Ratio Mathematica Volume 53, 2024

$(\odot, \vee)$-Derivations on EMV-algebras and EBL-algebras

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Abstract

In the paper, we will study $(\odot, \vee)$-derivations on EMV-algebras and EBL-algebras. Firstly, we define the notions of $(\odot, \vee)$-derivations on EMV-algebras and EBL-algebras, and we also introduce some classes of $(\odot, \vee)$-derivations on EMV-algebras and EBL-algebras, such as principal $(\odot, \vee)$-derivations and isotone $(\odot, \vee)$-derivations. In addition, we construct some lattice structures of $(\odot, \vee)$-derivations on EMV-algebras and give a classification of EBL-algebras with $(\odot, \vee)$-derivations.

Keywords: EMV-algebra; derivation; principal derivation; isotone derivation; EBL-algebra

2020 AMS subject classifications: 06D99 [1]

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[1]Received on November 20, 2024. Accepted on December 31, 2024. Published on December 31, 2024. DOI:10.23755/rm.v53i0.1661.

1 Introduction

MV-algebras were original introduced by Chang (1958) as algebraic structures of many-valued logic. Nowadays, this structure has been extended to BCK-algebras, hoops, quasi-algebras, etc. In addition, Georgescu and Iorgulescu (2001) introduced pseudo MV-algebras, which is a non-commutative extension of MV-algebras. Dvurecenskij and Zahiri (2019) defined EMV-algebras, which extended the notions of MV-algebras.

Hajek (2013) defined the notion of BL-algebras. Particularly, let A be an BL-algebra, for all $x \in A$, if $x^{--} = x$, then A is an MV-algebra. Similarly, Liu (2020) defined the notion of EBL-algebras.

The notion of derivations was defined by Leibniz-rule. Szasz (1975) defined the notion of derivations on a lattice (L,$\vee$,$\wedge$). In addition, Kolchin (1973) defined differential algebra, which is an algebraic structure with a derivation. Differential algebra has many generalizations, such as fields (Ritt 1932), commutative algebras (Singer 2009), non-commutative algebras (Guo and Keigher 2008) and MV-algebras (Hamal 2019). Alshehri (2010) defined the notion of derivations on MV-algebras A, which is a map d satisfying $d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y))$ for all $x, y \in A$. Torkzadeh and Abbasian (2013) introduced $(\odot, \vee)$-derivations on BL-algebras. Recently, Zhao, Gan and Yang (2024) introduced $(\odot, \vee)$-derivations on MV-algebras. $(\odot, \vee)$-derivations are indispensable tools in algebraic systems. The notion of $(\odot, \vee)$-derivations are helpful for studying algebraic structures and properties in algebraic systems. In the following, we will further study the $(\odot, \vee)$-derivations of EMV-algebras and EBL-algebras.

The paper is constructed as follows. In Section 2, we will introduce some foundational notions and properties related to EMV-algebras and EBL-algebras. In Section 3, we study $(\odot, \vee)$-derivations on EMV-algebras. In addition, we study some lattice structures of $(\odot, \vee)$-derivations on EMV-algebras. In Section 4, we introduce some important results of $(\odot, \vee)$-derivations on EBL-algebras. We also prove that ($\odot$, $\vee$)-derivations preserves ideals and filters.

2 Preliminaries

In this section, we will introduce some notions and statements on EMV-algebras and lattices which will be used in the following sections.

Definition 2.1. *(Dvurecenskij and Zahiri 2019) An EMV-algebra is an algebra* $(A, \vee, \wedge, \oplus, 0)$ *of type* $(2, 2, 2, 0)$ *satisfies the following conditions:*
(EMV1) $(A, \vee, \wedge, 0)$ *is a distributive lattice with lower bound 0;*
(EMV2) $(A, \oplus, 0)$ *is a commutative monoid;*
(EMV3) for each $m, n \in I(A)$ *such that* $m \leq n$*, there exists* $\lambda_{m,n}(x)$ *for each* $x \in$

$[m, n]$, and $([m, n], \vee, \wedge, \oplus, \lambda_{m,n}(x), m, n)$ is a MV-algebra, where $\lambda_{m,n}(x) = min\{z \in [m, n] | x \oplus z = n\}$ and $I(A)$ is the set of all idempotent elements of A;
(EMV4) $(A, \vee, \wedge, \oplus, 0)$ has enough idempotent elements, that is, for each $x \in A$, there exists $a \in I(A)$ such that $x \leq a$.

We will denote $\lambda_{0,a}$ by λ_a and denote the set of all idempotent elements of an algebra A by $I(A)$.

Proposition 2.1. *(Dvurecenskij and Zahiri 2019) Let $(A, \vee, \wedge, \oplus, 0)$ be an EMV-algebra. For all $a \in I(A)$ such that each $x, y \in [0, a]$, the following properties hold:*
(1) $\lambda_a(a) = 0$;
(2) $a \oplus x = a$;
(3) $x \odot y = \lambda_a(\lambda_a(x) \oplus \lambda_a(y))$;
(4) $\lambda_a(x) \oplus x = a$;
(5) $x \odot \lambda_a(x) = 0$;
(6) $x \vee y = (x \odot \lambda_a(y)) \oplus y$;
(7) $x \wedge y = x \odot (\lambda_a(x) \oplus y)$.

For all $a \in I(A)$ such that each $x, y \in [0, a]$, we will define a partial order $\leq$ on EMV-algebra $(A, \vee, \wedge, \oplus, 0)$ as follows:

$$x \leq y \Leftrightarrow \lambda_a(x) \oplus y = a.$$

In addition, we shall denote $x^0 = a, x^1 = x, ..., x^n = x^{n-1} \odot x$.

Lemma 2.1. *(Dvurecenskij and Zahiri 2019) Let A be an EMV-algebra. For all $x, y \in [0, a]$, the following statements are equivalent:*
(1) $x \leq y$;
(2) $\lambda_a(x) \oplus y = a$;
(3) $x \odot \lambda_a(y) = 0$;
(4) $y = x \oplus (y \odot \lambda_a(x))$;
(5) There is an element $z \in [0, a]$ such that $x \oplus z = y$.

Lemma 2.2. *(Dvurecenskij and Zahiri 2019) Let A be an EMV-algebra. For all $x, y, z \in [0, a]$, the following statements hold:*
(1) If $x \vee y$, $x \oplus y \in [0, a]$, then $x \odot y \leq x \wedge y \leq x, y \leq x \vee y \leq x \oplus y$;
(2) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$;
(3) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
(4) $x \leq y$ iff $\lambda_a(y) \leq \lambda_a(x)$;
(5) If $x \leq y$, then $x \wedge z \leq y \wedge z$; $x \vee z \leq y \vee z$; $x \oplus z \leq y \oplus z$; $x \odot z \leq y \odot z$;
(6) If $x \oplus y = 0$, we have $x = y = 0$; If $x \odot y = a$, we have $x = y = a$.

Lemma 2.3. *(Dvurecenskij and Zahiri 2019) Let A be an EMV-algebra. For all $x, y \in [0, a]$, the following statements are equivalent:*
(1) $x \in B(A)$;
(2) $x \oplus x = x$;
(3) $x \odot x = x$;
(4) $\lambda_a(x) \in B(A)$;
(5) $x \oplus y = x \vee y$;
(6) $x \odot y = x \wedge y$.

Definition 2.2. *(Dvurecenskij and Zahiri 2019) Let A be an EMV-algebra and I be a nonempty subset of A. I is called an ideal if for $x, y \in A$, the following conditions hold:*
(1) $0 \in I$;
(2) $x \oplus y \in I$ for all $x, y \in I$;
(3) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$.

Definition 2.3. *(Cignoli, d'Ottaviano and Mundici 2000) Let A be a lattice and I be a nonempty subset of A. I is called a lattice ideal if for $x, y \in A$, the following conditions hold:*
(1) $0 \in I$;
(2) $x \vee y \in I$ for all $x, y \in I$;
(3) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$.

A subset I of a poset A is called a downset if for all $x \in I$, $y \in A$ and $y \leq x$ implies $y \in I$.

Definition 2.4. *(Liu 2020) An EBL-algebra is an algebra $(A, \vee, \wedge, \odot, 0)$ of type $(2, 2, 2, 0)$ satisfying the following conditions:*
(EBL1) $(A, \vee, \wedge, 0)$ is a distributive lattice with lower bound 0;
(EBL2) $(A, \odot, 0)$ is a commutative semigroup;
(EBL3) for each $m, n \in I(A)$ such that $m \leq n$, there exists $x \overset{m,n}{\to} y$ for each $x, y \in [m, n]$, and $([m, n], \vee, \wedge, \odot, \overset{m,n}{\to}, m, n)$ is a BL-algebra, where $x \overset{m,n}{\to} y = \vee\{z \in [m, n] | x \odot z \leq y\}$ and $I(A)$ is the set of all idempotent elements of A;
(EBL4) $(A, \vee, \wedge, \odot, 0)$ has enough idempotent elements, that is, for each $x \in A$, there exists $a \in I(A)$ such that $x \leq a$.

Let $(A, \vee, \wedge, \odot, 0)$ be an EBL-algebra. For all $a \in I(A)$ such that each $x, y \in [0, a]$, we define a binary relation $\leq$ by $x \leq y \Leftrightarrow x \overset{a}{\to} y = a$. In addition, we define $x^{-a} = x \overset{a}{\to} 0$, $x^{--a} = (x \overset{a}{\to} 0) \overset{a}{\to} 0$, $x^{---a} = ((x \overset{a}{\to} 0) \overset{a}{\to} 0) \overset{a}{\to} 0$ and denote $x^0 = a, x^1 = x, ..., x^n = x^{n-1} \odot x$.

Proposition 2.2. *(Liu 2020) Let $(A, \vee, \wedge, \odot, 0)$ be an EBL-algebra. For all $a \in I(A)$ such that each $x, y, z \in [0, a]$, the following properties hold:*

(1) If $x \leq y$, then $y^{-a} \leq x^{-a}$;
(2) $x \odot y = 0$ iff $x \leq y^{-a}$;
(3) $x \leq x^{--a}$; $x^{---a} = x^{-a}$;
(4) $x \odot y \leq z$ iff $x \leq y \overset{a}{\rightarrow} z$;
(5) $x \wedge y = x \odot (x \overset{a}{\rightarrow} y)$;
(6) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$; $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
(7) If $x \in I([0, a])$, then $x^{--a} = x$:
(8) $x \odot y \leq x, y$.

Definition 2.5. *(Liu 2020) Let A be an EBL-algebra and I be a nonempty subset of A. I is called an ideal if for $x, y \in A$, the following conditions hold:*
(1) $0 \in I$;
(2) $y \in I$ and $x \leq y$ imply $x \in I$;
(3) $(x \overset{a}{\rightarrow} 0) \overset{a}{\rightarrow} y \in I$ for each $x, y \in I$, $a \in I(A)$.

Definition 2.6. *(Liu 2020) Let A be an EBL-algebra and F be a nonempty subset of A. F is called an filter if for $x, y \in A$, the following conditions hold:*
(1) $x \in F$ and $x \leq y$ imply $y \in F$;
(2) if $x, y \in F$, then $x \odot y \in F$.

Denote $N = \{n \geq 1 | n \text{ is an integer}\}$ by N and denote $\overbrace{x \odot x \odot \cdots x}^{n-times}$ by x^n.

3 ($\odot$, $\vee$)-derivations on EMV-algebras

In this section, we will introduce some notions and results of ($\odot$, $\vee$)-derivations on EMV-algebras, and characterize some classes of ($\odot$, $\vee$)-derivations on EMV-algebras. In addition, we also give some properties of fixed point sets of ($\odot$, $\vee$)-derivations. Finally, some lattice structures of ($\odot$, $\vee$)-derivations are characterized.

Definition 3.1. *Let A be an EMV-algebra. For all $a \in I(A)$ such that each $x, y \in [0, a]$, a map d is called an ($\odot$, $\vee$)-derivation on A if it satisfies the following condition:*

$$d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)).$$

In the following, we shall denote an arbitrary idempotent element on an EMV-algebra A by a.

For all $x \in [0, a]$, we define the identity map $Id_A(x) = x$ and the zero map $0_A(x) = 0$. Since $Id_A(x \odot y) = x \odot y = (x \odot y) \vee (x \odot y) = (Id_A(x) \odot y) \vee (x \odot Id_A(y))$ and $0_A(x \odot y) = 0 = 0 \vee 0 = (0_A(x) \odot y) \vee (0_A(y) \odot x)$, we get that Id_A and 0_A are ($\odot$, $\vee$)-derivations on A.

Example 3.1. *Let* $b \in [0, a]$. *For all* $x \in [0, a]$, *we define the map* $d_b : [0, a] \to [0, a]$ *by* $d_b := b \odot x$. *From* $d_b(x \odot y) = x \odot y \odot b = ((x \odot b) \odot y) \vee (x \odot (b \odot y)) = (d_b(x) \odot y) \vee (x \odot d_b(y))$, *it follows that* d_b *is an (*$\odot$*,* $\vee$*)-derivation, which is called a principal (*$\odot$*,* $\vee$*)-derivation.*

We will denote the set of all ($\odot$, $\vee$)-derivations on A by Der(A) and the set of all principal ($\odot$, $\vee$)-derivations on A by PDer(A).

In the following proposition, we will give some important properties of ($\odot$, $\vee$)-derivations which be employed in this paper.

Proposition 3.1. *Let* A *be an EMV-algebra and* $n \in N$. *For all* $d \in$*Der(*A*), the following statements hold for all* $x, y \in [0, a]$*:*
(1) $d(0) = 0$*;*
(2) $d(x) \odot \lambda_a(x) = x \odot d(\lambda_a(x)) = 0$*;*
(3) $d(x) \leq x$*;*
(4) $d(x) = d(x) \vee (x \odot d(a))$ *and* $x \odot d(a) \leq d(x)$*;*
(5) $d(\lambda_a(x)) \leq \lambda_a(x) \leq \lambda_a(d(x))$*;*
(6) If $d(x) \vee d(y)$, $d(x) \oplus d(y) \in [0, a]$, *then* $d(x) \odot d(y) \leq d(x \odot y) \leq d(x) \vee d(y) \leq d(x) \oplus d(y)$*;*
(7) $d(x^n) = x^{n-1} \odot d(x)$*;*
(8) $(d(x))^n \leq d(x^n)$*;*
(9) If I *is a downset of* A, *we have* $d(I) \subseteq I$, *where* $d(I) = \{d(x) | x \in I\}$*;*
(10) If $y \leq x$ *and* $d(x) = x$, *we have* $d(y) = y$.

Proof. (1) By the notion of ($\odot$, $\vee$)-derivations, we have $d(0) = d(0 \odot 0) = (d(0) \odot 0) \vee (0 \odot d(0)) = 0$.

(2) By (1) and Proposition 2.1 (5), we have $0 = d(0) = d(x \odot \lambda_a(x)) = (d(x) \odot \lambda_a(x)) \vee (x \odot d(\lambda_a(x)))$. It implies that $d(x) \odot \lambda_a(x) = x \odot d(\lambda_a(x)) = 0$.

(3) From $d(x) \odot \lambda_a(x) = 0$, it implies that $d(x) \leq x$ by Lemma 2.1.

(4) We have $d(x) = d(x \odot a) = (d(x) \odot a) \vee (x \odot d(a)) = d(x) \vee (x \odot d(a))$. It implies that $x \odot d(a) \leq d(x)$.

(5) By (3), we have $d(\lambda_a(x)) \leq \lambda_a(x)$ and $d(x) \leq x$. It implies that $\lambda_a(x) \leq \lambda_a(d(x))$ by Lemma 2.2 (4).

(6) By (3), we have $d(x) \odot d(y) \leq x \odot d(y)$ and $d(x) \odot d(y) \leq d(x) \odot y$. So $d(x) \odot d(y) \leq (d(x) \odot y) \vee (x \odot d(y)) = d(x \odot y)$. In addition, by Lemma 2.2 (1), we have $d(x) \odot y \leq d(x)$ and $x \odot d(y) \leq d(y)$. So $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)) \leq d(x) \vee d(y)$. Finally, by Lemma 2.2 (1) again, we have $d(x) \vee d(y) \leq d(x) \oplus d(y)$.

(7) In the following, we will make a mathematical induction for n:
(i) For $n = 1$, we have $d(x^1) = d(x) = a \odot d(x) = x^{1-1} \odot d(x)$;
(ii) For $n = 2$, we have $d(x^2) = d(x \odot x) = (d(x) \odot x) \vee (x \odot d(x)) = x \odot d(x) =$

$x^{2-1} \odot d(x)$;
(iii) Assume that $d(x^n) = x^{n-1} \odot d(x)$ holds. We have $d(x^{n+1}) = d(x^n \odot x) = (d(x^n) \odot x) \vee (x^n \odot d(x)) = (x^{n-1} \odot d(x) \odot x) \vee (x^n \odot d(x)) = x^n \odot d(x)$. Hence, $d(x^n) = x^{n-1} \odot d(x)$ holds.

(8) By (3) and Lemma 2.2 (5), we have $(d(x))^{n-1} \leq x^{n-1}$. In addition, by (7), we have $(d(x))^n = (d(x))^{n-1} \odot d(x) \leq x^{n-1} \odot d(x) = d(x^n)$.

(9) Let I be a downset of A and $y \in d(I)$. There exists $b \in I$ such that $y = d(b)$. By the notion of a downset, we have $y = d(b) \leq b \in I$. Hence, $d(I) \subseteq I$.

(10) Let $y \leq x$ and $d(x) = x$. We have $d(y) = d(x \wedge y) = d(x \odot (\lambda_a(x) \oplus y)) = (d(x) \odot (\lambda_a(x) \oplus y)) \vee (x \odot d(\lambda_a(x) \oplus y)) = (x \odot (\lambda_a(x) \oplus y)) \vee (x \odot d(\lambda_a(x) \oplus y)) = x \odot (\lambda_a(x) \oplus y) = x \wedge y = y$.□

In the following proposition, we will prove that if d is an ($\odot$, $\vee$)-derivation on EMV-algebra A, then $d = Id_A$ iff d is surjective.

Proposition 3.2. *Let A be an EMV-algebra and $d \in$Der(A). For all $x, y \in [0, a]$, the following statements are equivalent:*
(1) d=Id_A;
(2) $d(a) = a$;
(3) $d(b) = a$ for some $b \in [0, a]$;
(4) d is surjective;
(5) d satisfies that $d(x \oplus y) = (d(x) \oplus y) \wedge (x \oplus d(y))$.

Proof. Clearly, $(1) \Rightarrow (2) \Rightarrow (3)$ and $(1) \Rightarrow (4) \Rightarrow (3)$ hold.
$(2) \Rightarrow (1)$. Let $d(a) = a$. By Proposition 3.1 (10), we have $d(x) = x$. Hence, $d = Id_A$.
$(3) \Rightarrow (2)$. Let $d(b) = a$ for some $b \in [0, a]$. By Proposition 3.1 (3), we have $a = d(b) \leq b$. It implies that $b = a$. Hence, $d(a) = a$.
$(1) \Rightarrow (5)$. Let d=Id$_A$. We have $d(x \oplus y) = x \oplus y = (x \oplus y) \wedge (x \oplus y) = (d(x) \oplus y) \wedge (x \oplus d(y))$.
$(5) \Rightarrow (2)$. Let d satisfies that $d(x \oplus y) = (d(x) \oplus y) \wedge (x \oplus d(y))$. We have $d(a) = d(a \oplus a) = (d(a) \oplus a) \wedge (a \oplus d(a)) = a$.□

Proposition 3.3. *Let A be an EMV-algebra and $d \in$Der(A). For all $x \in [0, a]$, we define a map d^u on A by*

$$d^u(x) = \begin{cases} u, & x = a, \\ d(x), & x \neq a. \end{cases}$$

If $u \in [0, a]$ such that $u \leq d(a)$, then $d^u \in$ Der(A).

Proof. For all $x, y \in [0, a]$, we consider the following cases:
Case (i): If $x = y = a$, we have $d^u(a \odot a) = u = (u \odot a) \vee (a \odot u) = (d^u(a) \odot a) \vee (a \odot d^u(a))$;
Case (ii): If $x \neq a$ and $y \neq a$, we have $d^u(x) = d(x)$, $d^u(y) = d(y)$ and $x \odot y \neq a$. It implies that

$$\begin{aligned} & d^u(x \odot y) \\ = & \, d(x \odot y) \\ = & \, (d(x) \odot y) \vee (x \odot d(y)) \\ = & \, (d^u(x) \odot y) \vee (x \odot d^u(y)). \end{aligned}$$

Case (iii): If $x \neq a$ and $y = a$, we have $d^u(x) = d(x)$ and $d^u(y) = d^u(a) = u \leq d(a)$. It implies that $x \odot d^u(a) \leq x \odot d(a) \leq d(x)$ by Proposition 3.1 (4). So

$$\begin{aligned} & d^u(x \odot y) \\ = & \, d^u(x \odot a) \\ = & \, d^u(x) \\ = & \, d(x) \\ = & \, d(x) \vee (x \odot d^u(a)) \\ = & \, (d^u(x) \odot y) \vee (x \odot d^u(a)) \\ = & \, (d^u(x) \odot y) \vee (x \odot d^u(y)). \end{aligned}$$

Hence, $d^u \in$ Der(A).□

Corolary 3.1. *Let A be an EMV-algebra and $u \in [0, a]$. For all $x \in [0, a]$, we define a map $Id_A^{(u)}$ on A by:*

$$Id_A^{(u)}(x) = \begin{cases} u, & x = a, \\ x, & x \neq a. \end{cases}$$

Then $Id_A^{(u)} \in$ Der(A).

Proof. We have $Id_A \in$ Der(A) and $u \leq a = \mathrm{Id}_A(a)$. It implies that $Id_A^{(u)} \in$ Der(A) by Proposition 3.3.□

In the following, we will study the condition when an ($\odot$, $\vee$)-derivation on an EMV-algebra A is isotone.

Definition 3.2. *Let A be an EMV-algebra and $d \in$Der(A). d is called isotone if for all $x, y \in [0, a]$ such that $x \leq y$, it implies that $d(x) \leq d(y)$.*

By Definition 3.1, we have that Id_A and 0_A are ($\odot$, $\vee$)-derivations on A. For all $x, y \in [0, a]$ such that $x \leq y$, since $Id_A(x) = x \leq y = Id_A(y)$ and $0_A(x) = 0 \leq 0 = 0_A(y)$, we get that the identity map Id_A and the zero map 0_A are isotone.

Remark 3.1. *Let d be an ($\odot$, $\vee$)-derivation on EMV-algebra A. If it satisfies that $d(x \vee y) = d(x) \vee d(y)$, then d is isotone. Indeed, for all $x, y \in [0, a]$, if $x \leq y$, there* *exists $z \in [0, a]$ such that $y = x \vee z$. It implies that $d(y) = d(x \vee z) = d(x) \vee d(z)$. That is, $d(x) \leq d(y)$.*

Proposition 3.4. *Let A be an EMV-algebra. For all $b \in [0, a]$, we have principal ($\odot$, $\vee$)-derivation d_b is isotone.*

Proof. Let $x, y \in [0, a]$ such that $x \leq y$. By Lemma 2.2 (5), we have $d_b(x) = b \odot x \leq b \odot y = d_b(y)$. Hence, d_b is isotone.□

In the following proposition, we will show that if d is an ($\odot$, $\vee$)-derivation on an EMV-algebra A with $d(a) \in B(A)$, then d is isotone iff d is principal.

Proposition 3.5. *Let A be an EMV-algebra and d ∈Der(A) with $d(a) \in B(A)$. For all $x, y \in [0, a]$, the following statements are equivalent:*
(1) d is isotone;
(2) $d(x) \leq d(a)$;
(3) $d(x) = d(a) \odot x$;
(4) $d(x \wedge y) = d(x) \wedge d(y)$;
(5) If $d(x) \vee d(y) \in [0, a]$, then $d(x \vee y) = d(x) \vee d(y)$.

Proof. (1) ⇒ (2). It is obvious.
(2) ⇒ (3). Let $d(x) \leq d(a)$. By Lemma 2.3 (6) and Proposition 3.1 (4), we have $d(x) \leq d(a) \wedge x = d(a) \odot x \leq d(x)$. Hence, $d(x) = d(a) \odot x$.
(3) ⇒ (4). Let $d(x) = d(a) \odot x$. By Lemma 2.2 (2), we have

$$\begin{aligned} & d(x \wedge y) \\ &= d(a) \odot (x \wedge y) \\ &= (d(a) \odot x) \wedge (d(a) \odot y) \\ &= d(x) \wedge d(y). \end{aligned}$$

(4) ⇒ (1). Let $d(x \wedge y) = d(x) \wedge d(y)$. If $x \leq y$, we have $d(x) = d(x \wedge y) = d(x) \wedge d(y) \leq d(y)$.
(3) ⇒ (5). Let $d(x) = d(a) \odot x$. By Lemma 2.2 (3), we have

$$\begin{aligned} & d(x \vee y) \\ &= d(a) \odot (x \vee y) \\ &= (d(a) \odot x) \vee (d(a) \odot y) \\ &= d(x) \vee d(y). \end{aligned}$$

(5) ⇒ (1). Let $d(x \vee y) = d(x) \vee d(y)$. If $x \leq y$, we have $d(x) \leq d(x) \vee d(y) = d(x \vee y) = d(y)$.□

Clearly, if d satisfies the condition in Proposition 3.5, then d is principal.

We will denote the set of all isotone ($\odot$, $\vee$)-derivations with $d(a) \in B(A)$ by IDer(A).

Corolary 3.2. *Let A be an EMV-algebra. There exists a bijection between IDer(A) and $B(A)$.*

Proof. For all $d \in$ IDer(A), we define a map f: IDer(A) $\to B(A)$ by $f(d) = d(a)$. In addition, for all $b \in [0, a]$, we also define a map g: $B(A) \to$ IDer(A) by $g(b) = d_b$. By Proposition 3.5, we have $fg : b \mapsto d_b(a) = b \odot a = b$ and $gf : d(x) \mapsto d_{d(a)}(x) = d(a) \odot x = d(x)$. Hence, f is a bijection between IDer(A) and $B(A)$.□

Proposition 3.6. *Let A be an EMV-algebra and $d \in$Der(A). For all $x, y \in [0, a]$, we have $d \in$ IDer(A) iff $d(x \odot y) = d(x) \odot d(y)$.*

Proof. Let $d \in$ IDer(A). By Proposition 3.5, we have

$$\begin{aligned} & d(x \odot y) \\ = & d(a) \odot (x \odot y) \\ = & (d(a) \odot x) \odot (d(a) \odot y) \\ = & d(x) \odot d(y). \end{aligned}$$

Conversely, we have $d(a) = d(a) \odot d(a)$. It implies that $d(a) \in B(A)$. In addition, by Lemma 2.2 (1), we have $d(x) = d(x \odot a) = d(x) \odot d(a) \leq d(a)$. By Proposition 3.5 again, we get that d is isotone. Hence, $d \in$ IDer(A).□

Corolary 3.3. *Let A be an EMV-algebra and $d \in$ IDer(A). For all $x \in [0, a]$, we have $d^2(x) = d(x)$.*

Proof. By Proposition 3.5 (3) and Proposition 3.6, we have $d(d(x)) = d(a) \odot d(x) = d(a \odot x) = d(x)$.□

Proposition 3.7. *Let A be an EMV-algebra and $d \in$ IDer(A). For all $x, y \in [0, a]$, we have $d(x) \leq y$ iff $d(x) \leq d(y)$.*

Proof. Let $d(x) \leq y$. Since d is isotone, we have $d(d(x)) \leq d(y)$. In addition, by Corollary 3.3, we have $d(d(x)) = d(x)$. Hence, $d(x) \leq d(y)$. Conversely, let $d(x) \leq d(y)$. By Proposition 3.1 (3), we have $d(x) \leq d(y) \leq y$.□

Let A be an EMV-algebra and $d \in$Der(A). We will denote the set of all fixed point of d by Fix$_d(A)$. That is, Fix$_d(A)$=$\{x \in A | d(x) = x\}$.

Proposition 3.8. *Let A be an EMV-algebra and $d \in$Der(A). Then Fix$_d(A)$ is a downset.*

Proof. By the notion of downset and Proposition 3.1 (10), we can get that Fix$_d(A)$ is a downset.□

In the following proposition, we will study the relation between lattice ideals and fixed point sets of ($\odot$, $\vee$)-derivations on EMV-algebras.

Proposition 3.9. *Let A be an EMV-algebra. For all $x, y \in [0, a]$ such that $x \vee y \in [0, a]$, if $d \in$PDer(A), we have that Fix$_d(A)$ is a lattice ideal of A.*

Proof. Let $d \in$PDer(A). For all $x \in [0, a]$, there exists $u \in [0, a]$ such that $d(x) = u \odot x$. For all $x, y \in$ Fix$_d(A)$, we have $d(x) = x$ and $d(y) = y$. It implies that $d(x \vee y) = u \odot (x \vee y) = (u \odot x) \vee (u \odot y) = d(x) \vee d(y) = x \vee y$ by Lemma 2.2 (3). So Fix$_d(A)$ is closed under $\vee$. Hence, together with Fix$_d(A)$ is a downset, we can get that Fix$_d(A)$ is a lattice ideal of A.□

We define some binary operations on Der(A). For $d, d' \in$ Der(A), $(d \vee d')(x) = d(x) \vee d'(x)$, $(d \wedge d')(x) = d(x) \wedge d'(x)$ for all $x \in A$.

Proposition 3.10. *Let A be an EMV-algebra. For all $d, d' \in$ Der(A), we have $(d \vee d') \in$ Der(A).*

Proof. Let $d, d' \in$ Der(A). For all $x, y \in A$, by Lemma 2.2 (3), we have

$$\begin{aligned}
&(d \vee d')(x \odot y)\\
&= d(x \odot y) \vee d'(x \odot y)\\
&= ((d(x) \odot y) \vee (x \odot d(y))) \vee ((d'(x) \odot y) \vee (x \odot d'(y)))\\
&= ((d(x) \odot y) \vee (d'(x) \odot y)) \vee ((x \odot d(y)) \vee (x \odot d'(y)))\\
&= ((d(x) \vee d'(x)) \odot y) \vee (x \odot (d(y) \vee d'(y)))\\
&= ((d \vee d')(x) \odot y) \vee (x \odot (d \vee d')(y)).
\end{aligned}$$

Hence, $(d \vee d') \in$ Der(A).□

Theorem 3.1. *Let A be an EMV-algebra. For all $x, y \in [0, a]$, the following statements hold:*
(1) For all $u, v \in [0, a]$, we have $d_u \vee d_v = d_{u \vee v}$ and $d_u \wedge d_v = d_{u \wedge v}$;
(2) (PDer(A), $\vee$, $\wedge$, 0_A, Id$_A$) is a lattice;
(3) For all $d, d' \in$ IDer(A), we have $d \vee d', d \wedge d' \in$ IDer(A);
(4) (IDer(A), $\vee$, $\wedge$, 0_A, Id$_A$) is a lattice.

Proof. (1) Let $u, v \in [0, a]$. For all $x \in [0, a]$, we have

$$\begin{aligned}
&(d_u \vee d_v)(x)\\
&= d_u(x) \vee d_v(x)\\
&= (u \odot x) \vee (v \odot x)\\
&= (u \vee v) \odot x\\
&= d_{u \vee v}(x).
\end{aligned}$$

Similary, we have

$$\begin{aligned}
&(d_u \wedge d_v)(x)\\
&= d_u(x) \wedge d_v(x)\\
&= (u \odot x) \wedge (v \odot x)\\
&= (u \wedge v) \odot x\\
&= d_{u \wedge v}(x).
\end{aligned}$$

(2) By (1), we have that PDer(A) is closed under $\vee$ and $\wedge$. In addition, since 0_A, $\mathrm{Id}_A \in$ PDer(A), so (PDer(A), $\vee$, $\wedge$, 0_A, Id_A) is a lattice.

(3) Let $d, d' \in$ IDer(A). For all $x, y \in [0, a]$, we have $d(a), d'(a) \in B(A)$. It follows that $(d \vee d')(a) = d(a) \vee d'(a) \in B(A)$ by Lemma 2.2 (3). Similarly, $(d \wedge d')(a) \in B(A)$. In addition, since d and d' are isotone, we have that $d \vee d'$ and $d \wedge d'$ are isotone. Hence, $d \vee d', d \wedge d' \in$ IDer(A).

(4) From (3), we have that IDer(A) is closed under $\vee$ and $\wedge$. In addition, by 0_A, $\mathrm{Id}_A \in$ IDer(A), it implies that (IDer(A), $\vee$, $\wedge$, 0_A, Id_A) is a lattice.□

4 ($\odot$, $\vee$)-derivations on EBL-algebras

For an ($\odot$, $\vee$)-derivation d_1 on an EMV-algebra A_1, we have $d(x) \leq x$ for all $x \in A_1$. But this property holds only for part $y \in A_2$ on an EBL-algebra A_2. Let $d_2 \in$ Der(A_2). There exists $z \in [0, a]$ such that $z \leq d_2(z)$. In the following, we will introduce some notions and results of ($\odot$, $\vee$)-derivations on EBL-algebras.

Definition 4.1. *Let A be an EBL-algebra. For all $a \in I(A)$ such that $x, y \in [0, a]$, a map d is called an ($\odot$, $\vee$)-derivation on A if it satisfies the following condition:*

$$d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)).$$

We shall denote the set of all ($\odot$, $\vee$)-derivations on A by Der(A) and denote an arbitrary idempotent element on an EBL-algebra A by a.

Proposition 4.1. *Let A be an linearly ordered EBL-algebra and $k \in A$. For all $x, y \in A$, we define a map d by*

$$d(x) = \begin{cases} x, & x<k, \\ 0, & x \geq k. \end{cases}$$

If $x \odot y \geq k$ when $x, y \geq k$, then $d \in$ Der(A).

Proof. We will consider the following cases:
(i) For $x, y<k$, by Proposition 2.2 (8), we have

$$\begin{aligned} &d(x \odot y) \\ &= x \odot y \\ &= (x \odot y) \vee (x \odot y) \\ &= (d(x) \odot y) \vee (x \odot d(y)). \end{aligned}$$

(ii) For $x, y \geq k$, since $x \odot y \geq k$, we have

$$\begin{aligned} &d(x \odot y) \\ &= 0 \\ &= (0 \odot y) \vee (x \odot 0) \end{aligned}$$

$= (d(x) \odot y) \vee (x \odot d(y))$.

(iii) For $x<k$ and $y \geq k$, by Proposition 2.2 (8) again, we have

$$\begin{aligned} & d(x \odot y) \\ &= x \odot y \\ &= (x \odot y) \vee (x \odot 0) \\ &= (d(x) \odot y) \vee (x \odot d(y)). \end{aligned}$$

Hence, $d \in$ Der(A).□

In the following, we will give some different results than ($\odot$, $\vee$)-derivations in section 3.

Proposition 4.2. *Let A be an EBL-algebra and $n \in N$. For all $d \in$Der(A), the following statements hold for all $x, y \in [0, a]$:*
(1) If $x \leq y$, then $d(x) \leq y^{--a}$;
(2) $d(x) \leq x^{--a}$;
(3) If $x \odot y = 0$, then $d(x) \odot y = 0$;
(4) $x \odot d(a) \leq d(x)$;
(5) If $d(a) = a$, then $d(I([0, a])) = I([0, a])$;
(6) $d(x^{-a}) \leq (d(x))^{-a}$;
(7) If $d(x) = a$, then $x^{-a} = 0$;
(8) $d(a) = a$ iff $x \leq d(x)$.

Proof. (1) By Proposition 2.2 (1) and (2), we have $x \odot y^{-a} = 0$. So $0 = d(0) = d(x \odot y^{-a}) = (d(x) \odot y^{-a}) \vee (x \odot d(y^{-a}))$. It implies that $d(x) \odot y^{-a} = 0$. That is, $d(x) \leq y^{--a}$.

(2) It follows from (1).

(3) By Proposition 2.2 (2), we have $x \leq y^{-a}$. In addition, by Proposition 2.2 (1), we have $x^{--a} \leq y^{-a}$. So $d(x) \leq y^{-a}$. That is, $d(x) \odot y = 0$.

(4) The proof is similar to Proposition 3.1 (4).

(5) By (4), we have $x \odot d(a) = x \odot a = x \leq d(x)$. In addition, if $x \in I([0, a])$, by Proposition 2.2 (7), we have $d(x) \leq x$. Hence, $d(x) = x$ for all $x \in I([0, a])$.

(6) By (3) and Proposition 2.2 (1), we have $x^{---a} \leq (d(x))^{-a}$ and $d(x^{-a}) \leq x^{---a}$. Hence, $d(x^{-a}) \leq (d(x))^{-a}$.

(7) By (1) and Proposition 2.2 (1), we have $x^{-a} \leq a^{-a} = 0$. So $x^{-a} = 0$.

(8) Let $d(a) = a$. By (4), we have $x = x \odot a = x \odot d(a) \leq d(x)$. Conversely, we have $a \leq d(a)$. In addition, we have $d(a) = d(a \odot a) = (d(a) \odot a) \vee (a \odot d(a)) = d(a) \odot a$. It follows that $d(a) \leq a$. Hence, $d(a) = a$.□

Proposition 4.3. *Let A be an EBL-algebra and d be an ($\odot$, $\vee$)-derivation on A. If $d(a) \in I([0, a])$, then $d(d(a)) = d(a)$.*

Proof. By Proposition 4.2 (2) and Proposition 2.2 (7), we have $d(d(a)) \leq d(a)$. In addition, we have $d(d(a)) = d(d(a) \odot a) = (d(d(a)) \odot a) \vee d(a) =$

$d(d(a)) \vee d(a)$. It implies that $d(a) \leq d(d(a))$. Hence, $d(d(a)) = d(a)$ for all $d(a) \in I([0,a])$.□

Definition 4.2. *Let A be an EBL-algebra and d ∈Der(A). d is called isotone if $x \leq y$ for all $x, y \in [0,a]$, it implies that $d(x) \leq d(y)$.*

In the following, we will introduce some sufficient conditions for an isotone ($\odot$, $\vee$)-derivation on an EBL-algebra A.

Proposition 4.4. *Let A be an EBL-algebra and d be an ($\odot$, $\vee$)-derivation on A. For all $x \in [0,a]$, if $x \leq d(x)$, then d is an isotone ($\odot$, $\vee$)-derivation on A.*

Proof. Let $x \leq y$ for some $y \in [0,a]$. By Proposition 2.2 (5), we have

$$\begin{aligned}
& d(x) \\
&= d(x \wedge y) \\
&= d(y \odot (y \xrightarrow{a} x)) \\
&= (d(y) \odot (y \xrightarrow{a} x)) \vee (y \odot d(y \xrightarrow{a} x)) \\
&\leq d(y) \vee y \\
&= d(y).\Box
\end{aligned}$$

There exist some isotone ($\odot$, $\vee$)-derivations on EBL-algebras which does not satisfies $x \leq d(x)$.

Example 4.1. *The zero ($\odot$, $\vee$)-derivation 0_A is isotone, but it does not satisfies that $x \leq d(x)$ for all $x \in [0,a]$. Since $x \nleq d(x) = 0$ for all $x \neq 0$.*

Theorem 4.1. *Let A be an EBL-algebra and d be an ($\odot$, $\vee$)-derivation on A. For all $x, y \in [0,a]$, if $d(x) \leq x$, we have $d(x) = x$ iff $d(x \vee y) = (x \vee d(y)) \wedge (d(x) \vee y)$.*

Proof. Let $d(x) = x$ for all $x \in [0,a]$. Then $d(x \vee y) = x \vee y = (x \vee y) \wedge (x \vee y) = (x \vee d(y)) \wedge (d(x) \vee y)$. Conversely, take $y = x$, we have $d(x) = x \vee d(x)$. It implies that $x \leq d(x)$. Hence, we have $d(x) = x$.□

Definition 4.3. *An EBL-algebra $(A, \vee, \wedge, \odot, 0)$ with a ($\odot$, $\vee$)-derivation d is called isotone if d is isotone.*

In the following, we will give a classification of EBL-algebras with ($\odot$, $\vee$)-derivations.

Proposition 4.5. *Let A be an EBL-algebra and d be an ($\odot$, $\vee$)-derivation on A. For all $x \in [0,a]$, the following statements hold:*
(1) If $x \leq d(x)$, then A is an isotone EBL-algebra;
(2) If $d(x) \leq x$, then A could be either an isotone EBL-algebra or not an isotone EBL-algebra.

Proof. (1) Let $x \leq d(x)$. By Proposition 4.4, we have that A is an isotone EBL-algebra.

(2) By Example 4.1, we have that zero (⊙, ∨)-derivation 0_A is isotone which satisfies that $0 = d(x) \leq x$. Hence, A could be an isotone EBL-algebra. In addition, by Proposition 4.1, we have $d \in$ Der(A). But d is not isotone, there exists $0<y<k$ such that $d(y) = y>0 = d(k)$. Hence, A could be not an isotone EBL-algebra.□

Denote $d(I) = \{d(x)|x \in I\}$ by $d(I)$ and denote $d(F) = \{d(x)|x \in F\}$ by $d(F)$.

Definition 4.4. *Let d be an (⊙, ∨)-derivation on an EBL-algebra A and I be an ideal of A. We say d preserves I if $d(I) \subseteq I$.*

Definition 4.5. *Let d be an (⊙, ∨)-derivation on an EBL-algebra A and F be an filter of A. We say d preserves F if $d(F) \subseteq F$.*

In the following, we will show that (⊙, ∨)-derivations preserve ideals and filters.

Proposition 4.6. *Let A be an EBL-algebra and I be an ideal of A. For all $x \in I$, if $d(x) \leq x$, then $d(I) \subseteq I$.*

Proof. Let $y \in d(I)$. There exists $b \in I$ such that $y = d(b)$. Since $d(b) \leq b$, we have $y = d(b) \in I$ by Definition 2.5. Hence, $d(I) \subseteq I$.□

Proposition 4.7. *Let A be an EBL-algebra and F be an filter of A. For all $x \in F$, if $x \leq d(x)$, then $d(F) \subseteq F$.*

Proof. For all $y \in d(F)$, there exists $b \in F$ such that $y = d(b)$. Since $b \leq d(b)$, we have $y = d(b) \in F$ by Definition 2.6. That is, $d(F) \subseteq F$.□

5 Conclusion

In the paper, we generalized the notions of (⊙, ∨)-derivations to EMV-algebras and EBL-algebras. Some results on MV-algebras and BL-algebras are extended to EMV-algebras and EBL-algebras. Let d_1 be an (⊙, ∨)-derivation on an EMV-algebra A_1 and d_2 be an (⊙, ∨)-derivation on an EBL-algebra A_2. For all $x_1 \in [0, a_1]$, $x_2 \in [0, a_2]$, we have $d(x_1) \leq x_1$ and $d(x_2) \leq x_2^{--a_2}$, where a_1 and a_2 are arbitrary idempotent elements of A_1 and A_2, respectively. But there is no certain partial order relationship between $d(x_2)$ and x_2. Hence, (⊙, ∨)-derivations on EBL-algebras are more complex.

In the following study, there is a valued topic can be studied: In Proposition 4.5, if there is no partial order relationship between $d(x)$ and x, whether A is an isotone EBL-algebra or not?

Declarations

- **Conflict of interest** The authors declare no conflict of interest. No data is involved in this article.

- **Funding** This study was funded by Natural Science Foundation of Shandong Province [ZR2022MA043].

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