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Generalization of intuitionistic fuzzy n-normed spaces via ψ-extensions

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Abstract

The aim of this paper is to study the generalization of intuitionistic fuzzy normed spaces to intuitionistic fuzzy n-normed spaces. In this framework, we discuss intuitionistic fuzzy n-continuity and intuitionistic fuzzy n-boundedness. Furthermore, we introduce the intuitionistic fuzzy ψ-n-normed space, which generalizes the intuitionistic fuzzy n-normed space. Several results and properties in this new set-up are also presented and analyzed.

Keywords: Intuitionistic fuzzy n-normed space, Intuitionistic fuzzy $\psi - n$-normed space

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1 Introduction

The theory of fuzzy sets was first introduced by Zadeh (16) in 1965. Following Zadeh's pioneering work, many researchers have extended the concept into various branches of mathematics, leading to new theories such as fuzzy group theory, fuzzy differential equations, fuzzy topology, and fuzzy normed spaces (13), among others. In particular, the study of fuzzy normed spaces and their generalizations has attracted significant attention.

Atanassov (3) introduced the concept of intuitionistic fuzzy sets, which was further developed by Coker (4). Park (12) proposed the notion of intuitionistic fuzzy metric spaces, and Saadati and Park (13) introduced intuitionistic fuzzy normed spaces. In many situations where classical norms are inadequate, the framework of intuitionistic fuzzy norms proves to be more suitable. Extensive research has been conducted in this area, as seen in (7), (10), (11), (13).

More recently, M. Mursaleen (9) defined the structure of intuitionistic fuzzy 2-normed spaces and studied foundational results analogous to those in normed linear spaces. In this paper, we investigate continuity and boundedness within intuitionistic fuzzy 2-normed spaces. Additionally, T.K. Samanta and Sumit Mohinta (14) introduced intuitionistic fuzzy ψ-normed spaces and examined continuity and boundedness in that framework. Building upon these ideas, we introduce the notion of intuitionistic fuzzy ψ-n-normed spaces, which generalizes intuitionistic fuzzy n-normed spaces. This new framework provides a more flexible approach for handling situations where classical norms or n-norms may be insufficient due to inherent inexactness.

2 Preliminaries

We recall some notations and basic definitions used in this paper.

Definition 2.1. *(15) A binary operation* $*:[0,1]\times[0,1]\to[0,1]$ *is said to be a continuous t-norm if it satisfies the following conditions:*

(a) $*$ *is associative and commutative;*

(b) $*$ *is continuous;*

(c) $a*1=a$ *for all* $a\in[0,1]$*;*

(d) $a*b\le c*d$ *whenever* $a\le c$ *and* $b\le d$ *for each* $a,b,c,d\in[0,1]$.

Example 2.1. *Two typical examples of continuous t-norms are*

$$a*b=ab \;\; and \;\; a*b=\min\{a,b\}.$$

Definition 2.2. *(15) A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:*

(a) *$\diamond$ is associative and commutative;*

(b) *$\diamond$ is continuous;*

(c) *$a \diamond 0 = a$ for all $a \in [0,1]$;*

(d) *$a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.*

Example 2.2. *Two typical examples of continuous t-conorms are*

$$a \diamond b = \min\{a+b, 1\} \ \textit{ and } \ a \diamond b = \max\{a, b\}.$$

Definition 2.3. *(13) The five-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if V is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $V \times (0, \infty)$ satisfying the following conditions. For every $x, y \in V$ and $s, t > 0$,*

(a) $\mu(x,t) + \nu(x,t) \leq 1$;

(b) $\mu(x,t) > 0$;

(c) $\mu(x,t) = 1$ *if and only if* $x = 0$;

(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$;

(e) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$;

(f) $\mu(x, \cdot) : (0, \infty) \to [0,1]$ *is continuous;*

(g) $\lim_{t\to\infty} \mu(x,t) = 1$ *and* $\lim_{t\to 0} \mu(x,t) = 0$;

(h) $\nu(x,t) < 1$;

(i) $\nu(x,t) = 0$ *if and only if* $x = 0$;

(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$;

(k) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$;

(l) $\nu(x, \cdot) : (0, \infty) \to [0,1]$ *is continuous;*

(m) $\lim_{t\to\infty} \nu(x,t) = 0$ *and* $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm.

Example 2.3. *Let* $(V, \|\cdot\|)$ *be normed space over* $\mathbb{F}$. *Denote* $a * b = ab$ *and* $a \diamond b = \min\{a+b, 1\}, \forall a, b \in [0,1]$ *and let* μ_0 *and* ν_0 *be fuzzy sets on* $V \times (0, \infty)$ *defined as follows* $\mu_0(x,t) = \dfrac{t}{t + \|x\|}$, $\nu_0(x,t) = \dfrac{\|x\|}{t + \|x\|}$, *for all* $t \in \mathbb{R}^+$. *Then* $(V, \mu_0, \nu_0, *, \diamond)$ *is an intuitionistic fuzzy normed space over* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 2.4. *(9) Let* V *be a real vector space of dimension* d, *where* $n \leq d < \infty$. *An* n-norm *on* V *is a function*

$$\|\cdot, \cdot, \ldots, \cdot\| : V^n \longrightarrow \mathbb{R},$$

which satisfies, for every $x_1, x_2, \ldots, x_n, y_1 \in V$ *and* $\alpha \in \mathbb{R}$:

(a) $\|x_1, x_2, \ldots, x_n\| = 0$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(b) $\|x_1, x_2, \ldots, x_n\|$ *is invariant under permutation of its arguments;*

(c) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|$;

(d) $\|x_1 + y_1, x_2, \ldots, x_n\| \leq \|x_1, x_2, \ldots, x_n\| + \|y_1, x_2, \ldots, x_n\|$.

The pair $(V, \|\cdot, \cdot, \ldots, \cdot\|)$ *is then called an* n-normed space.

Definition 2.5. *(9) The five-tuple* $(V, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy 2-normed space (for short, IF 2-NS) if V is a vector space over* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ *is a continuous t-norm,* $\diamond$ *is a continuous t-conorm, and* μ, ν *are fuzzy sets on* $V \times \cdots \times V \times (0, \infty)$ *satisfying the following conditions. For every* $x, y, z \in V$ *and* $s, t > 0$,

(a) $\mu(x,y,t) + \nu(x,y,t) \leq 1$;

(b) $\mu(x,y,t) > 0$;

(c) $\mu(x,y,t) = 1$ *if and only if* x *and* y *are linearly dependent;*

(d) $\mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$;

(e) $\mu(x,y,t) * \mu(x,z,s) \leq \mu(x, y+z, t+s)$;

(f) $\mu(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous;*

(g) $\lim_{t\to\infty} \mu(x,y,t) = 1$ *and* $\lim_{t\to 0} \mu(x,y,t) = 0$;

(h) $\mu(x,y,t) = \mu(y,x,t)$;

(i) $\nu(x,y,t) < 1$;

(j) $\nu(x, y, t) = 0$ *if and only if* x *and* y *are linearly dependent;*

(k) $\nu(\alpha x, y, t) = \nu(x, y, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$*;*

(l) $\nu(x, y, t) \diamond \nu(x, z, s) \geq \nu(x, y + z, t + s)$*;*

(m) $\nu(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(n) $\lim_{t\to\infty} \nu(x, y, t) = 0$ *and* $\lim_{t\to 0} \nu(x, y, t) = 1$*,*

(o) $\nu(x, y, t) = \nu(y, x, t)$*.*

In this case $(\mu, \nu)_2$ *is called an intuitionistic fuzzy 2-norm on* V*. We denote it by* $(\mu, \nu)_2$*.*

Definition 2.6. *(9) The five-tuple* $(V, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy* n*-normed space (for short, IF* n*-NS) if* V *is a vector space over* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$*,* $*$ *is a continuous t-norm,* $\diamond$ *is a continuous t-conorm, and* μ, ν *are fuzzy sets on* $V \times \cdots \times V \times (0, \infty)$ *satisfying the following conditions. For every* $x_1, x_2, \ldots, x_n, y, z \in V$ *and* $s, t > 0$*,*

(a) $\mu(x_1, x_2, \ldots, x_n, t) + \nu(x_1, x_2, \ldots, x_n, t) \leq 1$*;*

(b) $\mu(x_1, x_2, \ldots, x_n, t) > 0$*;*

(c) $\mu(x_1, x_2, \ldots, x_n, t) = 1$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(d) $\mu(\alpha x_1, x_2, \ldots, x_n, t) = \mu(x_1, x_2, \ldots, x_n, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$*;*

(e) $\mu(x_1, x_2, \ldots, x_{n-1}, y, t) * \mu(x_1, x_2, \ldots, x_{n-1}, z, s)$
$\leq \mu(x_1, x_2, \ldots, x_{n-1}, y + z, t + s)$*;*

(f) $\mu(x_1, x_2, \ldots, x_n, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(g) $\lim_{t\to\infty} \mu(x_1, x_2, \ldots, x_n, t) = 1$ *and* $\lim_{t\to 0} \mu(x_1, x_2, \ldots, x_n, t) = 0$*;*

(h) $\mu(x_1, x_2, \ldots, x_n, t)$ *is invariant under permutation of* $x_1, x_2, \ldots, x_n$*;*

(i) $\nu(x_1, x_2, \ldots, x_n, t) < 1$*;*

(j) $\nu(x_1, x_2, \ldots, x_n, t) = 0$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(k) $\nu(\alpha x_1, x_2, \ldots, x_n, t) = \nu(x_1, x_2, \ldots, x_n, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$*;*

(l) $\nu(x_1, x_2, \ldots, x_{n-1}, y, t) \diamond \nu(x_1, x_2, \ldots, x_{n-1}, z, s)$
$\geq \nu(x_1, x_2, \ldots, x_{n-1}, y + z, t + s)$*;*

(m) $\nu(x_1, x_2, \ldots, x_n, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(n) $\lim_{t\to\infty} \nu(x_1, x_2, \ldots, x_n, t) = 0$ *and* $\lim_{t\to 0} \nu(x_1, x_2, \ldots, x_n, t) = 1$,

(o) $\nu(x_1, x_2, \ldots, x_n, t)$ *is invariant under permutation of* $x_1, x_2, \ldots, x_n$.

In this case $(\mu, \nu)_n$ *is called an intuitionistic fuzzy* n*-norm on* V. *We denote it by* $(\mu, \nu)_n$.

Example 2.4. *(9) Let* $(V, \|\cdot, \cdot, \ldots, \cdot\|)$ *be an* n*-normed space over* $\mathbb{F}$ *and let* $a*b = ab$ *and* $a \diamond b = \min\{a + b, 1\}$, *for all* $a, b \in [0, 1]$. *For every* $t > 0$, *consider*

$$\mu(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|},$$

$$\nu(x_1, x_2, \ldots, x_n, t) = \frac{\|x_1, x_2, \ldots, x_n\|}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then $(V, \mu, \nu, *, \diamond)$ *is an intuitionistic fuzzy* n*-normed space.*

Example 2.5. *(9) Let* $(V, \|\cdot, \cdot\|)$ *be a 2-normed space over* F *and let* $a*b = ab$ *and* $a \diamond b = \min\{a + b, 1\}$, *for all* $a, b \in [0, 1]$ *and every* $t > 0$, *consider* $\mu(x, y, t) = \dfrac{t}{t + \|x, y\|}$, $\nu(x, y, t) = \dfrac{\|x, y\|}{t + \|x, y\|}$. *Then* $(V, \mu, \nu, *, \diamond)$ *is an intuitionistic fuzzy 2-normed space.*

Definition 2.7. *(9) Let* $(V, \mu, \nu, *, \diamond)$ *be an intuitionistic fuzzy* n*-normed space and let* $r \in (0, 1),\ t > 0$ *and* $x \in V$. *The set*

$$B(x, r, t) = \{y \in V : \mu(y-x, x_2, \ldots, x_n, t) > 1-r,\ \ \nu(y-x, x_2, \ldots, x_n, t) < r\},$$

$\forall x_2, \ldots, x_n \in V$ *is called the open ball with center* x *and radius* r *with respect to* t.

Definition 2.8. *(9) Let* $(V, \mu, \nu, *, \diamond)$ *be an intuitionistic fuzzy* n*-normed space. A set* $U \subset V$ *is said to be an open set if each of its points is the centre of some open ball contained in* U. *The open set in an intuitionistic fuzzy* n*-normed space* $(V, \mu, \nu, *, \diamond)$ *is denoted by* $\mathbb{U}$.

Definition 2.9. *(9) Let* $(V, \mu, \nu, *, \diamond)$ *be an intuitionistic fuzzy* n*-normed space. A sequence* $\{x_n\}$ *in* V *is said to be Cauchy if for each* $r > 0$ *and each* $t > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that*

$$\mu(x_n - x_m, x_2, \ldots, x_n, t) > 1 - r \quad \textit{and} \quad \nu(x_n - x_m, x_2, \ldots, x_n, t) < r$$

for all $n, m \geq n_0$ *and for all* $x_2, \ldots, x_n \in V$.

Definition 2.10. *(9) Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy n-normed space. A sequence $\{x_k\}$ is said to be convergent to $L \in V$ with respect to the intuitionistic fuzzy n-norm $(\mu, \nu)_n$, if for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that*

$$\mu(x_k - L, x_2, \ldots, x_n, t) > 1 - \epsilon \quad \textit{and} \quad \nu(x_k - L, x_2, \ldots, x_n, t) < \epsilon$$

for all $k \geq k_0$ and for all $x_2, \ldots, x_n \in V$.

Definition 2.11. *(14) Let ψ be a function defined on the real field $\mathbb{R}$ into itself satisfying the following properties;*

(a) $\psi(-t) = \psi(t)$ *for all* $t \in \mathbb{R}$

(b) $\psi(1) = 1$

(c) ψ *is strictly increasing and continuous on* $(0, \infty)$

(d) $\lim_{\alpha \to 0} \psi(\alpha) = 0$ *and* $\lim_{\alpha \to \infty} \psi(\alpha) = \infty$.

Example 2.6. *(14) Consider $\psi(\alpha) = |\alpha|; \psi(\alpha) = |\alpha|^p, p \in \mathbb{R}^+$; $\psi(\alpha) = \frac{2\alpha^{2n}}{|\alpha|+1}$, $n \in \mathbb{N}^+$. The function ψ allows us to generalize fuzzy metric and normed space.*

Definition 2.12. *(14) The five-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy ψ-normed space if V is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, ν are fuzzy sets on $V \times (0, \infty)$ satisfying the following conditions. For every $x, y \in V$ and $s, t > 0$,*

(a) $\mu(x, t) + \nu(x, t) \leq 1$*;*

(b) $\mu(x, t) > 0$*;*

(c) $\mu(x, t) = 1$ *if and only if* $x = 0$*;*

(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{\psi(\alpha)})$ *for each* $\alpha \neq 0$*;*

(e) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$*;*

(f) $\mu(x, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(g) $\lim_{t \to \infty} \mu(x, t) = 1$ *and* $\lim_{t \to 0} \mu(x, t) = 0$*;*

(h) $\nu(x, t) < 1$*;*

(i) $\nu(x, t) = 0$ *if and only if* $x = 0$*;*

(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{\psi(\alpha)})$ *for each* $\alpha \neq 0$*;*

(k) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$;

(l) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ *is continuous;*

(m) $\lim_{t\to\infty} \nu(x,t) = 0$ *and* $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ,ν) *is called an intuitionistic fuzzy* ψ*-norm.*

3 Intuitionistic fuzzy n-normed space

Theorem 3.1. *In an intuitionistic fuzzy* n*-normed space* $(V,\mu,\nu,*,\diamond)$, *if* $\{x_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} x$ *and* $\{y_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} y$ *then* $\{x_n+y_n\}_{n=1}^{\infty}$ *is convergent to* $x+y$. *In other words, if* $(V,\mu,\nu,*,\diamond)$ *is an intuitionistic fuzzy* n*-normed space, then the addition is continuous in* $(V,\mu,\nu,*,\diamond)$.

Theorem 3.2. *In an intuitionistic fuzzy* n*-normed space* $(V,\mu,\nu,*,\diamond)$, *if* $\lambda_n, \lambda \in \mathbb{R}^+$, $\lambda_n \to \lambda$ *as* $n\to\infty$ *and* $\{x_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} x$ *as* $n \to \infty$, *then* $\{\lambda_n x_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} \lambda x$. *In other words, if* $(V,\mu,\nu,*,\diamond)$ *is an IF-*n*-NS, then the scalar multiplication is continuous in* $(V,\mu,\nu,*,\diamond)$.

Proof. The proofs of Theorems 3.1 and 3.2 follow directly from the definitions. □

Lemma 3.1. *Let* $\{x_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} x$ *as* $n\to\infty$ *in an intuitionistic fuzzy* n*-normed space* $(V,\mu,\nu,*,\diamond)$. *Then for every* $t>0$ *as* $n\to\infty$,

$$\mu(x_n, x_2, \ldots, x_n, t) \to \mu(x, x_2, \ldots, x_n, t),$$
$$\nu(x_n, x_2, \ldots, x_n, t) \to \nu(x, x_2, \ldots, x_n, t), \tag{1}$$

for all $x_2, \ldots, x_n \in V$.

Proof. Let $\{x_n\}_{n=1}^{\infty} \to^{(\mu,\nu)_n} x$ as $n\to\infty$ in $(V,\mu,\nu,*,\diamond)$. Then for $t>0$, $\forall k \in \mathbb{N}^+$, we have

$$\begin{aligned}\mu(x_n, x_2, \ldots, x_n, t) &= \mu(x_n - x + x, x_2, \ldots, x_n, \tfrac{t}{k+1} + \tfrac{kt}{k+1})\\ &\geq \mu(x_n - x, x_2, \ldots, x_n, \tfrac{t}{k+1}) * \mu(x, x_2, \ldots, x_n, \tfrac{kt}{k+1})\\ &\to 1 * \mu(x, x_2, \ldots, x_n, \tfrac{kt}{k+1}), \qquad (n\to\infty)\\ &= \mu(x, x_2, \ldots, x_n, \tfrac{kt}{k+1}).\end{aligned}$$

So,

$$\underline{\lim}_{n\to\infty} \mu(x_n, x_2, \ldots, x_n, t) \geq \mu(x, x_2, \ldots, x_n, \tfrac{kt}{k+1}), \qquad (k = 1, 2, \ldots).$$

Letting $k \to +\infty$ yields

$$\underline{\lim}_{n\to\infty}\mu(x_n, x_2, \ldots, x_n, t) \geq \mu(x, x_2, \ldots, x_n, t). \tag{2}$$

On the other hand, for all $k \in \mathbb{N}^+$,

$$\mu(x - x_n, x_2, \ldots, x_n, \tfrac{1}{k+1}) \to 1 > \tfrac{k}{k+1} > 0, \qquad (n \to \infty).$$

So there exists N such that

$$\mu(x - x_n, x_2, \ldots, x_n, \tfrac{1}{k+1}) > \tfrac{k}{k+1}, \qquad (\forall n > N).$$

Thus, for all $n > N$ and $t > 0$, we have

$$\begin{aligned}\mu(x_n, x_2, \ldots, x_n, t) * \tfrac{k}{k+1} &\leq \mu(x_n, x_2, \ldots, x_n, t) * \mu(x - x_n, x_2, \ldots, x_n, \tfrac{1}{k+1}) \\ &\leq \mu(x, x_2, \ldots, x_n, t + \tfrac{1}{k+1}).\end{aligned}$$

Hence,

$$\overline{\lim}_{n\to\infty}\mu(x_n, x_2, \ldots, x_n, t) * \tfrac{k}{k+1} \leq \mu(x, x_2, \ldots, x_n, t + \tfrac{1}{k+1}), \quad (k = 1, 2, \ldots).$$

Letting $k \to +\infty$ yields

$$\overline{\lim}_{n\to\infty}\mu(x_n, x_2, \ldots, x_n, t) \leq \mu(x, x_2, \ldots, x_n, t). \tag{3}$$

From (2) and (3), we conclude that

$$\lim_{n\to\infty} \mu(x_n, x_2, \ldots, x_n, t) = \mu(x, x_2, \ldots, x_n, t).$$

Similarly, one obtains

$$\lim_{n\to\infty} \nu(x_n, x_2, \ldots, x_n, t) = \nu(x, x_2, \ldots, x_n, t).$$

The proof is completed. □

Theorem 3.3. *In an intuitionistic fuzzy n-normed space $(V, \mu, \nu, *, \diamond)$, the mappings*

$$\mu, \nu : V^n \times (0, \infty) \to [0, 1]$$

are continuous.

Proof. Let $(x_1, x_2, \ldots, x_{n-1}) \in V^{n-1}$, $x \in V$ and $t > 0$ with

$$(x_n^{(j)}, x_1, x_2, \ldots, x_{n-1}, t_j) \longrightarrow (x, x_1, x_2, \ldots, x_{n-1}, t) \quad \text{as } j \to \infty$$

in $V^n \times (0, \infty)$. Then $x_n^{(j)} \to^{(\mu,\nu)_n} x$ as $j \to \infty$ in V, and $t_j \to t$ as $j \to \infty$ in $(0, \infty)$.

Thus, for every $\delta > 0$ such that $\delta < \min\{\frac{t}{2}, 1\}$, there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$, $t - \delta < t_j < t + \delta$,

$$\mu(x - x_n^{(j)}, x_1, \ldots, x_{n-1}, \delta) > 1 - \delta,$$

$$\nu(x - x_n^{(j)}, x_1, \ldots, x_{n-1}, \delta) < \delta. \tag{4}$$

Hence, for all $j \geq j_0$, we see from (4) that

$$\begin{aligned} \mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) &\geq \mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t - \delta) \\ &= \mu(x_n^{(j)} - x + x, x_1, \ldots, x_{n-1},\ \delta + t - 2\delta) \\ &\geq \mu(x_n^{(j)} - x, x_1, \ldots, x_{n-1}, \delta) * \mu(x, x_1, \ldots, x_{n-1}, t - 2\delta) \\ &\geq (1 - \delta) * \mu(x, x_1, \ldots, x_{n-1}, t - 2\delta), \end{aligned}$$

and similarly,

$$\begin{aligned} \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) &\leq \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t - \delta) \\ &= \nu(x_n^{(j)} - x + x, x_1, \ldots, x_{n-1},\ \delta + t - 2\delta) \\ &\leq \nu(x_n^{(j)} - x, x_1, \ldots, x_{n-1}, \delta) \diamond \nu(x, x_1, \ldots, x_{n-1}, t - 2\delta) \\ &\leq \delta \diamond \nu(x, x_1, \ldots, x_{n-1}, t - 2\delta). \end{aligned}$$

Thus, for all $j \geq j_0$,

$$\mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \ \geq\ (1 - \delta) * \mu(x, x_1, \ldots, x_{n-1}, t - 2\delta),$$

$$\nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \ \leq\ \delta \diamond \nu(x, x_1, \ldots, x_{n-1}, t - 2\delta).$$

This shows that

$$\underline{\lim}_{j\to\infty} \mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \ \geq\ (1 - \delta) * \mu(x, x_1, \ldots, x_{n-1}, t - 2\delta), \tag{5}$$

$$\overline{\lim}_{j\to\infty} \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \ \leq\ \delta \diamond \nu(x, x_1, \ldots, x_{n-1}, t - 2\delta). \tag{6}$$

Letting $\delta \to 0^+$ in (5) and (6), we get

$$\underline{\lim}_{j\to\infty} \mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \geq \mu(x, x_1, \ldots, x_{n-1}, t),$$

$$\overline{\lim}_{j\to\infty} \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \leq \nu(x, x_1, \ldots, x_{n-1}, t).$$

On the other hand, applying Lemma 3.1 for IF-n-NS, we obtain

$$\begin{aligned}\mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) &\leq \mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t+\delta)\\ &\longrightarrow\ \mu(x, x_1, \ldots, x_{n-1}, t+\delta),\\ \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) &\geq \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t-\delta)\\ &\longrightarrow\ \nu(x, x_1, \ldots, x_{n-1}, t-\delta).\end{aligned}$$

Hence,

$$\overline{\lim}_{j\to\infty}\mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \leq \mu(x, x_1, \ldots, x_{n-1}, t+\delta), \tag{7}$$

$$\underline{\lim}_{j\to\infty}\nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \geq \nu(x, x_1, \ldots, x_{n-1}, t-\delta). \tag{8}$$

Letting $\delta \to 0^+$ in (7) and (8), it follows that

$$\overline{\lim}_{j\to\infty}\mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \leq \mu(x, x_1, \ldots, x_{n-1}, t),$$

$$\underline{\lim}_{j\to\infty}\nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) \geq \nu(x, x_1, \ldots, x_{n-1}, t).$$

Combining the above inequalities, we conclude that

$$\lim_{j\to\infty} \mu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) = \mu(x, x_1, \ldots, x_{n-1}, t),$$

$$\lim_{j\to\infty} \nu(x_n^{(j)}, x_1, \ldots, x_{n-1}, t_j) = \nu(x, x_1, \ldots, x_{n-1}, t).$$

Therefore, the mappings $\mu, \nu : V^n \times (0, \infty) \to [0, 1]$ are continuous. $\square$

Definition 3.1. *A linear operator*

$$T : (V, \mu, \nu, *, \diamond)\ \longrightarrow\ (V, \mu', \nu', *, \diamond)$$

is said to be intuitionistic fuzzy n-bounded (shortly, IF-n-B) if there exist constants $h, k \in \mathbb{R} \setminus \{0\}$ such that

$$\mu'(Tx_1, x_2, \ldots, x_n, t)\ \geq\ \mu(hx_1, x_2, \ldots, x_n, t)$$

and

$$\nu'(Tx_1, x_2, \ldots, x_n, t)\ \leq\ \nu(kx_1, x_2, \ldots, x_n, t)$$

for every nonzero $x_1, x_2, \ldots, x_n \in V$ and for every $t > 0$.

Theorem 3.4. *Suppose that $(V, \mu, \nu, *, \diamond)$ and $(V, \mu', \nu', *, \diamond)$ are intuitionistic fuzzy n-normed spaces over $\mathbb{F}$ with*

(a) $1 \geq a \geq c \geq 0$ *and* $1 \geq b \geq c \geq 0$ *implies* $a * b \geq c$,

(b) $0 \le a \le c \le 1$ *and* $0 \le b \le c \le 1$ *implies* $a \diamond b \le c$.

If linear operators $T, T_1, T_2 : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ *are (IF-*n*-B) intuitionistic fuzzy* n*-bounded, then* $T_1 + T_2$ *and* cT $(c \in \mathbb{F})$ *are also IF-*n*-B.*

Proof. Since the linear operators $T, T_1, T_2 : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ are IF-n-B, there exist $k_1, k_2, h_1, h_2 \in \mathbb{R}^+$ such that

$$\begin{aligned}
&\mu^{'}(T_1x_1, x_2, \ldots, x_n, t) \ge \mu(h_1x_1, x_2, \ldots, x_n, t),\\
&\nu^{'}(T_1x_1, x_2, \ldots, x_n, t) \le \nu(k_1x_1, x_2, \ldots, x_n, t),\\
&\mu^{'}(T_2x_1, x_2, \ldots, x_n, t) \ge \mu(h_2x_1, x_2, \ldots, x_n, t),\\
&\nu^{'}(T_2x_1, x_2, \ldots, x_n, t) \le \nu(k_2x_1, x_2, \ldots, x_n, t),
\end{aligned}$$

for every nonzero $x_1, x_2, \ldots, x_n \in V$ and for every $t > 0$.

Put $h = \max\{h_1, h_2\}$ and $k = \max\{k_1, k_2\}$. Then for every $x_1, x_2, \ldots, x_n \in V$, we have

$$\begin{aligned}
\mu^{'}((T_1+T_2)x_1, x_2, \ldots, x_n, t) &= \mu^{'}(T_1x_1 + T_2x_1, x_2, \ldots, x_n, t)\\
&\ge \mu^{'}(T_1x_1, x_2, \ldots, x_n, \tfrac{t}{2}) * \mu^{'}(T_2x_1, x_2, \ldots, x_n, \tfrac{t}{2})\\
&\ge \mu(h_1x_1, x_2, \ldots, x_n, \tfrac{t}{2}) * \mu(h_2x_1, x_2, \ldots, x_n, \tfrac{t}{2})\\
&= \mu(x_1, x_2, \ldots, x_n, \tfrac{t}{2h_1}) * \mu(x_1, x_2, \ldots, x_n, \tfrac{t}{2h_2})\\
&\ge \mu(x_1, x_2, \ldots, x_n, \tfrac{t}{2h}) * \mu(x_1, x_2, \ldots, x_n, \tfrac{t}{2h})\\
&\ge \mu(x_1, x_2, \ldots, x_n, \tfrac{t}{3h})\\
&= \mu(3hx_1, x_2, \ldots, x_n, t).
\end{aligned}$$

Similarly, we obtain

$$\nu^{'}((T_1+T_2)x_1, x_2, \ldots, x_n, t) \le \nu(3kx_1, x_2, \ldots, x_n, t).$$

Hence $T_1 + T_2$ is IF-n-B. Analogous reasoning shows that cT is IF-n-B. □

Definition 3.2. *A linear operator* $T : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ *is said to be* intuitionistic fuzzy n-continuous *(shortly, IF-*n*-C) at a point* $x_1 \in V$ *if* $x_1^{(m)} \to^{(\mu,\nu)_n} x_1$ *as* $m \to \infty$ *in* $(V, \mu, \nu, *, \diamond)$ *implies that* $Tx_1^{(m)} \to^{(\mu^{'},\nu^{'})_n} Tx_1$ *as* $m \to \infty$ *in* $(V, \mu^{'}, \nu^{'}, *, \diamond)$.

An operator T *is said to be IF-*n*-C if it is intuitionistic fuzzy* n*-continuous everywhere in* V.

Definition 3.3. *A map* $T : (V, \mu, \nu, *, \diamond) \to (V, \mu^{'}, \nu^{'}, *, \diamond)$ *is said to be* intuitionistic fuzzy n-continuous *(shortly, IF-*n*-C) at a point* $x_{1,0} \in V$ *if, for any given*

$\epsilon > 0$ and $\alpha \in (0,1)$, there exist $\delta = \delta(\alpha,\epsilon) > 0$ and $\beta = \beta(\alpha,\epsilon) \in (0,1)$ such that for all $x_1 \in V$,

$$\mu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta) > \beta \;\Rightarrow\; \mu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) > \alpha,$$
$$\nu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta) < 1-\beta \;\Rightarrow\; \nu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) < 1-\alpha.$$

Remark 3.1. *The above definitions (3.2) and (3.3) are equivalent.*

Theorem 3.5. *Suppose that $(V,\mu,\nu,*,\diamond)$ and $(V,\mu^{'},\nu^{'},*,\diamond)$ are intuitionistic fuzzy n-normed spaces over $\mathbb{F}$. If linear operators $T_1,T_2 : (V,\mu,\nu,*,\diamond) \to (V,\mu^{'},\nu^{'},*,\diamond)$ are IF-n-C, then $c_1T_1 + c_2T_2$ is IF-n-C for all scalars $c_1,c_2 \in \mathbb{F}$.*

Proof. Let $x_1^{(m)} \to^{(\mu,\nu)_n} x_1$ as $m \to \infty$ in $(V,\mu,\nu,*,\diamond)$. Since T_1 and T_2 are IF-n-C, we have

$$T_1x_1^{(m)} \to^{(\mu^{'},\nu^{'})_n} T_1x_1, \quad T_2x_1^{(m)} \to^{(\mu^{'},\nu^{'})_n} T_2x_1 \quad \text{as } m \to \infty$$

in $(V,\mu^{'},\nu^{'},*,\diamond)$. By Theorems 3.1 and 3.2 (continuity of addition and scalar multiplication in IF-n-NS), we get for all $c_1,c_2 \in \mathbb{F}$:

$$c_1T_1x_1^{(m)} + c_2T_2x_1^{(m)} \to^{(\mu^{'},\nu^{'})_n} c_1T_1x_1 + c_2T_2x_1 \quad \text{as } m \to \infty,$$

which gives

$$(c_1T_1 + c_2T_2)x_1^{(m)} \to^{(\mu^{'},\nu^{'})_n} (c_1T_1 + c_2T_2)x_1 \quad \text{as } m \to \infty.$$

Hence, $c_1T_1 + c_2T_2$ is intuitionistic fuzzy n-continuous. □

Definition 3.4. *A linear operator $T : (V,\mu,\nu,*,\diamond) \to (V,\mu^{'},\nu^{'},*,\diamond)$ is said to be* strongly intuitionistic fuzzy n-continuous *at a point $x_1 \in V$ if, for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that*

$$\mu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) \geq \mu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta),$$
$$\nu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) < \nu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta),$$

for all $x_2, \ldots, x_n \in V$.

A linear operator T is said to be strongly IF-n-C if it is strongly IF-n-C at each point of V.

Definition 3.5. *A linear operator $T : (V,\mu,\nu,*,\diamond) \to (V,\mu^{'},\nu^{'},*,\diamond)$ is said to be* strongly intuitionistic fuzzy n-continuous *at a point $x_{1,0} \in V$ if, for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that*

$$\mu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) \geq \mu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta),$$
$$\nu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) < \nu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta),$$

for all $x_2, \ldots, x_n \in V$.

A linear operator T is said to be strongly IF-n-C if it is strongly IF-n-C at every point of V.

Theorem 3.6. *A linear operator $T : (V, \mu, \nu, *, \diamond) \to (V^{'}, \mu^{'}, \nu^{'}, *, \diamond)$ is strongly IF-n-C, then it is IF-n-C, but the converse is not necessarily true.*

Proof. Let a linear operator $T : (V, \mu, \nu, *, \diamond) \to (V^{'}, \mu^{'}, \nu^{'}, *, \diamond)$ be strongly IF-n-C. Let $x_{1,0} \in V$. Then for any given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $x_1 \in V$ and all $x_2, \ldots, x_n \in V$,

$$\mu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) \geq \mu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta),$$
$$\nu^{'}(Tx_1 - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) < \nu(x_1 - x_{1,0}, x_2, \ldots, x_n, \delta).$$

Let $\{x_1^{(m)}\}$ be a sequence in V such that $\{x_1^{(m)}\} \to^{(\mu,\nu)_n} x_{1,0}$. Then for all $x_2, \ldots, x_n \in V$ and $t > 0$,

$$\lim_{m\to\infty} \mu(x_1^{(m)} - x_{1,0}, x_2, \ldots, x_n, t) = 1, \quad \lim_{m\to\infty} \nu(x_1^{(m)} - x_{1,0}, x_2, \ldots, x_n, t) = 0.$$

By the strong IF-n-C property, we have for all m,

$$\mu^{'}(Tx_1^{(m)} - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) \geq \mu(x_1^{(m)} - x_{1,0}, x_2, \ldots, x_n, \delta),$$
$$\nu^{'}(Tx_1^{(m)} - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) < \nu(x_1^{(m)} - x_{1,0}, x_2, \ldots, x_n, \delta),$$

which implies

$$\lim_{m\to\infty} \mu^{'}(Tx_1^{(m)} - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) = 1,$$

$$\lim_{m\to\infty} \nu^{'}(Tx_1^{(m)} - Tx_{1,0}, x_2, \ldots, x_n, \epsilon) = 0.$$

Hence, $Tx_1^{(m)} \to^{(\mu^{'},\nu^{'})_n} Tx_{1,0}$ in $(V^{'}, \mu^{'}, \nu^{'}, *, \diamond)$. Therefore, T is IF-n-C.

Conversely, there exist operators which are IF-n-C but not strongly IF-n-C, showing that the converse is not true.

Example 3.1. *Let $(V = \mathbb{R}, \|\cdot, \ldots, \cdot\|)$ be an n-normed space over $\mathbb{F}$. Define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$.*

Let μ, ν, μ', ν' be fuzzy sets on $V^n \times (0, \infty)$ defined by

$$\mu(x_1, \ldots, x_n, t) = \frac{t}{t + \|x_1, \ldots, x_n\|},$$
$$\nu(x_1, \ldots, x_n, t) = \frac{\|x_1, \ldots, x_n\|}{t + \|x_1, \ldots, x_n\|},$$
$$\mu'(x_1, \ldots, x_n, t) = \frac{t}{t + k\|x_1, \ldots, x_n\|},$$
$$\nu'(x_1, \ldots, x_n, t) = \frac{k\|x_1, \ldots, x_n\|}{t + k\|x_1, \ldots, x_n\|},$$

for all $t > 0$ and $k > 0$.

Define a mapping $T : V \to V$ by $T(Y) = \frac{Y^4}{1+Y^2}$ for all $Y \in V$. Let $Y_0 \in V$, and let $\{Y_k\}$ be a sequence in V such that $\{Y_k\} \to Y_0$ in the intuitionistic fuzzy n-norm sense, that is, for all $t > 0$,

$$\lim_{k\to\infty} \mu(Y_k - Y_0, x_2, \ldots, x_n, t) = 1 \quad \text{and} \quad \lim_{k\to\infty} \nu(Y_k - Y_0, x_2, \ldots, x_n, t) = 0.$$

Then, for all $t > 0$,

$$\mu'(TY_k - TY_0, x_2, \ldots, x_n, t) = \frac{t}{t + k\|TY_k - TY_0, x_2, \ldots, x_n\|} \to 1,$$

and

$$\nu'(TY_k - TY_0, x_2, \ldots, x_n, t) = \frac{k\|TY_k - TY_0, x_2, \ldots, x_n\|}{t + k\|TY_k - TY_0, x_2, \ldots, x_n\|} \to 0,$$

as $k \to \infty$. Hence, T is IF-n-C.

However, to check strong IF-n-C, let $\epsilon > 0$ be given. Then

$$\mu'(TY - TY_0, x_2, \ldots, x_n, \epsilon) \geq \mu(Y - Y_0, x_2, \ldots, x_n, \delta)$$

would require

$$\delta \leq \frac{\epsilon \prod_{i=1}^{n} \|1 + Y_i^2\|}{k \prod_{i=1}^{n} \|Y_i - Y_{0,i}\| \, \|\text{polynomial terms of } Y_i, Y_{0,i}\|}.$$

Taking the infimum over all $Y \neq Y_0$ yields $\delta_1 = 0$, which is impossible.

Hence, T is IF-n-C but not strongly IF-n-C.

□

Theorem 3.7. *Let $(V, \mu, \nu, *, \diamond)$ and $(V, \mu', \nu', *, \diamond)$ be intuitionistic fuzzy n-normed spaces (IF-n-NS), and let $T : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *, \diamond)$ be a linear operator. Then the following conditions are equivalent:*

(a) *T is intuitionistic fuzzy n-bounded (IF-n-B).*

(b) *There exist constants $h, k \in \mathbb{R} - \{0\}$ such that*

$$\mu'(Tx_1, \ldots, Tx_n, t) \geq \mu(hx_1, \ldots, hx_n, t),$$
$$\nu'(Tx_1, \ldots, Tx_n, t) \leq \nu(kx_1, \ldots, kx_n, t),$$

for all $x_1, \ldots, x_n \in V$ (nonzero) and $t > 0$.

(c) *T is intuitionistic fuzzy n-continuous at some point $x_0 \in V$.*

(d) *T is intuitionistic fuzzy n-continuous (IF-n-C) everywhere.*

Proof. $(a) \Leftrightarrow (b)$: The result follows directly from the definition of IF-n-B.

$(c) \Leftrightarrow (d)$: Suppose T is intuitionistic fuzzy n-continuous at some point $x_0 \in V$. Let $\{x_n^{(i)}\}_{n=1}^{\infty} \to x_0$ in $(V, \mu, \nu, *, \diamond)$ for $i = 1, \ldots, n$. By the properties of IF-n-NS, we have

$$(x_n^{(1)} - x_1, \ldots, x_n^{(n)} - x_n) + x_0 \to x_0 \ \text{ as } \ n \to \infty.$$

Since T is linear, it follows that

$$T((x_n^{(1)} - x_1, \ldots, x_n^{(n)} - x_n) + x_0) = T(x_n^{(1)}, \ldots, x_n^{(n)}) \to T(x_0),$$

which shows that T is IF-n-C. The converse is obvious.

$(a) \Leftrightarrow (d)$: This follows from the generalization of results in (9) and the definitions of IF-n-B and IF-n-C. □

Definition 3.6. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy n-normed space. A subset D of V is said to be* compact *if any sequence in D has a subsequence converging to an element of D in the sense of $(\mu, \nu)_n$-convergence.*

Theorem 3.8. *Let $T : (V_1, \mu_1, \nu_1, *, \diamond) \to (V_2, \mu_2, \nu_2, *, \diamond)$ be a mapping and let D be a compact subset of V_1. If T is intuitionistic fuzzy n-continuous (IF-n-C) on V_1, then $T(D)$ is a compact subset of V_2.*

Proof. Let $\{y_n\}$ be a sequence in $T(D)$. Then for each n there exists $x_n \in D$ such that $T(x_n) = y_n$.

Since D is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an element $x_0 \in D$ such that

$$\{x_{n_k}\} \to^{(\mu_1, \nu_1)_n} x_0 \quad \text{in} \quad (V_1, \mu_1, \nu_1, *, \diamond).$$

Since T is intuitionistic fuzzy n-continuous at x_0, by Definition (3.2),

$$\{x_{n_k}\} \to^{(\mu_1, \nu_1)_n} x_0 \quad \Rightarrow \quad T\{x_{n_k}\} \to^{(\mu_2, \nu_2)_n} T(x_0).$$

Hence, the corresponding subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converges to $y_0 = T(x_0) \in T(D)$, which shows that $T(D)$ is compact in V_2. □

4 Intuitionistic fuzzy ψ-n-normed space

Definition 4.1. *The five-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy ψ-n-normed space, if V is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $V^n \times (0, \infty)$ satisfying the following conditions. For every $x_1, x_2, \ldots, x_n, z \in V$ and $t, s > 0$,*

(a) $\mu(x_1, \ldots, x_n, t) + \nu(x_1, \ldots, x_n, t) \leq 1$;

(b) $\mu(x_1, \ldots, x_n, t) > 0$;

(c) $\mu(x_1, \ldots, x_n, t) = 1$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(d) $\mu(\alpha x_1, x_2, \ldots, x_n, t) = \mu(x_1, x_2, \ldots, x_n, \frac{t}{\psi(\alpha)})$ *for each* $\alpha \neq 0$;

(e) $\mu(x_1, \ldots, x_{n-1}, y, t) * \mu(x_1, \ldots, x_{n-1}, z, s) \leq \mu(x_1, \ldots, x_{n-1}, y + z, t + s)$;

(f) $\mu(x_1, \ldots, x_n, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(g) $\lim_{t\to\infty} \mu(x_1, \ldots, x_n, t) = 1$ *and* $\lim_{t\to 0} \mu(x_1, \ldots, x_n, t) = 0$;

(h) $\mu(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t) = \mu(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n, t)$ *(symmetry);*

(i) $\nu(x_1, \ldots, x_n, t) < 1$;

(j) $\nu(x_1, \ldots, x_n, t) = 0$ *if and only if* $x_1, \ldots, x_n$ *are linearly dependent;*

(k) $\nu(\alpha x_1, x_2, \ldots, x_n, t) = \nu(x_1, x_2, \ldots, x_n, \frac{t}{\psi(\alpha)})$ *for each* $\alpha \neq 0$;

(l) $\nu(x_1, \ldots, x_{n-1}, y, t) \diamond \nu(x_1, \ldots, x_{n-1}, z, s) \geq \nu(x_1, \ldots, x_{n-1}, y + z, t + s)$;

(m) $\nu(x_1, \ldots, x_n, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*

(n) $\lim_{t\to\infty} \nu(x_1, \ldots, x_n, t) = 0$ *and* $\lim_{t\to 0} \nu(x_1, \ldots, x_n, t) = 1$;

(o) $\nu(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t) = \nu(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n, t)$ *(symmetry).*

In this case $(\mu, \nu)_n$ is called an intuitionistic fuzzy ψ-n-norm on V.

Definition 4.2. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. A sequence $\{x_k\}$ is said to be* convergent *to $x \in V$ with respect to the intuitionistic fuzzy ψ-n-norm $(\mu, \nu)_n$ if for every $r \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that*

$$\mu(x_k - x, x_2, \ldots, x_n, t) > 1 - r \quad \textit{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) < r$$

for all $k \geq k_0$ and all $x_2, \ldots, x_n \in V$.

Definition 4.3. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. A sequence $\{x_k\}$ in V is said to be* Cauchy *if for every $r \in (0,1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that*

$$\mu(x_k - x_m, x_2, \ldots, x_n, t) > 1 - r \quad \textit{and} \quad \nu(x_k - x_m, x_2, \ldots, x_n, t) < r$$

for all $k, m \geq k_0$ and all $x_2, \ldots, x_n \in V$.

Definition 4.4. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space, $r \in (0,1)$, $t > 0$, and $x \in V$. The set*

$$B(x, r, t) = \{y \in V : \mu(y-x, x_2, \ldots, x_n, t) > 1-r,\ \nu(y-x, x_2, \ldots, x_n, t) < r\}$$

, $\forall x_2, \ldots, x_n \in V$ is called the open ball *with center x and radius r with respect to t.*

Definition 4.5. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. A set $U \subset V$ is said to be* open *if each of its points is the center of some open ball contained in U. The collection of all open sets in $(V, \mu, \nu, *, \diamond)$ is denoted by $\mathbb{U}$.*

Theorem 4.1. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. A sequence $\{x_k\}$ converges to $x \in V$ if and only if*

$$\mu(x_k - x, x_2, \ldots, x_n, t) \to 1 \quad \textit{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) \to 0 \quad \textit{as } k \to \infty,$$

for all $x_2, \ldots, x_n \in V$ and $t > 0$.

Proof. Fix $t > 0$. Suppose $\{x_k\}$ converges to x in the intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$. Then for a given $r \in (0,1)$, there exists an integer $k_0 \in \mathbb{N}$ such that

$$\mu(x_k - x, x_2, \ldots, x_n, t) > 1 - r \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) < r$$

for all $k \geq k_0$ and all $x_2, \ldots, x_n \in V$. Thus, $1 - \mu(x_k - x, x_2, \ldots, x_n, t) < r$ and $\nu(x_k - x, x_2, \ldots, x_n, t) < r$, which implies

$$\mu(x_k - x, x_2, \ldots, x_n, t) \to 1 \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) \to 0 \quad \text{as } k \to \infty.$$

Conversely, if for each $t > 0$,

$$\mu(x_k - x, x_2, \ldots, x_n, t) \to 1 \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) \to 0 \quad \text{as } k \to \infty,$$

then for every $r \in (0,1)$, there exists an integer k_0 such that

$$1 - \mu(x_k - x, x_2, \ldots, x_n, t) < r \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) < r$$

for all $k \geq k_0$ and all $x_2, \ldots, x_n \in V$. Hence,

$$\mu(x_k - x, x_2, \ldots, x_n, t) > 1 - r \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, t) < r,$$

which shows that $\{x_k\}$ converges to x in $(V, \mu, \nu, *, \diamond)$. $\square$

Theorem 4.2. *The limit is unique for a convergent sequence $\{x_k\}$ in an intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$.*

Proof. Let $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} x_k = y$. Then

$$\lim_{k\to\infty} \mu(x_k - x, x_2, \ldots, x_n, t) = 1, \quad \lim_{k\to\infty} \nu(x_k - x, x_2, \ldots, x_n, t) = 0,$$

and

$$\lim_{k\to\infty} \mu(x_k - y, x_2, \ldots, x_n, t) = 1, \quad \lim_{k\to\infty} \nu(x_k - y, x_2, \ldots, x_n, t) = 0,$$

for all $x_2, \ldots, x_n \in V$ and $t > 0$.

Now, for any $s, t > 0$,

$$\begin{aligned}\nu(x - y, x_2, \ldots, x_n, s + t) &= \nu(x - x_k + x_k - y, x_2, \ldots, x_n, s + t) \\ &\geq \nu(x - x_k, x_2, \ldots, x_n, s) \diamond \nu(x_k - y, x_2, \ldots, x_n, t).\end{aligned}$$

Taking the limit as $k \to \infty$, we get

$$\nu(x - y, x_2, \ldots, x_n, s + t) = 0 \quad \Rightarrow \quad x = y.$$

Hence, the limit of a convergent sequence in an intuitionistic fuzzy ψ-n-normed space is unique. $\square$

Theorem 4.3. *In an intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$, every convergent sequence is a Cauchy sequence.*

Proof. Let $\{x_k\}$ be a convergent sequence in $(V, \mu, \nu, *, \diamond)$ with $\lim_{k\to\infty} x_k = x$. Let $r \in (0, 1)$ and $s, t > 0$. Then there exists an integer $k_0 \in \mathbb{N}$ such that

$$\mu(x_k - x, x_2, \ldots, x_n, s) > 1 - r \quad \text{and} \quad \nu(x_k - x, x_2, \ldots, x_n, s) < r \quad \forall k \geq k_0.$$

For $k, p \in \mathbb{N}$, we have

$$\begin{aligned}&\mu(x_{k+p} - x_k, x_2, \ldots, x_n, s + t) \\ &= \mu(x_{k+p} - x + x - x_k, x_2, \ldots, x_n, s + t) \\ &\geq \mu(x_{k+p} - x, x_2, \ldots, x_n, s) * \mu(x - x_k, x_2, \ldots, x_n, t) \\ &= \mu(x_{k+p} - x, x_2, \ldots, x_n, s) * \mu(x_k - x, x_2, \ldots, x_n, t) \\ &> (1 - r') * (1 - r') > 1 - r', \quad \forall k \geq k_0, \text{ Since, by choosing } r' * r' > r.\end{aligned}$$

Similarly,

$$\begin{aligned}&\nu(x_{k+p} - x_k, x_2, \ldots, x_n, s + t) \\ &= \nu(x_{k+p} - x + x - x_k, x_2, \ldots, x_n, s + t) \\ &\leq \nu(x_{k+p} - x, x_2, \ldots, x_n, s) \diamond \nu(x - x_k, x_2, \ldots, x_n, t) \\ &= \nu(x_{k+p} - x, x_2, \ldots, x_n, s) \diamond \nu(x_k - x, x_2, \ldots, x_n, t) \\ &< r' \diamond r' < r, \quad \forall k \geq k_0, \text{ Since, by choosing } r' \diamond r' < r.\end{aligned}$$

Hence, $\{x_k\}$ is a Cauchy sequence in the intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$. $\square$

Theorem 4.4. *In an intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$, a sequence $\{x_k\}$ is a Cauchy sequence if and only if*

$$\mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 1 \quad \textit{and} \quad \nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 0$$

as $k \to \infty$, for all $x_2, \ldots, x_n \in V$ and $t > 0$.

Proof. Fix $t > 0$. Suppose $\{x_k\}$ is a Cauchy sequence in the intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$. Then, for a given $r \in (0, 1)$, there exists an integer $k_0 \in \mathbb{N}$ such that

$$\mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) > 1 - r \quad \text{and} \quad \nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) < r$$

for all $k \geq k_0$ and all $x_2, \ldots, x_n \in V$. Thus,

$\mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 1$ and $\nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 0$ as $k \to \infty$.

Conversely, if for each $t > 0$,

$\mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 1$ and $\nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) \to 0$ as $k \to \infty$,

then for every $r \in (0, 1)$, there exists an integer $k_0 \in \mathbb{N}$ such that

$1 - \mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) < r$ and $\nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) < r \; \forall k \geq k_0$.

Hence,

$$\mu(x_{k+p} - x_k, x_2, \ldots, x_n, t) > 1 - r \quad \text{and} \quad \nu(x_{k+p} - x_k, x_2, \ldots, x_n, t) < r,$$

which shows that $\{x_k\}$ is a Cauchy sequence in $(V, \mu, \nu, *, \diamond)$. $\square$

Definition 4.6. *An intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$ is said to be* complete *if every Cauchy sequence in $(V, \mu, \nu, *, \diamond)$ is convergent.*

Theorem 4.5. *Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. A sufficient condition for $(V, \mu, \nu, *, \diamond)$ to be complete is that every Cauchy sequence in $(V, \mu, \nu, *, \diamond)$ has a convergent subsequence.*

Proof. Let $\{x_k\}$ be a Cauchy sequence in $(V, \mu, \nu, *, \diamond)$, and let $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ that converges to $x \in V$. Let $s, t > 0$ and $r \in (0, 1)$.

Since $\{x_k\}$ is Cauchy, there exists an integer $k_0 \in \mathbb{N}$ such that

$\mu(x_k - x_{k_j}, x_2, \ldots, x_n, s) > 1 - r$ and $\nu(x_k - x_{k_j}, x_2, \ldots, x_n, s) < r, \; \forall k, j \geq k_0$.

Also, since $\{x_{k_j}\}$ converges to x, we have

$$\mu(x_{k_j} - x, x_2, \dots, x_n, t) > 1 - r \text{ and } \nu(x_{k_j} - x, x_2, \dots, x_n, t) < r,\ \forall j \geq k_0.$$

Now,

$$\begin{aligned}
&\mu(x_k - x, x_2, \dots, x_n, s + t)\\
&= \mu(x_k - x_{k_j} + x_{k_j} - x, x_2, \dots, x_n, s + t)\\
&\geq \mu(x_k - x_{k_j}, x_2, \dots, x_n, s) * \mu(x_{k_j} - x, x_2, \dots, x_n, t)\\
&> (1 - r') * (1 - r') > 1 - r', \quad \forall k \geq k_0, \text{ Since, by choosing } r' * r' > r.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\nu(x_k - x, x_2, \dots, x_n, s + t)\\
&= \nu(x_k - x_{k_j} + x_{k_j} - x, x_2, \dots, x_n, s + t)\\
&\leq \nu(x_k - x_{k_j}, x_2, \dots, x_n, s) \diamond \nu(x_{k_j} - x, x_2, \dots, x_n, t)\\
&< r' \diamond r' < r, \quad \forall k \geq k_0, \text{ Since, by choosing } r' \diamond r' < r.
\end{aligned}$$

Hence, $\{x_k\}$ converges to x in $(V, \mu, \nu, *, \diamond)$. Therefore, the intuitionistic fuzzy ψ-n-normed space $(V, \mu, \nu, *, \diamond)$ is complete. □

Remark 4.1. *Straightforwardly, we get the results* $(3.1), (3.2), (3.1), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8)$ *are also holds in IFψ-n-NS.*

Theorem 4.6. *Every intuitionistic fuzzy ψ-n-normed space is an intuitionistic fuzzy n-normed space, but the converse is not true.*

Proof. Let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy ψ-n-normed space. By the definition of a ψ-n-normed space, take $\psi(\alpha) = |\alpha|$. Then the definition of an intuitionistic fuzzy n-normed space is satisfied, so $(V, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy n-normed space.

Conversely, let $(V, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy n-normed space. If $\psi(\alpha) \neq |\alpha|$, then the definition of an intuitionistic fuzzy ψ-n-normed space is not satisfied. Therefore, a general intuitionistic fuzzy n-normed space is not necessarily a ψ-n-normed space. □

5 Conclusion

In this paper, we have successfully extended the concept of intuitionistic fuzzy normed spaces to the framework of intuitionistic fuzzy n-normed spaces. We have explored key properties such as intuitionistic fuzzy n-continuity and n-boundedness, providing a foundation for further analysis in these spaces. Moreover, the introduction of intuitionistic fuzzy ψ-n-normed spaces has allowed us to generalize the existing structure of n-normed spaces and examine their convergence, Cauchy sequences, and completeness. The results obtained in this study offer a comprehensive understanding of these generalized spaces and pave the way for future research in fixed point theory, functional analysis, and applications involving fuzzy and intuitionistic fuzzy structures.

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