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Using the three-way matrix for AHP-group decision-making

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Abstract

The Analytic Hierarchy Process (AHP) is one of the most well-known and used multi-criteria methods in the field of decision problems. In many situations, the decision is not up to a single person but involves multiple decision-makers with different preferences with respect to the various elements of the problem. Applying a traditional AHP with certain preference aggregation mechanisms, the procedure is very burdened by the many pairwise comparison matrices due to the number of decision-makers. The aim of this paper is to summarize the problem of representing the different preferences between the criteria and the alternatives, for the various decision makers using the three-way data matrix. This matrix allows us to examine, in a contemporary and global manner, the preferences of several decision-makers in the decision-making context. Three-way principal component models can be used to treat the three-way matrix, particularly for the derivation of AHP priorities.

Keywords: AHP; three-way data matrix; decision-makers; preferences.

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1. Introduction

The Analytic Hierarchy Process (AHP) is a multi-criteria technique for solving decision problems. It is a high-performing technique, especially in contexts whose main goal is to rank several alternatives for the decision maker to choose the best suitable alternative. AHP was developed in the 1970s by T.S. Saaty and it helps the decision maker to identify the most suitable solution for his several needs by breaking down the decision problem into a hierarchy. A detailed exposition of AHP can be found in Saaty [1], [2], [3], [4], [5], [6], [7], while examples of the practical application of the process, both in public and private sectors, are provided in Saaty [8].

In AHP, the several elements (the evaluation adopted criteria, possible sub-criteria, alternatives object of the final decision) are compared pairwise and according to the properties of bottom-up procedures in the pairwise comparison matrices, in order to derive a priority for each level of the hierarchy. Then, through the derivation of global priorities, an ordering set of alternatives is obtained. This process is quite linear if the decision maker is a single individual; but when the decision involves groups of individuals there is the need to reconcile the different preferences.

There are many contributions given by literature to extend the typical AHP mechanisms when it comes to decisions belonging to more decision makers: considering procedures for aggregating the different preferences of the various actors involved is enough to ensure the regularity of the process. Expect for the rare and highly theoretical cases of consensus voting, the AHP procedure becomes complex in the presence of multiple decision-makers. There are contrasting preferences which have to be grouped, possible weighting of the different importance of the decision makers and a great number of pairwise comparison matrices to build, making the choice of the alternative that will represent the final decision of the problem more insidious and slower.

This is where our proposal of summarizing the problem of the different preferences fits in, about criteria and alternatives, for the various decision makers in a single three-way matrix, instead of the many pairwise comparison matrices. The three dimension of the matrix will be: decision maker, criterion, alternative. Each matrix element represents the position of each decision maker in relation to a precise criterion and to a precise alternative. To understand the relations among the several matrix dimensions, the models of the principal components can be used according to what literature suggests about the management of three-dimensional matrices.

This article is structured as follows: paragraph 1 describes the reference context, paragraph 2 includes a brief description of AHP technique, paragraph 3 deals with the problem of combining preferences in the group decisions, and paragraph 4 explains the possibility to use the three-way data matrix, paragraph 5 contains some concluding remarks.

2. AHP methodology

To identify the best alternative in AHP four steps are involved:

1. development of the hierarchy.
2. building pairwise comparison matrices.
3. derivation of local priorities.
4. derivation of global priorities.

The elements to be considered in the process of decision making are structured in a hierarchy consisting of at least three or more levels. The first level of hierarchy contains the specification of the overall goal of the decision; the second level involves the criteria, that is the rules that guide the decision-making process; finally at the third level, the base of the hierarchy, there are the alternatives, that is the actions to be evaluated.

In more complex hierarchies, each criterion can be broken down into more detailed sub-criteria; they will be at intermediate levels between the criterion of which they represent the detail (at a higher level) and the alternatives or other their breaking down (at a lower level). However, there must be at least a minimum of three levels in the hierarchy to solve the problem with AHP. To each hierarchy level the following properties must be applied: external dependence of the elements of one level from the elements of a higher level and internal independence among the elements of the same level.

The hierarchy can be viewed as a tree diagram: each rectangle represents a node and all the rectangles connected to this parent node are known as children nodes, which constitute comparison groups and represent equivalent or equal nodes. Figure 1 shows an example of a hierarchical structure of a decision-making problem for achieving a goal (level 1) with three criteria (level 2), each of them is structured into two sub-criteria (level 3) and four alternatives (level 4).

Next step is the building of the pairwise comparison matrices. These allow all the elements of the same hierarchical level to be compared pairwise, to evaluate their relative importance with reference to an element of the higher level. Therefore, there will be:

- 1 pairwise comparison matrix among r criteria.
- *r* pairwise comparison matrices among s alternatives.

The goal of the pairwise comparison of the elements is to establish which is more important in relation to the overlying level and to what extent. A generic pairwise comparison matrix A is represented in figure 2.

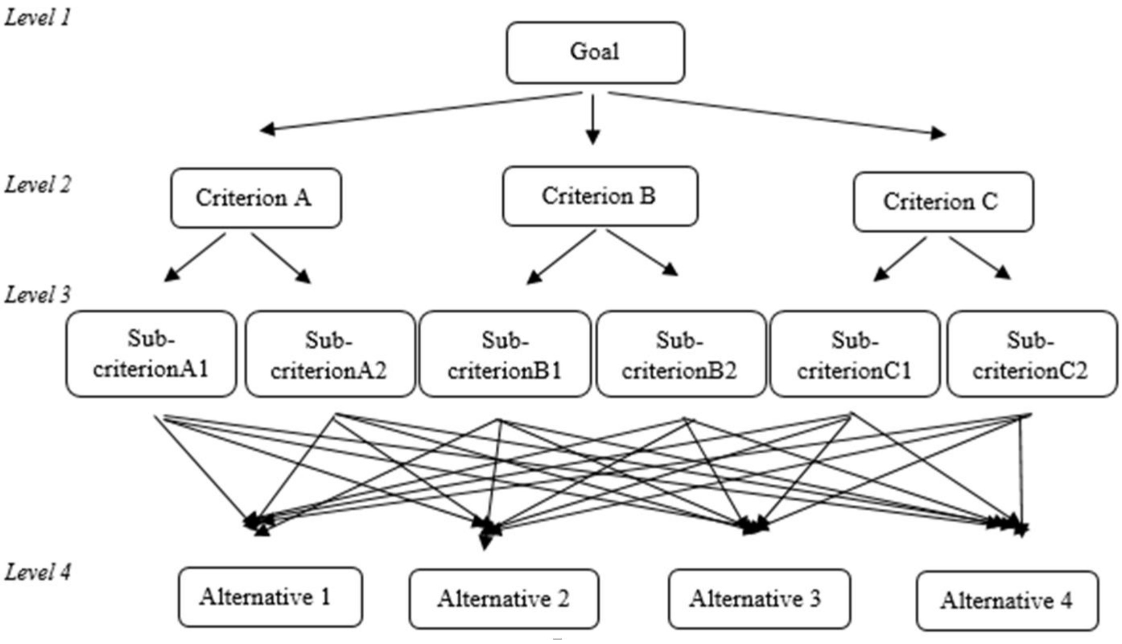

Figure 1. Hierarchical structure of a decision problem.

$$A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & 1 & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & 1 & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & 1 & \cdots \\ a_{n1} & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

Figure 2. Example of a pairwise comparison matrix.

The result of the single comparison is summarized in the a_{ij} coefficient, known as dominance coefficient, which represents an evaluation of the dominance of the *i-th* element over the *j-th* element. This means that if:

- $a_{ij} > 1 \Leftrightarrow$ *i-th* element $\succ$ *j-th* element;
- $a_{ij} = 1 \Leftrightarrow$ *i-th* element ~ *j-th* element;
- $a_{ij} < 1 \Leftrightarrow$ *i-th* element $\prec$ *j-th* element.

Each comparison reproduces the personal judgment of the decision maker. Saaty [9] has developed a semantic scale, which, using positive real numbers from 1 to 9, assigns a score to each compared pairwise, therefore creating a correspondence between numerical value and verbal judgment. Table 1 shows the quantification of the dominance coefficient according to Saaty's scale.

Intensity a_{ij}	*Definition*
1	Equal importance
3	Moderate importance
5	Strong importance
7	Very strong importance
9	Extreme importance
2, 4, 6, 8	Intermediate values between the two adjacent judgments
1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9	Reciprocals

Table 1. The fundamental Saaty's scale.

Therefore the pairwise comparison matrix is:

- square, with unitary elements on the main diagonal ($a_{ii}=1$) since the importance of each element compare to itself leads to an evaluation of indifference;
- positive, in fact all the numbers Saaty uses to quantify the comparisons are positive;
- reciprocal, since, for each value of *i* and of *j* the relation $a_{ij}=1/a_{ij}$ is verified. Coherently, if the element a_i is a certain number *q* times more important of the element a_j, then a_j will have the *1/q–th* of importance compared to a_i . As a consequence of the relation of reciprocity, the triangle which stands above the main diagonal is represented by reciprocal values of those contained in the triangle which stands underneath it;
- transitive, as in theory it should always be worth $a_{ik}a_{kj}=a_{ij}$. This means that a decision maker, for example, who considers that a_i to be 3 times more important of a_k and then considers a_k to be 2 times more important of a_j, should also consider ai to be 6 times more important of a_j;
- consistent due to the properties of reciprocity and transitivity.

The total number of comparisons contained in the matrix depends on the number of *n* elements to compare and is equal to n^2. Among these, only a certain number, equal to *(n-1)*, can be chosen freely, while the others are imposed by the properties of reciprocity and transitivity [10].

Starting from the built pairwise comparison matrices, we can determine the weight of each criterion (so-called derivation of local priorities). Ishizaka [10] has organized the several methods of derivation into two groups: the eigenvalues methods and the methods which minimize the distance between the matrix defined by the user and the closer consistent matrix. The goal, regardless of the method used, is to derive a priority for each level of the hierarchy.

Then, the weight of each alternative is to be established (so-called derivation of global priorities) to obtain an ordered system of them. The global weights allow to establish a preference order: the alternative will be the more preferable the greater its global weight is. For this purpose, the most used approach is that of additive aggregation; according to it the local priorities in relation to each criterion are multiplied by the weight of the criterion and added up. Alternatively, Lootsma [11] has introduced the method of the multiplicative aggregation to avoid reverse ordering which can occur after the introduction or the removal of alternatives from the problem.

The analysis is completed by the consistency control and the study of sensitivity. A matrix can be considered consistent when the rule of transitivity and the rule of reciprocity are met. The consistency shows a strong logical coherence among the expressed preferences and only when the matrix is consistent different methods of priorities derivation produce the same preference order.

However, it often happens that the matrices are not consistent as it is difficult to maintain this consistency of opinion, seen as a strict observance of the rules of transitivity and reciprocity, in all the pairwise comparisons. The causes that can lead to inconsistency of the judgments expressed by the decision maker can be various, among the most frequent there are [12]: material errors in reporting opinions, the lack of adequate information, indifference by the decision makers, inadequate structure of the decision problem, lack of consistency in the real world.

There are several consistency measurements and tolerance threshold to determine to what extent an inconsistent matrix is acceptable [13], [14], [15], [16], [17]. The most used method for the consistency measurement is based upon the Consistency Ratio *(CR)* given by the ratio between the Consistency Index *(CI)* and the Random Index *(RI)*. The *CI* takes two main elements of the matrix into account: the maximum eigenvalue and the n dimension; the *RI* repents the consistency index of a completely random index that is expected to contain highly inconsistent judgments. The AHP methodology accepts a low degree of inconsistency, as it is considered that it does not damage the validity of the obtained result; only when the inconsistency reaches too high level ($CR > 0.10$) it is necessary for the decision maker to revue his own opinions.

To validate the results obtained and verify their stability, the sensitivity analysis is finally carried out. This analysis allows to understand how they reached decision is grounded, and which are its drivers [18]. The analysis of sensitivity allows to verify which changes in the analysis model can cause significant differences in the final results. To carry out the analysis of sensitivity it is necessary to modify the input data, this can be done by acting on the criteria weights, on the weights themselves or the method of calculation used, and understand how these changes affects final performances. If the system of the alternatives does not change, the final results are said to be solid.

3. AHP for group decision-making

There are some circumstances where the decision-making process involves multiple decision makers. In these cases, the complexity of the decision increases because of the various preferences of the decision makers and/or for the different weight associated to the different importance they can have in the decision making process. The main problem in case of various decision makers is that they have potentially conflicting preferences, therefore there is the necessity to build the group's choice starting from single individual choices.

Literature offers several preferences aggregation techniques for the application of the AHP methodology to the group decision. Ishizaka & Labib [19] describe four different ways of combining multiple subjects' preferences. These techniques are shown in table 2.

		Mathematical aggregation	
		Yes	*No*
Aggregation on	*Judgments*	Geometric mean	Consensus vote
	Priorities	Weighted arithmetic mean	Consensus vote

Table 2. Preferences aggregation techniques.

The easiest way to obtain the combination of preferences of multiple subjects is the consensus vote, on judgements or priorities. Reaching a consensus vote on the judgements means reaching an agreement on the value of each element of the pairwise comparison matrices. In case of difficulties about the agreement, it is appropriate to base the consensus vote on the priorities already derived from each participant, even because in this way it is possible to identify the pair decision maker-preference and further discuss any possible disagreement.

Any time it is not possible to use the consensus vote, because the subjects involved are many or very distant, it is possible to perform aggregations in a mathematical way: the geometric mean of the single judgements of the pairwise comparison matrices or the weighted arithmetic mean of the priorities. Although the geometric mean is in compliance with transitive property, it does not meet Pareto optimality. Therefore, the priorities in the hierarchy of each decision maker can be aggregated due to the weighted arithmetic mean method.

Forman & Peniwati [20] describe two base techniques to aggregate individual preferences in a group preference, depending on whether the group wants to act as a single unit or as separate individuals:

- AIJ, Aggregating Individual Judgments;
- AIP, Aggregating Individual Priorities.

In the AIJ approach the individuals renounce to their own preferences by acting in concert and sharing their judgments, in order that the group will be a new “individual” and will act so. Individual judgments are aggregated synergistically, and their synthesis generates the group priorities. As the individual priorities are irrelevant, also the violation of the Pareto principle is negligible; on the contrary, the reciprocity of the judgements must be ensured and the geometric mean is the only appropriate procedure in the AIJ approach, as it is able to preserve the reciprocal property of the pairwise comparison matrices.

In the AIP approach the group is considered as a whole of individuals, with different values and hierarchies. This approach requires that the priorities derive from each individual hierarchy and so aggregate in group priorities. In AIP both the geometric mean and the arithmetic mean can be used.

There are further problems if the decision-makers do not have the same weight. The weights should indicate the experience of the decision-maker or the importance of the impact of the decision on those who will make it. The weights can be assigned by a superordinate decision-maker, accepted by all, or with a participatory approach based upon a pairwise comparison matrix with the relative importance of each member [21]. For those cases where the comparisons are missing, because the decision-maker does not have the skills to formulate a specific judgement, the aggregation methods that can be used are the linear programming [22] and the Bayesian approach [23].

4. Using and handling a three-way matrix for group decisions

The three-way data matrix is an instrument in which each elementary data, x_{ijk}, has three indices that correspond to a classification of it based on three elements:

$$X_{I*J*K} = x_{ijk} \qquad per\ i = 1,\ \ldots,\ I;\ j = 1,\ \ldots,\ J;\ k = 1,\ \ldots,\ K$$

The three-dimensionality of the data allows its geometric arrangement in a cube [24], [25], [26].

This study proposes the use of the three-way matrix in the AHP methodology in the context of group decisions. A decision-making process is typically characterized by the presence of criteria and alternatives compared; the introduction of a third element, that is the decision maker with his preference, can be comprehensively represented by a three-way data matrix. The three inputs "units x variables x occasions" of the matrix become "criteria x alternatives x decision makers". The generic element a_{ijk} of the matrix will express the *k-th* subject's preference for the *j-th* alternative with reference to the *i-th* criterion. The schematic drawing is shown in figure 3.

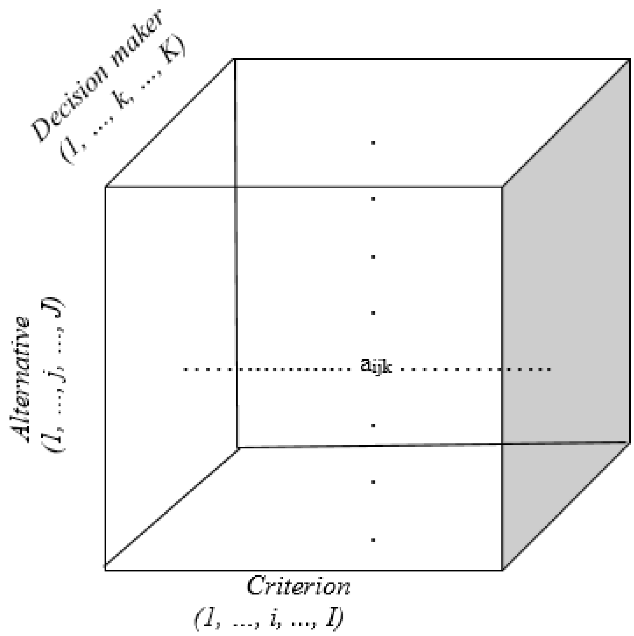

Figure 3. The three-way data matrix for group decision making.

Therefore, each dominance coefficient a_{ijk} will be identified according to three dimensions:

- *i (1, 2, 3, …, I)* which refers to criteria;
- *j (1, 2, 3, …, J)* which identifies the alternatives;
- *k (1, 2, 3, …, K)* which reproduces the decision makers.

The three-way data matrix can be used for group decisions to include the evaluation of *K* decision makers towards the criteria and the alternatives requested in the AHP. The arrangement of the elements (criteria, alternatives, decision makers) in rows, columns and tubes of the matrix is not subject to any precise rule since it is possible to represent them also in different order, for example according to the arrangement "decision makers x alternatives x criteria", leaving its meaning unchanged.

The matrix finishes all the pairwise comparisons which are necessary in a classic AHP. Its use can be justified through at least two important advantages:

- a greater convenience in managing the amount of data. This privilege will be the more noticeable the more *I*, *J* and *K* will increase, respectively number of criteria, alternatives and decision makers;
- a better usability of the information within the structure of the decision problem.

At the same time a three-way data matrix reduces the priority derivation and preference aggregation operations by not having to work on so many pairwise comparison matrices.

Once the preferences of the decision makers have been represented, with respect to criteria and alternatives, the matrix must be treated with specific methodologies that allow the mass of information hidden in the set of data contained in it to be summarized. To estimate the priorities to be used in the AHP, for example, any three-way Component Analysis model, such as Statis [27], [28], [29], Parafac/Candecomp [30], [31] or Tucker3 [32], can be applied.

5. Discussion and Conclusions

The aim of this work is to advance the proposal to use the three-way matrix in the AHP methodology for group decisions. This idea developed in consideration of the significant limitation of the AHP procedure, in the presence of a high number of pairwise comparisons to be analyzed when multiple subjects are involved in the decision. The issue has certainly not been treated in an exhaustive manner, since, in order to constitute a valid contribution to the analysis and functioning of the AHP method, its various aspects should be explored in greater depth.

A first future area of study could concern a comparison between the various models of the three-way Principal Components, to identify the one that is most effectively and efficiently suited to be used for the derivation of the eigenvalue which estimates the priorities. If the logic is to streamline the procedure, the processing times employed could be considered.

Another aspect worth to be examined concerns the control of the consistency of the matrix, through a necessary reformulation of the measures of the degree of coherence of the judgments expressed by the decision makers. The Consistency Index, mostly used for this purpose, is calculated taking into account the order n of the two-dimensional matrix, and would clearly not be applicable to the case of a three-dimensional matrix. It should therefore be redesigned with an adaptation or new consistency measurement indices should be studied.

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