Ratio Mathematica Volume 53, 2024

Some special cases on Stolarsky's means

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Abstract

In this paper we observe that one-parameter Stolarsky's means (SM) are deduced from both the Mean Value Theorem for derivatives (MVTD) and the Mean Value Theorem for definite integrals (MVTI), and we study their elementary properties as the parameter varies. In the subfamily of SM having a natural number as parameter, we geometrically interpret one of them in particular as a real elliptic cone. We link SM having the integer power of a prime number as a parameter to classical means **(**i.e., harmonic mean, geometric mean, arithmetic mean, power mean). Finally, from an extension of Flett's Theorem (FT), we derive the expression of a new mean that is a upper bound of the arithmetic mean.

Keywords: SM; MVTD; MVTI; FT.

2010 AMS subject classification: 26A24; 15A06; 26D07.

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1. Introduction

1.1 A look at the literature on Stolarsky's means

The literature on Stolarsky's Means (SM) originates from three mathematicians and statisticians: Otto Dunkel, Luigi Galvani and Renzo Cisbani[†]. Dunkel [11] in 1909 studied a particular power mean

$$y(x) = \left(\frac{a_1^x + \cdots + a_n^x}{n}\right)^{\frac{1}{x}} \quad (x \neq 0;\ \ a_i \in R_+) \tag{1}$$

which for $x = 1$ reduces to the arithmetic mean and for $x = -1$ coincides with the harmonic mean. He shows that $y(x)$ enjoys the following properties: a) it converges to the geometric mean for $x \to 0$, i.e., to

$$\sqrt[n]{a_1 \cdots a_n}$$

b) it tends, respectively, to a_1 for $x \to -\infty$ and to a_n for $x \to -\infty$ where $a_1 = \min a_i$ and $a_n = \max a_i$ $(i \in \{1, \cdots, n\})$; c) it is continuous throughout R if we extend $y(x)$ by continuity in $x = 0$; d) it is increasing in R. On the other hand, Galvani [15], in 1929, generalizes Dunkel's proof of the convergence of $y(x)$ to the geometric mean for $x \to 0$ by observing that if (a_n) is a subsequence of positive real numbers converging to some λ then $y(x) \to \lambda$, i.e

$$\lambda = \sqrt[n]{a_1 \cdots a_n}$$

Moreover, he, dividing into n parts of equal amplitude h the interval $[a; b]$, where $a, b \in R_+;\ \ a < b;\ h = \frac{b-a}{n};\ n \in N$, and considering $a_1, \cdots, a_n$ as an arithmetic progression of first term a and reason h through the position

$$a_1 = a;\ a_2 = a + h;\ \cdots;\ a_n = a + (n-1)h$$

observes that the expression (1) of $y(x)$, if rewritten in the following ways $(x \neq 0)$

$$\left(\frac{a^x + (a+h)^x + \cdots + (a + (n-1)h)^x}{n}\right)^{1/x} =$$

[†] Both Galvani and Cisbani were students of Corrado Gini, a jurist, statistician and demographer well known for "the famous inequality measure that bears his name and his work in the early days of balanced sampling". Galvani is also credited with contributions on the Theory of Sampling (Gini and Galvani, 1929; Galvani (1951)). See the article by Matti Langel - Ives Tille´. Corrado Gini, a pioneer in balanced sampling and inequality theory METRON - International Journal of Statistics. 2011, vol. LXIX, no. 1, pp. 45-65

$$= \left(\frac{a^x h + (a+h)^x h + \cdots + (a+(n-1)h)^x}{b-a} \right)^{1/x} \tag{2}$$

for n $\rightarrow \infty$ tends to

$$y(x) = \left(\frac{\int_a^b t^x dt}{b-a} \right)^{1/x} = \begin{cases} \left[\dfrac{b^{x+1} - a^{x+1}}{(x+1)(b-a)} \right]^{1/x} & for\ x \neq -1 \\ \dfrac{b-a}{\ln b - \ln a} & for\ x = 1 \end{cases}$$

Ultimately Galvani, from the formula

$$y(x) = \left[\frac{b^{x+1} - a^{x+1}}{(x+1)(b-a)} \right]^{1/x} \ (x \neq 0, x \neq -1) \tag{3}$$

for $x \rightarrow -1$, for $x \rightarrow 0$, at x = 1, 2 deduces the following four means, respectively:

$$L(a,b) = \frac{b-a}{\ln b - \ln a} \ (a \neq b); L(a,a) = a \tag{4}$$

$$I(a,b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} \tag{5}$$

$$A(a,b) = \frac{a+b}{2} \tag{6}$$

$$\eta_3 = \sqrt{\frac{a^2 + ab + b^2}{3}} \tag{7}$$

The first three means are known in the literature as the logarithmic mean *L*, the identical mean *I*, and the arithmetic mean *A*, respectively, while the fourth mean, η_3, is completely unknown and not used in Statistics. Cisbani[‡] [10], in 1938, realizes that further relationships can be obtained from Dunkel's formula (1) if the numbers $a_1, \cdots, a_n$ relate not only in arithmetic progression but also in harmonic progression, quadratic progression, ... More generally, Cisbani adopts the formula $\left(a^j + ih\right)^{1/j}$ $(j = \cdots, -1, 0, 1, 2, \dots)$ to divide the interval into n parts and defines the expression

$$\left[\frac{1}{n}\sum\left(a^i + ih\right)^{\frac{x}{j}}\right]^{1/x} \quad (j \neq 0;\ x \neq 0) \tag{8}$$

from which for $n \to \infty$ deduces the relationship

$$y_j = \left[\frac{1}{b^j - a^j}\int_{a^j}^{b^j} t^{\frac{x}{j}}dt\right]^{\frac{1}{j}} \tag{9}$$

which, when solved, gives the following means:

$$\left[\frac{b^{x+j} - a^{x+j}}{\left(\frac{x}{j} + 1\right)(b^j - a^j)}\right]^{\frac{1}{x}} \quad (x \neq -j)$$

$$\left[\frac{b^j - a^j}{j(\ln b - \ln a)}\right]^{\frac{1}{j}} \quad (x = -j)$$

$$\frac{1}{e^{\frac{1}{j}}}\left[\frac{b^{b^j}}{a^{a^j}}\right]^{\frac{1}{b^j - a^j}} \quad (x = 0)$$

Moreover, Cisbani in his article explores some possible forms that the formula (9) can take, drawing up a double-entry (x,j) table in which he reports as many as 49 means. One has to wait until 1975 for Galvani and Cisbani's formulas to be rediscovered (or taken up again) by K.Stolarsky [47][48], who, after rewriting them as follows:

$$u(\alpha) = \left[\frac{x^\alpha - y^\alpha}{\alpha(x-y)}\right]^{\frac{1}{\alpha-1}} \ (\alpha \neq 0;\ \alpha \neq\ 1);\ u(\alpha,\beta) = \left[\frac{\beta(x^\alpha - y^\alpha)}{\alpha(x^\beta - y^\beta)}\right]^{\frac{1}{\alpha-\beta}} \tag{10}$$

[‡] Substantially, both Galvani and Cisbani, using the definition of Riemann integral, prove MVTI in some special cases. They then in order to deduce MS go from the discrete to the continuous following an inverse path to what will be done by later authors who either start from MVTD or MVTI.

shows that for $x \neq y$, $u(\alpha, \beta)$ is strictly increasing with respect to both α and β. Subsequently, the strands of research on SM have differed. Some authors have focused their attention on a particular SM, namely the logarithmic mean, which is also important because it plays an important role in some engineering applications, such as the study of conduction of heat in liquids flowing in pipes [2] and in statistics, in the calculation of relative changes (or growth rates) [28][49]. In particular, Stolarsky [47] and Pittenger [37] extend to three variables and n variables respectively the pair of inequalities

$$G < L < A \tag{11}$$

already proven in two variables by B. Ostle and H. L. Terwilliger [34] (right side of (11)) in 1957 and by B.C. Carlson [4] (the left side of (11)) in 1966 between the geometric mean G, the logarithmic mean L and the arithmetic mean A. The inequality (11) is important because it represents a refinement of the well-known inequality $G < A$ in that there is a third mean, the logarithmic mean L, which interpolates G and A. Inequality (11) has been reproved many time (see also [44], [45] for other references). Subsequently, Tung Po Lin in 1974 [27] found an interesting relationship between the logarithmic mean L and the means of powers M_p of parameter p:

$$M_p = \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}} \tag{12}$$

establishing that

$$M_0 < L < M_{1/3} \tag{13}$$

where M_0, $M_{1/3}$ are respectively the power mean of parameter $p = 0$ and $p = 1/3$:

$$M_0 = \sqrt{xy};\ M_{1/3} = \left(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{2}\right)^3$$

Subsequently, Peter R Mercer in 2017 [30] uses Cauchy's Theorem of Mean Value, avoiding passages to the limit, to deduce the expression of the logarithmic mean (4), the inequalities (9) and some inequalities by Tung-Po Lin in 1974 [27]; moreover, P.R. Mercer in 2017 [30] also obtains the following refinement of (11):

$$A(a,b)^{1/3} G(a,b)^{2/3} < L(a,b) < \frac{1}{3} A(a,b) + \frac{2}{3} G(a,b) \tag{14}$$

Finally, G. Jameson and P.R. Mercer in 2019 [22] demonstrate inequalities (13) and (14) with a single method. In detail, they first establish that the pair of inequalities in the two variables a and b (9) is equivalent to thefollowing pair of inequalities in a variable x

$$x^{1/2} \leq \frac{x-1}{\log x} \leq \frac{1}{2}(x+1)$$

and then they, using the power series of the hyperbolic functions sinh and cosh, achieve the above inequalities. Some authors, in particular A. Witkowski in 2004 [50] and F. Qi in 1998 [38] and in 2000 [39], have related SM to notions of monotonocity; others authors, including F.Qi in 2002 [41]; A. Witkowski in 2006 [51]; J.Sándor in 2015[46]; J.Jaksetic, J. Pecaric, and A.U.Rehman in 2010 [23]; J.Pecaric and G.Roqia 2010 [36], have linked SM to notion of log-convexity; instead the property of Schur-convexity of SM was studied by Elezovic and J. Pečarić in 2005 [12], F. Qi in 2005[40], K.Muraliand and M. Nagaraja in 2016 [32]; Zhen-Hang Yang, Feng Qi in 2024 [52]. Some authors, however, have investigated the relationships between Gini Means [16] and SM finding profound analogies between the two types of means [22][33][5][1]. There has also been an attempt to represent SM with power series by H.W. Gould and M. E. Mays 1984 [17]. SM have recently been used in Numerical Analysis to improve the convergence of Newton's third-order Method by Herceg in 2013 [19], in Statistical Mechanics for the Fokker-Plank operator by Heida in 2021 [18] and in Information Theory to define divergence measures by Cerone and Dragomir in 2004 [7]. We first show that the link of SM with classical means is not limited to correspondence with a few parameters as seems to emerge from the literature, and we prove new formulas by elementary methods by developing new notable products from the formula

$$b^n - a^n \ (n \in N;\ \ a, b\ \in R_+, a\ <\ b)$$

We also characterize both from the perspective of Euclidean geometry and through the methods of linear algebra the mean $\sqrt{\frac{a^2+ab+b^2}{3}}$ introduced by Galvani and Cisbani. We also look for new ways to show that SM are between a minimum and a maximum value. SM, as we will show later, are deduced from the MVTD (or MVTI) [38][43]. Flett's Theorem (FT) represents the main extension of the MVTD. We will prove that it is impossible to deduce SM from the FT but it will be permissible to do so from a particular extension of it.

1.2 Preliminaries

Let a and b be two fixed real numbers, with $a < b$. If the function f is continuous in *[a;b]* and differentiable in $]a;b[$, then the MVTD states that there exists at least one η [§]such that [**]

$$\frac{f(b)-f(a)}{b-a} = f'(\eta) \tag{15}$$

If f' is invertible on $]a;b[$ then

$$\eta = (f')^{-1}\left[\frac{f(b)-f(a)}{b-a}\right] \tag{16}$$

If, on the other hand, g is continuous in [a;b], the MVTI asserts that there exists at least $\eta_* \in [a;b]$ such that

$$\frac{1}{b-a}\int_a^b g(x)\ dx = g(\eta_*) \tag{17}$$

If g is invertible on *[a;b]* then

$$\eta_* = g^{-1}\left[\frac{1}{b-a}\int_a^b g(x)\ dx\right] \tag{18}$$

In general $\eta \neq \eta_*$[††]. It is convenient to highlight the dependence of η and η_* on a and b by making use of the notations $\eta(a,b)$ and $\eta_*(a,b)$ [42].
Definition [4] A continuous function M: $R_+^2 \to R$ is said to be a mean of two numbers a and b if and only if for any $a, b, \ \lambda \in R_+$,

M1) $min\{a,b\} \leq M(a,b) \leq max\{a,b\}$

M2) $M(a,b) = M(b,a)$

M3) $M(\lambda a, \lambda b) = \lambda M(a,b)$

Let α be real different from *0* and *1*. Applying MVTD to the function $f(x) = x^\alpha$ we have that there exists at least one $\eta(a,b)$, with $a < \eta(a,b) < b$, such that[‡‡]

[§] Under the additional hypothesis that *f* is either strictly convex or strictly concave in *[a;b]* the point η is unique [20]

[**] We observe that

$$\frac{f(b)-f(a)}{b-a} = \frac{-[f(a)-f(b)]}{-(a-b)} = \frac{f(a)-f(b)}{a-b} = f'(\eta)$$

[††] We observe that if $f(x) = g(x) = e^x$ then $\eta = \eta_* = \ln\left(\frac{e^b-e^a}{b-a}\right)$ [25][29][35]

[‡‡] It is known in general that if α > 1 then $f(x) = x^\alpha$ is strictly convex at $]0;+\infty[$,, while if 0 < α < 1 then 0 < α < 1 is strictly concave in $]0;+\infty[$- Having fixed α we can take expression (1) as uniquely determined.

$$\frac{b^\alpha - a^\alpha}{b - a} = \alpha \cdot \eta(a,b)^{\alpha-1}$$

whence

$$\eta_\alpha(a,b) = \left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]^{\frac{1}{\alpha-1}} \tag{19}$$

The latter result is also obtained by applying the MVTI to the function $g(x) = x^{\alpha-1}$:

$$\frac{1}{b-a}\int_a^b x^{\alpha-1}\,dx = g(\eta_*);\ \frac{1}{b-a}\cdot\left[\frac{x^\alpha}{\alpha}\right]_a^b = g(\eta_*);\ \frac{1}{b-a}\cdot\frac{b^\alpha - a^\alpha}{\alpha} = \eta_*^{\alpha-1}(a,b)$$

It is immediately seen that for $\eta_\alpha(a,b)$ the properties M2 and M3 apply:

$$\eta_\alpha(a,b) = \left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]^{\frac{1}{\alpha-1}} = \left[\frac{-(a^\alpha - b^\alpha)}{-\alpha(a-b)}\right]^{\frac{1}{\alpha-1}} = \left[\frac{a^\alpha - b^\alpha}{\alpha(a-b)}\right]^{\frac{1}{\alpha-1}} = \eta_\alpha(b,a)$$

$$\eta_\alpha(\lambda a,\lambda b) = \left[\frac{(\lambda b)^\alpha - (\lambda a)^\alpha}{\alpha(\lambda b - \lambda a)}\right]^{\frac{1}{\alpha-1}} = \left[\frac{\lambda^\alpha(a^\alpha - b^\alpha)}{\alpha\lambda(b-a)}\right]^{\frac{1}{\alpha-1}} = \lambda\left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]^{\frac{1}{\alpha-1}}$$
$$= \lambda\eta_\alpha(a,b)$$

Further below we will also prove the M1 property. As α varies, we obtain an infinite $S_\alpha = \{\eta_\alpha\}$ family of means called Stolarsky means (one-parameter) in [42] or generalized logarithmic means in [47] and [48]. Applying Cauchy's Mean Value Theorem[§§] [26] [42] to the functions $f(x) = x^\alpha$; $g(x) = x^\beta$ $(\alpha \neq \beta;\ 0 < a < b)$ in the interval $[a;b]$ there exists at least an η in $]a;b[$ such that:

$$\frac{b^\alpha - a^\alpha}{b^\beta - b^\beta} = \frac{\alpha\eta^{\alpha-1}}{\beta\eta^{\beta-1}};\ \frac{\beta}{\alpha}\cdot\frac{b^\alpha - a^\alpha}{b^\beta - b^\beta} = \eta^{\alpha-\beta}$$

[§§] The Cauchy’s mean value theorem (for derivatives) is also known as Extended Mean-Value Theorem whence***

$$\eta_{\alpha,\beta}(a,b) = \left[\frac{\beta}{\alpha}\cdot\frac{b^{\alpha}-a^{\alpha}}{b^{\beta}-b^{\beta}}\right]^{\frac{1}{\alpha-\beta}} \quad (\alpha \neq \beta) \qquad (20)$$

defined as Stolarky means††† relative to two parameters, α and β, in two variables, a and b. Furthermore, it is observed that $\eta_{\alpha,1}(a,b) = \eta_{\alpha}(a,b)$.

2. Classical means deduced from SM. Trends and properties of SM as the parameter α changes; inequalities.

We will start this section by connecting the SM to the classic means right away.

2.1 Examples of classical means deduced from SM by assigning the parameter α a particular value.

The classical means, i.e., the geometric mean, arithmetic mean, ... can be deduced by applying MVTD or MVTI to appropriate functions, or from the expression of $\eta_{\alpha}(a,b)$ by assigning a particular value to the parameter α or by performing on $\eta_{\alpha}(a,b)$ a limit operation, as illustrated in the following table 1:

Mean	MVTD	MVTI	Parameter
Arithmetic A	$f(x) = x^2$	$g(x) = x$	$\alpha = 2$
Geometric G	$f(x) = x^{-1}$	$g(x) = x^{-2}$	$\alpha = -1$
Identical I	$f(x) = x \cdot \ln x$	$g(x) = \ln x$	$\alpha \to 1$
Logarithmic L	$f(x) = \ln x$	$g(x) = \dfrac{1}{x}$	$\alpha \to 0$
Power mean	$f(x) = x^{1/2}$	$g(x) = x^{-1/2}$	$\alpha = 1/2$

*** The geometric meaning of the radical of η_{α} is as follows: it is the slope, divided by α, of the secant line s the diagram of the function $f(x) = x^{\alpha}$ at the points $A(a; f(a)) = (a; a^{\alpha})$; $B(b; f(b) = (b; b^{\alpha}))$, or is the slope, divided by α, of the tangent line t to the diagram of the $f(x) = x^{\alpha}$ at the point $C(\eta_{\alpha}; f(\eta_{\alpha}))$ and parallel to s.

††† Stolarsky [47][48] introduces them without deducing them from Cauchy's Theorem but generalizing the logarithmic mean.

Table 1 A comparison between MVTD and MVTI assigning a particular value to the parameter α.

For example, for α = *-1* we obtain the geometric mean

$$\eta_{-1}(a,b) = \left[\frac{b^{-1}-a^{-1}}{-(b-a)}\right]^{-\frac{1}{2}} = \left[-\frac{\frac{1}{b}-\frac{1}{a}}{(b-a)}\right]^{-\frac{1}{2}} = \left[-\frac{\frac{a-b}{ab}}{(b-a)}\right]^{-\frac{1}{2}}$$

$$= \left(\frac{b-a}{ab}\cdot\frac{1}{b-a}\right)^{-\frac{1}{2}} = \left[\left(\frac{1}{ab}\right)^{-1}\right]^{1/2} = \sqrt{ab}$$

for α = *2* we have the arithmetic mean

$$\eta_2(a,b) = \left[\frac{b^2-a^2}{2(b-a)}\right]^{\frac{1}{2-1}} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$$

For $\alpha = 1/2$ we generate the power mean

$$\eta_{\frac{1}{2}}(a,b) = \left[\frac{b^{\frac{1}{2}}-a^{\frac{1}{2}}}{\frac{(b-a)}{2}}\right]^{\frac{1}{\frac{1}{2}-1}} = \left[\frac{b^{\frac{1}{2}}-a^{\frac{1}{2}}}{\frac{(b-a)}{2}}\right]^{-2} = \left[\frac{\left(b^{\frac{1}{2}}-a^{\frac{1}{2}}\right)\left(b^{\frac{1}{2}}+a^{\frac{1}{2}}\right)}{\frac{(b-a)\left(b^{\frac{1}{2}}+a^{\frac{1}{2}}\right)}{2}}\right]^{-2} =$$

$$= \left[\frac{b-a}{\frac{(b-a)\left(b^{\frac{1}{2}}+a^{\frac{1}{2}}\right)}{2}}\right]^{-2} = \left(\frac{2}{b^{\frac{1}{2}}+a^{\frac{1}{2}}}\right)^{-2} = \left[\left(\frac{2}{b^{\frac{1}{2}}+a^{\frac{1}{2}}}\right)^{-1}\right]^{2} = \left(\frac{b^{\frac{1}{2}}+a^{\frac{1}{2}}}{2}\right)^{2}$$

When performing some operations with the limits on $\eta_\alpha(a,b)$ better to think $\eta_\alpha(a,b)$ as a function "exponential-power" $f(\alpha)^{g(\alpha)} = e^{g(\alpha)\ln f(\alpha)}$ where

$$f(\alpha) = \frac{b^\alpha - a^\alpha}{\alpha(b-a)};\ g(\alpha) = \frac{1}{\alpha-1}$$

and concern the functions $b^\alpha;\ a^\alpha$ as exponential functions of base b and a respectively and exponent α $(\alpha \neq 0;\ \alpha \neq 1)$.

2.2 The Stolarsky mean $\eta_\alpha(a,b)$ tends to the logarithmic mean L(a,b) for that α tends to 0

For $\alpha \to 0$ we have that $\eta_\alpha(a,b)$ tends to the logarithmic mean L(a,b), i.e.

$$\lim_{\alpha\to 0}\eta_\alpha(a,b) = \lim_{\alpha\to 0}\left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]^{\frac{1}{\alpha-1}} = \frac{a\ -\ b}{ln\ a\ -\ ln\ b} = L(a,b)$$

Indeed, the function $\left[\frac{b^\alpha-a^\alpha}{\alpha(b-a)}\right]^{\frac{1}{\alpha-1}}$ is presented as an exponential-power function $f(\alpha)^{g(\alpha)}$ where

$$f(\alpha) = \frac{b^\alpha - a^\alpha}{\alpha(b-a)};\ g(\alpha) = \frac{1}{\alpha-1}$$

From continuity in $\alpha = 0$ of the exponential functions $b^\alpha;\ a^\alpha$ we have

$$\lim_{\alpha\to 0}(b^\alpha - a^\alpha) = b^0 - a^0 = 1 - 1 = 0$$

whence the limit

$$\lim_{\alpha\to 0}\frac{b^\alpha - a^\alpha}{\alpha(b-a)}$$

is of the type $0/0$. Applying the Rule of Hospital to it and remembering that $0/0$.

$$D\ b^\alpha = \ b^\alpha \ln b\,;\ D\ a^\alpha = \ a^\alpha \ln a;\ \ D\alpha \ = \ 1$$

we obtain

$$\lim_{\alpha\to 0}\frac{b^\alpha \ln b - a^\alpha \ln a}{1\cdot(b-a)} = \frac{1\cdot\ln b - 1\cdot\ln a}{b-a} = \frac{\ln b - \ln a}{b-a}$$

Because

$$\lim_{\alpha\to 0}\left(\frac{1}{\alpha-1}\right) = -1$$

then

$$\lim_{\alpha\to 0}\left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]^{\frac{1}{\alpha-1}} = \lim_{\alpha\to 0} e^{g(\alpha)\ln f(\alpha)} = \lim_{\alpha\to 0} e^{\left(\frac{1}{\alpha-1}\right)\cdot\ln\left[\frac{b^\alpha-a^\alpha}{\alpha(b-a)}\right]}$$

from which

$$= e^{-\ln\left(\frac{\ln b - \ln a}{b-a}\right)} = e^{\ln\left(\frac{b-a}{\ln b - \ln a}\right)} = \frac{a\ -\ b}{ln\ a\ -\ ln\ b} > 0$$

2.3 The Stolarsky mean $\eta_{\alpha}(a,b)$ tends to the identic mean I(a,b) for that α tends to 1

For $\alpha \to 1$ we have that $\eta_{\alpha}(a,b)$ tends to the identic mean I(a,b), i.e.

$$\lim_{\alpha \to 1} \eta_{\alpha}(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} = I(a,b) > 0$$

Indeed, thought of $\eta_{\alpha}(a,b)$ still as an exponential-power function $f(\alpha)^{g(\alpha)}$, and observed that

$$\lim_{\alpha \to 1}\left[\frac{b^{\alpha} - a^{\alpha}}{\alpha(b-a)}\right] = \frac{b-a}{1 \cdot (b-a)} = 1;\ \lim_{\alpha \to 1}\left(\frac{1}{\alpha - 1}\right) = \infty$$

then the limit of the function $f(\alpha)^{g(\alpha)}$ occurs in the indeterminate form of the type 1^{∞}, which we rewrite as follows

$$\lim_{\alpha \to 1} e^{\left(\frac{1}{\alpha-1}\right)\cdot \ln\left[\frac{b^{\alpha}-a^{\alpha}}{\alpha(b-a)}\right]} = \lim_{\alpha \to 1} e^{\frac{\ln\left[\frac{b^{\alpha}-a^{\alpha}}{\alpha(b-a)}\right]}{\alpha-1}}$$

Applying the L'Hospital Rule to the exponent $e^{\frac{\ln\left[\frac{b^{\alpha}-a^{\alpha}}{\alpha(b-a)}\right]}{\alpha-1}}$, which is of the type 0/0, we have

$$\frac{\left[\frac{\alpha(b-a)}{b^{\alpha}-a^{\alpha}}\right] \cdot \frac{(b^{\alpha}\ln b - a^{\alpha}\ln a)[\alpha(b-a)] - (b^{\alpha}-a^{\alpha})(b-a)}{\alpha^2(b-a)^2}}{1}$$

which, properly simplified, leads to the formula

$$\frac{(b^{\alpha}\ln b - a^{\alpha}\ln a)[\alpha(b-a)] - (b^{\alpha}-a^{\alpha})(b-a)}{\alpha(b-a)(b^{\alpha}-a^{\alpha})} = \left[\frac{b^{\alpha}\ln b - a^{\alpha}\ln a}{b^{\alpha}-a^{\alpha}} - \frac{1}{\alpha}\right]$$

The latter, for $\alpha \to 1$, gives the expression

$$\frac{b\ln b - a\ln a}{b-a} - 1 = \frac{\ln b^b - \ln a^a}{b-a} = \frac{\ln(b^b/a^a)}{b-a} - 1 = \ln(b^b/a^a)^{1/b-a} - 1$$

Therefore, the limit of $\eta_\alpha(a,b)$, for $\alpha \to 1$, leads to

$$e^{\left[\ln\left(b^b/a^a\right)^{1/b-a}-1\right]} = e^{\ln\left(b^b/a^a\right)^{1/b-a}} \cdot e^{-1} = \frac{1}{e} \cdot \left(b^b/a^a\right)^{1/b-a}$$

2.4 The SM $\eta_\alpha(a,b)$ lies between a and b

We show that for $\eta_\alpha(a,b)$ M1 holds true, i.e.

$$\lim_{\alpha\to-\infty} \eta_\alpha(a,b) = a = min\{a,b\}; \ \lim_{\alpha\to+\infty} \eta_\alpha(a,b) = b = max\{a,b\}$$

from which

$$\eta_\alpha(a,b) = \left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]^{\frac{1}{\alpha-1}} \tag{21}$$

Let's start with the left limit:

Reviewing as above the functions b^α; a^α as exponential functions of basis b and a respectively and exponent α, and observing that

$$\lim_{\alpha\to-\infty} b^\alpha = \lim_{\alpha\to-\infty} a^\alpha = 0; \lim_{\alpha\to-\infty} \frac{1}{\alpha} = 0 \ ; \lim_{\alpha\to-\infty} g(\alpha) = \lim_{\alpha\to-\infty} \left(\frac{1}{\alpha-1}\right) = 0$$

$$\lim_{\alpha\to-\infty} f(\alpha) = \lim_{\alpha\to-\infty} \frac{1}{b-a} \cdot (b^\alpha - a^\alpha) \cdot \frac{1}{\alpha} = 0$$

we have that the limit of $\eta_\alpha(a,b) = [f(\alpha)]^{g(\alpha)}$ for $\alpha \to -\infty$ occurs of the type 0^0.
Written the function as

$$e^{\frac{\ln\left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]}{\alpha-1}}$$

to the exponent, which is of the type ∞/∞, the De L'Hospital Rule can be applied. Similarly to the above we get

$$\lim_{\alpha\to-\infty} \frac{b^\alpha \ln b - a^\alpha \ln a}{b^\alpha - a^\alpha} - \frac{1}{\alpha} = \lim_{\alpha\to-\infty} \frac{a^\alpha\left(\frac{b^\alpha}{a^\alpha}\ln b - \ln a\right)}{a^\alpha\left(\frac{b^\alpha}{a^\alpha} - 1\right)} - \frac{1}{\alpha} =$$

$$\lim_{\alpha\to-\infty} \frac{\left(\frac{b}{a}\right)^\alpha \ln b - \ln a}{\left(\frac{b}{a}\right)^\alpha - 1} - \frac{1}{\alpha} = \ln a$$

having assumed, as done above, *0 < a < b*, whence *b/a >1* and $\lim_{\alpha\to-\infty}\left(\frac{b}{a}\right)^{\alpha}=0$. Therefore, our limit is

$$e^{\ln a}=a$$

i.e. the $min\{a,b\}$.

Finally, we prove that

$$\lim_{\alpha\to+\infty}\left[\frac{b^{\alpha}-a^{\alpha}}{\alpha(b-a)}\right]^{\frac{1}{\alpha-1}}=b$$

Retracing the procedure used for the previous limit, placed $t=\left(\frac{b}{a}\right)^{\alpha}$, we have

$$\mathrm{B}\lim_{\alpha\to+\infty}\frac{\left(\frac{b}{a}\right)^{\alpha}\ln b-\ln a}{\left(\frac{b}{a}\right)^{\alpha}-1}-\frac{1}{\alpha}=\lim_{t\to+\infty}\frac{(\ln b)t-\ln a}{t-1}=\ln b$$

from which

$$e^{\ln b}=b$$

i.e. the $\max\{a,b\}$.
It is also possible to provide a second proof, which does not employ the concept of a limit, of the inequalities $a<\eta_{\alpha}<b$, applying the following proposition, stated and proved in [24]

Proposition 1

If *x* > 0 , y > 0 and *r > 1* then

$$ry^{r-1}<\frac{x^{r}-y^{r}}{x-y}<rx^{r-1} \tag{22}$$

If $0<r<1$ then

$$rx^{r-1}<\frac{x^{r}-y^{r}}{x-y}<ry^{r-1} \tag{23}$$

Indeed, $a=y$; $b=x$; $\alpha=r$; if $\alpha>1$ then, from Proposition 1,

$$\alpha a^{\alpha-1}<\frac{b^{\alpha}-a^{\alpha}}{b-a}<\alpha b^{\alpha-1}$$

from which

$$a^{\alpha-1}<\frac{b^{\alpha}-a^{\alpha}}{\alpha(b-a)}<b^{\alpha-1}$$

whence

$$a = (a^{\alpha-1})^{\frac{1}{\alpha-1}} < \eta_\alpha = \left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]^{\frac{1}{\alpha-1}} < (b^{\alpha-1})^{\frac{1}{\alpha-1}} = b$$

i.e.

$$a < \eta_\alpha < b \tag{24}$$

Similarly, we reason for $0 < \alpha < 1$.

2.5 Diagram of SM as the parameter α varies.

It is shown [47] that η_α is increasing at α, from which it follows, by the Limit Theorem of monotone functions, that a and b coincide with the lower and upper extremes, respectively, of the codomain of $\eta_\alpha(a, b)$. Furthermore, the following Stolarsky inequality [47] holds

$$G(x,y) \le L(x,y) \le I(x,y) \le A(x,y)$$

It follows from the above that $\eta_\alpha(a,b)$, regarded as a function of the variable α alone, has as its domain the set $A = R - \{0,1\}$; it is positive in A; it is continuous in A since it is composed of continuous functions; it is increasing in A; in the plane $\alpha - \eta$ admits $\eta = a$ and $\eta = b$ as horizontal asymptotes; it has in $\alpha = 0$ and $\alpha = 1$ eliminable discontinuities and its codomain is between a and b (Fig 1):

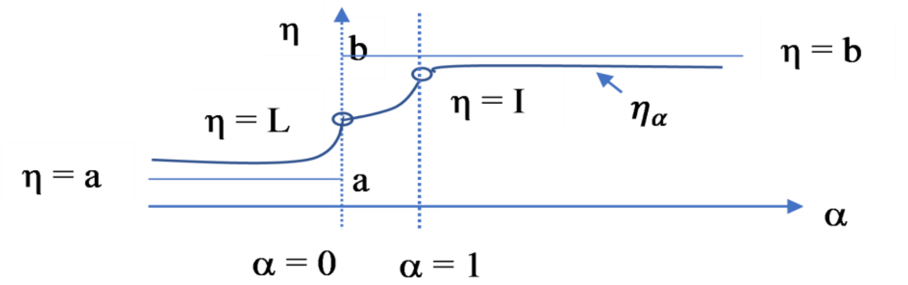

Figure 1 In this figure, we illustrate the diagram of SM η_α as parameter a varies.

Having said $\bar{\eta}_\alpha$ the extension by continuity of \η_α in $\alpha = 0$ and $\alpha = 1$, then $\bar{\eta}_\alpha$ is defined in R and sends R in $]a;b[$.
From now on, we will no longer distinguish η from η_* and denote them both by the notation $\eta_\alpha(a,b)$ or simply η_α .

3. Deduction of classical mean by MVTD‡‡‡

We illustrate some cases below:

case: $f(x) = x^2$ (arithmetic mean of a and b)

$$\frac{b^2 - a^2}{b - a} = 2\eta;\ \frac{(b-a)(b+a)}{b-a} = 2\eta;\ \ \eta = \frac{a+b}{2}$$

case: $f(x) = \frac{1}{x}$ (geometric mean of a and b)

$$\frac{b \cdot \ln b - a \cdot \ln a}{b - a} = \ln \eta + 1;\ \ \frac{\ln b^b - \ln a^a}{b - a} - 1 = \ln \eta;\ \ \ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} - 1$$

$$= \ln \eta;\ \ \eta = \frac{1}{e} \cdot \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

case: $f(x) = x \cdot \ln x$ (identic mean of a and b)

$$\frac{b \cdot \ln b - a \cdot \ln a}{b - a} = \ln \eta + 1;\ \ \frac{\ln b^b - \ln a^a}{b - a} - 1 = \ln \eta;\ \ \ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} - 1$$

$$= \ln \eta;\ \ \eta = \frac{1}{e} \cdot \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

case: $f(x) = \ln x$ (logarithmic mean of a and b)

Because $f'(\eta) = \frac{1}{\eta}$, we have

$$\frac{\ln b - \ln a}{b - a} = \frac{1}{\eta};\ \frac{-(\ln a - \ln b)}{-(a - b)} = \frac{1}{\eta};\ \eta\ =\ \frac{a\ -\ b}{ln\ a\ -\ ln\ b}$$

case: $f'(\eta) = \frac{1}{2\sqrt{\eta}}$ (arithmetic mean of $a, \sqrt{ab}, \sqrt{ab}, b$)

$$\frac{\sqrt{b} - \sqrt{a}}{b - a} = \frac{1}{2\sqrt{\eta}};\ \frac{\sqrt{b} - \sqrt{a}}{(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})} = \frac{1}{2\sqrt{\eta}};\ \frac{\sqrt{b} + \sqrt{a}}{2} = \sqrt{\eta};$$

‡‡‡ Look in this regard at the following link:
https://demonstrations.wolfram.com/StolarskyApproximationsToPopularMeans/

$$\eta = \left(\frac{\sqrt{b}+\sqrt{a}}{2}\right)^2 = \frac{a+2\sqrt{ab}+b}{4} = A\left(a,\sqrt{ab},\sqrt{ab},b\right)$$

4. Deduction of classical means by MVTI

Applying the MVTI for $g(x)=x$, being $g(\eta)=\eta$, we obtain the arithmetic mean

$$\eta = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b =$$

$$= \frac{1}{b-a}\left[\frac{(b-a)(b+a)}{2}\right];\ \eta = \frac{a+b}{2} = A(a,b)$$

Furthermore, if we place $g(x)=x^{-2}$, we get $g(\eta)=\eta^{-2}$, from which applying MVTI we have

$$\eta^{-2} = \frac{1}{b-a}\int_a^b x^{-2}\,dx =$$

$$= \frac{1}{b-a}\cdot[-x^{-1}]_a^b = \frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{ab}$$

$$\eta = \sqrt{ab} = G(a,b)$$

The identical mean I(a,b) can be obtained by applying the MVTI to the function $g(x) = \ln x$:

$$\frac{1}{b-a}\int_a^b \ln x\,dx = \ln\eta\,;\ \frac{1}{b-a}[x\cdot\ln x - x]_a^b = \ln\eta\,;$$

$$\frac{b\ln b - a\ln a - (b-a)}{b-a} = \ln\eta\,;\ \frac{\ln(b^b/a^a)}{b-a} - 1 = \ln\eta\,;\ \ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} - \ln e = \ln\eta$$

$$\ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} + \ln\frac{1}{e} = \ln\eta\ ;\ \ln\left[\frac{1}{e}\cdot\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}\right] = \ln\eta$$

from which

$$I(a,b) = \eta = \frac{1}{e}\cdot\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} =$$

$$\frac{1}{e}\cdot\left(\frac{b^b}{a^a}\right)^{\frac{1}{-(a-b)}} = \frac{1}{e}\cdot\left[\left(\frac{b^b}{a^a}\right)^{-1}\right]^{\frac{1}{a-b}} = \frac{1}{e}\cdot\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}$$

For the logarithmic mean, we start with the function $g(x) = \frac{1}{x}$, whence

$$\frac{1}{b-a}\cdot\int_a^b \frac{1}{x}dx=\frac{1}{\eta};\ \frac{1}{b-a}\cdot[\ln x]_a^b=\frac{1}{\eta};\ \frac{\ln b-\ln a}{b-a}=\frac{1}{\eta};\ \eta=\frac{b-a}{\ln b-\ln a}$$

Finally, the mean power can be obtained from the function $g(x)=x^{-1/2}$:

$$\frac{1}{b-a}\cdot\int_a^b x^{-\frac{1}{2}}dx=\frac{1}{\sqrt{\eta}};\ \frac{1}{b-a}\cdot\left[\frac{\sqrt{x}}{1/2}\right]_a^b=\frac{1}{\sqrt{\eta}};$$

$$\frac{2(\sqrt{b}-\sqrt{a})}{b-a}=\frac{1}{\sqrt{\eta}};\ \eta=\left(\frac{\sqrt{b}+\sqrt{a}}{2}\right)^2$$

5. Some SM with natural n as parameter

As natural n varies, we consider the subfamily family T_n of SM given by

$$\eta_n(a,b)=\left[\frac{b^n-a^n}{n(b-a)}\right]^{\frac{1}{n-1}}$$

From the well-known identity[§§§]

$$b^n-a^n=(b-a)(b^{n-1}+b^{n-2}a+\cdots+ba^{n-2}+a^{n-1}) \tag{25}$$

we get

$$\eta_n(a,b)=\left[\frac{(b-a)(b^{n-1}+b^{n-2}a+\cdots+ba^{n-2}+a^{n-1})}{n(b-a)}\right]^{\frac{1}{n-1}}=$$

$$=\left(\frac{b^{n-1}a^0+b^{n-2}a^1+\cdots+b^1a^{n-2}+b^0a^{n-1}}{n}\right)^{\frac{1}{n-1}} \tag{26}$$

Expression (5) can be thought of as the root of index n-1 of an arithmetic mean of n terms

$$\eta_n(a,b)=\sqrt[n-1]{A(a^{n-1}b^0,a^{n-2}b^1,\cdots,a^1b^{n-2},a^0b^{n-1})} \tag{27}$$

from which the statistical significance of η_n emerges more clearly.

[§§§] Relation (25) is easily proved by induction. The second factor $b^{n-1}+b^{n-2}a+\cdots+ba^{n-2}+a^{n-1}$ is a homogeneous polynomial of degree n-1 in the variables a and b formed by n-1 terms of the type $b^{n-1-j}a^j$ $(j=0,\cdots,n-1)$ in which the powers of b decreases from n-1 to 0 and the powers of a increases from 0 to n-1.

It is possible to obtain a further development of the difference between two powers with integer exponent $a^n - b^n$, and thus another expression of η_n, inspired by [6]. In fact, placing $Q_0 = b - a$, we have

$$a^n = (b + Q_0)^n = b^n + nb^{n-1}Q_0 + \frac{n(n-1)}{2!}b^{n-2}Q_0^2 + \cdots + Q_0^n$$

whence

$$a^n - b^n = Q_0\left(nb^{n-1} + \frac{n(n-1)}{2!}b^{n-2}Q_0 + \cdots + Q_0^{n-1}\right)$$

and then

$$\eta_n(a,b) = \left[\frac{nb^{n-1} + \frac{n(n-1)}{2!}b^{n-2}Q_0 + \cdots + b^0Q_0^{n-1}}{n}\right]^{\frac{1}{n-1}}$$

The latter relationship can also be related as an arithmetic mean of n terms:

$$\eta_n(a,b) = \sqrt[n-1]{A\left(nb^{n-1}, \frac{n(n-1)}{2!}b^{n-2}Q_0\,, \cdots, b^0Q_0^{n-1}\right)}$$

Later we will see that the second factor of (25) can be appropriately factorized in some special cases (when n is an integer power of 2, or 3, or 5, ..., $p^h (p\ prime\ number;\ h \in Z)$ and the SM can be characterized as classical means.
Earlier we provided as many as two demonstrations of the inequality (17) for every α real number and different from 0 and 1. Relative to the case $\alpha = n \in N$, based on (25) and inspired by [6] it is possible to provide a further demonstration that $a < \eta_n(a,b) < b$ ($\forall n \in N$). In fact, the following holds

Theorem 1 Let 0 < a < b. Then

$$a < \eta_n < b \ (\forall n \in N)$$

Proof. We first prove the left-hand inequality, viz, i.e.

$$a < \eta_n \ (\forall n \in N)$$

From (25) we derive that

$$b^{n-1} + b^{n-2}a^1 + \cdots + a^{n-1} > (n-1)a^{n-1} \tag{28}$$

whence

$$\frac{b^{n-1} + b^{n-2}a^1 + \cdots + a^{n-1}}{n} > \frac{(n-1)}{n} \cdot a^{n-1}$$

therefore

$$\eta_n > \left(\frac{n-1}{n}\cdot a^{n-1}\right)^{\frac{1}{n-1}} = a\cdot\left(\frac{n-1}{n}\right)^{\frac{1}{n-1}} = a\cdot e^{\left(\frac{1}{n-1}\right)\cdot\ln\left(\frac{n-1}{n}\right)}$$

From

$$\lim_{n\to+\infty}\left(\frac{1}{n-1}\right) = 0;\ \lim_{n\to+\infty}\ln\left(\frac{n-1}{n}\right) = \ln 1 = 0$$

we get

$$\lim_{n\to+\infty}\left[a\cdot e^{\left(\frac{1}{n-1}\right)\cdot\ln\left(\frac{n-1}{n}\right)}\right] = a\cdot e^{0\cdot 0} = a\cdot 1 = a$$

whence the assertion is reached.
As for the proof of the second inequality, we immediately observe that

$$\eta_n = \left[\frac{b^n-a^n}{n(b-a)}\right]^{\frac{1}{n-1}} < \left[\frac{b^n-a^n}{(n-1)(b-a)}\right]^{\frac{1}{n-1}}$$

Since $0 < a < b$, reasoning in a similar way as above, we arrive at the following inequality

$$b^{n-1}+b^{n-2}a^1+\cdots+a^{n-1} < (n-1)b^{n-1} \tag{29}$$

from which

$$\frac{b^n-a^n}{(n-1)(b-a)} < \frac{(n-1)b^{n-1}}{(n-1)} = b^{n-1}$$

therefore

$$\eta_n < [b^{n-1}]^{\frac{1}{n-1}} = b\ (n\in N)$$

In the literature for the development of the difference of two powers b^n-a^n $(0<a<b;\ n\in N)$ only the formula (25) is known. We will show that b^n-a^n, in the special case where$n=p^h$ $(p\ prime\ number;\ h\in N)$, can be written as a product of homogeneous polynomials in a and b, all of positive sign, with a recursive relation.
We now characterize the development of b^n-a^n in the particular case where $n=p^h$ $(p\ prime\ number;\ h\in N)$ as a product of homogeneous polynomials. The following holds

Theorem 2 Let p be a prime number fixed in N and $h\in N$, then

$$b^{p^h}-a^{p^h} = \prod_{i=0}^{h} Q_i \tag{30}$$

where

$$Q_0 = b - a \tag{31}$$

$$Q_i = \sum_{j=0}^{p-1} a^{(p-1-j)\cdot p^{i-1}} b^{j\cdot p^{i-1}} \; (1 \le i \le h) \tag{32}$$

Proof .
The relation (30) is true for $h = 0$. We admit that (30) is true for h (inductive hypothesis). We observe that between the two differences of consecutive powers[****] $\left(b^{p^{h+1}} - a^{p^{h+1}}\right)$ and $\left(b^{p^h} - a^{p^h}\right)$ the following relationship exists:

$$b^{p^{h+1}} - a^{p^{h+1}} = \left(b^{p^h} - a^{p^h}\right) \sum_{j=0}^{p-1} a^{(p-1-j)\cdot p^h} b^{j\cdot p^h}$$

By inductive hypothesis

$$b^{p^h} - a^{p^h} = \prod_{i=0}^{h} Q_i$$

and being

$$\sum_{j=0}^{p-1} a^{(p-1-j)\cdot p^h} b^{j\cdot p^h} = Q_{h+1}$$

then

$$b^{p^{h+1}} - a^{p^{h+1}} = \prod_{i=0}^{h+1} Q_i$$

hence the assertion. We apply the previous result to characterize η_{p^h}.

Corollary 1 Let p be a fixed prime number in N and $n = p^h$ $(h \in N)$. Then

$$\eta_{p^h} = \left[\frac{b^{p^h} - a^{p^h}}{p^h(b-a)}\right]^{\frac{1}{p^h-1}} = \sqrt[p^h-1]{\prod\nolimits_{i=1}^{h} A_i} \qquad (1 \le i \le h), \tag{33}$$

[****] In the sense that the exponents of p, that is, h and $h+1$, are consecutive natural numbers.

where

$$A_i\left(a^{(p-1)\cdot p^{i-1}}b^0, a^{(p-2)\cdot p^{i-1}}b^{1\cdot p^{i-1}}, \dots, a^0 b^{(p-1)\cdot p^{i-1}}\right) \tag{34}$$

Proof.
We have

$$\eta_{p^h} = \left[\frac{b^{p^h} - a^{p^h}}{p^h(b-a)}\right]^{\frac{1}{p^h-1}} = \sqrt[p^h-1]{\frac{\prod_{i=1}^h Q_i}{p^h}}$$

$$= \sqrt[p^h-1]{\frac{\prod_{i=1}^h Q_i}{p^h}} = \sqrt[p^h-1]{\prod\nolimits_{i=1}^h A_i}$$

where the difference of powers $b^{p^h} - a^{p^h}$ is developed as the product of *h+1* homogeneous polynomials Q_i in a and b, of which the first factor is (b-a) while the remaining factors have degree $(p-1)\cdot p^i$ $(1 \le i \le h)$ and are each formed by p terms:

$$b^{p^h} - a^{p^h} = \prod_{i=0}^{h} Q_i$$

$$Q_0 = b - a$$

$$Q_i = \sum_{j=0}^{p-1} a^{(p-1-j)\cdot p^{i-1}} b^{j\cdot p^{i-1}} \ (1 \le i \le h))$$

$$A_i\left(a^{(p-1)\cdot p^{i-1}}b^0, a^{(p-2)\cdot p^{i-1}}b^{1\cdot p^{i-1}}, \dots, a^0 b^{(p-1)\cdot p^{i-1}}\right)(1 \le i \le h))$$

Corollary 2 Let p be a fixed prime number in N and $h \in N$. Then

$$b^{-p^h} - a^{-p^h} = \frac{\prod_{i=0}^h Q_i}{a^{p^h} b^{p^h}}$$

where

$$Q_0 = b - a$$

$$Q_i = \sum_{j=0}^{p-1} a^{(p-1-j)\cdot p^{i-1}} b^{j\cdot p^{i-1}} \ (1 \le i \le h)$$

Proof.
We proceed as in Theorem 2.

Corollary 3 Let p be a fixed prime number in N and $h \in N$. Then the following identity is valid

$$\eta_{-p^h} = \left[\frac{b^{-p^h} - a^{-p^h}}{-p^h(b-a)}\right]^{\frac{1}{-p^h-1}} = G(a,b) \cdot \sqrt[p^h+1]{\frac{G^{p^h-1}(a,b)}{\prod_{i=1}^{h} A_i}}$$

Proof.
By Corollary 2 we obtain

$$\eta_{-p^h} = \left[\frac{b^{-p^h} - a^{-p^h}}{-p^h(b-a)}\right]^{\frac{1}{-p^h-1}} =$$

$$= \sqrt[p^h+1]{a^{p^h} b^{p^h} \cdot \frac{p^h}{\prod_{i=1}^{h} Q_i}}$$

$$= \sqrt[p^h+1]{\frac{G^{2\cdot p^h}(a,b)}{\prod_{i=1}^{h} A_i}} = G(a,b) \cdot \sqrt[p^h+1]{\frac{G^{p^h-1}(a,b)}{\prod_{i=1}^{h} A_i}}$$

where

$$Q_i = \sum_{j=0}^{p-1} a^{(p-1-j)\cdot p^{i-1}} b^{j\cdot p^{i-1}} \; (1 \leq i \leq h)$$

$$A_i\left(a^{(p-1)\cdot p^{i-1}} b^0, a^{(p-2)\cdot p^{i-1}} b^{1\cdot p^{i-1}}, \ldots, a^0 b^{(p-1)\cdot p^{i-1}}\right)(1 \leq i \leq h))$$

and also

$$\eta_{-p^h}(a,b) = \sqrt[p^h+1]{G^2(a,b) \cdot \prod_{i=1}^{h} H_i}$$

where

$$H_i\left(a^{(p-1)\cdot p^{i-1}} b^0, a^{(p-2)\cdot p^{i-1}} b^{1\cdot p^{i-1}}, \ldots, a^0 b^{(p-1)\cdot p^{i-1}}\right)(1 \leq i \leq h))$$

6. Statistical significance of η_{2^h} (h ∈ N)

6.1 Statistical significance of η_4

We can also reason for the case $\alpha = 4$ in the same way as for the previous cases, limiting ourselves to applying only the general formula of SM:

$$\eta_4(a,b) = \left[\frac{b^4 - a^4}{4(b-a)}\right]^{\frac{1}{3}} = \left[\frac{(b-a)(b+a)(b^2+a^2)}{4(b-a)}\right]^{\frac{1}{3}} =$$

$$= \left[\frac{(a+b)(a^2+b^2)}{4}\right]^{\frac{1}{3}} = \left[\frac{(a+b)(a^2+b^2)}{4}\right]^{\frac{1}{3}}$$

$$= \left[\left(\frac{a+b}{2}\right)^2 \cdot \left(\frac{a^2+b^2}{a+b}\right)\right]^{\frac{1}{3}} = [A^2(a,b) \cdot C(a,b)]^{\frac{1}{3}}$$

i.e.

$$\eta_4(a,b) = \sqrt[3]{A^2 \cdot C}$$

where

$$C(a,b) = \frac{a^2+b^2}{a+b}$$

is the contra-harmonic mean of *a* and *b*, with *0 < a < b*.
Otherwise

$$\left[\frac{(a+b)(a^2+b^2)}{4}\right]^{\frac{1}{3}} = \left[\frac{a+b}{2} \cdot \frac{a^2+b^2}{2}\right]^{\frac{1}{3}} = \sqrt[3]{A(a,b) \cdot QM(a^2,b^2)}$$

and so on.
We illustrate the formula

$$\eta_4(a,b) = \sqrt[3]{A^2 \cdot C}$$

in space $AC\eta$ space, where A, C and η represent the three coordinate axes respectively. We have (Fig. 2)

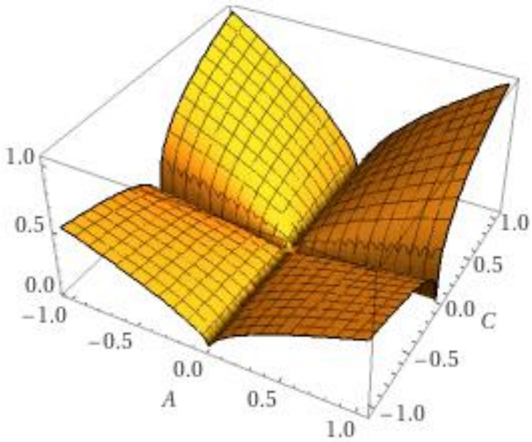

Figure 2 Geometric interpretation of η_4 (Wolfram Alpha)

6.2 Statistical significance of η_8

For $\alpha = 8$ we get

$$\eta_8(a,b) = \left[\frac{b^8 - a^8}{8(b-a)}\right]^{\frac{1}{7}} = \left[\frac{(b-a)(b+a)(b^2+a^2)(b^4+a^4)}{8(b-a)}\right]^{\frac{1}{7}} =$$

$$= \sqrt[7]{A^2(a,b)\cdot C(a,b)\cdot QM^2(a^2,b^2)}$$

6.3 Statistical significance of η_{2^h}

We saw earlier that from the well-known formula concerning the difference of two squares

$$b^2 - a^2 = (b-a)(b+a)$$

one can also deduce the formulas

$$b^4 - a^4 = (b-a)(b+a)(b^2+a^2)$$

$$b^8 - a^8 = (b-a)(b+a)(b^2+a^2)(b^4+a^4)$$

More generally, $\forall\ h \geq 1$, one can prove by induction the formula

$$b^{2^h} - a^{2^h} = (b-a)(a+b)(a^2+b^2)\cdots\left(a^{2^{h-1}} + b^{2^{h-1}}\right) \tag{35}$$

It is true for $h = 1$:

$$b^{2^1} - a^{2^1} = b^2 - a^2 = (b-a)(b+a)$$

We admit that it is true for some *h* (inductive hypothesis) and prove that it is true for *h+1*. We observe[††††] that the following decomposition applies:

$$b^{2^{h+1}} - a^{2^{h+1}} = \left(b^{2^h}\right)^2 - \left(a^{2^h}\right)^2 = \left(b^{2^h} - a^{2^h}\right) \cdot \left(b^{2^h} + a^{2^h}\right)$$

For the inductive hypothesis, developing the first factor to the right of the second equality gives:

$$b^{2^{h+1}} - a^{2^{h+1}} = (b-a)(a+b)(a^2+b^2)\cdots\left(a^{2^{h-1}} + b^{2^{h-1}}\right)\left(a^{2^h} + b^{2^h}\right)$$

From the above observation and taking into account that

$$a^{2^{h-1}} = a^{2^1 \cdot 2^{h-2}} = \left(a^{2^{h-2}}\right)^2;\; b^{2^{h-1}} = b^{2^1 \cdot 2^{h-2}} = \left(b^{2^{h-2}}\right)^2$$

is easily deduced by induction, $\forall h \geq 2$, that

$$\eta_{2^h}(a,b) = \left[\frac{b^{2^h} - a^{2^h}}{2^h(b-a)}\right]^{\frac{1}{2^h-1}}$$

$$= \sqrt[2^h-1]{A^2(a,b) \cdot C(a,b) \cdot QM^2(a^2,b^2) \cdots QM^2\left(a^{2^{h-2}}, b^{2^{h-2}}\right)}$$

or

$$= \sqrt[2^h-1]{A(a,b) \cdot QM^2(a^2,b^2) \cdots QM^2\left(a^{2^{h-2}}, b^{2^{h-2}}\right)}$$

remarkable result that tells us that the Stolarsky means of parameter α equal to $2^h (h \geq 2)$ can be written explicitly in terms of some classical means, more precisely they can be traced to a radical having as its index $2^h - 1$, and as its radical a product in which the powers of the averages of *A*, *C* and *QM* appear as factors, or only of *A* and *QM*.

Developing the definition of SM based on (35) gives:

$$\eta_{2^h} = \sqrt[2^h-1]{\frac{(b-a)(a+b)(a^2+b^2)\cdots\left(a^{2^{h-1}} + b^{2^{h-1}}\right)}{2^h(b-a)}} =$$

$$= \sqrt[2^h-1]{\frac{(a+b)(a^2+b^2)\cdots\left(a^{2^{h-1}} + b^{2^{h-1}}\right)}{2^h}}$$

[††††] Indeed $\left(b^{2^h}\right)^2 = b^{2 \cdot 2^h} = b^{2^1 \cdot 2^h} = b^{2^{h+1}}$ e $\left(a^{2^h}\right)^2 = a^{2 \cdot 2^h} = a^{2^1 \cdot 2^h} = a^{2^{h+1}}$

$$= \sqrt[2^h-1]{\left(\frac{a+b}{2}\right)\cdot\left(\frac{a^2+b^2}{2}\right)\left(\frac{a^4+b^4}{2}\right)\cdots\left(\frac{a^{2^{h-1}}+b^{2^{h-1}}}{2}\right)}$$

$$= \sqrt[2^h-1]{A(a,b)\cdot QM^2(a^2,b^2)\cdots QM^2\left(a^{2^{h-2}},b^{2^{h-2}}\right)}$$

This formula is true, as we have seen, for $h = 2$. We admit that it is true for a certain $h \in N$ and we show that it is also true for $h+1$. It suffices to observe that

$$\left(a^{2^{h-1}}\right)^2+\left(a^{2^{h-1}}\right)^2 = a^{2\cdot 2^{h-1}}+b^{2\cdot 2^{h-1}} = a^{2^h}+b^{2^h}$$

and that

$$\frac{a^{2^h}+b^{2^h}}{2} = \left[\frac{\left(a^{2^{h-1}}\right)^2+\left(a^{2^{h-1}}\right)^2}{2}\right]^2 = QM^2\left(a^{2^{h-1}},b^{2^{h-1}}\right)$$

or

$$= \sqrt[2^h-1]{\frac{(a+b)(a^2+b^2)\cdots\left(a^{2^{h-1}}+b^{2^{h-1}}\right)}{2^h}}$$

$$= \sqrt[2^h-1]{\left(\frac{a+b}{2}\right)^2\cdot\left(\frac{a^2+b^2}{a+b}\right)\cdot\left(\frac{a^4+b^4}{2}\right)\cdots\left(\frac{a^{2^{h-1}}+b^{2^{h-1}}}{2}\right)}$$

$$= \sqrt[2^h-1]{\left(\frac{a+b}{2}\right)^2\cdot\left(\frac{a^2+b^2}{a+b}\right)\cdot\left[\frac{(a^2)^2+(b^2)^2}{2}\right]\cdots\left[\frac{\left(a^{2^{h-2}}\right)^2+\left(b^{2^{h-2}}\right)^2}{2}\right]}$$

$$= \sqrt[2^h-1]{A^2(a,b)\cdot C(a,b)\cdot QM^2(a^2,b^2)\cdots QM^2\left(a^{2^{h-2}},b^{2^{h-2}}\right)}$$

This formula is true, as we have seen, for $h = 2$. We admit that it is true for a certain $h \in N$ and we show that it is also true for $h+1$. It suffices to observe that

$$\left(a^{2^{h-1}}\right)^2+\left(a^{2^{h-1}}\right)^2 = a^{2\cdot 2^{h-1}}+b^{2\cdot 2^{h-1}} = a^{2^h}+b^{2^h}$$

and that

$$\frac{a^{2^h}+b^{2^h}}{2} = \left[\frac{\left(a^{2^{h-1}}\right)^2+\left(a^{2^{h-1}}\right)^2}{2}\right]^2 = QM^2\left(a^{2^{h-1}},b^{2^{h-1}}\right)$$

7. Statistical significance of η_{-2} , η_{-4} and η_{-2^h} $(h \in N)$

The case where the parameter α is worth -2 deserves special attention because it allows us to arrive at two interesting functional relationships, one between the harmonic mean H and the geometric mean *G* and another between the geometric mean *G* and the arithmetic mean *A*.
Let us begin with the first relationship. By positing $\alpha = -2$ in Stolarsky's general means formula $\eta_\alpha(a,b) = \left[\frac{b^\alpha - a^\alpha}{\alpha(b-a)}\right]^{\frac{1}{\alpha-1}}$ (or also by applying MVTD to the function $f(x) = x^{-2}$, or by applying the MVTI to the function $g(x) = x^{-3}$) we have the following functional relationship between the harmonic mean H and the geometric mean G of two positive numbers a and b with a < b[‡‡‡‡]

$$\eta_{-2} = \sqrt[3]{H(a,b) \cdot G^2(a,b)}$$

Indeed

$$\eta_{-2}(a,b) = \left[\frac{b^{-2} - a^{-2}}{-2(b-a)}\right]^{\frac{1}{-2-1}} = \left[\frac{\left(\frac{1}{b} - \frac{1}{a}\right)\left(\frac{1}{b} + \frac{1}{a}\right)}{-2(b-a)}\right]^{-\frac{1}{3}}$$

$$= \left[\frac{\left(\frac{b-a}{ab}\right)\left(\frac{1}{b} + \frac{1}{a}\right)}{2(b-a)}\right]^{-\frac{1}{3}} = \left[\frac{\left(\frac{1}{b} + \frac{1}{a}\right)}{2} \cdot \frac{1}{ab}\right]^{-\frac{1}{3}} = \sqrt[3]{\left(\frac{2}{\frac{1}{a} + \frac{1}{b}}\right)(ab)}$$

whence the assertion being

$$H(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}; \; G(a,b) = \sqrt{ab}$$

We observe that you also have

$$\eta_{-2} = \sqrt[3]{\frac{2}{a+b} \cdot (ab)^2} = \sqrt[3]{\frac{G^4}{A}} = G \cdot \sqrt[3]{\frac{G}{A}}$$

Instead, applying the MVTI to the function $f(x) = x^{-3}$ is done as follows:

$$\frac{1}{b-a}\int_a^b x^{-3}\,dx = \eta^{-3}$$

[‡‡‡‡] The latter condition is necessary to be able to apply the MVTI.

$$\frac{1}{b-a}\left[\frac{x^{-2}}{-2}\right]_a^b = \eta^{-3};\ \ \frac{1}{b-a}\cdot\frac{b^{-2}-a^{-2}}{-2} = \eta^{-3}$$

$$\frac{1}{b-a}\cdot\frac{\frac{1}{b^2}-\frac{1}{a^2}}{-2} = \eta^{-3};\ \ \frac{\left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{b}+\frac{1}{a}\right)}{-2(b-a)} = \eta^{-3}$$

$$\frac{\left(\frac{b-a}{ab}\right)\left(\frac{1}{b}+\frac{1}{a}\right)}{2(b-a)} = \eta^{-3};\ \frac{1}{ab}\cdot\ \frac{\frac{1}{a}+\frac{1}{b}}{2} = \eta^{-3};\ \ \eta^3 = \left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right)(ab)$$

$$\eta_{-3} = \sqrt[3]{\frac{2}{\frac{1}{a}+\frac{1}{b}}}\cdot\sqrt[3]{ab} = \sqrt[3]{H(a,b)}\cdot\sqrt[3]{G^2(a,b)} = \sqrt[3]{G^2(a,b)\cdot H(a,b)}$$

Finally, applying MVTD to the function $f(x) = x^{-2}$ we start from the following relationship:

$$\frac{b^{-2}-a^{-2}}{b-a} = -2\eta^{-3}$$

and continue as above.

The case $\alpha = -4$ has strong similarities with the case $\alpha = -2$. In fact, developing the general formula of SM gives:

$$\eta_{-4}(a,b) = \left[\frac{b^{-4}-a^{-4}}{-4(b-a)}\right]^{\frac{1}{-4-1}} = \left[\frac{\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left(\frac{1}{b^2}+\frac{1}{a^2}\right)}{-4(b-a)}\right]^{-\frac{1}{5}}$$

$$= \left[\frac{\left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{b}+\frac{1}{a}\right)\left(\frac{1}{b^2}+\frac{1}{a^2}\right)}{-4(b-a)}\right]^{-\frac{1}{5}} = \left[\frac{\left(\frac{a-b}{ab}\right)\left(\frac{1}{b}+\frac{1}{a}\right)\left(\frac{1}{b^2}+\frac{1}{a^2}\right)}{-4(b-a)}\right]^{-\frac{1}{5}} =$$

$$= \left[\frac{1}{4}\cdot\frac{b-a}{ab}\cdot\frac{1}{b-a}\cdot\left(\frac{1}{b}+\frac{1}{a}\right)\left(\frac{1}{b^2}+\frac{1}{a^2}\right)\right]^{-\frac{1}{5}} =$$

$$= \left[\frac{1}{ab}\cdot\frac{\frac{1}{a}+\frac{1}{b}}{2}\cdot\frac{\frac{1}{a^2}+\frac{1}{b^2}}{2}\right]^{-\frac{1}{5}} = \left[(ab)\cdot\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right)\cdot\left(\frac{2}{\frac{1}{a^2}+\frac{1}{b^2}}\right)\right]^{\frac{1}{5}}$$

$$= \sqrt[5]{G^2(a,b)\cdot H(a,b)\cdot H(a^2,b^2)}$$

or

$$\eta_{-4}(a,b)=\left[\frac{1}{-4(b-a)}\cdot\frac{(a-b)(a+b)(a^2+b^2)}{a^4b^4}\right]^{-\frac{1}{5}}=\left[a^4b^4\cdot\frac{2}{a+b}\cdot\frac{2}{a^2+b^2}\right]^{\frac{1}{5}}$$

$$=\sqrt[5]{\frac{G^8(a,b)}{A(a,b)\cdot A(a^2,b^2)}}=G\sqrt[5]{\frac{G^3(a,b)}{A(a,b)\cdot A(a^2,b^2)}}$$

For $n=-2^h(h\in N)$ we have, reasoning by induction,

$$\eta_{-2^h}=\sqrt[2^h+1]{G^2(a,b)\cdot H(a,b)\cdot H(a^2,b^2)\cdots H\left(a^{2^{h-1}},b^{2^{h-1}}\right)}$$

or

$$\eta_{-2^h}=\sqrt[2^h+1]{\frac{G^{2^{h+1}}(a,b)}{A(a,b)\cdot A(a^2,b^2)\cdots A\left(a^{2^{h-1}},b^{2^{h-1}}\right)}}$$

We observe that $2^{h+1}>2^h+1$, and being $2^{h+1}-2^h-1=2^h-1$, we have that

$$2^{h+1}=(2^h+1)+(2^h-1)$$

Whence

$$\eta_{-2^h}=G(a,b)\cdot\sqrt[2^h+1]{\frac{G^{2^h-1}(a,b)}{A(a,b)\cdot A(a^2,b^2)\cdots A\left(a^{2^{h-1}},b^{2^{h-1}}\right)}}$$

8. The mean η_3

Applying MVTD to the function $f(x)=x^3$ we have

$$\frac{b^3-a^3}{b-a}=3\eta^2;\ \frac{(b-a)\cdot(b^2+ba+a^2)}{b-a}=3\eta^2;\eta=\sqrt{\frac{a^2+ab+b^2}{3}}$$

Instead, applying the MVTI to the function $g(x)=x^2$ gives [42]

$$\frac{1}{b-a}\cdot\int_a^b x^2\,dx=\eta^2;\ \frac{1}{b-a}\cdot\left[\frac{x^3}{3}\right]_a^b=\eta^2;\ \frac{1}{b-a}\cdot\frac{b^3-a^3}{3}=\eta^2;\ \eta=\sqrt{\frac{a^2+ab+b^2}{3}}$$

Or more simply by placing *n = 3* in the expression of the $\eta_n(a,b)$ we have

$$\eta_3(a,b)=\sqrt{\frac{a^2+ab+b^2}{3}}$$

The radical that appears in the expression of η_3 always has meaning for every real *a* and *b*. We show that the function

$$\varphi(a, b) = a^2 + ab + b^2$$

admits absolute minimum at the point (0;0), and thus

$$\varphi(a, b) = a^2 + ab + b^2 \geq 0;\ \varphi(0,0) = 0$$

for every real a and b. We begin by determining any zeros of the partial prime derivatives of $\varphi(a, b)$:

$$\begin{cases} \varphi_a = 2a + b \\ \varphi_b = a + 2b \end{cases};\ \varphi_a(0,0) = \varphi_b(0,0) = 0$$

The only zero of the prime partial derivatives is O(0;0). We calculate the Hessian of the function $\varphi(a, b)$:

$$H(a, b) = \begin{vmatrix} \varphi_{aa} & \varphi_{ab} \\ \varphi_{ba} & \varphi_{bb} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$$

The question arises as to whether $\eta_3 \in]a; b[$. In the case where $0 < a < b$,, the point η_3 is definitely internal to [a;b]. In fact,

$$\sqrt{\frac{a^2 + ab + b^2}{3}} > \sqrt{\frac{a^2}{3}} = \sqrt{3}\,|a| = \sqrt{3}\,a > a;$$

$$\sqrt{\frac{a^2 + ab + b^2}{3}} < \sqrt{\frac{3b^2}{3}} = |b| = b.$$

We observe that, for each positive real *a* and *b*, with *a* < *b*, the inequality applies

$$0 < A(a, b) < \eta_3(a, b)$$

that is, the Stolarsky mean η_3 of parameter $\alpha = 3$ is strictly greater than the arithmetic mean A.
In fact, if absurdly

$$\frac{a + b}{2} \geq \sqrt{\frac{a^2 + ab + b^2}{3}}$$

then

$$\frac{a^2 + 2ab + b^2}{4} \geq \frac{a^2 + ab + b^2}{3}$$

from which

$$3a^2 + 6ab + 3b^2 \geq 4a^2 + 4ab + 4b^2$$

i.e.

$$a^2 + 2ab + b^2 \leq 0$$

vs

$$a^2 + 2ab + b^2 = (a + b)^2 > 0$$

The expression of η_3 can be related to that of quadratic mean *QM* and geometric mean *G*. Indeed

$$\sqrt{\frac{a^2 + ab + b^2}{3}} = \sqrt{\frac{a^2 + \left(\sqrt{ab}\right)^2 + b^2}{3}} = QM(a, G(a,b), b)$$

where

$$QM(x_1, \cdots, x_n) = \sqrt{\frac{{x_1}^2 + \cdots + {x_n}^2}{n}}$$

is the quadratic mean of n real not necessarily positive terms $x_1, \cdots, x_n$.

9. Geometric interpretation of η_3

The mean η_3 can be interpreted by Euclidean Geometry or by Linear Algebra.

9.1 Geometric interpretation using Euclidean Geometry in space.

Recall [9] that, by a well-known result of Euclidean Geometry in space, the volume of truncated square-based straight pyramid is

$$V_{\text{truncated square-based straight pyramid}} = \frac{a^2 + ab + b^2}{3} \cdot h$$

where a, b, h are the edge of the upper base, the edge of the lower base and the height of the truncated square-based straight pyramid, with a < b, respectively (Fig. 3).

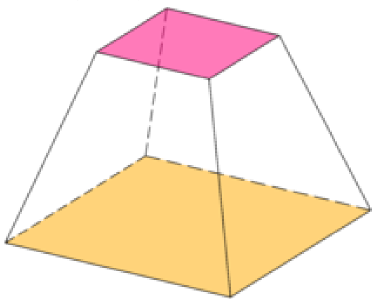

Figure 3 Truncated based straight pyramid

Comparing the radical of the $\eta_3(a,b)$ with the volume of the truncated straight pyramid with a square base, one deduces the equality of the two expressions if and only if h = 1. A similar expression is given for the volume for a straight cone having for its top base a circle of radius *r*, for its top base a circle of radius *R*, with *r* < *R*, and for its height *h* (Fig. 4):

$$V_{\text{truncated circle−based straight cone}} = \frac{r^2 + rR + R^2}{3} \cdot \pi h$$

In the latter case, the equality between the radicand of $\eta_3(a, b)$ and the volume of the truncated straight cone is achieved by posing $h = 1/\pi$.

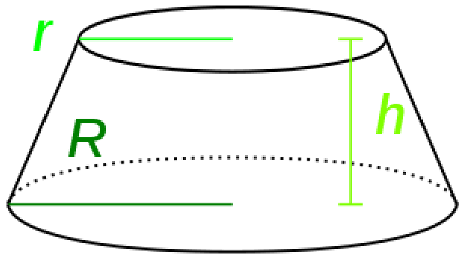

Figure 4 The truncated straight cone

In general, the expression of the truncated straight pyramid having A and B as the area of the bases and h for height is given by

$$V = H(A, B) \cdot h$$

where for heronian mean $H(x, y)$ of two positive numbers x and y we define the expression[§§§§]

$$H(x, y) = \frac{x + \sqrt{xy} + y}{3}$$

similarly we reason for an interpretation in terms of heronian mean of the volume of the truncated right cone having circles as its bases.
Ultimately, using the language of Euclidean Solid Geometry, $\eta_3(a, b)$ can be regarded as the square root of the volume of a truncated straight pyramid with a square base of height $h = 1$ or of the volume of the truncated right cone having circles as bases and height $h = 1/\pi$.
Wanting to further emphasize the juxtaposition of the $\eta_3(a, b)$ with the basic concepts of Euclidean Solid Geometry, it may be mentioned that the measurement of the volume of a truncated straight pyramid with a square base is one of the oldest results in the History of Mathematics as the instructions for determining it are found in the famous Moscow mathematical papyrus written around 1890 BCE [31].

[§§§§] In turn, the Heronian mean can be thought of as a particular weighted average of the arithmetic mean and geometric mean of two positive numbers x and y:

$$H(x, y) = \frac{2}{3} \cdot \frac{x + y}{2} + \frac{1}{3} \cdot \sqrt{xy}$$

9.2 Geometric interpretation of the $\eta_3(a,b)$ by linear algebra

For simplicity, let us pose $\eta_3(a,b) = z$, a = x e b = y. Then the previous formula becomes

$$z = \sqrt{\frac{x^2 + xy + y^2}{3}}$$

from which

$$x^2 + xy + y^2 - 3z^2 = 0 \qquad (36)$$

The surfaces generated by z (in blue) and the rooting (in purple) can thus be visualized using Geogebra 3D (Fig. 5):

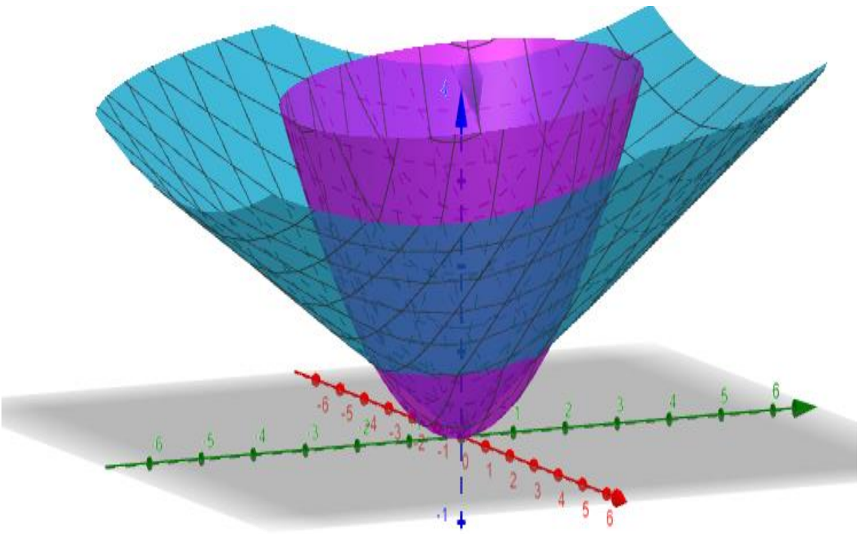

Figure 5 The surface *z* in blue and the rooting in purple (Geogebra 3D)

Instead, (36) generates a two-pitch cone with vertex at *O(0,0,0)* (Fig. 6)

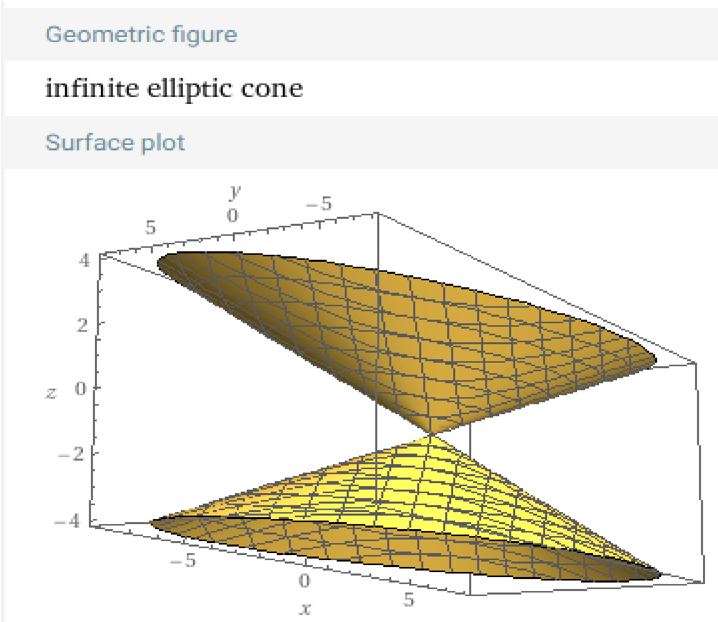

Figure 6 The infinite elliptic cone (Wolfram Alpha)

We think of the elements of R^3 as points, and each point of R^3 we identify with the vector x of its coordinates, the origin of the reference system corresponds to the null vector 0, while the coordinate axes are the straight lines generated by the ortho-normal basis $\{e_1, e_2, e_3\}$, where $e_1 = [1,0,0]^T, e_2 = [0,1,0]^T, e_3 = [0,0,1]^T$ are two-by-two orthogonal verses.

Recall that a quadric is defined as the locus of points in R^3 that satisfy the following equation in the *x,y,z* coordinates: $a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2b_1x + 2b_2y + 2b_3z + c = 0$
and that it depends on the 10 coefficients $a_{11},\ a_{22}, a_{33}, a_{12}, a_{13}, a_{23}, b_1, b_2, b_3, c$.
Let's say $\boldsymbol{x}^T = [x, y, z]$ the row vector formed by the variables x,y,z, A the symmetric matrix[*****] given by the coefficients of the second-degree terms,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

[*****] The square matrix A is defined as symmetrical if $A^T = A$, where A^T is the transposed matrix of *A*.

$\boldsymbol{b}^T = [b_1, b_2, b_3]$ the row vector of the coefficients of the first-degree terms in x, y, z and c the constant, the equation of the quadric can be written in a more compact form as follows:

$$q(\boldsymbol{x}) = \boldsymbol{x^T A x} + 2\boldsymbol{b}^T\boldsymbol{x} + c \tag{37}$$

q being the sum of the quadratic form[†††††] $\boldsymbol{x^T A x}$,of the linear form $2\boldsymbol{b}^T\boldsymbol{x}$ and the constant c. Comparing (36) with (37), the square of η_3 looks like a quadric $q(x)$, where $x = [x, y, z]^T$ having

$$A = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix};\ \boldsymbol{b}^T = [0, 0,0];\ c = 0$$

The following block matrix is used to classify the quadric q arising from the expression of η_3

$$B = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix}$$

whereby *q(x)* takes the form

$$q(x) = [x^T \quad 1]B\begin{bmatrix} x \\ 1 \end{bmatrix} = z^T B z$$

where z is the vector obtained by adding the number 1 to *x* as the last component. *B* is said to be the matrix associated with the polynomial q(*x*). Therefore

$$B = \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is immediately seen that $det(A) = -9/4 \neq 0$ whence rank(A) = 3. Moreover *rank(B) = 3* since *det (B) = 0* being the fourth row (the fourth column) of B consisting of all 0s. It is concluded that *q(x)* is a real cone. Since b is the null vector the cone q(*x*) already has its vertex in the origin of the Cartesian reference[‡‡‡‡‡] . To write the cone in reduced form, we need to eliminate the second-degree term of "mixed" type *xy* that appears in (37) by performing a rotation *x = QX* where *Q* is an orthogonal matrix with determinant equal to 1. The matrix *Q* will have for columns the normalized eigenvectors relative to the eigenvalues of *A*. To obtain *det(Q) = 1* eventually we will change the sign

[†††††] A quadratic form in two variables x and y is defined as a homogeneous polynomial of second degree of the type

$$q(x, y) = ax^2 + 2bxy + cy^2 = [x \quad y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}[x \quad y]$$

[‡‡‡‡‡] In fact, since b = 0, the linear system Aw=b corresponds to the homogeneous system Aw=0, and since *rank(A)=rank(A|B)=3* , the system Aw=0 admits as its only solution w = 0. Therefore, the transformation $x = X - w$, which allows the elimination of any first-degree terms in the quadric equation, is the translation of amplitude *w = 0*, i.e., the identical affine transformation *x = X*. A foregone consideration because of the absence of linear terms in x, y and z in the equation of the quadric we are studying.

to the components of one of the columns of *Q*. For this purpose we determine the eigenvalues of the real symmetric matrix A[§§§§§] . We obtain

$$p(\lambda) = |\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda - 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 3 \end{vmatrix} = 0$$

Developing the determinant according to the third line gives:

$$(\lambda + 3)\left[(\lambda - 1)(\lambda\ - 1) - \frac{1}{4}\right] = 0$$

$$(\lambda + 3)\left(\lambda^2 - 2\lambda + 1 - \frac{1}{4}\right) = 0$$

$$(\lambda + 3)\left(\lambda^2 - 2\lambda + \frac{3}{4}\right) = 0$$

$$(\lambda + 3)\left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{3}{2}\right) = 0$$

The eigenvalues then are all real, as we expected, and distinct[******] and are given by

$$\lambda_1 = -3\ ;\ \lambda_2 = \tfrac{1}{2}\ ;\ \lambda_3 = \tfrac{3}{2}$$

9.2.1 Eigenvectors and main axes of the quadric

We solve the system $\lambda I - A = 0$ for the different eigenvalues found above.

$$\begin{cases} (\lambda - 1)x - \frac{1}{2}y = 0 \\ -\frac{1}{2}x + (\lambda - 1)y = 0 \\ (\lambda + 3)z = 0 \end{cases}$$

V_1 eigenspace related to the eigenvalue $\lambda_1 = -3$:

[§§§§§] The eigenvalues of a real symmetric matrix are real numbers.

[******] Recall that given a real symmetric matrix *A*, distinct eigenvalues of *A* correspond to two-by-two orthogonal eigenvectors.

$$\begin{cases} -4x - \frac{1}{2}y = 0 \\ -\frac{1}{2}x - 4y = 0 \\ (-3+3)z = 0; 0 \cdot z = 0 \end{cases} ; \begin{cases} -8x - y = 0;\ 8x + y = 0 \\ -x - 4y = 0;\ x + 4y = 0 \\ 0 \cdot z = 0 \end{cases}$$

We step down the matrix of coefficients of the unknowns:

$$\begin{bmatrix} 8 & 1 & 0 \\ 1 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & 1 & 0 \\ 0 & 63 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix of coefficients of unknowns has rank = 2. Therefore, the number of free variables is 1 (= number of unknowns - matrix rank = 3-2). An arbitrary value k is assigned to z.

$$\begin{cases} 8x + y = 0 \\ x + 8y = 0 \end{cases}$$

$$\begin{cases} y = -8x\ ; \\ x - 64x = 0;\ -63x = 0;\ x = 0 \rightarrow y = 0 \\ z = k \end{cases}$$

By writing *(0;0;k) = k(0;0;1)* we have that the eigenvector for *k = 1* corresponding to the eigenvalue $\lambda = -3$ is

$$v_1 = (0{,}0{,}1)$$

Therefore the eigenspace V_1 generated by $\lambda_2 = \frac{1}{2}$ is the line of R^3 passing through the points

$$O = (0{,}0{,}0)\ ;\ P_1 = (0{,}0{,}1).$$

V_2 eigenspace related to the eigenvalue $\lambda_2 = \frac{1}{2}$:

$$\begin{cases} -\frac{1}{2}x - \frac{1}{2}y = 0; x + y = 0 \\ -\frac{1}{2}x - \frac{1}{2}y = 0;\ -x + y = 0; x + y = 0 \\ \left(\frac{1}{2} + 3\right)z = 0;\ z = 0 \end{cases}$$

Stepwise reduction of the matrix of coefficients of unknowns

$$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The coefficient matrix has *rank 2*. Therefore, the homogeneous system possesses only one free variable. We assign y an arbitrary value k, $y = k$.
Since $x = -y$ and $z = 0$ then we have *(-y;y;0) = k(-1;1;0)* from which the eigenvector $v_2 = (-1,1,0)$.
The eigenspace V_2 generated by the eigenvector v_2 is the line of R^3 passing through the points $O = (0,0,0)$ and $P_2 = (-1,1,0)$.
The V_3 eigenspace related to the eigenvalue $\lambda_3 = 3/2$:

$$\begin{cases} \left(\frac{3}{2} - 1\right)x - \frac{1}{2}y = 0 \\ -\frac{1}{2}x + \left(\frac{3}{2} - 1\right)y = 0;\ -x + y = 0 \\ \left(\frac{3}{2} + 3\right)z = 0 \end{cases}$$

$$\begin{cases} \frac{1}{2}x - \frac{1}{2}y = 0 \\ -\frac{1}{2}x + \frac{1}{2}y = 0; \\ \frac{9}{2}z = 0 \end{cases} \begin{cases} x - y = 0 \\ -x + y = 0 \\ z = 0 \end{cases}$$

Stepwise reduction of the matrix of coefficients of unknowns:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix, reduced to steps, has two nonzero rows, and therefore *rank 2*. The number of free variables is 1. We assign y an arbitrary value, $y = k$. Since $x = y$ and $z = 0$ then *(y;y;0) = k (1;1;0)*
Eigenvector for $k = 1$:

$$v_3 = (1,1,0)$$

The eigenspace V_3 generated by v_3 is the line of R^3 passing through the points $O = (0,0,0)$ and $P_3 = (1,1,0)$.
Moreover, the space R^3 is direct sum of subspaces of R^3 two by two orthogonal to each other :

$$R^3 = V_1 \oplus V_2 \oplus V_3$$

We represent in R^3 using Geogebra 3D the vectors $\{v_1, v_2, v_3\}$ (Fig.7) from which the eigenspaces generated by them can be deduced:

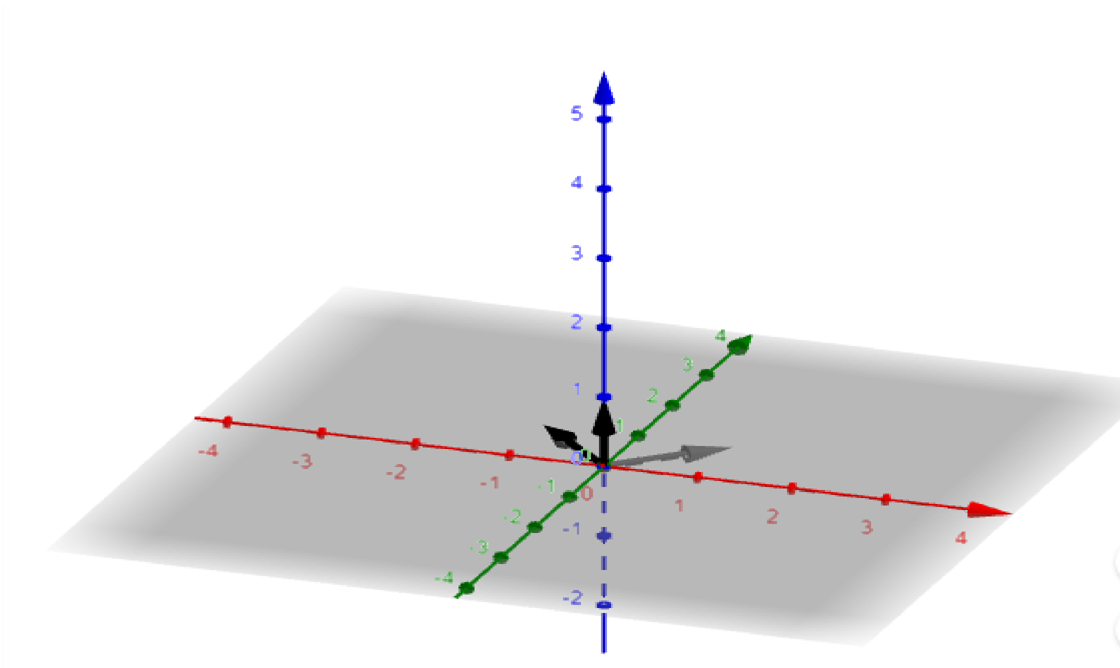

Figure 7 Representation in R^3 dei vettori v_1, v_2 and v_3.

Rotation matrix and reduced form of the quadric

The matrix S

$$S = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

having for columns the eigenvectors of A v_1, v_2, v_3 found above allows A to be diagonalized but in general is not an orthogonal matrix, as we shall now verify: Developing the determinant of S according to the third row gives:

$$\det(S) = \begin{vmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = (+1) \cdot \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

Since $\det(S) = -2 \neq 0$ then S is invertible and its inverse is

$$S^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

We verify that the eigenvectors of A forming S, v_1, v_2, v_3, are two by two orthogonal

$$v_1 \cdot v_2 = 0 \cdot (-1) + 0 \cdot 1 + 1 \cdot 0 = 0$$

$$v_1 \cdot v_3 = 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 0$$

$$v_2 \cdot v_3 = (-1) \cdot 1 + 1 \cdot 1 + 0 \cdot 0 = -1 + 1 = 0$$

and that they are not all of unified standard:

$$\|v_1\| = 1;\ \|v_2\| = \sqrt{2};\ \|v_3\| = \sqrt{2}$$

The matrix S is not orthogonal[††††††] : the product of the matrix by its transpose does not coincide with the identity matrix I:

$$SS^T = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We normalize the eigenvectors v_1, v_2, v_3 by dividing each eigenvector by its norm:

$$u_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; u_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix};\ u_3 = \frac{v_3}{\|v_3\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

The matrix S^* formed by the orthonormal base $\{u_1, u_2, u_3\}$ of R^3:

$$S^* = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$$

is orthogonal:

$$det(S^*) = \begin{vmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1$$

[††††††] A square matrix Q of order n is defined as orthogonal if $Q^T Q = I$, where I is the identity matrix of order n. Since $(Q^T Q)^T = I^T \Leftrightarrow QQ^T = I$ it is obtained that Q orthogonal if $QQ^T = I$. It follows from the definition that the orthogonal matrix Q admits inverse Q^{-1} e $Q^{-1} = Q^T$. Furthermore $det(Q) = \pm 1$.

but it does not have determinant equal to 1 and therefore is not a rotation matrix Q. We change the sign of the third column of S^* and obtain

$$Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$$

whence $det(Q) = \frac{1}{2} + \frac{1}{2} = 1$. By means of the transformation x =QX we can eliminate the mixed term in xy from equation (11). The subspaces of R^3 generated by $\{u_1, u_2, u'_3\}$,, where $u'_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$ represent the principal axes.

Trace of the rotation matrix Q and rotation angle

$$Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$$

$$Trace(Q) = 0 + \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$$

From

$$\cos\theta = \frac{1}{2}\,(Trace(Q) - 1) = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - 1\right)$$

obtained from Euler's Theorem, we derive the angle of rotation.

We show that a section of the cone q(x) is a real ellipse. We section the quadric Q by a plane *z = k,* with *k >0,* the radical being nonnegative:

$$\frac{x^2 + xy + y^2}{3} = k$$

We illustrate the case for k = 1, i.e., cutting the quadric with a z = 1 plane yields the conic (Fig. 8)

$$q'(x, y) = x^2 + xy + y^2 - 3 = 0$$

Like the quadric studied above, *q'(x,y)* can also be written in a more compact form

$$q(\boldsymbol{x}) = \boldsymbol{x^T A x} + 2\boldsymbol{b}^T\boldsymbol{x} + c \tag{38}$$

where

$$A' = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix};\ b = [0 \quad 0]^T;\ c' = -3\ (for\ k = 1;\ c' = \ -3k\ in\ the\ general\ case)$$

is the matrix of the coefficients of the terms of the second degree and B'

$$B' = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

is the block matrix

$$B' = \begin{bmatrix} A' & b' \\ b'^T & c' \end{bmatrix}$$

Since

$$det(B') = (-3)\left(1 - \frac{1}{4}\right) = -\frac{9}{4} \neq 0$$

we can state that *q'* is nondegenerate conic having the center *C* in the origin O*(0,0)* and *b* is the null vector. Also

$$A' = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$det(A') = 1 - \frac{1}{4} = \frac{3}{4} \neq 0$$

from which *rank(A) = 2* and *rank(B) = 3*. In the general case

$$x^2 + xy + y^2 - 3k = 0$$

the conic is nondegenerate if $k \neq 0$ since

$$det(B') = (-3k)\left(1 - \frac{1}{4}\right) = -\frac{9}{4}k < 0\ (\forall k \neq 0)$$

We determine the eigenvalues of the symmetric matrix *A* by considering its characteristic polynomial

$$p_A = |\lambda I - A'| = \begin{vmatrix} \lambda - 1 & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - \frac{1}{4} = \lambda^2 - 2\lambda + 1 - \frac{1}{4} = \lambda^2 - 2\lambda + \frac{3}{4}$$

$$\Delta = 4 - 3 = 1$$

$$\lambda = \frac{2 \pm 1}{2}$$

$$\lambda_1 = \frac{1}{2}\,; \;\; \lambda_2 = \frac{3}{2}$$

It is necessary to classify the conic. We define

$I_1 = trace(A') = \lambda_1 + \lambda_2$; $I_2 = det(A') = \lambda_1\lambda_2$; $I_3 = det(B')$ serve that rank(A') = 2 and rank(B') = 3 $(\forall k \neq 0)$, which is equivalent to saying that $I_2 \neq 0;\ I_3 \neq 0$ Furthermore,

$$I_1 = \lambda_1 + \lambda_2 = \frac{1}{2} + \frac{3}{2} = 2;\;\; I_2 = \lambda_1\lambda_2 = \frac{3}{4};\;\; I_3 = \det(B) = -\frac{9}{4}k < 0\ (\forall k > 0)$$

Being $I_1 > 0;\ I_1 I_3 < 0$, the equation of the conic represents a real ellipse.

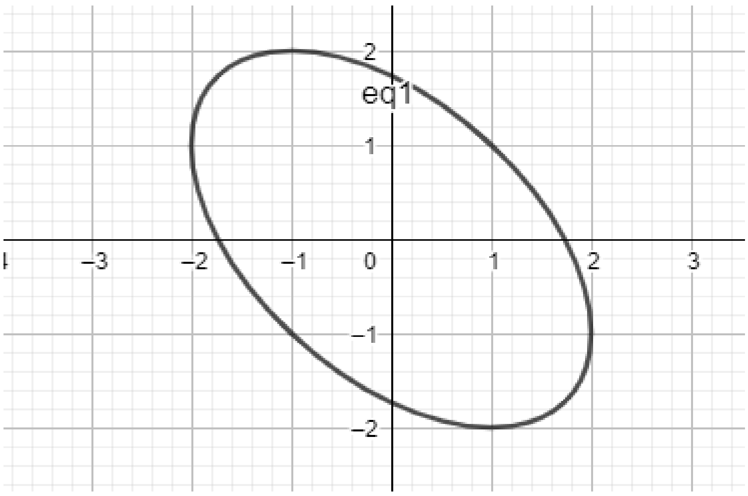

Figure 8 Representation of ellipse of equation $x^2+xy+y^2-3=0$

By a well-known theorem on the canonical form of a polynomial of the second degree, the canonical form of the conic is then

$$\lambda_1 X^2 + \lambda_2 Y^2 + \tilde{c} = 0$$

$$\frac{1}{2}X^2 + \frac{3}{2}Y^2 + \tilde{c} = 0$$

from the formula

$$-\frac{9}{4}k = I_3 = \det(A)\,\tilde{c} = \frac{3}{4}\tilde{c}$$

we derive that

$$\tilde{c} = -3k(\forall k \neq 0)$$

Therefore, the equation of the conic can be written as

$$\frac{1}{2}X^2 + \frac{3}{2}Y^2 - 3k = 0;\ \frac{X^2}{2} + \frac{Y^2}{\frac{2}{3}} = 3k$$

Dividing all the terms in the equation by $-\tilde{c}$ we obtain

$$\frac{X^2}{6k} + \frac{Y^2}{2k} = 1\ (\forall k \neq 0)$$

that is, the equation of the ellipse in canonical form:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

where

$$a = \sqrt{6k};\ b = \sqrt{2k};\ k \neq 0$$

For $k = 1$ we have the ellipse (Fig. 9)

$$\frac{X^2}{6} + \frac{Y^2}{2} = 1$$

with the foci lying on the x-axis, having minor semi-axis equal to $\sqrt{2}$; semi-major axis equal to $\sqrt{6}$; semi focal distance equal to 2 and eccentricity $\frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$

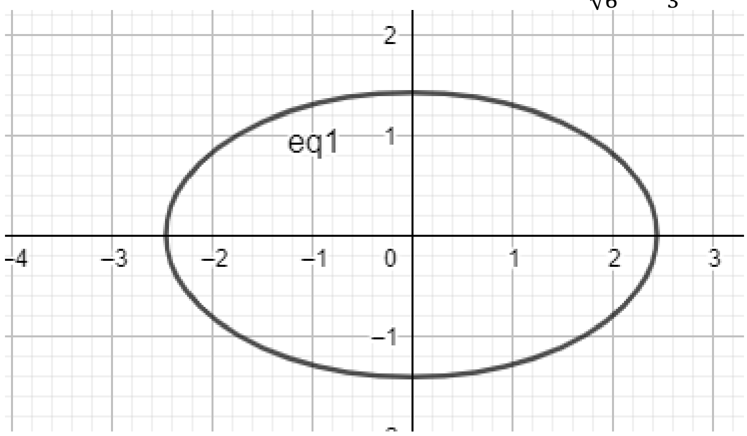

Figure 9 Representation of the ellipse of equation $x^2/6+y^2/2=1$ (Geogebra)

The matrix A' is diagonalized through the matrix formed by the eigenvectors relative to

$$\lambda_1 = \frac{1}{2}\ ;\ \lambda_2 = \frac{3}{2}$$

Indeed, the system

$$\begin{cases}(\lambda-1)x+\dfrac{1}{2}y=\\ \dfrac{1}{2}+(\lambda-1)y=0\end{cases}$$

for $\lambda_1 = \frac{1}{2}$ gives the solution *(x,y) = k(-1,1)* while for $\lambda_2 = \frac{3}{2}$ leads to the solution *(x,y) = k (1,1)* and thus to the matrix, for *k =1*,

$$S' = \begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}$$

formed by the eigenvectors

$$w_1 = \begin{bmatrix}-1\\ 1\end{bmatrix};\ w_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$$

The matrix *S'* is not an orthogonal matrix. In fact *det(S') = -2* although the vectors are orthogonal to each other

$$w_1 \cdot w_2 = (-1)(1) + (1)(1) = -1 + 1 = 0$$

Being $\|w_1\| = \|w_2\| = \sqrt{2}$, we have that the vectors

$$u = \frac{w_1}{\|w_1\|}, v = \frac{w_2}{\|w_2\|}$$

represent an orthonormal basis of R^2 and the subspaces of R^2 generated by each of them are the lines to them, called the principal axes of the ellipse (Fig.10, Geogebra).

The matrix

$$Q' = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$$

is orthogonal since

$$det(Q') = -\frac{1}{2} - \frac{1}{2} = -1$$

But it is not a rotation matrix. By appropriately modifying Q' by changing the sign to an element of *Q'*:

$$Q'' = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

we have that $\det(Q'') = 1$. Whence *Q''* represents the rotation matrix sought.

The transformation $Z = Q'X$ allows the ellipse to be rotated *135°* clockwise with respect to the *x*-axis (Fig.11). In fact, posed

$$\sin\theta = -1/\sqrt{2} \ ; \ cos\ \theta = 1/2$$

we have that $\tan\theta = -1$, from which $\theta = 45° + 90° = 135°$ (or $45°$ counterclockwise)

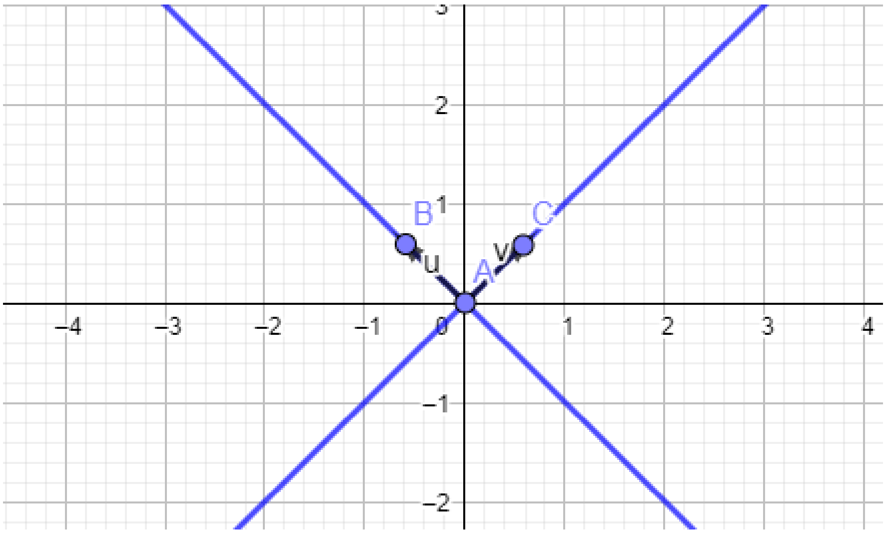

Figure 10 Representation of subspaces generated by vectors *u* and *v*

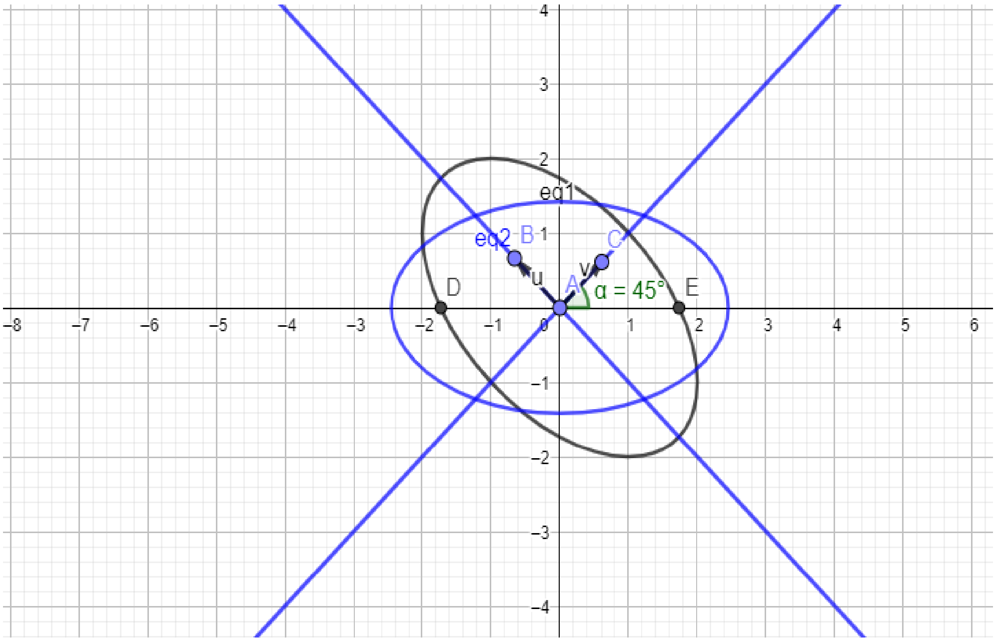

Figure 11 Representation of ellipses $x^2+xy+y^2-3=0$, $x/6+y/2=1$ and subspaces generated by vectors *u* and *v* (Geogebra)

In Figure 10 the ellipse in black corresponds to the equation of the conic written in noncanonical form while the ellipse in blue represents the equation of the ellipse in canonical form.
As k varies, we obtain the following family of ellipses (Fig. 12; Wolfram Alpha)

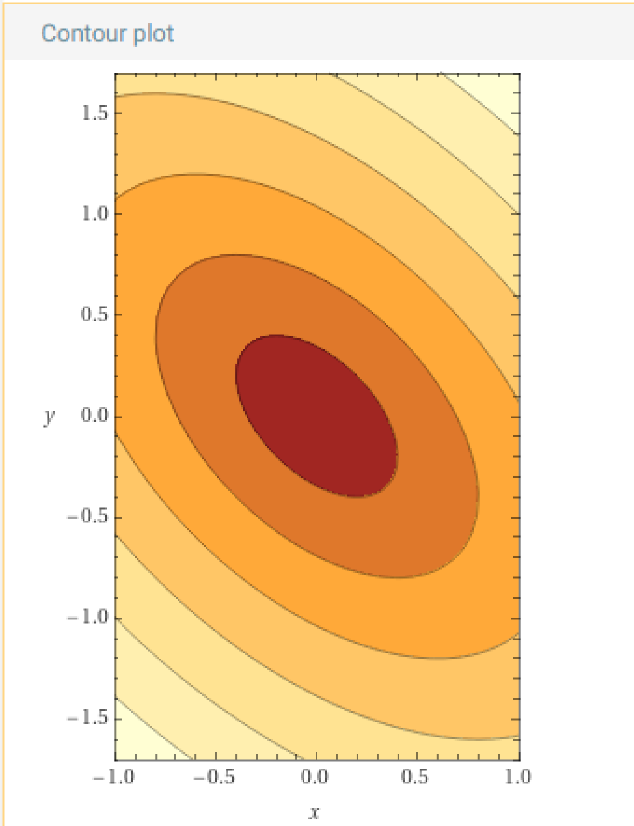

Figure 12 Family of ellipses obtained at the variation of *k*

10. A new mean η_3^F originated from an extension of Flett's theorem

One of the possible extensions of MVTD is the Flett's theorem [14][42] which, under the assumptions of a function *f* derivable in *[a,b]*, with *a* < *b* and , asserts the existence of at least one ξ, with *a* < ξ < *b* such that

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a}$$

and one η, with $a < \eta < b$ such that

$$f'(\eta) = \frac{f(b) - f(\eta)}{b - \eta}$$

We observe that it is not possible to generate Stolarsky's means by means of Flett's theorem because the condition $f'(a) = f'(b)$ is not verified for the function $f(x) = x^\alpha$. In fact, $\alpha a^{\alpha-1} = \alpha b^{\alpha-1}$ if, and only if, $a = b$ against the fact that by hypothesis a < b. Moreover, the function $f(x) = x^\alpha$ does not even satisfy the assumption of the Flett's theorem analogue for integral calculus, which states that if f is a function defined and continuous in [a,b] and *f(a) = f(b)* then there exists Nor does the function satisfy the assumption of the Flett's theorem analogue for integral calculus, which states that if f is a function defined and continuous in *[a,b]* and *f(a) = f(b)* then there is at least one η, with $a < \eta < b$, such that

$$(\eta - a)f(\eta)\int_a^\eta f(x)\ dx$$

In fact, the condition *f(a) = f(b),* equivalent to the equality leading to the relation $a = b$ against the fact that $a < b$ by hypothesis, is not verified. However, there is an extension of the Flett's theorem [42, p. 165] that removes the equality condition of the prime derivatives on the extremes of the defining interval. It states that if *f* is derivable in [a,b], there exists at least one ξ, with $a < \xi < b$, such that

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a} + \frac{1}{2}\cdot\frac{f'(b) - f'(a)}{b - a}(\xi - a)$$

We suppose $f(x) = x^\alpha$, the previous relationship becomes.

$$\alpha\xi^{\alpha-1} = \frac{\xi^\alpha - a^\alpha}{\xi - a} + \frac{1}{2}\cdot\frac{\alpha b^{\alpha-1} - \alpha a^{\alpha-1}}{b - a}(\xi - a)$$

For $\alpha = 3$ the above formula takes the following form:

$$3\xi^2 = \frac{\xi^3 - a^3}{\xi - a} + \frac{3}{2}\cdot\frac{b^2 - a^2}{b - a}(\xi - a)$$

$$3\xi^2 = \frac{(\xi - a)\left(\xi^2 + \xi a + a^2\right)}{\xi - a} + \frac{3}{2}\cdot\frac{(b - a)(b + a)}{b - a}\ (\xi - a)$$

$$3\xi^2 = \xi^2 + \xi a + a^2 + \frac{3}{2}\cdot(b + a)(\xi - a)$$

$$6\xi^2 = 2\xi^2 + 2\xi a + 2a^2 + 3(b + a)(\xi - a)$$

$$4\xi^2 - (2a + 3a + 3b)\xi - 2a^2 + 3ab + 3a^2 = 0$$

$$4\xi^2 - (5a + 3b)\xi + 3ab + a^2 = 0$$

We assume $A = 4, B = -(5a + 3b), C = 3ab + a^2$, then the discriminant of the second-degree equation in the unknown ξ is

$$\Delta = [-(5a + 3b)]^2 - 4 \cdot 4 \cdot (3ab + a^2) =$$
$$= 25a^2 + 9b^2 + 30ab - 48ab - 16a^2 =$$
$$= 9a^2 - 18ab + 9b^2$$

being $a < b$ by assumption. Whence

$$\xi = \frac{(5a + 3b) \pm 3|a - b|}{8} = \frac{5a + 3b \ \pm 3(b - a)}{8}$$

which leads to two solutions

$$\xi_+ : \frac{5a + 3b + 3b - 3a}{8} = \frac{2a + 6b}{8} = \frac{a + 3b}{4}$$
$$\xi_- : \frac{5a + 3b - 3b + 3a}{8} = \frac{8a}{8} = a$$

The second solution is discarded because it coincides with a while the first is accepted because it falls strictly between a and b. In fact,

$$a < b; 4a < \ a + 3b < 4b; \ a = \frac{4a}{4} < \frac{a + 3b}{4} < \frac{4b}{4} = b$$

Therefore, the η_3 assumes the form

$$\eta_3^F = \frac{a + 3b}{4}$$

where the letter F is intended to recall Flett's theorem. A simple geometric interpretation can also be obtained for this mean by recourse to the notion of the area of a trapezoid having the bases measuring a and 3b, respectively, and the height h equal to ½.
We can observe that $A(a, b) < \ \eta_3^F(a, b)$. In fact, if by absurdity it were

$$\eta_3^F < \ A(a, b) \ (0 < a < b)$$

we would have

$$\frac{a + 3b}{4} < \frac{a + b}{2}; \ a + 3b \ < \ 2a + 2b; \ b < a$$

against the fact that we have assumed $a < b$. Therefore η_3^F can also be regarded as a particular weighted arithmetic mean with weights $\frac{1}{2}$ and $\frac{3}{2}$:

$$\eta_3^F = A\left(\frac{a}{2}, \frac{3b}{2}\right)$$

11. Statistical significance of $\eta_{3^h}(h \in N)$

We note that for $\alpha = 9$

$$b^9 - a^9 = (b-a)(a^2 + ab + b^2) \cdot (a^6 + a^3b^3 + b^6)$$

for $\alpha = 27$

$$b^{27} - a^{27} = (b-a)(a^2 + ab + b^2) \cdot (a^6 + a^3b^3 + b^6) \cdot (a^{18} + a^9b^9 + b^{18})$$

for $\alpha = 81$

$$b^{27} - a^{27} = (b-a)(a^2 + ab + b^2) \cdot (a^6 + a^3b^3 + b^6) \cdot (a^{18} + a^9b^9 + b^{18}) \cdot (a^{54} + a^{27}b^{27} + b^{54})$$

from which

$$\eta_9(a,b) = \sqrt[8]{QM^2(a, GM^2(a,b), b) \cdot QM^2(a^3, GM^2(a^3,b^3), b^3)}$$

$$\eta_{27}(a,b) = \sqrt[26]{QM^2(a, GM^2(a,b), b) \cdot QM^2(a^3, GM^2(a^3,b^3), b^3) \cdot QM^2(a^9, GM^2(a^9,b^9), b^9)}$$

$$\eta_{81}(a,b) = \sqrt[80]{QM^2(a, GM^2(a,b), b) \cdot QM^2(a^3, GM^2(a^3,b^3), b^3) \cdot} \cdot \sqrt[80]{QM^2(a^9, GM^2(a^9,b^9), b^9) \cdot QM^2(a^{27}, GM^2(a^{27},b^{27}), b^{27})}$$

…

More generally, from the formula valid for every positive natural *h*:

$$b^{3^h} - a^{3^h} = (b-a)(a^2 + ab + b^2) \cdot\cdot (a^6 + a^3b^3 + b^6) \cdots \left(a^{2\cdot 3^{h-1}} + a^{3^{h-1}} b^{\cdot 3^{h-1}} + b^{2\cdot 3^{h-1}}\right)$$

Reasoning by induction and assuming that the previous formula up to *h* is true, since

$$b^{3^{h+1}} - a^{3^{h+1}} = \left(b^{3^h} - a^{3^h}\right) \cdot \left(a^{2\cdot 3^h} + a^{3^h} b^{3^h} + b^{2\cdot 3^h}\right)$$

we have that

$$b^{3^{h+1}} - a^{3^{h+1}} = (b-a)(a^2 + ab + b^2) \cdots \left(a^{2\cdot 3^h} + a^{3^h} b^{3^h} + b^{2\cdot 3^h}\right)$$

Therefore, we obtain that $\eta_{3^h}(a,b)$ admits the following expression

$$\sqrt[3^h - 1]{QM^2(a, GM^2(a,b), b) \cdots QM^2\left(a^{3^{h-1}}, GM^2\left(a^{3^{h-1}}, b^{\cdot 3^{h-1}}\right), b^{3^{h-1}}\right)}$$

result that tells us that the SM of parameter equal to a power of base 3 and positive integer exponent can be traced back to the root of a monomial in which appear as factors the square of the mean *QM* having as arguments appropriate powers of *a*, *ab* and *b*.

12. Statistical significance of η_{-3}, η_{-9}, η_{-3^h} $(h \in N)$

For the case $\alpha = -3$ we can repeat the steps of the case $\alpha = -2$ and connect η_{-3} to G and H. We will simply apply the general formula of SM, proceeding similarly by MVTD and MVTI:

$$\eta_{-3}(a,b) = \left[\frac{b^{-3}-a^{-3}}{-3(b-a)}\right]^{\frac{1}{-3-1}} = \left[\frac{\left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{b^2}+\frac{1}{ab}+\frac{1}{a^2}\right)}{-3(b-a)}\right]^{-\frac{1}{4}}$$

$$= \left[\frac{\left(\frac{b-a}{ab}\right)\left(\frac{1}{b^2}+\frac{1}{ab}+\frac{1}{a^2}\right)}{3(b-a)}\right]^{-\frac{1}{4}}$$

$$= \left[(ab)\left(\frac{3}{\frac{1}{b^2}+\frac{1}{ab}+\frac{1}{a^2}}\right)\right]^{\frac{1}{4}} = \sqrt[4]{G^2(a,b)\cdot H(a^2,ab,b^2)}$$

From

$$b^{-3}-a^{-3} = \frac{(a-b)(a^2+ab+b^2)}{a^3b^3}$$

we have also

$$\eta_{-3}(a,b) = \left[\frac{\frac{(b-a)(a^2+ab+b^2)}{a^3b^3}}{3(b-a)}\right]^{-\frac{1}{4}} = \left[a^3b^3\cdot\frac{3}{a^2+ab+b^2}\right]^{\frac{1}{4}}$$

$$= \sqrt[4]{\frac{G^6(a,b)}{A(a^2,ab,b^2)}} = G(a,b)\cdot\sqrt[4]{\frac{G^2(a,b)}{A(a^2,G^2(a,b),b^2)}}$$

Being

$$b^{-9}-a^{-9} = \left(\frac{1}{b^3}\right)^3-\left(\frac{1}{a^3}\right)^3 = \left(\frac{1}{b^3}-\frac{1}{a^3}\right)\cdot\left(\frac{1}{b^6}+\frac{1}{b^3a^3}+\frac{1}{a^6}\right) =$$

$$= \left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{b^2}+\frac{1}{ba}+\frac{1}{a^2}\right)\cdot\left(\frac{1}{b^6}+\frac{1}{b^3a^3}+\frac{1}{a^6}\right)$$

we obtain

$$\eta_{-9}(a,b) = \sqrt[10]{G^2(a,b)\cdot H(a^2,ab,b^2)\cdot H(a^6,a^3b^3,b^6)}$$

or

$$\eta_{-9}(a,b)=\left[\frac{\frac{(b-a)(a^2+ab+b^2)}{a^9b^9}}{9(b-a)}\right]^{-\frac{1}{10}}$$

$$=\left[a^9b^9\cdot\frac{3}{a^2+ab+b^2}\cdot\frac{3}{a^6+a^3b^3+b^6}\right]^{\frac{1}{10}}$$

$$=\sqrt[10]{\frac{G^{18}(a,b)}{A(a^2,ab,b^2)\cdot A(a^6,a^3b^3,b^6)}}=G(a,b)\cdot\sqrt[10]{\frac{G^{9}(a,b)}{A(a^2,ab,b^2)\cdot A(a^6,a^3b^3,b^6)}}$$

from which, more generally, reasoning by induction, we arrive at the following formula valid for any natural h:

$$\eta_{-3^h}(a,b)=\sqrt[3^h+1]{G^2(a,b)\cdot H(a^2,ab,b^2)\cdots H\left(a^{2\left(3^{h-1}\right)},(ab)^{3^{h-1}},b^{2\left(3^{h-1}\right)}\right)}$$

or

$$\eta_{-3^h}(a,b)=\sqrt[3^h+1]{\frac{G^{2\cdot 3^h}(a,b)}{A(a^2,ab,b^2)\cdots A\left(a^{2\left(3^{h-1}\right)},(ab)^{3^{h-1}},b^{2\left(3^{h-1}\right)}\right)}}$$

Therefore. being $2\cdot 3^h>3^h+1$ e $2\cdot 3^h-3^h-1=3^h-1$, we have also that $2\cdot 3^h=(3^h+1)+(3^h-1)$, from which

$$\eta_{-3^h}(a,b)=G(a,b)\cdot\sqrt[3^h+1]{\frac{G^{3^h-1}(a,b)}{A(a^2,ab,b^2)\cdots A\left(a^{2\left(3^{h-1}\right)},(ab)^{3^{h-1}},b^{2\left(3^{h-1}\right)}\right)}}$$

13. Statistical significance of η_{5^h} ($h\in N$)

We have that

$$\eta_{5^h}=\left[\frac{b^{5^h}-a^{5^h}}{5^h(b-a)}\right]^{\frac{1}{5^h-1}}=$$

$$=\sqrt[5^h-1]{\frac{\prod_{i=1}^h Q_i}{5^h}}$$

$$=\sqrt[5^h-1]{\prod_{i=1}^h A_i}$$

where the difference of powers $b^{5^h} - a^{5^h}$ is developed as the product of *h+1* homogeneous Q_i polynomials in a and b, of which the first factor is *(b-a)* while the remaining factors have degree $4 \cdot 5^i$ $(1 \le i \le h)$ and are each formed by 5 terms:

$$b^{5^h} - a^{5^h} = \prod_{i=0}^{h} Q_i$$

$$Q_0 = b - a$$

$$Q_i = \sum_{j=0}^{4} a^{(4-j)\cdot 5^{i-1}} b^{j\cdot 5^{i-1}} \; (1 \le i \le h))$$

$$A_i\left(a^{4\cdot 5^{i-1}}b^0, a^{3\cdot 5^{i-1}}b^{1\cdot 5^{i-1}}, a^{2\cdot 5^{i-1}}b^{2\cdot 5^{i-1}}, a^{1\cdot 5^{i-1}}b^{3\cdot 5^{i-1}}, a^0 b^{4\cdot 5^{i-1}}\right)(1 \le i \le h))$$

14. SM with two-parameter α and β $(\boldsymbol{\alpha = 2^h\beta}\,;\boldsymbol{h} \in \boldsymbol{Z})$ and classical means

SM with two parameters

$$\eta_{\alpha,\beta}(a,b) = \left[\frac{\beta}{\alpha}\cdot\frac{b^\alpha - a^\alpha}{b^\beta - b^\beta}\right]^{\frac{1}{\alpha-\beta}} \; (\alpha \ne \beta)$$

can also be traced back to classical averages. Let us illustrate the previous statement with some examples.

Case : $\alpha = 2\beta$

$$\eta_{\alpha,\beta}(a,b) = \left[\frac{\beta}{2\beta}\cdot\frac{b^{2\beta} - a^{2\beta}}{b^\beta - b^\beta}\right]^{\frac{1}{2\beta-\beta}} = \left[\frac{1}{2}\cdot\frac{b^{2\beta} - a^{2\beta}}{b^\beta - b^\beta}\right]^{\frac{1}{\beta}} = \left[\frac{b^\beta + a^\beta}{2}\right]^{\frac{1}{\beta}}$$

Case : $\alpha = 4\beta$

$$\eta_{4\beta,\beta}(a,b) = \left[\frac{\beta}{4\beta}\cdot\frac{b^{2\beta} - a^{2\beta}}{b^\beta - b^\beta}\right]^{\frac{1}{2\beta-\beta}} = \left[\frac{1}{4}\cdot\frac{b^{4\beta} - a^{4\beta}}{b^\beta - b^\beta}\right]^{\frac{1}{\beta}}$$

$$= \left[\frac{a^\beta + b^\beta}{2}\cdot\frac{\left(a^\beta\right)^2 + \left(b^\beta\right)^2}{2}\right]^{\frac{1}{3\beta}} =$$

$$= \sqrt[3\beta]{A(a^\beta, b^\beta)\cdot QM(a^\beta, b^\beta)}$$

Case: $\alpha = 8\beta$

$$\eta_{8\beta,\beta}(a,b)=\left[\frac{\beta}{8\beta}\cdot\frac{b^{8\beta}-a^{8\beta}}{b^{\beta}-b^{\beta}}\right]^{\frac{1}{7}}=\left[\frac{a^{\beta}+b^{\beta}}{2}\cdot\frac{\left(a^{\beta}\right)^{2}+\left(b^{\beta}\right)^{2}}{2}\cdot\frac{\left(a^{\beta}\right)^{4}+\left(b^{\beta}\right)^{4}}{2}\right]^{\frac{1}{7\beta}}$$

$$=\sqrt[7\beta]{A(a^{\beta},b^{\beta})\cdot QM^{2}(a^{\beta},b^{\beta})\cdot QM^{2}(a^{2\beta},b^{2\beta})}$$

From

$$a^{2^{h-2}\cdot\beta}=\left(a^{\beta}\right)^{2^{h-2}};\ b^{2^{h-2}\cdot\beta}=\left(b^{\beta}\right)^{2^{h-2}}$$

we have

$$\eta_{2^{h}\beta,\beta}(a,b)=$$

$$=\sqrt[\left(2^{h}-1\right)\beta]{A(a^{\beta},b^{\beta})\cdot QM^{2}(a^{\beta},b^{\beta})\cdot QM^{2}(a^{2\beta},b^{2\beta})\cdots QM^{2}\left(a^{2^{h-2}\cdot\beta},b^{2^{h-2}\cdot\beta}\right)}$$

Case : $\alpha=-2\beta$

$$\eta_{-2\beta,\beta}(a,b)\ =\left[\frac{\beta}{-2\beta}\cdot\frac{b^{-2\beta}-a^{-2\beta}}{b^{\beta}-a^{\beta}}\right]^{\frac{1}{-2\beta-\beta}}\ =\left[\left(-\frac{1}{2}\right)\cdot\frac{\left(\frac{1}{b^{\beta}}\right)^{2}-\left(\frac{1}{a^{\beta}}\right)^{2}}{b^{\beta}-a^{\beta}}\right]^{\frac{1}{-3\beta}}$$

$$=\left[a^{\beta}b^{\beta}\cdot\frac{2}{\frac{1}{a^{\beta}}+\frac{1}{b^{\beta}}}\right]^{\frac{1}{3\beta}}$$

$$=\sqrt[3\beta]{G^{2}(a^{\beta},b^{\beta})\cdot H(a^{\beta},b^{\beta})}$$

Case : $\alpha=-4\beta$

$$\eta_{-4\beta,\beta}(a,b)\ =\left[\frac{\beta}{-4\beta}\cdot\frac{b^{-4\beta}-a^{-4\beta}}{b^{\beta}-a^{\beta}}\right]^{\frac{1}{-4\beta-\beta}}\ =\left[\left(-\frac{1}{4}\right)\cdot\frac{\left(\frac{1}{b^{2\beta}}\right)^{2}-\left(\frac{1}{a^{2\beta}}\right)^{2}}{b^{\beta}-a^{\beta}}\right]^{\frac{1}{-5\beta}}$$

$$=\left[\left(-\frac{1}{4}\right)\cdot\frac{\left(\frac{1}{b^{2\beta}}-\frac{1}{a^{2\beta}}\right)\left(\frac{1}{b^{2\beta}}+\frac{1}{a^{2\beta}}\right)}{b^{\beta}-a^{\beta}}\right]^{\frac{1}{-5\beta}}$$

$$= \left[\left(-\frac{1}{4}\right) \cdot \frac{\left(\frac{1}{b^\beta} - \frac{1}{a^\beta}\right)\left(\frac{1}{b^\beta} + \frac{1}{a^\beta}\right)\left(\frac{1}{b^{2\beta}} + \frac{1}{a^{2\beta}}\right)}{b^\beta - a^\beta} \right]^{-\frac{1}{5\beta}}$$

$$= \sqrt[5\beta]{G^2(a^\beta, b^\beta) \cdot H(a^\beta, b^\beta) \cdot H(a^{2\beta}, b^{2\beta})}$$

In the case that $\alpha = -2^h\beta(h \in N)$ we have, by reasoning by induction,

$$\eta_{-2^h\beta,\beta} = \sqrt[(2^h+1)\beta]{G^2(a^\beta, b^\beta) \cdot H(a^\beta, b^\beta) \cdot H(a^{2\beta}, b^{2\beta}) \cdots H\left(a^{(2^{h-1})\beta}, b^{(2^{h-1})\beta}\right)}$$

Similarly for $\alpha = 9\beta$

$$b^{9\beta} - a^{9\beta} =$$

$$= \left(b^{3\beta}\right)^3 - \left(a^{3\beta}\right)^3 = \left(b^{3\beta} - a^{3\beta}\right)\left(b^{6\beta} + b^{3\beta}b^{3\beta} + b^{6\beta}\right)$$

$$= \left(b^\beta - a^\beta\right)\left(b^{2\beta} + b^\beta b^\beta + a^{q2\beta}\right)\left(b^{6\beta} + b^{3\beta}b^{3\beta} + b^{6\beta}\right)$$

for $\alpha = 27\beta$

$$b^{27\beta} - a^{27\beta} = \left(b^\beta - a^\beta\right)\left(a^{2\beta} + a^\beta b^\beta + b^{2\beta}\right) \cdot \left(a^{6\beta} + a^{3\beta}b^{3\beta} + b^{6\beta}\right) \cdot \left(a^{18\beta} + a^{9\beta}b^{9\beta} + b^{18\beta}\right)$$

for $\alpha = 81\beta$

$$b^{81\beta} - a^{81\beta} = \left(b^\beta - a^\beta\right)\left(a^{2\beta} + a^\beta b^\beta + b^{2\beta}\right) \cdot \left(a^{6\beta} + a^{3\beta}b^{3\beta} + b^{6\beta}\right) \cdot \left(a^{18\beta} + a^{9\beta}b^{9\beta} + b^{18\beta}\right) \cdot \left(a^{54\beta} + a^{27\beta}b^{27\beta} + b^{54\beta}\right)$$

from which

$$\eta_{9\beta,\beta}(a,b) = \sqrt[8\beta]{QM^2(a^\beta, GM^2(a^\beta, b^\beta), a^\beta) \cdot QM^2(a^{3\beta}, GM^2(a^{3\beta}, b^{3\beta}), b^{3\beta})}$$

And more generally

$$\eta_{3^h\beta,\beta}(a,b) =$$

$$= \sqrt[(3^h-1)\beta]{QM^2(a^\beta, GM^2(a^\beta, b^\beta), a^\beta) \cdots QM^2\left(a^{(3^{h-1})\beta}, GM^2\left(a^{(3^{h-1})\beta}, b^{(3^{h-1})\beta}\right), b^{(3^{h-1})\beta}\right)}$$

15. Economic significance of SM [8]

Let X be a subset of R^n. A function $f: X \rightarrow R$ is said to be homogeneous of degree r if

$$f(jx_1, \dots, jx_n) = j^r f(x_1, \dots, x_n)$$

where *j* is a positive constant and $(jx_1, \dots, jx_n) \in X$ $\left(\forall x \in X;\ x = (x_1, \dots, x_n)\right)$.

In Linear Algebra the set *X*, constant *j*, *x* are called the cone, scalar and vector of R^n, respectively. The constant r can also be equal to 0. If $r = 1$ the f is said to be linearly homogeneous. Consider an economy where a single good is produced in quantity *Y* using capital *K* and labor *L* as the only factors of production. Both *L* and *K* are considered positive. The main field of application in economics of linearly homogeneous functions is given by the notion of aggregate production function, which we will write in the form

$$Y = F(K, L)$$

that is, a production function is a relationship F between the inputs employed *K* and *L* and the output *Y*. Let *F* be linearly homogeneous. Placed

$$j = \frac{1}{L}$$

we have

$$\frac{Y}{L} = \frac{1}{L}F(K, L) = F\left(\frac{K}{L}, \frac{L}{L}\right) = F\left(\frac{K}{L}, 1\right)$$

By positing $Y/L = y$, $K/L = k$, we obtain the production function $y = f(k)$ in intensive form, that is, the output per unit of effective labor *y* is a function f of capital per unit of effective labor *k*. Let us consider the Cobb-Douglas production function

$$Y = AK^{\alpha}L^{\alpha-1}\ (\alpha \in]0; 1[)$$

meaning by *A* the state of technology. We observe that.

$$Y = AK^{\alpha}L^{1-\alpha} = AL\left(\frac{K}{L}\right)^{\alpha} = ALk^{\alpha}$$

from which, dividing by L the first and last terms of the equalities above, we have

$$y = f(k) = Ak^{\alpha}(\alpha \in]0; 1[)$$

If we set $A = 1$, f becomes $y = f(k) = k^{\alpha}$.

The Cobb-Douglas function written in intensive form k^{α} can be thought of as a power function of base *k* and exponent α. It is usually assumed that *F*, in *X*, in addition to being linearly homogeneous, is also of class C^2, that is, *F* admits continuous prime and second derivatives with respect to both variables *K* and *L*. To avoid running into problems with the partial derivatives of *F*, it is usually convenient to consider the set *X* as an open set of R^n. For *f(k)* one can repeat what has already been said for the power function

$x^{\alpha}(\alpha \neq 0;\ \alpha \neq 1)$ to verify that it satisfies the definition of Stolarsky averages. The ratio y capital/labor is called labor productivity and is introduced to study the growth of an economy. The assumption of linear homogeneity on F allows for constant returns to scale. For example, in the particular case where both factors of production K and L are doubled, output also doubles, i.e.

$$2Y = F(2K, 2L)$$

16. Probabilistic significance of SM

We recall that a continuous random variable X is said to have a Uniform distribution over the interval *[a,b]* if its density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & for\ a \leq x \leq b \\ 0 & otherwise \end{cases}$$

The r^{th} moment of X is

$$\mu_r = EX^r = \int_{-\infty}^{+\infty} x^r f(x)dx = \int_a^b \frac{x^r}{b-a}dx = \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \quad (r \in N)$$

from which we have that

$$\mu_1 = EX = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

$$\mu_2 = EX^2 = \frac{b^3 - a^3}{2(b-a)} = \frac{b^2 + ab + a^2}{3}$$

Comparing EX^r and μ_2 with the expressions of the formula (19) of SM and the formula (7) of η_3 , we can conclude that SM and Uniform distribution are closely linked.

17. Conclusions and final remarks

We have shown by elementary methods that for $n = p^h$ $(p\ prime\ number;\ h \in N)$ the Stolarky means η_{p^h} can be traced explicitly to classical means and that for $n = 3$ the Stolarky mean η_3 can be interpreted as a real quadric. Both results form an ideal bridge to the pioneering works published by the two Italian mathematicians and statisticians, L. Galvani (1927) and R. Cisbani (1938) from which many authors have been inspired to introduce and/or further explore Stolarky means. Interesting and a harbinger of further research insights is our attempt to generate, in the n = 3. case, means of the type of SM from an extension of Flett's Theorem, which in turn can be considered a mean-value theorem. The formulas and considerations given in this paper are original, including the steps to deduce by means of limits the elementary properties of classical means from MVTD and MVTI and the inequalities obtained by applying identities contained in [6] and [24]. Finally, the connection of SM to Cobb-Douglas may lead to new interpretations of some economic growth models with chaos in the future.

References

[1] H. Alzer and S. Ruscheweyh. On the Intersection of Two-Parameter Mean Value Families. Proceedings of the American Mathematical Society, Vol. 129, No. 9, 2655-2662, Sep., 2001.

[2] R. Bhatia. The logarithmic mean. Resonance, Volume 13, 583–594, 2008.

[3] F. Burk. The Geometric, Logarithmic, and Arithmetic Mean Inequality. The American Mathematical Monthly, Vol. 94, No. 6, 527-528, 1987.

[4] B.C. Carlson. Some inequalities for hypergeometric functions. Proc. Amer. Math. Soc., 17, 32-39, 1966.

[5] B.C. Carlson. The logarithmic mean. The American Mathematical Monthly, Vol. 79, No. 6, 615-618, 1972.

[6] G.D. Birkhoff and H.S. Vanvinder. On the Integral Divisors of $a^n - b^n$. Annals of Mathematics Vol. 5 No. 4, 173-180, 1904.

[7] P. Cerone and S. Dragomir. Stolarsky and Gini Divergence Measures in Information Theory. Research report collection, 7 (2), 2004.

[8] A.C. Chiang and K. Wainwright. Fundamental Methods of Mathematical Economics, 4th edition, McGraw Hill, 383-387, 2004.

[9] Chao Pin Chen. Stolarsky and Gini means. Research Group in Mathematical Inequalities and Applications, vol.11, no. 4, article11, 2008.

[10] R. Cisbani. Contributi alla teoria delle medie. Metron V. 13 N. 2, 23-34, 1938.

[11] O. Dunkel. Generalized Geometric Means and Algebraic Equations, Annals of Mathematics, Second Series, Vol. 11, No. 1, 21-32, Oct., 1909.

[12] N. Elezovic and J. Pečarić, A note on Schur-convex functions, Rocky Mountain J. Math. 30, 853-856, 2000.

[13] H. Eves. Great Moments in Mathematics (Before 1650), Mathematical Association of America, 11-13, 1980.

[14] T. M. Flett. A Mean Value Theorem. The Mathematical Gazette , Vol. 42, No. 339, 38-39, 1958.

[15] L. Galvani. Dei limiti a cui tendono alcune medie. Bollettino dell'Unione Matematica Italiana, Serie 1, Vol. 6, n. 4, 173-179, 1927.

[16] C. Gini. Di una formula delle medie. Metron, 13, 3-22, 1938.

[17] H. W. Gould and m. E. Mays. Series Expansions of Means. Journal of Mathematical Analysis and Applications 101, 611-621, 1984.

[18] M Heida, M Kantner, A Stephan. Consistency and convergence for a family of finite volume discretizations of the Fokker–Planck operator, ESAIM: MathematicalModelling …, 2021 - esaim-m2an.org.

[19] Djordje Herceg, Dragoslav Herceg. Third-order modifications of Newton's method based on Stolarsky and Gini means/Journal of Computational and Applied mathematics, volume 245 june 2013, 53-61.

[20] J.B Hiriart-Urruty. A note on the mean value theorem for convex functions. Bollettino UMI (5) 17-B , 765-775, 1980.

[21] C.O. Imoru. The power mean and the logarithmic mean. Internat. J. Mah. Math. Sci. Vol. 5 No.2, 337-343, 1982.

[22] G.Jameson and P.R. Mercer. The Logarithmic Mean Revisited. The American Mathematical Monthly, Vol. 126, No. 7, 641-645, 2019.

[23] J. Jaksetic, J. Pecaric, and A.U.Rehman. On Stolarsky and related means. Mathematical Inequalities & Applications, Volume 13, No. 4, 899–909, 2010.

[24] N. D. Kazarinoff. Analytic inequalities, Holt, Rinehart and Winston, 1960.

[25] Kung Kuen Tse. A Note on the Mean Value Theorems. Advances in Pure Mathematics, Scientific Research Publishing, Vol. 11, No. 5, 395-399, 2021.

[26] E.B. Leach and M.C. Sholander. Extended mean values II. Journal of Mathematical Analysis and Applications 92, 207-223, 1983.

[27] T.P. Lin. The power mean and the logarithmic mean. Amer. Math. Monthly 81, 879-883, 1974.

[28] G. Lorenzen. A Unified Approach to the Calculation of Growth Rates. The American Statistician, Vol. 44, No. 2, 148-150, 1990.

[29] M.M. Mays. Function Which Parametrize Means, American Mathematical Monthly, Volume 90, Issue 10, 677-683, Dec., 1983.

[30] P.R. Mercer. 101.07 Cauchy's mean value theorem meets the logarithmic mean. The Mathematical Gazette, Vol. 101, No. 550, 108-115, 2017.

[31] Uta Merzbach and Carl Boyer. A History of Mathematics, Third Edition, 16-17.

[32] K. Muraliand and M. Nagaraja. Schur convexityof Stolarsky's extended mean values. Journal of Mathematical Inequalities, Volume10, Number 3, 725–735, 2016.

[33] E. Neuman and Z. Páles. On comparison of Stolarsky and Gini means. J. Math. Anal. Appl. 278, 274–284, 2003.

[34] B. Ostle-H.L.Terwilliger. A comparison of two means. Proc. Montana Acad. Sci., 17, 69-70, 1957.

[35] C. Palmisani. Un confronto tra i teoremi del valor medio. Periodico di Matematica (IV) Vol. V, 29-42, Marzo 2023.

[36] J.Pecaric and G.Roqia. Generalization of Stolarsky Type Means. Journal of Inequalities and Applications, 2010.

[37] A.O. Pittenger. The Logarithmic Mean in n Variables. The American Mathematical Monthly, Volume 92, Issue 2, 99-104, 1982.

[38] F. Qi. Generalized weighted mean values with two parameters. Proc. Roy. Soc. London Ser. A 454 , 2723–2732, 1998.

[39] F. Qi, J.-Q. Mei and S.-L. Xu. New proofs of weighted power mean inequalities and monotonicity for generalized weighted mean values. Math. Inequal. Appl. 3, 377–383, 2000.

[40] F. Qi. A note on Schur-convexity of extended mean values. Rocky Mountain J. Math. 35, no.5, 1787–1793, 2005.

[41] F.Qi. Logarithmic convexity of extended mean values. Proc. Amer. Math. Soc. 130, 1787-1796, 2002.

[42] P.K. Sahoo and T. Riedel. Mean value theorems an functional equations, World Scientific Publishing Company, Singapore, *1998.*

[43] J. The Schur-convexity of Stolarsky and Gini means. Banach J. Math. Anal. 1, no. 2, 212–215, 2007.

[44] J.Sándor. A note on the logarithmic mean. The American Mathematical Monthly, Vol. 123, No. 1, 112, 2016

[45] J.Sándor. A basic logarithmic inequality, and the logarithmic mean. Notes on Number Theory and Discrete Mathematics ISSN 1310–5132 Vol. 21, No. 1, 31–35, 2015.

[46] J.Sándor. A note on log-convexity of power means. Annales Mathematicae et Informaticae 45, 107–110, 2015.

[47] K. B. Stolarsky. Generalizations of the logarithmic mean. Mathematics Magazine , Vol. 48, No. 2 , 87-92, Mar. 1975.

[48] K. B. Stolarsky. The Power and Generalized Logarithmic Means. The American Mathematical Monthly , Vol. 87, No. 7, 545-548, 1980.

[49] Leo Törnqvist, Pentti Vartia and Yrjö O. Vartia. How Should Relative Changes Be Measured? The American Statistician, Vol. 39, No. 1, 43-46, 1985.

[50] A. Witkowski. Monotonicity of generalized weighted mean values. Colloquium mathematicum, Vol. 99, No. 2, 2004.

[51] A. Witkowski. Convexity of weighted Stolarsky means. Journal of Inequalities in Pure and Applied Mathematics, Volume7, Issue 2, Article73, 2006.

[52] Zhen-Hang Yang, Feng Qi. Bivariate homogeneous functions of two parameters: monotonicity, convexity, comparisons, and functional inequalities. Journal of Mathematical Analysis and Applications, Nov. 2024 (in press).