y = 0$, then $r(x^-) = 1 = r(1)$ and so $x = 0$, which is a contradiction. If $y = x$, in view of Proposition 3.3 (1), $x^- \vee r(0) = r(x^-) = x^-$. Thus, $r(0) \le x^-$ and $x \le r(0)^- = r(0)^{n-1}$ imply that

$$x = x \odot x \le r(0)^{n-1} \odot r(0)^{n-1} = r(0)^n \odot r(0)^{n-2} = 0 \odot r(0)^{n-2} = 0.$$

This contraction to our assumption. Thus, $0 < y < x$. From $x \in B(M)$, we have

$$\begin{aligned} y \wedge (x \odot y^-) &= r(x^-)^\sim \wedge (x \wedge r(x^-)) = r(x^-)^\sim \wedge r(x^-) \\ &= (x \wedge r(0)^\sim) \wedge (x^- \vee r(0)) \\ &= (x \wedge x^- \wedge r(0)) \vee (x \wedge r(0)^\sim \wedge r(0)) \\ &= x \wedge r(0)^\sim \wedge r(0), \qquad (x \in B(M), x \wedge x^- = 0) \\ &= x \wedge r(0)^{n-1} \wedge r(0), \qquad (r \text{ is } n\text{-strict}) \\ &= x \wedge r(0)^{n-1}. \end{aligned}$$

We claim that $x \wedge r(0)^{n-1} \neq 0$. Otherwise, since $x \in B(M)$, $r(0) \leq x^-$ and so $x \leq r(0)^- = r(0)$. It follows that $x = x \odot x \leq r(0)^{n-1} \odot r(0)^{n-1} = 0$ which is absurd.
(3) If $x \in M \setminus B(M)$, then similarly to (2), we can show that $0 < r(x^-)^\sim < x$. Note that by Proposition 3.1 (1), $x \vee r(0) \leq r(x)$. Set $y := r(x^-)^\sim$. We will show that

$$0 \neq y \wedge (x \odot y^-) = r(x^-)^\sim \wedge (x \odot r(x^-)).$$

By Theorem 4.2,

$$r(x^-) = ((x^-) + (n-1)u)/n = (u - x + (n-1)u)/n = (nu - x)/n = u - x/n.$$

(i) Given $p \in G$ such that p/n exists, then $p/n + u = (p + nu)/n$. Indeed,

$$n(p/n + u) = n(p/n) + nu = p + nu,$$

so $p/n + u = (p + nu)/n$ (G enjoys unique extraction of roots). Then

$$n(p/n - u) = n(p/n) - nu = p - nu.$$

and hence $p/n - u = (p - nu)/n$.
(ii) For every $p \in G$, if p/n exists, then $(-p)/n$ exists, and $(-p)/n = -p/n$, since $n(p/n) = p$. Thus, $-p = n(-p/n)$.
(iii) Given $a \in M$, in the ℓ-group G by (i), we have

$$r(a^-)^\sim = u - ((nu - a)/n)u + (a - nu)/n = (a - nu + nu)/n = a/n.$$

That is, a/n exists and belongs to M. Also,

$$\begin{aligned} r(x^-)^\sim \wedge (x \odot r(x^-)) &= x/n \wedge ((x + r(x^-) - u) \vee 0) \\ &= x/n \wedge ((x + u - x/n - u) \vee 0) \\ &= x/n \wedge (x - x/n) = x/n \wedge (x + (-x/n)) \\ &= x/n \wedge (x + (-x)/n) = x/n \wedge (n-1)x/n = x/n. \end{aligned}$$

Since $x \neq 0$, if we choose $y = r(x^-)^\sim$, then $y \wedge (x \odot y^-) = x/n \neq 0$. $\square$

5 Discussion and Conclusion

This paper aims to extend the concept of square roots in pseudo MV-algebras of [Dvurecenskij and Zahiri, 2022] and to achieve this goal, it is necessary to introduce the notion of n-roots. The research systematically explores the properties, classifications, and connections with related algebraic structures of n-roots in pseudo MV-algebras, providing a solid foundation for the development of non-commutative algebraic logic. First, the paper clarifies the definitions of n-roots and weak n-roots, proving that weak n-roots are injective and unique, while n-roots need to satisfy additional conditions related to inverse operations. It is revealed through the decomposition theorem that pseudo MV-algebras with n-roots can be decomposed into a Boolean algebra and a strict pseudo MV-algebra, a result based on the categorical equivalence between pseudo MV-algebras and unital ℓ-groups as well as the properties of normal prime ideals. Moreover, Boolean algebras are successfully characterized among MV-algebras by using n-roots.

In the study of strict n-roots, the paper provides a comprehensive characterization of each strict n-root. It is demonstrated that every MV-algebra with a strict n-root can be described by a unital n-divisible ℓ-group, and strict n-roots are detailedly characterized. Additionally, the paper establishes a connection between n-strict pseudo MV-algebras and strongly atomless pseudo MV-algebras, proving that in the variety of MV-algebras with n-roots, the concepts of n-strict MV-algebras and strongly atomless MV-algebras coincide. Compared with the square roots defined on pseudo MV-algebras, the n-th roots investigated in this paper exhibit a higher degree of generality. Furthermore, the paper verifies that the class of pseudo MV-algebras with n-roots forms a variety, and discusses properties such as the preservation of n-roots under homomorphic mappings and the n-root invariance of ideals. These findings not only improve the element classification theory of pseudo MV-algebras but also provide a new algebraic tool for handling the "detailed splitting" and "inverse operation" of proposition truth values in non-commutative fuzzy reasoning systems, which is of great significance for applications in fuzzy systems, quantum logic, and ordered algebraic structures.

Future research will focus on fully characterizing these n-roots, further exploring the interplay of divisibility, atomicity, and logical completeness in non-classical algebraic models, and promoting the practical application of pseudo MV-algebras with n-roots in more fields.

Declarations

Conflict of interest The authors declare no conflict of interest. No data is involved in this article.

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