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The structure behind the Collatz algorithm

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Abstract

The Collatz algorithm gives the operations to apply to both even and odd numbers so that starting with any natural number and applying these operations repeatedly, the number one is eventually reached. The Collatz conjecture claims that this is true no matter what starting number is used. It happens that the natural numbers can be organised into a hierarchical structure of infinite geometric sequences and the Collatz algorithm is the means of ascending through this hierarchy.
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1 Introduction

Starting with any natural number, the Collatz algorithm [1] instructs us to divide by 2 if the number is even and to multiply by 3 and add 1 if the number is odd. Applying the algorithm to the even number 36 we get

$$18, 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

Applying it to the odd number 75 produces

$$226, 113, 340, 170, 85, 256, 128, 64, 32, 16, 8, 4, 2, 1.$$

In both cases the number 1 is reached. We note that when nearing 1 the geometric sequence $1, 2, 4, 8, 16$ occurs in the first and the same sequence with a few extra terms $1, 2, 4, 8, 16, 32, 64, 128, 256$ occurs in the second case. Both occur reversed. It is no coincidence that this sequence occurs at this stage of the application of the algorithm.

The strategy we employ in this paper is to construct a hierarchical structure for the natural numbers using the inverses of the two operations used in the Collatz algorithm. Consequently, the algorithm merely becomes the means of navigating upwards through the hierarchy to reach the number 1.

We shall show that the natural numbers can be partitioned into an infinite set of infinite geometric sequences all having the same common ratio 2 and all starting with an odd number. These sequences can be organised into a hierarchical structure by using the inverse of the operation of multiplying by 3 and adding 1. Repeatedly dividing by 2 is the way to move backwards along any one of these sequences. Multiplying by 3 and adding 1 is the way to move from the initial term of any sequence to an even term of the sequence that is one level above it in the hierarchy. Using this hierarchical structure, not only can we move from any term back to the number 1 but we can also move from any number to any other number and in addition we can find a number located at any specified level within the hierarchy.

2 Partitioning the set of natural numbers

First we shall show that the set of all even numbers can be partitioned into an infinite number of infinite geometric sequences all with common ratio 2 and with distinct first terms.

Definition 2.1. *The set of natural numbers* $\mathbb{N} = \{1,\ 2,\ 3,\ \dots\}$.

In this paper the word 'number' shall mean 'natural number' unless stated otherwise.

Proposition 2.1. *The set of even numbers* $\{2n,\ n \in \mathbb{N}\} = \bigcup_{i\in\mathbb{N}} \{(2i-1)2^j,\ j \in \mathbb{N}\}$

Proof. Consider the even numbers:

$$2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16, \ldots$$

First extract the powers of two, $\{2^k,\ k \in \mathbb{N}\}$

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ \ldots$$

This leaves $6,\ 10,\ 12,\ 14,\ 18,\ 20,\ 22,\ 24,\ 26,\ 28,\ 30,\ 34,\ \ldots$
Next extract 3 times the powers of two, $\{2^k \times 3,\ k \in \mathbb{N}\}$

$$6,\ 12,\ 24,\ 48,\ \ldots$$

This leaves $10,\ 14,\ 18,\ 20,\ 22,\ 26,\ 28,\ 30,\ 34,\ \ldots$
Next extract 5 times the powers of two, $\{2^k \times 5,\ k \in \mathbb{N}\}$

$$10,\ 20,\ 40,\ 80,\ \ldots$$

This leaves $14,\ 18,\ 22,\ 26,\ 28,\ 30,\ 34,\ 36,\ 38,\ 42,\ \ldots$
By continuing in this manner we partition the even numbers into an infinite number of disjoint infinite geometric sequences:

$$2,\ 4,\ 8,\ 16,\ \ldots$$

$$6,\ 12,\ 24,\ 48,\ \ldots$$

$$10,\ 20,\ 40,\ 80,\ \ldots$$

$$\vdots$$

Therefore $\{2n,\ n \in \mathbb{N}\} = \bigcup_{i\in\mathbb{N}} \{(2i-1)2^j,\ j \in \mathbb{N}\}$ □

Next we complete the partitioning of the set of all natural numbers.

Proposition 2.2. *The set of all natural numbers can be partitioned into an infinite set of infinite geometric sequences.* $\mathbb{N} = \bigcup_{i\in\mathbb{N}} \{(2i-1)2^{j-1},\ j \in \mathbb{N}\}$

Proof. To each of the infinite geometric sequences, $\{(2i-1)2^j,\ j \in \mathbb{N}\}$, prepend the corresponding odd number to get $\{(2i-1)2^{j-1},\ j \in \mathbb{N}\}$.

$$\therefore\ \mathbb{N} = \bigcup_{i\in\mathbb{N}} \{(2i-1)2^{j-1},\ j \in \mathbb{N}\}$$ □

We make the following observations regarding the elements of these sequences.

1. Every element of every sequence can be expressed in the form $2^{k-1}a$ for some $k \in \mathbb{N}$ and some a an odd number.

2. Every element which is an even number can be expressed in the form $2^k a$ for some $k \in \mathbb{N}$ and some a an odd number.

3. Where the initial term a in any sequence is congruent to $1 \pmod 3$ then the even elements in that sequence are alternately congruent to $2 \pmod 3$ and to $1 \pmod 3$.

4. Where the initial term a in any sequence is congruent to $2 \pmod 3$ then the even elements in that sequence are alternately congruent to $1 \pmod 3$ and to $2 \pmod 3$.

5. Where the initial term a in any sequence is congruent to $0 \pmod 3$, i.e., is a multiple of 3, then the even elements in that sequence are also multiples of 3.

Definition 2.2. *We shall refer to the initial term of each sequence as the* ***head*** *of the sequence.*

3 Mapping even numbers exceeding a multiple of 3 by 1, to odd numbers

It is those even elements in each sequence which are congruent to $1 \pmod 3$ that are of most interest to us. Each of these elements can be expressed in one of 2 ways:

$$\begin{cases} 2^{2k}a, k \in \mathbb{N} \quad \text{when } a \equiv 1 \pmod 3 \\ 2^{2k-1}a, k \in \mathbb{N} \quad \text{when } a \equiv 2 \pmod 3 \end{cases}$$

because an even power of 2 is congruent to $1 \pmod 3$ and an odd power to $2 \pmod 3$.

Let A be the set of all even numbers congruent to $1 \pmod 3$ and let B be the set of all odd numbers. If $x \in A$ then $(x-1)$ is odd and a multiple of 3. Hence $(x-1)/3 \in B$. We now have the following:

Proposition 3.1. *The mapping*

$$\theta : A \to B : x \to (x-1)/3 \tag{1}$$

is a bijection.

Proof. First, we show that θ is injective. Suppose that $x, y \in A$ and that $\theta(x) = \theta(y)$.
Thus, $(x-1)/3 = (y-1)/3$.
Therefore $x = y$.
Hence, each element in A maps to a distinct element in B.
Next we show that θ is surjective.
Suppose there exists $m \in B$.
Then $3m+1$ is even and congruent to 1 (mod 3).
Therefore $3m+1 \in A$.
Now $\theta(3m+1) = m$.
Hence, each element in B is the image of an element in A.
It follows that θ is a bijection. $\square$

Because, in Section 2, we have partitioned the set of even numbers into an infinite number of geometric sequences of the form $\{2^k a, k \in \mathbb{N}\}$ where a is an odd number, the elements of set A are distributed among those sequences for which a is congruent to 1 (mod 3) or 2 (mod 3). See observations 3, 4 and, 5 in Section 2.

4 Constructing the hierarchy of sequences

1. Begin with 1 and construct the geometric sequence using $u_{n+1} = u_n \times 2$ to yield the sequence $\{1, 2, 4, 8, 16, \dots\}$

2. Every alternate even term of this sequence is congruent to 1 (mod 3) being of the form $2^{2k}, k \in \mathbb{N}$.

3. Apply the mapping θ to each of the terms congruent to 1 (mod 3) to generate an odd number which shall be the head of a new geometric sequence, i.e. $\theta(2^{2k}) = (2^{2k}-1)/3$. This produces the odd numbers $1, 5, 21, 85, 341, \dots$

4. For each of these odd numbers a — except 1 whose sequence is already generated — generate the geometric sequence $\{u_1 = a,\ u_{n+1} = u_n \times 2,\ n \in \mathbb{N}\}$ with a as head.

5. For each new sequence whose head is congruent to 1 (mod 3) or 2 (mod 3) apply the mapping θ to those terms which are even and congruent to 1 (mod 3).

6. Continue this process indefinitely.

In effect, that subset of the set A in Section 3 which is contained in each generated sequence whose head is congruent to 1 (mod 3) or 2 (mod 3) is mapped to a subset of the set of odd numbers. The set A is built up sequence by sequence.

Example 4.1. *Take the odd number* 5*, produced above by the even number* 16*. It generates the sequence* $\{5, 10, 20, 40, 80, 160, \dots\}$*. The even number* 10 *produces the odd number* 3*; the even number* 40 *produces the odd number* 13*; the even number* 160 *produces the odd number* 53 *and so on. Each of these odd numbers generates a geometric sequence with the odd number as head.*

5 Some patterns to the odd numbers generated

The odd numbers produced by any given sequence follow a recursive pattern. Let m be an even number congruent to 1 (mod 3) in a sequence. The number m produces the odd number $(m-1)/3$. The next number in the sequence congruent to 1 (mod 3) is $4m$. This produces the odd number $(4m-1)/3$. This can be rewritten in the form $4\big((m-1)/3\big)+1$ so we have the recursive pattern

$$u_{n+1} = 4u_n + 1 \tag{2}$$

for the odd numbers generated.

The pattern can be adapted to ascertain whether any given odd number is the first odd number to be generated from a sequence. If, on subtracting 1 and dividing by 4, we get another odd number then the odd number is not the first generated. If we get an even number then it is the first or if there is a remainder on dividing by 4 then it is also the first.

Example 5.1. *The odd number* 561 *is the first odd number generated from a sequence since the quotient on dividing by* 4 *is even. We can check that it is generated by the even number* 1684 *in the sequence whose head is* 421*. The odd number* 563 *is the first odd number generated from a sequence since there is a remainder on dividing by* 4*. We can check that it is generated by the even number* 1690 *in the sequence whose head is* 845*. The odd number* 565 *is not the first odd number generated from a sequence since the quotient on dividing by* 4 *is odd. We can check that it is generated by the even number* 1696 *in the sequence whose head is* 53 *and that the first odd number generated is* 35*.*

Because of the recursive pattern, the congruence of the odd numbers produced cycles through the values $0, 1, 2$ in some order depending on the congruence of the first odd number produced. Let b be the first odd number produced from some sequence. The next one is $4b+1$ and since $4b$ is congruent to b (mod 3), $4b+1$ is congruent to $(b+1)$ (mod 3). Thus, the cycle goes

$$\begin{cases} 2,0,1 & \text{when } b \equiv 2 \pmod 3 \\ 0,1,2 & \text{when } b \equiv 0 \pmod 3 \\ 1,2,0 & \text{when } b \equiv 1 \pmod 3 \end{cases}$$

Definition 5.1. *Let us define the even number congruent to* $1 \pmod 3$ *as the* ***parent*** *and the sequence that contains it as the* ***parent sequence*** *and the odd number produced by applying the mapping* θ (1) *to the parent as the* ***child*** *and the sequence which has the child as its head as the* ***child sequence****.*

The heads of the child sequences produced by the parents in the parent sequence can be pre-determined by the head of the parent sequence. Let a be the head of the parent sequence. When a is congruent to 2 (mod 3), the first parent is $2a$ and the first child is $(2a-1)/3$. Hence, the children are given recursively from (2) by:

$$u_{n+1} = 4u_n + 1, \quad u_1 = (2a-1)/3$$

Example 5.2. *Consider the parent sequence* $\{5, 10, 20, 40, \ldots\}$*. Its head* 5 *is congruent to* $2 \ (\text{mod } 3)$*. The heads of the child sequences produced by this parent sequence are* $3, 13, 53, \ldots$

When a is congruent to 1 (mod 3), the first parent is $4a$ and the first child is $(4a-1)/3$. Hence, the children are given recursively by:

$$u_{n+1} = 4u_n + 1, \quad u_1 = (4a-1)/3$$

Example 5.3. *Consider the parent sequence* $\{7, 14, 28, 56, 112, \ldots\}$*. Its head* 7 *is congruent to* $1 \pmod 3$*. The heads of the child sequences produced by this parent sequence are* $9, 37, 149, \ldots$

We can solve the recurrence relation (2) to obtain

$$u_n = 4^{n-1}u_1 + \frac{4^{n-1}-1}{3}$$

When $a \equiv 1 \pmod 3$ then $u_1 = \dfrac{4a-1}{3}$. Inserting this and simplifying we get

$$u_n = \frac{4^n a - 1}{3} \quad \text{when } a \equiv 1 \pmod 3 \tag{3}$$

When $a \equiv 2 \pmod 3$ then $u_1 = \dfrac{2a-1}{3}$. Inserting this and simplifying we get

$$u_n = \frac{4^{n-1}.2a - 1}{3} \quad \text{when } a \equiv 2 \pmod 3 \tag{4}$$

In both Equations (3) and (4) the value of n in the index of the power of 4 tells us the position of the child among the offspring of the parent sequence with head a. Given any random odd number we can use either of (3) or (4) to determine the head of its parent sequence and its ordinal position.

Example 5.4. *Which child of its parent sequence is* 277 *and what is the head of the parent sequence?*

$$\begin{aligned} \textit{Let } 277 &= \frac{4^n a - 1}{3} \\ 832 &= 4^n a \\ &= 4^3 \times 13 \end{aligned}$$

Thus, 277 *is the third child of the parent sequence whose head is* 13*.*

We now show that every odd natural number is a child of a parent sequence.

Proposition 5.1. *Every odd natural number is a child of a parent sequence.*

Proof. Consider $\dfrac{4^n a - 1}{3}$, where a is odd and $a \equiv 1 \pmod 3$. Since $4^n \equiv 1 \pmod 3$ and $a \equiv 1 \pmod 3$ the numerator is congruent to $0 \pmod 3$. It is also an odd number. Hence, $\dfrac{4^n a - 1}{3}$ is odd. A similar argument applies to $\dfrac{4^{n-1}.2a - 1}{3}$. Let q be an odd natural number. Then q can be written in the form $\dfrac{4^n a - 1}{3}$ or $\dfrac{4^{n-1}.2a - 1}{3}$ for some value of a and of n. Hence, q is a child of the parent sequence whose head is a. $\square$

6 Levels and level numbers

We started with the sequence $\{1, 2, 4, 8, 16, \dots\}$. Let us designate this sequence as being at Level 1. Level 1 contains one infinite geometric sequence.

The infinite number of sequences generated by the odd numbers produced by those even numbers congruent to 1 (mod 3) contained in the first sequence we designate as being at Level 2. Level 2 contains an infinite number of infinite geometric sequences.

Those sequences which are generated from the sequences at Level 2 we designate as being at Level 3. Level 3 and each subsequent Level contain a countably infinite family of infinite geometric sequences.

Every sequence in the hierarchy is at a certain level as are the individual elements of that sequence. Thus, we have the following definition:

Definition 6.1. *The **Level** of a number x is the count of the heads of the sequences which must be traversed in order to move from x to the number 1. The count includes the number 1 and the head of the sequence containing x. The Level of a sequence is the same as the Level of any element of that sequence.*

Example 6.1. *The number 18 is at Level 7. The heads of the sequences traversed are*

$$9, 7, 11, 17, 13, 5, 1.$$

The sequence $\{9, 18, 36, \dots\}$ is also at Level 7.

Definition 6.2. *Let us define the **antecedents** of a number x as the list of the heads of the sequences which must be traversed in order to move from x to the number 1.*

Figure 1 shows the output from code written in the Python programming language to calculate the level and antecedents of a specified number. Not only can

```
To find the level and antecedents of a number:

Type any positive whole number: 1122

1122 is at level 14
Antecedents of 1122: [561, 421, 79, 119, 179, 269, 101, 19, 29, 11, 17, 13, 5, 1]
```

Figure 1: A screenshot showing the level and antecedents of a specified number.

we use the hierarchy of sequences to obtain the level and antecedents of any number but we can also use it to move from any number to any other number as shown in Figure 2. Note that in Figure 2 the path between the two numbers x and y say, contains two adjacent even numbers. These two are elements of a sequence whose head is the first common antecedent of x and y when traversing upwards. In addition we can find a number at a specified level (Figure 3).

7 Interpreting the Collatz algorithm

The Collatz algorithm tells us how to move from any given number through the hierarchy of geometric sequences to reach the number 1 at the head of the topmost sequence of the hierarchy.

```
***** Travel from any number to any other number *****

Starting number: ? 561

Finishing number: ? 563

561 is at level 14
563 is at level 14

Outbound journey from 561 to 563
[561, 421, 79, 238, 952, 317, 845, 563]

Return journey from 563 to 561
[563, 845, 317, 952, 238, 79, 421, 561]
```

Figure 2: A screenshot showing the path between two numbers.

```
***** To find a number at a given level *****

Type the level number you require ? 42

The number 3569835538891 is at level 42
```

Figure 3: A screenshot showing a number at a specified level.

The instruction to divide an even number by 2 simply means "Move backwards towards the head of the sequence containing the even number".

The instruction to multiply an odd number by 3 and add 1 means "Move up from the odd number to its parent, an even number, which lies on a sequence one level above the sequence containing the odd number".

Given any even number we can determine (a) the head of the sequence which contains it, (b) its level number, (c) its antecedents.

Given any odd number we can determine (a) its parent, (b) the head of its parent sequence, (c) its position among the children of its parent sequence, (d) its level number, (e) its antecedents.

No odd number exists in isolation. Each is connected to its parent, an even number congruent to $1 \pmod 3$, through the mapping θ (1) and the parent is connected to the head of the parent sequence through the Collatz algorithm for even numbers. Thus, we associate a triple of numbers with each odd number — (child, parent, head of the parent sequence). Consider the triple (x_1, y_1, z_1). Then, z_1 becomes the child in the triple (z_1, y_2, z_2). Next, z_2 becomes the child in (z_2, y_3, z_3). This process continues all the way up to the triple $(1, 4, 1)$ at the top level. The latter triple stops the ascent from one level to the level above it. Instead, it loops around indefinitely.

Definition 7.1. *Let us define C_e as the operation of dividing an even number by* 2 *repeatedly until we get an odd number and C_o as the operation of multiplying an odd number by* 3 *and adding* 1.

The operation C_e starts with an even number and produces an odd number and both are at the same Level ℓ. The operation C_o starts with an odd number at Level ℓ and produces an even number at Level $\ell - 1$.The compound operation $C_o \circ C_e$ starts with an even number at Level ℓ and produces an even number at Level $\ell - 1$. Likewise the compound operation $C_e \circ C_o$ starts with an odd number at Level ℓ and produces an odd number at Level $\ell - 1$. Thus, we have the following:

Proposition 7.1. *Repeated use of $C_o \circ C_e$ reduces the Level number to one. Likewise, repeated use of $C_e \circ C_o$ achieves the same result.*

Proof. Suppose we start with an even number at Level ℓ. Each application of $C_o \circ C_e$ reduces ℓ by 1. Ultimately it reaches 1.

Now, suppose we start with an odd number at some Level ℓ. As before, each application of $C_e \circ C_o$ reduces ℓ by 1 and ultimately it reaches 1. □

8 Accounting for the rise and fall of the antecedents of a number

If we consider the head of a parent sequence and map it to the heads of the child sequences generated by the parent sequence, we get a one to many mapping. But if we confine the mapping to the head of the first child sequence we get a one to one mapping. The head of the parent sequence is an odd number congruent to 1 (mod 3) or to 2 (mod 3) as described in Section 4.

Consider the odd number congruent to 2 (mod 3). It can be expressed in the form $6n - 1$. The first even number in the sequence generated by $6n - 1$ that is congruent to 1 (mod 3) is $2(6n - 1)$. This even number produces its child, the odd number $4n - 1$. Now $6n - 1 > 4n - 1 \; \forall \; n \in \mathbb{N}$ so the first child is always smaller than the head of the parent sequence. Subsequent children are all bigger due to the recursive formula $u_{n+1} = 4u_n + 1$ from Section 5.

Now consider the odd number congruent to 1 (mod 3). It can be expressed in the form $6n - 5$. The first even number in the sequence generated by $6n - 5$ that is congruent to 1 (mod 3) is $4(6n - 5)$. This even number produces its child, the odd number $8n - 7$. Now $6n - 5 < 8n - 7 \; \forall \; n > 1$ so the first child is always bigger than the head of the parent sequence for $n > 1$. Subsequent children are all bigger too.

It follows that the heads of child sequences are all bigger than the head of a parent sequence except when the head of the parent sequence is congruent to 2 (mod 3) and the child sequence is the first one. It is this that causes the antecedents of a number to fluctuate up and down. We shall illustrate with an example.

Example 8.1. *The antecedents of the number* 29 *are* $11, 17, 13, 5, 1$. *Looking at this in the opposite direction so that the heads of the parent sequences come first we have* $1, 5, 13, 17, 11$. *The head of the first parent sequence is* 1 *which is congruent to* $1 \pmod 3$ *and* 5 *is the head of its second child sequence so* $5 > 1$ *as expected. Taking* 5 *as the head of the parent sequence,* 13 *is the head of its second child sequence and* $13 > 5$ *as expected. Taking* 13 *as the head of the parent sequence,* 17 *is the head of its first child sequence so* $17 > 13$ *as expected since* 13 *is congruent to* $1 \pmod 3$. *Taking* 17 *as the head of the parent sequence,* 11 *is the head of its first child sequence so* $11 < 17$ *as expected since* 17 *is congruent to* $2 \pmod 3$. *Taking* 11 *as the head of the parent sequence,* 88 *is the second parent and* 29 *is its child so* $29 > 11$ *as expected.*

9 A related hierarchical structure

To illustrate the difficulty with the Collatz conjecture we look at a related algorithm which is equivalent to the Collatz algorithm for negative integers but applied to the natural numbers. We construct a hierarchical structure of the same geometric sequences as before, starting with $\{1, 2, 4, 8, 16, \dots\}$. Instead of making the even elements which are one more than a multiple of 3 be the parent elements we make those even elements which are one less than a multiple of 3 be the parents. Hence, it is the elements $2, 8, 32, \dots$ of the first sequence that are the parents. We generate each child by adding one to the parent and dividing by three to get $1, 3, 11, \dots$. The inverse operation to ascend the hierarchy is to multiply by 3 and subtract 1.

The properties of this other structure are very similar to those we developed already. The heads of the child sequences of a parent sequence follow a recursive pattern as before and each child can be expressed in terms of the head of the parent sequence as before. Every odd natural number can be expressed in a form similar to Proposition 5.1 so that every odd natural number is a child of a parent sequence.

Despite all these similarities the natural numbers split into 3 separate hierarchies and in only one of these can the algorithm return numbers to 1. In another, numbers can only return to cycle through the heads 5 and 7 indefinitely — more fully, through $5, 14, 7, 20, 5$. In the third hierarchy the numbers can only return to cycle through $17, 25, 37, 55, 41, 61, 91$. The reason for this is that one of the antecedents of a number turns out to be that number itself. This happens to each of the numbers listed here. Despite the result corresponding to Proposition 5.1 being true, in this instance it is no guarantee that all odd natural numbers can return to the head of the first sequence.

10 Conclusion

If we consider the even numbers $2, 4, 6, 8, 10, 12, \ldots$ we see that they are written in an orderly way. But if we write the same numbers in the same order in terms of powers of two, $2^1, 2^2, 2^1 \times 3, 2^3, 2^1 \times 5, 2^2 \times 3, \ldots$, they appear to be much less orderly. It suggests that they can be split up as follows:

$$\begin{aligned}
&2^1, 2^2, 2^3, \ldots \\
&2^1 \times 3, 2^2 \times 3, 2^3 \times 3, \ldots \\
&2^1 \times 5, 2^2 \times 5, 2^3 \times 5, \ldots
\end{aligned}$$

This is what we did in Section 2, partitioning both the even numbers and the natural numbers into infinite geometric sequences with common ratio two.

In Section 3 we set up a bijection between the set of even numbers congruent to $1 \pmod 3$ and the set of odd numbers by using the inverse of the Collatz algorithm for odd numbers. This enabled us in Section 4 to generate the heads of new child sequences from the even numbers congruent to $1 \pmod 3$ in the current sequence. The child sequences were then generated from the heads by using the inverse of the Collatz algorithm for even numbers. In this way the hierarchy of sequences was built up.

Following on from this in Section 5 we showed that the heads of the child sequences followed a recursive pattern and hence, we were able to ascertain which child was the first child of its parent sequence and also what was the position of a given child among the offspring of the parent sequence.

In Section 6 we showed how the sequences were ordered within the hierarchy by defining the level of both a sequence and its elements within the hierarchy. We demonstrated that the antecedents and level of any number can be obtained; we showed that it is possible to move from any number to any other number within the hierarchy and that, given a particular level, we can obtain a number situated at that level.

Importantly, in Section 7 we proved that both parts of the Collatz algorithm applied in succession to either an even number or an odd number reduce the Level number by one and repeated applications reduce the Level to Level 1.

In Section 8 we gave an explanation of why the antecedents of a number rise and fall in magnitude.

In this paper we have shown how the natural numbers can be partitioned into an infinite number of infinite geometric sequences and that these sequences can be organised into a hierarchy. We provided the mechanism for creating the hierarchy and for ascending from any sequence to the head of the topmost sequence. But this alone does not confirm the Collatz conjecture as the example in Section 9 illustrates. It is necessary to show that the hierarchy stays intact and does not split into two or more hierarchies as happened in the example in Section 9. To keep it intact the head of any sequence cannot appear again among its antecedents or, equivalently, among its descendants.

References

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