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# $\eta$-Ricci solitons in a LP-Sasakian manifolds admitting quarter-symmetric metric connection

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**Abstract**

The objective of this paper is to investigate the $\eta$-Ricci solitons in a LP-Sasakian manifolds admitting quarter-symmetric metric connection satisfying certain curvature conditions. We study the Ricci solitons in a LP-Sasakian manifolds admitting quarter-symmetric metric connection satisfying certain conditions. Finally, we have given an example of 3-dimensional $\eta$-Ricci solitons in a LP-Sasakian manifolds with respect to quarter-symmetric metric connection.

**Keywords**: LP-Sasakian manifolds; quarter-symmetric metric connection; $\eta$-Ricci solitons; $\eta$-Einstein manifold

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# 1 Introduction

In 1989, K. Matsumoto (18) introduced the notion of Lorentzian Para-Sasakian manifold (LP-Sasakian manifold). In 1992, the same notion was independently studied by I. Mihai and R. Rosca (19) and they obtained several results on this manifold. LP-Sasakian manifold have been also studied by various authors (2; 27; 28; 16). The quarter-symmetric connection generalizes the semi-symmetric connection. In 1975, Golab (13) defined and studied quarter-symmetic connection in a differentiable manifold. After Golab (13), Rastogi (25) continued the systematic study of quarter-symmetric metric connection. In 1980, Mishra and Pandey (20) studied quarter-symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifold. In 1982, Yano and Imai (32) studied quarter-symmetric metric connection in Hermition and Kaehlerian manifold. A quarter-symmetric metric connection have been studied by many authors (21; 24; 29; 12) in several ways to a different extent.

Ricci solitons in contact metric manifolds have been studied by Bagewadi et al.(1; 4; 5), Bejan and Crasmareanu (6), Chandra et al. (9), Chen and Deshmukh (10), Chenxu and Meng (14) and Nagaraja and Premalatha (22). Later on, $\eta$-Ricci soliton have been studied by various authors (23; 31; 7; 8) and many others. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}gij(t) = -2Rij.$$

A Ricci soliton $(g, V, \lambda)$ on a Riemannian manifold $(M, g)$ is generalization of an Einstein metric such that it satisfies the following condition (15)

$$(L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) \quad = \quad 0, \tag{1}$$

where $S$ is the Ricci tensor, $L_V$ is the Lie derivative operator along the vector field $V$ on $M$ and $\lambda$ is a real number. The Ricci soliton is said to be shrinking, steady, or expanding according as $\lambda$ is negative, zero or positive.

Ricci solitons, in the context of general relativity, have been studied by M. Ali and Z. Ahsan (3). As a generalization of Ricci soliton, the notion of $\eta$-Ricci soliton was introduced by J. T. Cho and M. Kimura (11). They have studied Ricci soliton of real hypersurfaces in a non-flat complex space form and defined $\eta$-Ricci soliton, which satisfies the equation

$$(L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) \quad = \quad 0, \tag{2}$$

where $\lambda$ and $\mu$ are real numbers. In particular, $\mu = 0$, then the notion of $\eta$-Ricci soliton $(g, V, \lambda, \mu)$ reduce to the notion of $(g, V, \lambda)$. Recently $\eta$-Ricci solitons have been studied by various authors such as A. Singh and S. Kishor (30), A. M. Blaga (8), D. G. Prakasha and B. S. Hadimani (23) and many others.

The present work is organized as follows: After introduction, in section-2, we recall basic formulas and results of LP-Sasakian manifold. Section-3 deals with curvature tensor in a LP-Sasakian manifolds admitting quarter-symmetric metric connection $(QSMC)$. Section-4 is concerned with the study of $\eta$-Ricci solitons on LP-Sasakian manifolds admitting quarter-symmetric metric connection $(QSMC)$ with example. In section-5, we study $\eta$-Ricci soliton on LP-Sasakian manifold admitting quarter-symmetric metric connection $(QSMC)$ satisfying $\widetilde{R}(X,Y).\widetilde{S}=0$. Section-6 Example of 3-dimensional LP-Sasakian manifold admitting quarter-symmetric metric connection $(QSMC)$.

# 2 Preliminaries

An $n$-dimensional smooth manifold $M$ together with a $(1,1)$-tensor field $\phi$, a contravariant vector field $\xi$, $\eta$ is 1-form and Lorentzian like metric $g$ is called a LP-Sasakian manifold if

$$\phi^2 X = X + \eta(X)\xi, \qquad \eta(\xi) = -1, \tag{3}$$

$$g(X,\xi) = \eta(X), \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \tag{4}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{5}$$

$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{6}$$

$$\nabla_X \xi = \phi(X), \tag{7}$$

$\forall\ X, Y \in \chi(M)$, where $\chi(M)$ is a set of all smooth vector fields on $M$ and $\nabla$ is the Levi-Civita connection with respect to $g$. If we put

$$\phi(X,Y) = g(\phi X, Y), \tag{8}$$

$\forall\ X, Y \in M$, then $\phi(X,Y)$ is a symmetric $(0,2)$ tensor field. Since the 1-form $\eta$ is closed in a LP-Sasakian manifold, so we have

$$(\nabla_X \eta)Y = \phi(X,Y), \quad \phi(X,\xi) = 0 \tag{9}$$

$\forall\ X, Y \in \chi(M)$. In a LP-Sasakian manifold $M$ the following relations holds

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{10}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{11}$$

$$R(\xi, X)\xi = -R(X, \xi)\xi = X + \eta(X)\xi, \tag{12}$$

$$\eta(R(X, Y)Z) = [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{13}$$

$$S(X, \xi) = (n-1)\eta(X), \quad Q\xi = (n-1)\xi, \tag{14}$$

$\forall\ X, Y, Z \in \chi(M)$, $g(QX, Y) = S(X, Y)$ and $R$ is the curvature tensor, $S$ is the Ricci tensor, $Q$ is the Ricci operator in a LP-Sasakian manifold with the Levi-Civita connection $\nabla$.

**Definition 2.1.** *A LP-Sasakian manifold is said to be a generalized $\eta$-Einstein manifold if its Ricci tensor S of type $(0, 2)$ satisfies the following condition*

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z) + cg(\phi Y, Z). \tag{15}$$

*on $M$, where $a, b$ and $c$ are smooth functions on $M$. If $c = 0$, then the manifold is called an $\eta$-Einstein manifold.*

# 3 Curvature tensor in a LP-Sasakian manifolds admitting quarter-symmetric metric connection $(QSMC)$

Let $M$ be a LP-Sasakian manifold with Levi-Civita connection $\nabla$, A quarter-symmetric metric connection $\widetilde{\nabla}$ in a LP-Sasakian manifold (17) is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{16}$$

A linear connection $\widetilde{\nabla}$ in a Riemannian manifold $M$ is said to be a quater symmetric metric connection if the torsion tensor $\widetilde{T}$ with respect to connection $\widetilde{\nabla}$ satisfies

$$\widetilde{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y], \tag{17}$$

where $\eta$ is a 1-form and $\phi$ is $(1, 1)$ tensor field. A linear connection $\widetilde{\nabla}$ is said to be a metric connection. if $\widetilde{\nabla}_g = 0$. i.e.,

$$(\widetilde{\nabla}_X g)(Y, Z) = 0. \tag{18}$$

$\forall\ X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on $M$, then $\widetilde{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection.

The curvature tensor $\widetilde{R}$ of LP-Sasakian manifold with a quarter-symmetric metric connection $\widetilde{\nabla}$is defined by

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z. \tag{19}$$

Using (3), (7), (9), (16) and (19), we have

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + [g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X] \\ &\quad +\eta(Z)[\eta(Y)X - \eta(X)Y] + [\eta(X)g(Y,Z) \\ &\quad -\eta(Y)g(X,Z)]\xi. \end{aligned}\tag{20}$$

$\forall\ X,Y,Z \in \chi(M)$ and $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the Riemannian curvature tensor of the Levi-Civita connection $\nabla$.
Contracting above equation (20), we have

$$\widetilde{S}(Y,Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z) - g(\phi Y,Z)\psi. \tag{21}$$

where $\psi = trace\phi$ and is defined by $\psi = \sum_{i=1}^{n} g(\phi e_i, e_i)$.
Again contracting (21), we have

$$\widetilde{r} = r - (n-1) - \psi^2, \tag{22}$$

where $r$ is the scalar curvature of the connection $\nabla$.
Replacing $Z$ by $\xi$ in (20) and using (3), (4), we have

$$\widetilde{R}(X,Y)\xi = 0. \tag{23}$$

Replacing $X$ by $\xi$, $Y$ by $X$ and $Z$ by $Y$ in (21) and using (3), (4), (11), we have

$$\widetilde{R}(\xi,X)Y = 0. \tag{24}$$

Putting $Y$ = $\xi$ in (24) and using (3), we have

$$\widetilde{R}(\xi,X)\xi = 0. \tag{25}$$

Putting $Y = X$ and $Z = \xi$ in (21) and using (3), (4), we have

$$\widetilde{S}(X,\xi) = 0. \tag{26}$$

Replacing $Y$ by $\xi$ in (16) and using (3), (4), (7), we have

$$\widetilde{\nabla}_X\xi = 0. \tag{27}$$

$\forall\, X, Y \in \chi(M)$.
Now, we express the Lie derivative along $\xi$ on $M$ with a quarter-symmetric metric connection as follows:

$$(\widetilde{L}_\xi g)(Y,Z) = \widetilde{L}_\xi g(Y,Z) - g(\widetilde{L}_\xi Y, Z) - g(Y, \widetilde{L}_\xi Z). \tag{28}$$

By virtue of (17), (28) takes the form

$$\begin{aligned}(\widetilde{L}_\xi g)(Y,Z) &= \widetilde{\nabla}_\xi g(Y,Z) - g(\widetilde{\nabla}_\xi Y - \widetilde{\nabla}_Y \xi - \phi Y, Z) \\ &\quad -g(Y, \widetilde{\nabla}_\xi Z - \widetilde{\nabla}_Z \xi - \phi Z).\end{aligned} \tag{29}$$

which after computation gives

$$(\widetilde{L}_\xi g)(Y,Z) = 2g(\phi Y, Z). \tag{30}$$

$\forall\, Y, Z \in \chi(M)$.
Hence, we have the following:

**Theorem 3.1.** *Every $n$-dimensional $\eta$-Ricci soliton LP-Sasakian manifold admitting quarter-symmetric metric connection is irregular.*

**Theorem 3.2.** *In a $\eta$-Ricci soliton LP-Sasakian manifold, $\xi$ and $g$ are parallel with respect to the connection $\widetilde{\nabla}$.*

**Theorem 3.3.** *In a $\eta$-Ricci soliton LP-Sasakian manifold, the integral curves of the vector field $\xi$ are geodesic with respect to the connection $\widetilde{\nabla}$.*

# 4 $\eta$-Ricci solitons on LP-Sasakian manifolds admitting quarter-symmetric metric connection $(QSMC)$

Suppose that a LP-Sasakian manifolds with quarter-symmetric metric connection admits an $\eta$-Ricci solitons $(g, \xi, \lambda, \mu)$, then it holds (2) and we have

$$(\widetilde{L}_\xi g)(Y,Z) + 2\widetilde{S}(Y,Z) + 2\lambda g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0. \tag{31}$$

By virtue of (30), (31) takes the form

$$\widetilde{S}(Y,Z) = -g(\phi Y, Z) - \lambda g(Y,Z) - \mu\eta(Y)\eta(Z). \tag{32}$$

Replacing $Y$ by $\xi$ and $Z$ by $\xi$ in (32) and using (3), (4), (26), we have

$$\lambda = \mu = 0. \tag{33}$$

Hence, we have the following:

**Theorem 4.1.** *If $(g, \xi, \lambda, \mu)$ is an η-Ricci soliton in a LP-Sasakian manifolds with quarter-symmetric metric connection, then the scalars $\lambda$ and $\mu$ are equal.*

**Example 4.1.** *Let $M = \{(x, y, z) \in \mathcal{R}^3 : z > 0\}$ be a 3-dimensional manifold, where $(x, y, z)$ are the standard coordinates in $\mathcal{R}^3$. The vector fields (17)*

$$e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad e_3 = \frac{\partial}{\partial z} = \xi,$$

*are linearly independent at every point of $M$ and hence form a basis of $T_pM$.*

The Lorentzian like (semi-Riemannian) metric $g$ on $M$ is defined by

$$\begin{aligned} g(e_1, e_1) = 1, \quad g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1, \\ g(e_1, e_2) = 0, \quad g(e_2, e_3) = 0, \quad g(e_1, e_3) = 0, \end{aligned} \tag{34}$$

Suppose $\eta$ be the 1-form defined as $\eta(X)$=$g(X, e_3)$=$g(X, \xi)$, $\forall\ X \in \chi(M)$. Let $\phi$ be the $(1, 1)$-tensor field defined by

$$\phi(e_1) = -e_1, \quad \phi(e_2) = -e_2, \quad \phi(e_3) = 0. \tag{35}$$

By linearity property of $\phi$ and $g$, we have

$$\eta(e_3) = -1, \quad \phi^2 X = X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$\forall\ X, Y \in \chi(M)$.
Now by direct computations, we obtain

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -e_2, \quad [e_3, e_1] = e_1. \tag{36}$$

Koszul's formula is given as

$$\begin{aligned} 2g\left(\nabla_X Y, Z\right) &= X g\left(Y, Z\right) + Y g\left(Z, X\right) - Z g\left(X, Y\right) \\ &\quad - g\left(\left[Y, Z\right], X\right) + g\left(\left[Z, X\right], Y\right) + g\left(\left[X, Y\right], Z\right), \end{aligned}$$

for arbitrary vector fields $X, Y, Z \in \chi(M)$.
Using Koszul's formula, we can easily calculate

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1, \\ &\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = -e_2, \\ &\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \end{aligned} \tag{37}$$

Let $X = \sum_{i=1}^{3} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3 \in \chi(M)$.

Also one can easily verify that $\nabla_X \xi = \phi X$, and $(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$.
Thus, the manifold $M$ is a LP-Sasakian manifold. From (16) and (37), we find

$$\begin{aligned} &\widetilde{\nabla}_{e_1} e_1 = 0, \quad \widetilde{\nabla}_{e_1} e_2 = 0, \quad \widetilde{\nabla}_{e_1} e_3 = 0, \\ &\widetilde{\nabla}_{e_2} e_1 = 0, \quad \widetilde{\nabla}_{e_2} e_2 = 0, \quad \widetilde{\nabla}_{e_2} e_3 = 0, \\ &\widetilde{\nabla}_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \widetilde{\nabla}_{e_3} e_3 = 0. \end{aligned} \tag{38}$$

It is known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

By using the above results, we easily calculate the components of the curvature tensors $R(e_i, e_j)e_k$ ; $i, j, k = 1, 2, 3$ of $\nabla$ as follows:

$$\begin{aligned} &R(e_1,e_2)e_1 = -e_2, \quad R(e_1,e_2)e_2 = e_1, \quad R(e_1,e_2)e_3 = 0, \\ &R(e_2,e_3)e_1 = 0, \quad R(e_2,e_3)e_2 = -e_3, \quad R(e_2,e_3)e_3 = -e_2, \\ &R(e_1,e_3)e_1 = -e_3, \quad R(e_1,e_3)e_2 = 0, \quad R(e_1,e_3)e_3 = -e_1, \end{aligned} \tag{39}$$

The curvature tensor $\widetilde{\nabla}$ can be calculated by using (19), (36) and (38), we have

$$\widetilde{R}(e_i, e_j)e_k = 0; \forall i, j, k = 1, 2, 3. \tag{40}$$

From these curvatures tensors, we obtain

$$S(e_1, e_1) = S(e_2, e_2) = 2, \quad S(e_3, e_3) = -2. \tag{41}$$

$$\widetilde{S}(e_1, e_1) = \widetilde{S}(e_2, e_2) = \widetilde{S}(e_3, e_3) = 0. \tag{42}$$

Therefore from (41) and (42) we get $r = 6$ and $\widetilde{r} = 0$, respectively. Thus it can be seen that the equation (22) is satisfied, where $\psi = \sum_{i=1}^{3} g(\phi e_i, e_i) = -2$. Now from equations (8), (17) and (18), we find

$$\phi(e_1, e_1) = \phi(e_2, e_2) = -1, \quad \phi(e_3, e_3) = 0. \tag{43}$$

In view of (17), the torsion tensor $\widetilde{T}$ with respect to $\widetilde{\nabla}$ as follows:

$$\begin{aligned} &\widetilde{T}(e_1, e_1) = 0, \quad \widetilde{T}(e_2, e_2) = 0, \quad \widetilde{T}(e_3, e_3) = 0, \\ &\widetilde{T}(e_1, e_2) = 0, \quad \widetilde{T}(e_1, e_3) = e_1, \quad \widetilde{T}(e_2, e_3) = e_2, \end{aligned} \tag{44}$$

and also

$$(\widetilde{\nabla}_X g)(Y,Z) = 0. \tag{45}$$

for any vector fields $X, Y, Z \in \chi(M)$ respectively. Thus the connection $\widetilde{\nabla}$ defined on $M$ is a quarter-symmetric metric connection.
Now from (32), we find $\widetilde{S}(e_1, e_1) = 1 - \lambda$ and $\widetilde{S}(e_3, e_3) = \lambda - \mu$, therefore from equation (42) it follows that $\lambda = 1$ and $\mu = 1$. Thus $(g, \xi, \lambda, \mu)$ for $\lambda = 1$ and $\mu = 1$ defines an $\eta$-Ricci soliton in a LP-Sasakian manifold admitting quarter-symmetric metric connection.
By using the values of $\lambda$ and $\mu$ in (32), we get

$$\widetilde{S}(Y,Z) = -g(\phi Y, Z) - g(Y,Z) - \eta(Y)\eta(Z). \tag{46}$$

Thus the manifold $M$ (defined in example 4.1) admitting an $\eta$-Ricci soliton $(g, \xi, \lambda, \mu)$ with a quarter-symmetric metric connection is a generalized $\eta$-Einstein manifold of the form (46).
Let $(g, V, \lambda, \mu)$ be an $\eta$-Ricci soliton in a LP-Sasakian manifold with quarter-symmetric metric connection, then we have

$$(\widetilde{L}_V g)(Y,Z) + 2\widetilde{S}(Y,Z) + 2\lambda g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0, \tag{47}$$

where $\widetilde{L}_V$ is the Lie derivative along the vector field $V$ on $M$ with a $QSMC$. Now, let $(g, V, \lambda, \mu)$ be an $\eta$-Ricci soliton in a LP-Sasakian manifold with a quarter-symmetric metric connection such that $V$ is pointwise collinear with $\xi$, i.e., $V = b\xi$, where $b$ is a function. Then (47) holds and we have

$$\begin{aligned} &bg(\widetilde{\nabla}_Y \xi, Z) + (Yb)\eta(Z) + bg(Y, \widetilde{\nabla}_Z \xi) + 2bg(\phi Y, Z) \\ &+(Zb)\eta(Y) + 2\widetilde{S}(Y,Z) + 2\lambda g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0. \end{aligned} \tag{48}$$

which in view of (27) takes the form

$$\begin{aligned} &(Yb)\eta(Z) + 2bg(\phi Y, Z) + (Zb)\eta(Y) \\ &+2\widetilde{S}(Y,Z) + 2\lambda g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0. \end{aligned} \tag{49}$$

Putting $Z = \xi$ in (49) and using (3), (4) and (26), we have

$$-(Yb) + (\xi b)\eta(Y) + 2\lambda\eta(Y) - 2\mu\eta(Y) = 0. \tag{50}$$

Again putting $Y = \xi$ in (50) and using (3), we get

$$(\xi b) + \lambda - \mu = 0. \tag{51}$$

Combinning the equations (50) and (51), we get

$$db = (\lambda - \mu)\eta. \tag{52}$$

Now applying d on (52), we find

$$(\lambda - \mu)d\eta = 0. \tag{53}$$

i.e,

$$\lambda = \mu, \quad d\eta \neq 0. \tag{54}$$

Using (54) in (52), we find $db = 0$,.i.e., $b$ is constant. Therefore (49) reduce to

$$\widetilde{S}(Y, Z) = -\lambda g(Y, Z) - bg(\phi Y, Z) - \mu\eta(Y)\eta(Z). \tag{55}$$

In view of (21), (55) takes the form

$$S(Y, Z) = -\lambda g(Y, Z) - (n - 1 + \mu)\eta(Y)\eta(Z) - (\psi - b)g(\phi Y, Z). \tag{56}$$

Hence, we have the following theorem:

**Theorem 4.2.** *If $(g, V, \lambda, \mu)$ is an $\eta$-Ricci soliton in a LP-Sasakian manifold with a quarter-symmetric metric connection such that $V$ is pointwise collinear with $\xi$, then $V$ is a constant multiple of $\xi$ and the manifold is a generalized $\eta$-Einstein manifold.*

**Corollary 4.1.** *If $(g, V, \lambda, \mu)$ is an $\eta$-Ricci soliton in a LP-Sasakian manifold with a quarter-symmetric metric connection such that $V$ is pointwise collinear with $\xi$, then the Ricci soliton on $M$ is steady, shrinking and expanding according as $\mu = 0$, $\mu < 0$ and $\mu > 0$, respectively.*

## 5 $\eta$-Ricci solitons on LP-Sasakian manifold admitting quarter-symmetric metric connection $(QSMC)$ satisfying $\widetilde{R}(X, Y).\widetilde{S} = 0$.

Let $M$ be an $n$-dimensional LP-Sasakian manifolds with quarter-symmetric metric connection admits an $\eta$-Ricci soliton. Now we have

$$\widetilde{S}(\widetilde{R}(X, Y)U, V) + \widetilde{S}(U, \widetilde{R}(X, Y)V) = 0. \tag{57}$$

$\forall\ X, Y, U, V \in \chi(M)$.
Replacing $X$ by $\xi$ in (57), we get

$$\widetilde{S}(\widetilde{R}(\xi, Y)U, V) + \widetilde{S}(U, \widetilde{R}(\xi, Y)V) = 0. \tag{58}$$

In view of (24), (58) becomes

$$g(Y,U)\widetilde{S}(\xi,V)+\eta(U)\widetilde{S}(Y,V)+g(Y,V)\widetilde{S}(U,\xi)+\eta(V)\widetilde{S}(U,Y)=0. \quad (59)$$

Replacing $U$ by $\xi$ and using (3), (25) reduces to

$$\widetilde{S}(Y,V)=0. \quad (60)$$

In view of (21), (60) takes the form

$$S(Y,V)=-(n-1)\eta(Y)\eta(V)+g(\phi Y,V)\psi. \quad (61)$$

From (30), (31) and (60), we find

$$g(\phi Y,V)+\lambda g(Y,V)+\mu\eta(Y)\eta(V)=0. \quad (62)$$

Putting $V=Y=\xi$ in (62) and using (3) and (4), it follows that

$$\lambda=\mu. \quad (63)$$

Hence, we have the following theorem:

**Theorem 5.1.** *If $(g,\xi,\lambda,\mu)$ is an $\eta$-Ricci soliton in a LP-Sasakian manifold with a quarter-symmetric metric connection satisfying $\widetilde{R}(X,Y).\widetilde{S}=0$, then the manifold is a class of generalized $\eta$-Einstein manifold and the scalars $\lambda$ and $\mu$ are equal.*

**Corollary 5.1.** *If $(g,\xi,\lambda,\mu)$ is an $\eta$-Ricci soliton in a LP-Sasakian manifold with a quarter-symmetric metric connection satisfying $\widetilde{R}(X,Y).\widetilde{S}=0$, then the Ricci soliton on $M$ is steady, shrinking and expanding according as $\mu=0$, $\mu<0$ and $\mu>0$, respectively.*

# 6 Example of 3-dimensional LP-Sasakian manifold with a quarter-symmetric metric connection

**Example 6.1.** *Let $M=\{(x_1,y_1,z_1)\in\mathcal{R}^3 : z_1>0\}$ be a 3-dimensional manifold, where $(x_1,y_1,z_1)$ are the standard coordinates in $\mathcal{R}^3$. The vector fields (17)*

$$e_1=coshz_1\frac{\partial}{\partial x_1}+sinhz_1\frac{\partial}{\partial y_1},\quad e_2=sinhz_1\frac{\partial}{\partial x_1}+coshz_1\frac{\partial}{\partial y_1},\quad e_3=\frac{\partial}{\partial z_1}=\xi,$$

*are linearly independent at every point of $M$ and hence form a basis of $T_pM$.*

The Lorentzian like (semi-Riemannian) metric $g$ on $M$ is defined by

$$\begin{aligned} g(e_1,e_1)=1, \quad g(e_2,e_2)=1, \quad g(e_3,e_3)=-1, \\ g(e_1,e_2)=0, \quad g(e_2,e_3)=0, \quad g(e_1,e_3)=0, \end{aligned} \tag{64}$$

Suppose $\eta$ be the 1-form defined as $\eta(X)$=$g(X,e_3)$=$g(X,\xi)$, $\forall\ X \in \chi(M)$. Let $\phi$ be the $(1,1)$-tensor field defined by

$$\phi(e_1)=-e_2, \quad \phi(e_2)=-e_1, \quad \phi(e_3)=0. \tag{65}$$

By linearity property of $\phi$ and g, we have

$$\eta(e_3)=-1, \quad \phi^2X=X+\eta(X)\xi, \quad g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y),$$

$\forall\ X,Y \in \chi(M)$.
Now by direct computations, we obtain

$$[e_1,e_2]=0, \quad [e_2,e_3]=-e_1, \quad [e_3,e_1]=e_2. \tag{66}$$

Using Koszul's formula, we can easily calculate

$$\begin{aligned} &\nabla_{e_1}e_1=0, && \nabla_{e_1}e_2=-e_3, && \nabla_{e_1}e_3=-e_2, \\ &\nabla_{e_2}e_1=-e_3, && \nabla_{e_2}e_2=0, && \nabla_{e_2}e_3=-e_1, \\ &\nabla_{e_3}e_1=0, && \nabla_{e_3}e_2=0, && \nabla_{e_3}e_3=0. \end{aligned} \tag{67}$$

Also one can easily verify that $\nabla_X\xi=\phi X$, and $(\nabla_X\phi)Y=g(X,Y)\xi+\eta(Y)X+2\eta(X)\eta(Y)\xi$.
Thus, the manifold $M$ is a LP-Sasakian manifold. From (16) and (67), we find

$$\begin{aligned} &\widetilde{\nabla}_{e_1}e_1=0, && \widetilde{\nabla}_{e_1}e_2=0, && \widetilde{\nabla}_{e_1}e_3=0, \\ &\widetilde{\nabla}_{e_2}e_1=0, && \widetilde{\nabla}_{e_2}e_2=0, && \widetilde{\nabla}_{e_2}e_3=0, \\ &\widetilde{\nabla}_{e_3}e_1=0, && \nabla_{e_3}e_2=0, && \widetilde{\nabla}_{e_3}e_3=0. \end{aligned} \tag{68}$$

It is known that

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z.$$

By using the above results, we easily calculate the components of the curvature tensors $R(e_i,e_j)e_k$ ; $i,j,k=1,2,3$ of $\nabla$ as follows:

$$\begin{aligned} &R(e_1,e_2)e_1=e_2, && R(e_1,e_2)e_2=-e_1, && R(e_1,e_2)e_3=0, \\ &R(e_2,e_3)e_1=0, && R(e_2,e_3)e_2=-e_3, && R(e_2,e_3)e_3=-e_2, \\ &R(e_1,e_3)e_1=-e_3, && R(e_1,e_3)e_2=0, && R(e_1,e_3)e_3=-e_1, \end{aligned} \tag{69}$$

$\widetilde{R}(e_i, e_j)e_k$; $i, j, k = 1, 2, 3$ of $\widetilde{\nabla}$ can be calculated by using (19), (66) and (68). Thus, we have

$$\begin{aligned} &\widetilde{R}(e_1, e_2)e_1 = e_2, \quad \widetilde{R}(e_1, e_2)e_2 = 0, \quad \widetilde{R}(e_1, e_2)e_3 = 0, \\ &\widetilde{R}(e_2, e_3)e_1 = 0, \quad \widetilde{R}(e_2, e_3)e_2 = 0, \quad \widetilde{R}(e_2, e_3)e_3 = 0, \\ &\widetilde{R}(e_1, e_3)e_1 = 0 \quad \widetilde{R}(e_1, e_3)e_2 = 0, \quad \widetilde{R}(e_1, e_3)e_3 = 0, \end{aligned} \tag{70}$$

From these curvature tensors, we obtain

$$S(e_1, e_1) = S(e_2, e_2) = 0, \quad S(e_3, e_3) = -2. \tag{71}$$

$$\widetilde{S}(e_1, e_1) = \widetilde{S}(e_2, e_2) = \widetilde{S}(e_3, e_3) = 0. \tag{72}$$

Therefore from (71) and (72) we get $r = 2$ and $\widetilde{r} = 0$, respectively. Thus it can be seen that the equation (22) is satisfied, where $\psi = \sum_{i=1}^{3} g(\phi e_i, e_i) = 0$.
Now from equation (32), we have

$$\widetilde{S}(e_1, e_1) = -\lambda, \quad \widetilde{S}(e_3, e_3) = \lambda - \mu. \tag{73}$$

Therefore from (72) it follows that $\lambda = \mu = 0$. Hence, Ricci soliton $(g, \xi, \lambda, \mu)$ in a LP-Sasakian manifold admitting a quarter-symmetric metric connection $\widetilde{\nabla}$ is steady, which verifies corollaries (4.1) and (5.1).

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