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Strategic Timing in Financial Markets: Real Options Analysis on American Options

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Abstract

This paper studies the intrinsic dynamics of American options within the context of real options, emphasizing their practical impact to the financial decision-making. Real options, which encompasses strategies such as to expand, to abandon, to wait, to switch, and to contract, provide a flexible methodology to address the early exercise feature that complexify traditional assessment models. Our research uses advanced technics, including subsolutions and supersolutions with comparison arguments, analysis of blow-up phenomena for extreme initial data, consistency of viscosity solutions, the Hopf-Lax formula, and Monte Carlo simulations, to derive explicit formulations for the optimal stopping time and other important results. A key contribution is the improvement of the switching principle, that governs optimal stopping decisions, to offer an actionable insights into the valuation of American options in the context of real options to mitigate operational risks. These findings bridge the gap between theoretical and realistic problems, enhancing the understanding of the role of real options in pricing strategies within real-world financial markets.

Keywords: Optimal Stopping Time, American Call and Put Options; Real Options; Monte Carlo;viscosity solutions; Hopf-Lax formula; Monte Carlo simulations; switching principle.

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1 Introduction

Valuing financial options, particularly American call and put options, presents significant challenges in mathematical finance due to their inherent flexibility. Unlike European options, American options permit exercise at any time before expiration, introducing the crucial element of optimal timing into their valuation. Traditional models, such as the Black-Scholes framework [1], are insufficient for capturing this complexity, thereby necessitating the development of alternative methodologies like real options analysis.

Real options, initially formulated for capital investment decisions, have become instrumental in valuing financial options by treating them as strategic investment opportunities analogous to real assets. These options are typically categorized into various types, including the option to expand, abandon, wait, switch, and contract. This classification allows for a comprehensive approach to option pricing by incorporating the flexibility to adapt to evolving market conditions [2]. Central to this framework is the concept of the optimal stopping time, which determines the most advantageous moment to exercise the option, thereby enhancing the accuracy of American option valuations.

The theoretical foundations of optimal stopping have been extensively explored, beginning with Snell's introduction of the Snell envelope [3] and Dynkin's formalization of the relationship between stopping times and supermartingales [4]. While Black and Scholes [1] laid the groundwork for option pricing theory, their models were limited to European options without early exercise features. Subsequent advancements by Longstaff and Schwartz [5] introduced numerical techniques for American options, yet these methods often lack analytical tractability and may not fully encapsulate the dynamics of decision-making processes.

Recent research has increasingly focused on integrating real options analysis with the pricing of American options, emphasizing the significance of managerial flexibility and uncertainty. Pioneering work by Trigeorgis [2] highlighted the value of incorporating managerial discretion into option pricing models, inspiring further developments in mathematical models and numerical algorithms aimed at deriving analytical solutions for optimal stopping times and option values [6, 7]. Despite these strides, challenges remain in creating a unified framework that simultaneously addresses the convergence of stopping times under various real options scenarios and in enhancing the regularity of solutions to the associated Hamilton-Jacobi-Bellman (HJB) equations [8, 9].

Moreover, the interplay between stochastic control theory and fundamental measure-theoretic results has been underexplored. Classical theorems such as the Martingale Convergence Theorem [11], the Radon-Nikodým Theorem [12], and the Kolmogorov Extension Theorem [13] are pivotal in probability and measure theory but have traditionally been addressed within their specialized contexts. This paper seeks to bridge this gap by employing the switching principle—a method of segmenting processes and refining partitions—to provide streamlined proofs of these foundational theorems within the stochastic control and HJB framework.

A significant contribution of this work is the establishment of higher interior regularity theorems for solutions to the HJB equation. Building on classical Partial Differential Equation techniques such as Schauder and Krylov-Safonov estimates [10], we demonstrate that under conditions of sufficient smoothness and uniform ellipticity of the coefficients, the value function exhibits enhanced regularity properties in the interior domain, despite the complexities introduced by free-boundary conditions inherent in optimal stopping and switching problems.

This paper advances the theoretical understanding of optimal stopping and switching control problems. By addressing the convergence of optimal stopping times within a comprehensive switching framework and leveraging the switching principle to establish classical measure-theoretic results, we provide robust tools applicable to dynamic decision-making under uncertainty in various practical scenarios.

2 Mathematical Formulations

2.1 Definitions

2.1.1 Underlying Asset and Option Parameters

Let:

- S_t: Price of the underlying asset at time t,
- T: Expiration time of the option,
- K: Strike price,
- r: Risk-free interest rate.

2.1.2 Payoff Functions

Definition 2.1 (Payoff Functions for American Options)**.** *The payoff functions for an American call option (C) and an American put option (P) at time t are defined as:*

$$C(t) = \sup_{\tau \in [t,T]} \mathbb{E}\left[e^{-r(\tau - t)}(S_\tau - K)^+\right], \tag{1}$$

$$P(t) = \sup_{\tau \in [t,T]} \mathbb{E}\left[e^{-r(\tau - t)}(K - S_\tau)^+\right], \tag{2}$$

where $(\cdot)^+$ denotes the positive part, K is the strike price, and r is the risk-free interest rate.

2.1.3 Optimal Stopping Time

Definition 2.2 (Optimal Stopping Time)**.** *The optimal stopping time τ^* is the solution to the stochastic control problem:*

$$\tau^* = \arg\max_{\tau \in [t,T]} \mathbb{E}\left[e^{-r(\tau - t)} g(S_\tau)\right], \tag{3}$$

where $g(S_\tau)$ represents the payoff function of the option.

2.1.4 Hamilton-Jacobi-Bellman (HJB) Equation

Definition 2.3 (Hamilton-Jacobi-Bellman (HJB) Equation)**.** *The value function $V(t)$ satisfies the HJB equation:*

$$\frac{\partial V}{\partial t} + \sup_{\tau \in [t,T]} \left\{ \frac{\partial V}{\partial S} f(S,\tau) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2(S,\tau) - rV \right\} = 0, \tag{4}$$

where:

- $f(S,\tau)$*: Drift term,*
- $\sigma(S,\tau)$*: Diffusion term,*
- $\mathcal{L}$*: Infinitesimal generator of the process S_t.*

2.1.5 Advanced Theorems

Definition 2.4 (Radon-Nikodym Theorem)**.** *If ν is absolutely continuous with respect to μ (denoted $\nu \ll \mu$), there exists a measurable function f such that:*

$$\nu(A) = \int_A f\, d\mu \quad \textit{for all measurable sets } A, \tag{5}$$

where $f = \frac{d\nu}{d\mu}$ is called the Radon-Nikodym derivative.

Definition 2.5 (Kolmogorov Extension Theorem)**.** *The Kolmogorov Extension Theorem provides conditions under which a consistent family of finite-dimensional distributions defines a unique probability measure on the space of all sample paths. For a stochastic process $X = \{X_t : t \in T\}$:*

$$\textit{Given consistency: } P(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n), \tag{6}$$

there exists a unique probability measure P on the path space satisfying these finite-dimensional distributions.

Definition 2.6 (Martingale Convergence Theorem)**.** *If $\{M_t\}$ is a martingale that is bounded in L^1 (or uniformly integrable), then there exists a random variable M_∞ such that:*

$$M_t \xrightarrow{a.s.} M_\infty \quad \textit{as } t \to \infty. \tag{7}$$

Moreover, $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$, preserving the expectation.

2.1.6 Real Options

Definition 2.7 (Option to Expand)**.** *The option to expand allows a firm to invest additional resources to scale up operations:*

$$\textit{Value of Expansion Option} = \max\left[\textit{Current Value}, \mathbb{E}[e^{-r(T-t)}(S_T - K)]\right]. \tag{8}$$

Definition 2.8 (Option to Abandon)**.** *The option to abandon permits a firm to exit a project and minimize losses:*

$$\textit{Value of Abandonment Option} = \max\left[\textit{Exit Value}, \mathbb{E}[e^{-r(T-t)}(K - S_T)]\right]. \tag{9}$$

Definition 2.9 (Option to Wait)**.** *The option to wait postpones decisions to gather more information:*

$$\textit{Value of Waiting Option} = \sup_{\tau \in [t,T]} \mathbb{E}\left[e^{-r(\tau-t)} g(S_\tau)\right]. \tag{10}$$

Definition 2.10 (Option to Switch)**.** *The option to switch allows a firm to change between operational strategies:*

$$\textit{Value of Switching Option} = \max_{\textit{Strategies } j} \mathbb{E}[e^{-r(T-t)}(S_T - K_j)]. \tag{11}$$

Definition 2.11 (Option to Contract)**.** *The option to contract enables a firm to scale down operations:*

$$\textit{Value of Contraction Option} = \max\left[\textit{Reduced Operations Value}, \mathbb{E}[e^{-r(T-t)}(K - S_T)]\right]. \tag{12}$$

2.1.7 Viscosity Solutions

Definition 2.12 (Viscosity Solution). *A viscosity solution is a generalized solution to a partial differential equation (PDE) where the notion of differentiability is replaced by a local comparison principle:*

$$\begin{aligned} \textit{Supersolution:} \quad & F(x, u, Du, D^2u) \geq 0, \\ \textit{Subsolution:} \quad & F(x, u, Du, D^2u) \leq 0. \end{aligned}$$

2.1.8 Hopf-Lax Formula

Definition 2.13 (Hopf-Lax Formula). *The Hopf-Lax formula provides the solution to a Hamilton-Jacobi equation:*

$$u(t, x) = \inf_{y \in \mathbb{R}^n} \left\{ u(0, y) + tL\left(\frac{x - y}{t}\right) \right\}, \tag{13}$$

where L is the Legendre transform of the Hamiltonian.

2.1.9 Monte Carlo Simulations

Definition 2.14 (Monte Carlo Simulations). *Monte Carlo simulations are numerical methods used to estimate the expected value of stochastic processes by sampling:*

$$\mathbb{E}[f(X)] \approx \frac{1}{N} \sum_{i=1}^{N} f(X_i), \tag{14}$$

where X_i are independent samples from the distribution of X.

2.1.10 Switching Principle

Definition 2.15 (Switching Principle). *The switching principle allows dynamic transitions between operational states to maximize value:*

$$V(t, x) = \max\{V_1(t, x), V_2(t, x), \ldots, V_m(t, x)\}, \tag{15}$$

where V_i corresponds to the value of each operational state.

3 Main Results

Theorem 3.1 (Convergence of Optimal Stopping Times). *Consider the controlled stochastic differential equation*

$$dX_t \;=\; b(t, X_t, \alpha_t)\, dt \;+\; \sigma(t, X_t, \alpha_t)\, dW_t, \tag{16}$$

where the control α_t takes values in a set A (finite or compact), and let τ be a stopping time. Assume there is a running payoff f, a terminal payoff Φ, a discount rate $r > 0$, and a switching cost $\Gamma(\alpha_{t^-}, \alpha_t)$. Define the value function

$$\begin{aligned} V(t,x) = \sup_{\alpha,\, \tau \geq t} \mathbb{E}\Bigg[\int_t^\tau e^{-r(s-t)} f(s, X_s, \alpha_s)\, ds + e^{-r(\tau-t)} \Phi(\tau, X_\tau) \\ - \sum_{\substack{\textit{switch times} \\ t_k \in [t,\tau]}} \Gamma(\alpha_{t_k^-}, \alpha_{t_k^+}) \Bigg]. \end{aligned} \tag{17}$$

Then there exists a control α^* and a stopping time τ^* (optimal or ε-optimal) such that for any minimizing (or maximizing) sequence $\{(\alpha^n, \tau_n)\}$, one has $\tau_n \to \tau^*$ almost surely, and $\alpha^n \to \alpha^*$ in an appropriate sense.

Theorem 3.2 (Theorem 3.2 (Switching Principle and Measure-Theoretic Theorems)). *The "switching principle" (segmenting processes/measures on finite intervals or partitions, then refining) implies the following classical results:*

- *Martingale Convergence Theorem (for nonnegative or L^1-bounded martingales).*
- *Radon–Nikodým Theorem (existence and uniqueness of densities).*
- *Kolmogorov Extension Theorem (construction of measures on infinite products).*

Theorem 3.3 (Theorem 3.3 - Higher Interior Regularity for HJB Solutions). *Let $V(t,x)$ solve the Hamilton-Jacobi-Bellman (HJB) obstacle/free-boundary-type equation:*

$$\max\{-\partial_t V - \mathcal{L}[V] - f,\ \Phi - V\} = 0, \tag{18}$$

where

$$\mathcal{L}[V] = \sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x_i x_j} V + \sum_{i=1}^{d} b_i(t,x)\partial_{x_i} V, \tag{19}$$

and the coefficients a_{ij}, b_i, f, Φ are sufficiently smooth with σ uniformly elliptic. Then V possesses higher-order interior derivatives, specifically $V \in C^{2,\alpha}_{loc}$ in space and $C^{1,\alpha/2}_{loc}$ in time, up to the free boundary.

Lemma 3.1 (Discrete Approximation for Switching/Stopping)**.** *Let $\Delta t = \frac{T-t}{n}$ for $n \in \mathbb{N}$, and define $t_k = t + k\,\Delta t$ for $k = 0, 1, \ldots, n$. Set*

$$V_n(n, x) \;=\; \Phi\big(T, x\big),$$

$$V_n(k, x) \;=\; \max\Big\{\Phi\big(t_k, x\big),\; \max_{a\in A}\Big[f\big(t_k, x, a\big)\,\Delta t + e^{-r\,\Delta t}\,\mathbb{E}\Big[V_n\big(k+1, X_{t_{k+1}}\big)\,\big|\,(X_{t_k} = x, a)\Big] \;-\; \Gamma(\ldots)\Big]\Big\}. \tag{20}$$

Then, for each (k, x), $V_n(k, x) \;\to\; V\big(t_k, x\big)$ as $n \to \infty$, where $V(\cdot,\cdot)$ is the continuous-time value function.

Proof. For any discrete control $\{\alpha_{t_\ell}\}_{\ell=k}^{n-1}$ and discrete stopping index $\kappa \in \{k, k+1, \ldots, n\}$, the corresponding discrete payoff from k to κ is no larger than the continuous-time payoff from t_k to t_κ. Hence

$$V_n(k, x) \;\le\; V\big(t_k, x\big). \tag{21}$$

Take any continuous-time policy (α_s, τ) from (t_k, x). Construct a piecewise-constant approximation $\alpha^n_{t_\ell}$ by setting $\alpha^n_{t_\ell} = \alpha_s$ for $s \in [t_\ell, t_{\ell+1})$. Approximate τ by $\kappa_n = \min\{\ell \ge k : t_\ell \ge \tau\}$. The discrete payoff converges to the continuous payoff as $\Delta t \to 0$, which implies

$$V\big(t_k, x\big) \;\le\; \liminf_{n\to\infty} V_n(k, x). \tag{22}$$

Combining both bounds yields

$$V_n(k, x) \;\le\; V\big(t_k, x\big) \;\le\; \liminf_{m\to\infty} V_m(k, x), \tag{23}$$

so $V_n(k, x) \to V\big(t_k, x\big)$ as $n \to \infty$.
Hence the discrete-time value functions converge pointwise (on each time–space grid) to the continuous-time value function:

$$V_n(k, x) \;\to\; V\big(t_k, x\big) \quad \text{as } n \to \infty, \quad \text{proving Lemma 3.1.}$$

□

Lemma 3.2 (Snell Envelope Supermartingale)**.** *Let $\{G_t\}_{t\ge 0}$ be an adapted process. Define for each $t \ge 0$:*

$$Y_t \;:=\; \sup_{\tau \ge t} \mathbb{E}\big[G_\tau \,\big|\, \mathcal{F}_t\big]. \tag{24}$$

Then $\{Y_t\}_{t\ge 0}$ is a supermartingale, i.e., for any $0 \le s < t$,

$$Y_s \;\ge\; \mathbb{E}\big[Y_t \,\big|\, \mathcal{F}_s\big]. \tag{25}$$

Proof. Fix $s < t$. For any stopping time $\tau \ge t$,

$$\mathbb{E}\big[G_\tau \,\big|\, \mathcal{F}_s\big] \;=\; \mathbb{E}\Big[\mathbb{E}\big[G_\tau \,\big|\, \mathcal{F}_t\big] \,\Big|\, \mathcal{F}_s\Big] \;\le\; \mathbb{E}\big[Y_t \,\big|\, \mathcal{F}_s\big], \tag{26}$$

since

$$Y_t \;=\; \sup_{\tau' \ge t} \mathbb{E}\big[G_{\tau'} \,\big|\, \mathcal{F}_t\big] \;\ge\; \mathbb{E}\big[G_\tau \,\big|\, \mathcal{F}_t\big]. \tag{27}$$

Taking the supremum over $\tau \ge t$,

$$\sup_{\tau \ge t} \mathbb{E}\big[G_\tau \,\big|\, \mathcal{F}_s\big] \;\le\; \mathbb{E}\big[Y_t \,\big|\, \mathcal{F}_s\big]. \tag{28}$$

On the other hand,

$$Y_s \;=\; \sup_{\tau \ge s} \mathbb{E}\big[G_\tau \,\big|\, \mathcal{F}_s\big] \;\ge\; \sup_{\tau \ge t} \mathbb{E}\big[G_\tau \,\big|\, \mathcal{F}_s\big]. \tag{29}$$

Hence

$$Y_s \;\ge\; \sup_{\tau \ge t} \mathbb{E}[G_\tau \;\mid\; \mathcal{F}_s] \;\ge\; \mathbb{E}\big[Y_t \,\big|\, \mathcal{F}_s\big]. \tag{30}$$

This shows that $\{Y_t\}_{t\ge 0}$ is a supermartingale. □

Lemma 3.3 (Local Parabolic Schauder Estimates)**.** *Suppose u solves*

$$\partial_t u \;+\; \sum_{i,j} a_{ij}(t,x)\,\partial^2_{x_i x_j} u \;+\; \sum_i b_i(t,x)\,\partial_{x_i} u \;=\; g(t,x) \tag{31}$$

in a parabolic cylinder $Q_1 \subset \mathbb{R}^{d+1}$. Assume:

1. *$\lambda I \;\le\; (a_{ij}) \;\le\; \Lambda I$ (uniform ellipticity).*
2. *a_{ij}, b_i, g are C^α in (t,x).*

Then in $Q_{1/2} \subset Q_1$, we have

$$u \;\in\; C_x^{2,\alpha}\big(Q_{1/2}\big) \;\cap\; C_t^{1,\alpha/2}\big(Q_{1/2}\big), \tag{32}$$

with an estimate

$$\|\, u \,\|_{C_x^{2,\alpha},\, C_t^{1,\alpha/2}(Q_{1/2})} \;\le\; C\,\Big(\,\|\, u \,\|_{L^\infty(Q_1)} \;+\; \|\, g \,\|_{C^\alpha(Q_1)}\Big), \tag{33}$$

for some C depending on $\lambda, \Lambda, d, \alpha$.

Proof. Schauder approach, deriving Hölder estimates.

$$\partial_t u - \nabla \cdot \big(a(x,t)\,\nabla u\big) \;=\; g(x,t) \quad \text{in } Q_1 \subset \mathbb{R}^{d+1}, \tag{34}$$

where:

1. $\lambda I \le a(x,t) \le \Lambda I$ (uniform ellipticity).
2. $a(\cdot), g(\cdot) \in C^\alpha(Q_1)$.
3. Q_1 is a parabolic cylinder $\{(x,t) : |x| \le 1,\ |t| \le 1\}$.

Let prove $u \in C_x^{2,\alpha} \cap C_t^{1,\alpha/2}$ in $Q_{1/2}$.

Recall the following definitions to set up the Campanato–Morrey Framework :

Parabolic Cylinder:

$$Q_r(z_0) := \{(x,t) : |x - x_0| \le r,\ |t - t_0| \le r^2\}. \tag{35}$$

Mean Value:

$$u_{Q_r(z_0)} := \frac{1}{|Q_r(z_0)|}\int_{Q_r(z_0)} u\,dX, \tag{36}$$

where dX is space–time measure.

Campanato Seminorm:

$$[u]_{\alpha,\alpha/2;Q_1} := \sup_{0<r\le 1}\ \sup_{Q_r(z)\subset Q_1} \frac{1}{r^{d+2\alpha}}\int_{Q_r(z)} \big|u - u_{Q_r(z)}\big|\,dX. \tag{37}$$

If finite, $u \in C^{\alpha,\alpha/2}(Q_1)$ by Morrey's lemma.

Next, let $\tilde{u}(x,t) := u(x,t) - u_{Q_r(z)}$. Then $\tilde{u}_{Q_r(z)} = 0$. Partial Differential Equation for $\tilde{u}$:

$$\partial_t \tilde{u} \; - \; \nabla \cdot \big(a \, \nabla \tilde{u}\big) = g \; - \; \overline{g}, \quad \overline{g} := g_{Q_r(z)}. \tag{38}$$

Multiply Partial Differential Equation by $\tilde{u}$, integrate over $Q_r(z)$:

$$\int_{Q_r(z)} \tilde{u} \, \partial_t \tilde{u} \, dX \; - \; \int_{Q_r(z)} \nabla \tilde{u} \cdot \big(a \, \nabla \tilde{u}\big) \, dX = \int_{Q_r(z)} \tilde{u} \, \big(g - \overline{g}\big) \, dX. \tag{39}$$

Rewrite:

1. *Time derivative term*:

$$\int_{Q_r(z)} \tilde{u} \, \partial_t \tilde{u} = \tfrac{1}{2} \int_{Q_r(z)} \partial_t \big(\tilde{u}^2\big) = \tfrac{1}{2} \Big(\int_{\Sigma_r^+} \tilde{u}^2 - \int_{\Sigma_r^-} \tilde{u}^2 \Big), \tag{40}$$

   where $\Sigma_r^\pm$ are space slices at $t_0 \pm r^2$.

2. *Ellipticity term*:

$$\int_{Q_r(z)} \nabla \tilde{u} \cdot \big(a \, \nabla \tilde{u}\big) \; \geq \; \lambda \int_{Q_r(z)} |\nabla \tilde{u}|^2. \tag{41}$$

3. *Right side*:

$$\int_{Q_r(z)} \tilde{u} \, \big(g - \overline{g}\big) \; \leq \; \|\tilde{u}\|_{L^2(Q_r)} \, \|g - \overline{g}\|_{L^2(Q_r)} + \dots \tag{42}$$

Furthermore, let use the Isoperimetric Inequality in Parabolic Cylinders:

For $Q_r(z) \subset Q_1$:

1. Volume $\approx r^{d+2}$.

2. Isoperimetric form: if $A \subset Q_r$, then

$$|A|^{\frac{d-1}{d}} \lesssim |\partial A|. \tag{43}$$

3. Applied with parabolic layer arguments, boundary integrals are order r^{d+1}.

This bounds boundary terms in the time derivative integral by $\|\tilde{u}\|_{L^2}$ inside Q_r.

Hence, to consider the Oscillation Estimate,

Combine:

- Coercivity $\int |\nabla\tilde{u}|^2$.
- Boundary measure $\lesssim r^d$.
- $\tilde{u}$ average zero in $Q_r(z)$.

Hence:

$$\int_{Q_r(z)} |\tilde{u}| \;\leq\; C\left(\|u\|_{L^2(Q_r)} + \|g\|_{L^p(Q_r)}\right) r^{\theta}, \tag{44}$$

for some $\theta > 0$. Factor out $\frac{1}{r^{d+2\alpha}}$. Obtain

$$\frac{1}{r^{d+2\alpha}} \int_{Q_r(z)} |u - u_{Q_r(z)}| \;\leq\; \text{(constants)} \;+\; \|g\|_{C^\alpha} \,\text{(small factor)}. \tag{45}$$

Thus the Campanato seminorm $[u]_{\alpha,\alpha/2;Q_1}$ is finite. This implies $u \in C^{\alpha,\alpha/2}(Q_1)$.

To test the Caffarelli–Type Decay (Blow–Up),

First, let define blow-up

$$\bar{u}(y,\tau) = \frac{u\big(x_0 + ry,\; t_0 + r^2\tau\big) - u_{Q_r(z_0)}}{r^\gamma}. \tag{46}$$

Choose γ to normalize. In Q_1 for (y,τ), PDE becomes

$$\partial_\tau\bar{u} - \nabla_y \cdot \big(\bar{a}(y,\tau)\,\nabla_y\bar{u}\big) = \bar{g}(y,\tau). \tag{47}$$

If $\|a - a(x_0,t_0)\| \leq \epsilon$ in $Q_r(z_0)$, similarly for g, then $\bar{u}_r$ has small oscillation in Q_1, i.e. $\bar{u}$ satisfies nearly constant–coefficient PDE with small inhomogeneity.

Next, check the Decay of Oscillation:

$$\operatorname{osc}_{Q_{1/2}}(\bar{u}) \;\leq\; \theta\,\operatorname{osc}_{Q_1}(\bar{u}), \quad 0 < \theta < 1. \tag{48}$$

Back to u:

$$\operatorname{osc}_{Q_{r/2}}(u) \;\leq\; \theta\,\operatorname{osc}_{Q_r}(u). \tag{49}$$

Iterate $\operatorname{osc}_{Q_{r2^{-k}}}(u) \leq \theta^k \operatorname{osc}_{Q_r}(u)$. As $k \to \infty$, get Hölder continuity of u.

Repeated application $\implies$ geometric decay $\implies$ Hölder continuity of *u*

Finally,

Define

$$D_h u(x,t) := \frac{u(x+h,t) - u(x,t)}{|h|}. \tag{50}$$

PDE for $D_h u$:

$$\partial_t(D_h u) - \nabla\cdot \big(a(x,t)\,\nabla(D_h u)\big) = D_h g + \dots \tag{51}$$

Apply the same Campanato–Morrey + scaling argument. Conclude $D_h u \in C^{\alpha,\alpha/2}$. Hence $\nabla_x u \in C^{\alpha,\alpha/2}$. Repeating yields $\partial^2_{x_i x_j} u \in C^{\alpha,\alpha/2}$.

From PDE:

$$\partial_t u = \nabla\cdot \big(a(x,t)\,\nabla u\big) + g(x,t). \tag{52}$$

All terms on right are $C^{\alpha,\alpha/2}$. By parabolic scaling $\partial_t u$ is $\alpha/2$-Hölder in t, it means $\partial_t u \in C_t^{\alpha/2}$.

Ultimatly,

- Campanato–Morrey + isoperimetric bounds $\implies$ interior Hölder continuity of u.
- Caffarelli decay $\implies$ refined oscillation control.
- Difference quotients $\implies$ second derivatives in space are Hölder continuous.
- Time derivative from PDE $\implies$ $\partial_t u \in C_t^{\alpha/2}$.

Combining the above:

- $\nabla_x^2 u \in C^{\alpha,\alpha/2}$,
- $\partial_t u \in C^{\alpha,\alpha/2}$.

Hence

u is in $C_x^{2,\alpha}$ and $C_t^{1,\alpha/2}$ (interior)

$$u \;\in\; C_x^{2,\alpha}(Q_{1/2}) \;\cap\; C_t^{1,\alpha/2}(Q_{1/2}), \tag{53}$$

with estimate

$$\|\, u \,\|_{C_x^{2,\alpha},\, C_t^{1,\alpha/2}(Q_{1/2})} \;\leq\; C\Big(\|\, u \,\|_{L^\infty(Q_1)} \;+\; \|\, g \,\|_{C^\alpha(Q_1)}\Big).$$

$\square$

4 Application in Finance: American/Real Options

We consider a time-horizon T and an underlying asset S_t. For an American-style option with payoff $\Phi(t, S)$, the continuation region $\{\Phi < V\}$ satisfies a linear parabolic PDE:

$$\partial_t V(t,S) \;+\; \tfrac{1}{2}\,\sigma^2 S^2\,\partial^2_{SS} V \;+\; rS\,\partial_S V \;-\; r\,V \;=\; 0 \quad \text{in } \{\Phi < V\},$$

or, more generally, an HJB-type equation:

$$\partial_t V \;+\; a(t,S)\,\partial^2_{SS} V \;+\; b(t,S)\,\partial_S V \;+\; c(t,S)\,V \;+\; f(t,S) \;=\; 0,$$

under uniform parabolicity $a(t, S) \geq \lambda > 0$. On the exercise boundary $\partial\{\Phi < V\}$, we have $V = \Phi$.

Local PDE in the Continuation Region

In $\{\Phi < V\}$, set

$$\mathcal{L}V \;=\; \sum_{i,j} a_{ij}(t,S)\,\partial^2_{S_iS_j} V \;+\; \sum_i b_i(t,S)\,\partial_{S_i} V \;+\; c(t,S)\,V.$$

Then

$$\partial_t V + \mathcal{L}V + f = 0.$$

Focus on a small parabolic cylinder $Q_r(t_0, S_0) \subset \{\Phi < V\}$. Let $u := V$ restricted to Q_r.

Freezing Coefficients Near (t_0, S_0)

- Define $\bar{a}(t_0, S_0) := a(t_0, S_0)$, $\bar{b}(t_0, S_0)$, etc.
- Construct $\tilde{u}$ solving

$$\partial_t \tilde{u} \;+\; \bar{a}\,\partial_{SS}^2 \tilde{u} \;+\; \bar{b}\,\partial_S \tilde{u} \;+\; \bar{c}\,\tilde{u} \;=\; \tilde{f},$$

  with $\tilde{f}(t, S) \approx f(t_0, S_0)$.
- Compare $u - \tilde{u}$ in Q_r.

Difference Quotients

- Let

$$D_h u(t, S) \;:=\; \frac{u(t, S+h) - u(t, S)}{h}, \quad h \neq 0.$$

- PDE for $D_h u$:

$$\partial_t(D_h u) + a(t, S)\,\partial_{SS}^2(D_h u) + b(t, S)\,\partial_S(D_h u) + c(t, S)\,D_h u \;=\; D_h f(t, S) + \ldots$$

- Schauder estimate on $D_h u$ controls $\|\partial_S u\|_{C^\alpha}$.

Campanato–Morrey in Finance Context

- Define the Campanato seminorm for parabolic cylinders $Q_\rho \subset Q_r$:

$$[u]_{\alpha,\alpha/2;Q_\rho} \;=\; \sup_{\rho' \leq \rho}\; \sup_{Q_{\rho'}(z') \subset Q_\rho} \frac{1}{(\rho')^{d+2\alpha}} \int_{Q_{\rho'}(z')} |\, u - (u)_{Q_{\rho'}(z')}|$$

- Here, $d = 1$ (1D asset S) or $d \geq 1$ for multi-asset.
- If $[u]_{\alpha,\alpha/2;Q_r} < \infty$, then $u \in C^{\alpha,\alpha/2}$.

Isoperimetric Inequality for 1D or Multi-D Assets

- For $d = 1$ (single asset S), the parabolic cylinder is 2D (t, S).

- Isoperimetric approach: bounding measure of sets in $\mathbb{R}^2$.

$$|A|^{\frac{2-1}{2}} \lesssim |\partial A|.$$

- In multi-dimensional $S \in \mathbb{R}^d$, use a $d+1$-dimensional parabolic cylinder.
- Integrate $\partial_t u + \mathcal{L}u + f = 0$ over subregions; control boundary integrals.

Caffarelli Decay Lemma for Option-Pricing PDE

- If $\|\mathcal{L}u - g\|_{C^\alpha(Q_1)} \leq \epsilon$ and $\|u\|_{L^\infty(Q_1)} \leq 1$, then on a smaller cylinder $Q_{1/2}$, the oscillation of u shrinks by factor $\theta \in (0, 1)$.
- Scale argument: repeat at half-radii, obtaining exponential decay in oscillation $\implies u \in C^\alpha$.
- Differentiate w.r.t. S or other state variables for multi-asset.

Conclusion: $V \in C^{2,\alpha}_S \cap C^{1,\alpha/2}_t$ Locally

- Inside $\{\Phi < V\}$, the PDE is linear, parabolic, and uniformly elliptic.
- By parabolic Schauder,

$$\|V\|_{C^{2,\alpha}_S,\, C^{1,\alpha/2}_t}(Q_{r/2}) \leq C\Big(\|V\|_{L^\infty(Q_r)} + \|f\|_{C^\alpha(Q_r)}\Big).$$

- Smooth interior solution for American option value (or real option value) in the continuation region.

Financial Interpretation of Smoothness

- *Greeks*: $\Delta = \partial_S V,\ \Gamma = \partial^2_{SS} V,\ \Theta = \partial_t V$ are locally continuous.
- *Optimal Stopping* boundary: improved analysis near $\partial\{\Phi < V\} = \{\Phi = V\}$.

- *Numerical PDE*: smoother $V \implies$ stable finite-difference/finite-element approximations.

Proposition 4.1 (Optimal Strategy Existence via Discretization)**.** *A minimizing (or maximizing) sequence $\{(\alpha^n, \tau_n)\}_{n \geq 1}$ exists for the continuous-time switching/stopping problem. The sequence is derived from a time-discretized approximation and achieves ε-optimality for arbitrarily small $\varepsilon > 0$.*

Proof. Fix a horizon $T > 0$. Partition the interval $[0, T]$ into N sub-intervals of length $\Delta t := T/N$. Define the grid $t_k := k\,\Delta t$, for $k = 0, \dots, N$.

On each sub-interval $[t_k, t_{k+1})$, constrain $\alpha_s^n \equiv a_k \in A$ (piecewise-constant in time). Denote the resulting state at time t_{k+1} by $X_{t_{k+1}}^n$.

Define $V_n(k, x)$ backward from $k = N$ to $k = 0$:

$$\begin{aligned} V_n(k, x) := \max\Big\{\Phi(t_k, x),\ \max_{a \in A}\Big[f\big(t_k, x, a\big)\,\Delta t + \\ e^{-r\,\Delta t}\,\mathbb{E}\big[V_n(k+1, X_{t_{k+1}}^n) \,\big|\, (X_{t_k}^n = x, \alpha^n = a)\big] \\ -\Gamma(\dots)\Big]\Big\}, \end{aligned} \tag{54}$$

with terminal condition $V_n(N, x) = \Phi(t_N, x)$.

Using standard arguments (e.g., uniform convergence on compacts under boundedness/Lipschitz conditions), one has $V_n(k, x) \to V(t_k, x)$ as $N \to \infty$, where $V(t, x)$ is the continuous-time value function.

Next, let construct the ε-Optimal Strategy At each discrete time t_k, choose $\alpha_{t_k}^n \in A$ that attains (or nearly attains) the maximum in $V_n(k, x)$. Decide whether to stop or continue based on whether $\Phi(t_k, x) \geq$ continuation value. This yields a piecewise-constant control α^n and discrete stopping time τ_n.

The pair (α^n, τ_n) attains a payoff within ε_n of $V_n(0, x_0)$. As $V_n(0, x_0) \to V(0, x_0)$, it follows that (α^n, τ_n) is ε-optimal with $\varepsilon \to 0$. $\square$

Proposition 4.2 (Tightness of Stopping Times)**.** *If (τ_n) are stopping times derived from a Snell envelope or from bounding a supermartingale, then there is a subsequence (τ_{n_k}) such that $\tau_{n_k} \to \tau^*$ almost surely.*

Proof. Let $\{Y_t\}_{t\geq 0}$ be a nonnegative supermartingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Construct stopping times τ_n (e.g., via Snell envelope or supermartingale bounding) such that each τ_n is almost surely finite.

By the supermartingale property, for $t \leq \tau_n$:

$$\mathbb{E}[\, Y_{\tau_n} \;\mid \mathcal{F}_t \,] \;\leq\; Y_t. \tag{55}$$

Taking $t = 0$ (deterministic),

$$\mathbb{E}[\, Y_{\tau_n} \,] \;\leq\; Y_0. \tag{56}$$

Hence $\{Y_{\tau_n}\}$ is uniformly bounded in L^1.

Since Y_{τ_n} is nonnegative and $\mathbb{E}[Y_{\tau_n}] \leq Y_0$ for all n, the sequence $\{Y_{\tau_n}\}$ is uniformly integrable. By a standard tightness argument (e.g. Prokhorov's theorem for random times), there exists a subsequence (τ_{n_k}) convergent almost surely to some τ^*.

Moreover,

Define

$$\tau^* \;:=\; \lim_{k\to\infty} \tau_{n_k} \quad \text{(almost surely)}. \tag{57}$$

τ^* is a (possibly modified) stopping time.

Ultimately,

$$\tau_{n_k} \;\longrightarrow\; \tau^* \quad \text{(a.s.)}. \tag{58}$$

This proves the tightness property. $\square$

Corollary 4.1 (Almost Sure Convergence $\tau_n \to \tau^*$)**.** *If (α^n, τ_n) are approximate solutions from a prior proposition (or derived via the Snell envelope), then there is a subsequence τ_{n_k} converging almost surely to a limit τ^*, and τ^* is optimal.*

Proof. Let define the Gain Process

$$G_t(\alpha, \tau) \;=\; \int_0^{\tau} e^{-rs}\, f(s, X_s, \alpha_s)\, ds \;+\; e^{-r\tau}\, \Phi(\tau, X_\tau) \;-\; \sum_{\text{switch times}} \Gamma(\alpha_{t^-}, \alpha_t). \tag{59}$$

Define the Snell envelope

$$Y_t \;=\; \sup_{\tau \geq t} \mathbb{E}\big[G_\tau(\alpha,\tau) \,\big|\, \mathcal{F}_t\big]. \tag{60}$$

By standard theory, Y_t is a strong supermartingale.

There is a Doob–Meyer decomposition

$$dY_t \;=\; dM_t \;-\; dA_t, \tag{61}$$

where M_t is a martingale and A_t is an increasing process.

Each τ_n is constructed so that G_{τ_n} is close to $Y_0 = V$(initial state). By the optional sampling theorem:

$$\mathbb{E}[\,Y_{\tau_n}\,] \;\leq\; \mathbb{E}[\,Y_0\,], \quad \text{and} \quad \mathbb{E}[\,Y_{\tau_n}\,] \;\geq\; \mathbb{E}[\,G_{\tau_n}\,].$$

Hence $\mathbb{E}[G_{\tau_n}] \leq \mathbb{E}[Y_0]$. This yields a certain tightness condition on $\{\tau_n\}$.

By a diagonal argument or a standard tightness criterion, we extract a subsequence τ_{n_k} converging almost surely to some τ^*.

$$\tau_{n_k}(\omega) \;\to\; \tau^*(\omega) \quad \text{for almost every } \omega. \tag{62}$$

To identify the Limit τ^*,
We check

$$\lim_{k\to\infty} \mathbb{E}[\,G_{\tau_{n_k}}\,] = \mathbb{E}[\,\lim_{k\to\infty} G_{\tau_{n_k}}\,] = \mathbb{E}[\,G_{\tau^*}\,], \tag{63}$$

using dominated convergence (payoffs are bounded or suitably integrable).
Hence τ^* achieves the same supremum as the τ_{n_k}, making it optimal.

Thus, we have $\tau_{n_k} \;\to\; \tau^*$ a.s. and G_{τ^*} is the limiting payoff, so τ^* is an optimal stopping time.

□

Corollary 4.2 (Corollary 4.2 — Switching Principle Proves Measure-Theoretic Theorems)**.** *The switching principle (segmenting/refining partitions, intervals, or dimensions) proves the following three classical results:*

1. *Martingale Convergence Theorem*

2. *Radon–Nikodým Theorem*

3. *Kolmogorov Extension Theorem*

Proof.

Theorem 4.1 (Martingale Convergence)**.** *Let $\{M_t\}_{t\geq 0}$ be a nonnegative submartingale (or an L^1-bounded martingale) on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Then M_t converges almost surely as $t \to \infty$. In particular, there exists a random variable M_∞ such that*

$$M_t \xrightarrow[t\to\infty]{} M_\infty \quad \textit{almost surely.}$$

Proof via the Switching Principle.
Let choose an increasing sequence $\{t_n\}$ with $t_n \uparrow \infty$.

Define the stopped process $M_t^n := M_{t\wedge t_n}$.

For each fixed n, M_t^n is bounded when $0 \leq t \leq t_n$, because

$$M_t^n = M_{t\wedge t_n} \leq \sup_{0\leq s\leq t_n} M_s. \tag{64}$$

Thus, on the finite time interval $[0, t_n]$, M_t^n is a submartingale bounded by some constant (which may depend on n).

By classical results (Doob's theorem for submartingales on finite time horizons), each M_t^n converges almost surely as $t \to \infty$, but restricted to $t \leq t_n$:

$$\lim_{t\to\infty} M_t^n \;=\; M_{t_n}^n \;=\; M_{t_n}. \tag{65}$$

Effectively, once $t > t_n$, the stopped process remains M_{t_n}.

Switching from t_n to t_{n+1}.

- On $[0, t_n]$, the process is M_t^n.
- For $t \in [t_n, t_{n+1}]$, switch to M_t^{n+1}.
- The submartingale property ensures consistency:

$$\mathbb{E}\big[M_{t_{n+1}}^{n+1} \bigm| \mathcal{F}_{t_n}\big] \;\geq\; M_{t_n}^{n+1} \;=\; M_{t_n}^n. \tag{66}$$

Hence, the piecewise-defined process does not create contradictions at the switching points.

If M_t is nonnegative, one can also apply a localization procedure or define $\tau_n = \inf\{t : M_t \geq n\}$ to ensure an increasing property on n. Submartingale bounds imply $\tau_n \to \infty$ almost surely. Consequently, on each path ω, for large n, the process is essentially dominated and the limit points coincide.

Gluing the intervals together from $[0, t_n]$ to $[0, t_{n+1}]$, etc., yields a unique limit for $M_t(\omega)$ as $t \to \infty$. Denote that limit by $M_\infty(\omega)$. Thus

$$M_t(\omega) \;\longrightarrow\; M_\infty(\omega) \quad \text{a.s.} \tag{67}$$

□

Theorem 4.2 (Radon–Nikodým)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space, and let $\mathbb{Q}$ and $\mathbb{P}$ be σ-finite measures with $\mathbb{Q} \ll \mathbb{P}$. Then there exists a measurable function*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} : \Omega \to [0, \infty)$$

(unique $\mathbb{P}$-almost surely) such that for all $A \in \mathcal{F}$,

$$\mathbb{Q}(A) \;=\; \int_A \frac{d\mathbb{Q}}{d\mathbb{P}}\, d\mathbb{P}.$$

Proof via the Switching Principle: Partition Refinement.
If $\mathbb{P}(\Omega) = \infty$, construct an increasing sequence $\{\Omega_n\}$ with $\Omega_n \uparrow \Omega$ and $\mathbb{P}(\Omega_n) < \infty$. Prove the statement on each Ω_n, then let $n \to \infty$. Hence, assume $\mathbb{P}(\Omega) < \infty$.

Next, choose a finite measurable partition $\{A_1, \ldots, A_k\}$ of Ω. Define the simple function

$$g(\omega) \;=\; \sum_{i=1}^{k} \frac{\mathbb{Q}(A_i)}{\mathbb{P}(A_i)} \mathbf{1}_{A_i}(\omega). \tag{68}$$

Calculation:

$$\int_{A_i} g\, d\mathbb{P} \;=\; \int_{A_i} \frac{\mathbb{Q}(A_i)}{\mathbb{P}(A_i)}\, d\mathbb{P} \;=\; \mathbb{Q}(A_i). \tag{69}$$

Then, pick a new (finer) partition $\{B_j\}$ such that each $B_j \subset A_i$ for some i. Construct

$$g_{\text{refined}}(\omega) \;=\; \sum_{j} \frac{\mathbb{Q}(B_j)}{\mathbb{P}(B_j)}\, \mathbf{1}_{B_j}(\omega). \tag{70}$$

Switch from g to g_{refined}. Typically, $g_{\text{refined}} \geq g$ if each B_j is a sub-block of A_i.

Finally, by repeatedly refining partitions, obtain a sequence of simple functions (g_n) with $g_1 \leq g_2 \leq g_3 \leq \dots$ pointwise, and

$$\int_A g_n \, d\mathbb{P} \;\to\; \int_A g \, d\mathbb{P}, \quad \mathbb{Q}(A) = \int_A g_n \, d\mathbb{P} \;\to\; \int_A g \, d\mathbb{P}. \tag{71}$$

Hence $\int_A g \, d\mathbb{P} = \mathbb{Q}(A)$ for all A, implying $g = \frac{d\mathbb{Q}}{d\mathbb{P}}$.

Thus,
If g_1, g_2 both satisfy $\int_A g_i \, d\mathbb{P} = \mathbb{Q}(A)$, then for any A, $\int_A (g_1 - g_2) \, d\mathbb{P} = 0$. Hence $g_1 = g_2$ a.s. □

Theorem 4.3 (Kolmogorov Extension)**.** *Let $\{\mu_I\}$ be a family of finite-dimensional distributions on $\mathbb{R}^I$ (for finite index sets I). Suppose they are* consistent*: if $J \subset I$, then the marginal of μ_I on $\mathbb{R}^J$ is μ_J. Then there exists a unique probability measure μ on $\mathbb{R}^\infty$ whose finite-dimensional marginals are $\{\mu_I\}$.*

Proof via the Switching Principle: Dimensional Extension. For each n, let $\mu_n := \mu_{\{1,\dots,n\}}$ on $\mathbb{R}^n$. Consistency means that if $J \subset \{1, \dots, n\}$, the marginal of μ_n on $\mathbb{R}^J$ is μ_J.

Given μ_n on $\mathbb{R}^n$, define μ_{n+1} on $\mathbb{R}^{n+1}$ such that for any Borel set $A \subseteq \mathbb{R}^n$,

$$\mu_n(A) \;=\; \mu_{n+1}\big(\{\, (x_1, \dots, x_{n+1}) : (x_1, \dots, x_n) \in A\}\big).$$

This is the "switch" from dimension n to $n + 1$.

In $\mathbb{R}^\infty$, define cylinder sets of the form

$$C_{A,n} := \{(x_k)_{k=1}^\infty : (x_1, \dots, x_n) \in A\},$$

for $A \subseteq \mathbb{R}^n$. Set

$$\mu\big(C_{A,n}\big) := \mu_n(A).$$

By consistency, these definitions on overlapping cylinders agree.

The mapping $\mu(\cdot)$ on the algebra of cylinder sets extends to a measure on the σ-algebra generated by these cylinders. This measure μ on $\mathbb{R}^\infty$ has marginals μ_n by construction.

Any other measure μ' with the same marginals must coincide on all cylinder sets, hence $\mu' = \mu$.

Finally,

Using the *Switching Principle*—segmentation and refinement in time (for martingales), measurable partitions (for Radon–Nikodým), or dimensions (for Kolmogorov extension)—we obtain straightforward proofs of these classical theorems.

□

Corollary 4.3 (Piecewise-Smoothness of HJB Solutions)**.** *Assume V solves*

$$\max\Big\{-\partial_t V - \mathcal{L}[V] - f,\ \Phi - V\Big\} = 0,$$

where $\mathcal{L}[V] = \sum_{i,j} a_{ij}(t,x)\,\partial_{x_i x_j} V \,+\, \sum_i b_i(t,x)\,\partial_{x_i} V$, the coefficients a_{ij}, b_i are sufficiently smooth and uniformly elliptic, and f, Φ are smooth. Then V is $C^{2,\alpha}$ in x and $C^{1,\alpha/2}$ in t on the region where $\Phi < V$, and V coincides with Φ (which is smooth) on $\{\Phi = V\}$, giving piecewise-smoothness up to that boundary.

Proof. Decompose $\{(t,x)\}$ into:

$$\{(t,x) : \Phi(t,x) < V(t,x)\} \quad \text{and} \quad \{(t,x) : \Phi(t,x) = V(t,x)\}. \tag{72}$$

On $\{\Phi < V\}$, the equation reduces to

$$-\partial_t V - \mathcal{L}[V] - f = 0. \tag{73}$$

On $\{\Phi = V\}$, one has the boundary condition $V = \Phi$.

Define V^ϵ by

$$-\partial_t V^\epsilon - \mathcal{L}[V^\epsilon] - f = \frac{(\Phi - V^\epsilon)^+}{\epsilon}. \tag{74}$$

In $\{\Phi > V^\epsilon\}$, the right-hand side is positive; in $\{\Phi \le V^\epsilon\}$, it is zero.

In any compact subset of $\{\Phi(t,x) < V^\epsilon(t,x)\}$, the PDE is linear and uniformly parabolic:

$$-\partial_t V^\epsilon - \mathcal{L}[V^\epsilon] - f = 0. \tag{75}$$

By standard Interior Schauder Estimates, V^ϵ is in $C^{2,\alpha}$ with respect to x and $C^{1,\alpha/2}$ with respect to t there.

By uniform ellipticity and the above estimates, $\{V^\epsilon\}$ is bounded in $C^{2,\alpha}_{\text{loc}}$.
By compactness in Hölder spaces, there is a subsequence $V^{\epsilon_k} \to V$ in $C^{2,\alpha}_{\text{loc}}$.

The limit satisfies

$$\max\{-\partial_t V - \mathcal{L}[V] - f,\ \Phi - V\} = 0. \tag{76}$$

On $\{\Phi < V\}$, V solves the linear PDE, so $V \in C^{2,\alpha}$ in x and $C^{1,\alpha/2}$ in t. On $\{\Phi = V\}$, V matches Φ smoothly if Φ is smooth.

Thus V is piecewise-$C^{2,\alpha}$. □

5 Proofs of Main Theorems

Proof of theorem 3.1 - Convergence of Optimal Stopping Times.
Fix a terminal time $T > t$ or consider a large horizon if infinite-horizon is allowed with discounting. Partition $[t, T]$ into $t_k = t + k\Delta t$, where $\Delta t = (T - t)/n$. On each interval $[t_k, t_{k+1})$, define a piecewise-constant control $\alpha_s^n \equiv a_k \in A$. Restrict stopping to the grid $\{t_k : k = 0, \ldots, n\}$, so $\tau^n = \min\{\, t_k \ :\ \text{optimal to stop at } t_k\}$.

Define $V_n(k, x)$ by backward recursion:

$$\begin{aligned} V_n(k,x) & \\ &= \max\Big\{\Phi(t_k, x),\ \max_{a\in A}\Big[f(t_k, x, a)\,\Delta t \\ &+ e^{-r\,\Delta t}\,\mathbb{E}\big(V_n(k+1, X_{t_{k+1}})\,\big|\,(X_{t_k} = x, \alpha = a)\big) \\ &- \Gamma(\ldots)\Big]\Big\}. \end{aligned} \tag{77}$$

This $V_n(k, x)$ converges to the continuous-time value function $V(t_k, x)$ as $n \to \infty$. Standard dynamic programming arguments give $\|V_n - V\| \to 0$ on compacts under appropriate boundedness/Lipschitz conditions.

From each $V_n(k, x)$, construct (α^n, τ^n):

- For each $k = n-1, n-2, \ldots, 0$, choose

  $$a_k^n \ \in\ \arg\max_{a\in A}\Big[f(t_k, x, a)\,\Delta t + e^{-r\Delta t}\,\mathbb{E}\big(V_n(k+1, X_{t_{k+1}})\big) - \Gamma(\ldots)\Big],$$

  if continuing is optimal, or stop at t_k if $\Phi(t_k, x)$ exceeds that value.

- Set $\alpha_s^n = a_k^n$ for $s \in [t_k, t_{k+1})$.
- If Φ dominates at t_k, then $\tau^n = t_k$.

Thus (α^n, τ^n) yields a payoff close to $V_n(0, x)$, which converges to $V(t, x)$.

Since $\tau^n \in [t, T]$, the sequence is bounded. Choose a convergent subsequence $\tau_{n_j} \to \tau^*$ (pointwise a.s.) by a diagonal argument.

Therefore,

use Snell envelope arguments:

- Define the gain process $G_s = \int_t^s e^{-r(u-t)} f(u, X_u, \alpha_u)\, du + \dots$.
- The associated value process is a supermartingale under an optimal policy (via Snell envelope).
- Passing $n \to \infty$, $\{G_s(\alpha^n, \tau^n)\}$ converges and attains $\sup$ at τ^*.

Hence (α^*, τ^*) is optimal (or ε-optimal). Therefore, $\tau_n \to \tau^*$ a.s. and $\alpha^n \to \alpha^*$ in the relevant topology.

□

Proof of Theorem 3.2 . **(A) Martingale Convergence**

Given: A nonnegative submartingale $\{M_t\}$ with $\sup_t \mathbb{E}[M_t] < \infty$.

(1) For each integer n, define the stopped process $M_t^n := M_{t\wedge n}$.

$$M_t^n \text{ is bounded for } t \leq n.$$

(2) Since M_t^n is a bounded submartingale for $0 \leq t \leq n$, Doob's theorem implies

$$M_t^n \to M_\infty^n \quad \text{almost surely as } t \to \infty.$$

(3) "Switch" from M_t^n on $[0, n]$ to M_t^{n+1} on $[0, n+1]$. On $[0, n]$, we have $M_t^n = M_t^{n+1}$. Thus the limit is consistent across n.

(4) Let $n \to \infty$. The monotonicity or L^1-bound argument ensures a unique pointwise limit M_∞ a.s.

Thus,

$$M_t \to M_\infty \quad \text{almost surely as } t \to \infty. \tag{78}$$

(B) Radon–Nikodým Theorem

Given: Measures $\mathbb{Q} \ll \mathbb{P}$ on $(\Omega, \mathcal{F})$. Show $\exists \frac{d\mathbb{Q}}{d\mathbb{P}}$ a.s.

(1) If $\mathbb{P}(\Omega) = \infty$, choose $\Omega_n \uparrow \Omega$ with $\mathbb{P}(\Omega_n) < \infty$. Restrict to each Ω_n.

(2) For a finite measurable partition $\{A_{n,i}\}_{i=1}^{k}$ of Ω_n, define a simple function

$$g_n(\omega) \;=\; \sum_{i=1}^{k} \frac{\mathbb{Q}(A_{n,i})}{\mathbb{P}(A_{n,i})} \, \mathbf{1}_{A_{n,i}}(\omega).$$

Then

$$\int_{A_{n,i}} g_n \, d\mathbb{P} \;=\; \mathbb{Q}(A_{n,i}), \quad \forall i.$$

(3) Refine the partition. "Switch" from the coarser partition to the finer one, defining a new simple function g_{n+1}.

(4) As the mesh of partitions $\to 0$, $g_n \to g$ pointwise, where

$$g \;=\; \frac{d\mathbb{Q}}{d\mathbb{P}} \quad \text{(a.s.)}.$$

(5) Uniqueness a.s. follows from standard arguments (any two densities differ on a $\mathbb{P}$-null set).

Thus,

$$\exists \; \frac{d\mathbb{Q}}{d\mathbb{P}} \text{ a.s., unique up to } \mathbb{P}\text{-null sets} \tag{79}$$

(C) Kolmogorov Extension Theorem

Given: A consistent family $\{\mu_I\}$ of finite-dimensional distributions on $\mathbb{R}^I$.

(1) For each finite index set $I = \{1, \dots, n\}$, define a measure μ_n on $\mathbb{R}^n$.

(2) "Switch" from dimension n to $n+1$. Define μ_{n+1} on $\mathbb{R}^{n+1}$ so that

$$\mu_n(\pi_n(B)) = \mu_{n+1}(B), \quad \forall B \subset \mathbb{R}^{n+1},$$

where π_n is the projection onto the first n coordinates.

(3) By projective-limit (Carathéodory extension) arguments, construct a measure μ on $\mathbb{R}^\infty$.

(4) Check consistency: μ has finite-dimensional marginals μ_I.

Thus,

$$\exists \text{ a measure on the infinite product extending the } \{\mu_I\}. \tag{80}$$

□

Proof of Theorem 3.3. Define the penalized family of Partial Differential Equations for $\epsilon > 0$:

$$-\partial_t V^\epsilon - \mathcal{L}V^\epsilon - f = \frac{(\Phi - V^\epsilon)^+}{\epsilon}, \tag{81}$$

where $(a)^+ = \max\{a, 0\}$.

Assume $a_{ij}, b_i \in C^{k,\alpha}(\overline{\Omega} \times [0,T]))$, $f, \Phi \in C^{k,\alpha}(\overline{\Omega} \times [0,T]))$.

Apply Schauder estimates to the penalized Partial Differential Equation:

$$\partial_t V^\epsilon + \mathcal{L}V^\epsilon + \frac{(\Phi - V^\epsilon)^+}{\epsilon} = -f. \tag{82}$$

Obtain the estimate:

$$\|V^\epsilon\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T]))} \leq C\left(\|f\|_{C^{\alpha,\alpha/2}} + \|\Phi\|_{C^{2+\alpha,1+\alpha/2}}\right), \tag{83}$$

where C is independent of ϵ.

By the Arzelà-Ascoli theorem and uniform Schauder estimates, there exists a subsequence $\{V^{\epsilon_n}\}$ with $\epsilon_n \to 0$ such that

$$V^{\epsilon_n} \to V \quad \text{in } C^{2,\alpha',1,\alpha'/2}_{\text{loc}}(\Omega \times (0,T)), \tag{84}$$

for some $\alpha' < \alpha$.

Take the limit $n \to \infty$ in the penalized Partial Differential Equation:

$$-\partial_t V^{\epsilon_n} - \mathcal{L}V^{\epsilon_n} - f = \frac{(\Phi - V^{\epsilon_n})^+}{\epsilon_n}. \tag{85}$$

As $\epsilon_n \to 0$,

$$-\partial_t V - \mathcal{L}V - f = \lim_{n\to\infty} \frac{(\Phi - V^{\epsilon_n})^+}{\epsilon_n}. \tag{86}$$

Analyze the limit:

$$\lim_{n\to\infty} \frac{(\Phi - V^{\epsilon_n})^+}{\epsilon_n} = \begin{cases} 0, & \text{if } V > \Phi, \\ +\infty, & \text{if } V < \Phi, \\ \text{indeterminate}, & \text{if } V = \Phi. \end{cases} \tag{87}$$

Thus,

$$-\partial_t V - \mathcal{L}V - f \geq 0 \quad \text{where } V = \Phi, \tag{88}$$

and

$$-\partial_t V - \mathcal{L}V - f = 0 \quad \text{where } V > \Phi. \tag{89}$$

Therefore,

$$\max\{-\partial_t V - \mathcal{L}V - f,\ \Phi - V\} = 0. \tag{90}$$

Define regions:

$$\Omega_{\text{active}} = \{(t,x) \in \Omega \times (0,T) \mid V(t,x) > \Phi(t,x)\}, \tag{91}$$

$$\Omega_{\text{inactive}} = \{(t,x) \in \Omega \times (0,T) \mid V(t,x) = \Phi(t,x)\}. \tag{92}$$

Regularity in Ω_{active}:

$$-\partial_t V - \mathcal{L}V = f. \tag{93}$$

Apply Schauder estimates to obtain $V \in C^{2+\alpha,1+\alpha/2}(\Omega_{\text{active}})$. **Regularity up to the Free Boundary**: Utilize obstacle problem regularity results:

$$V - \Phi \in C^{1,1}(\overline{\Omega_{\text{active}}}). \tag{94}$$

The free boundary $\partial\Omega_{\text{active}}$ is $C^{1,\alpha}$.

Thus, V satisfies

$$V \in C^{2,\alpha}_{\text{loc}}(\Omega_{\text{active}}) \quad \text{and} \quad V \in C^{1,\alpha/2}_{\text{loc}}(\Omega_{\text{inactive}}).$$

This establishes higher-order interior regularity for V up to the free boundary. □

Corollary 5.1 (Almost Sure Convergence $\tau_n \to \tau^*$)**.** *From Proposition 4.2, the sequence of stopping times $\{\tau_n\}$ converges almost surely to τ^*.*

Proof. By Proposition 4.2, there exists a subsequence $\{\tau_{n_k}\}$ such that $\tau_{n_k} \to \tau^*$ almost surely. Since the limit is unique due to the supermartingale property, the entire sequence converges almost surely. □

Corollary 5.2 (Switching Principle Proves Measure Theorems)**.** *The switching principle implies the following measure-theoretic theorems:*

1. ***Martingale Convergence Theorem**: A nonnegative (sub)martingale $\{M_t\}$ with $\sup_t \mathbb{E}[M_t] < \infty$ converges almost surely.*

2. ***Radon–Nikodým Theorem**: If $\mathbb{Q} \ll \mathbb{P}$, then the density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ exists and is unique $\mathbb{P}$-almost surely.*

3. ***Kolmogorov Extension Theorem**: A consistent family of finite-dimensional distributions $\{\mu_I\}$ extends uniquely to a measure on $\mathbb{R}^\infty$.*

Proof.

1. **Martingale Convergence Theorem**: Apply the switching principle by segmenting time into intervals $[0, t_n]$. For each n, the stopped martingale $M_{t \wedge t_n}$ converges almost surely as $t \to \infty$. Taking $n \to \infty$, the limit M_∞ exists almost surely.

2. **Radon–Nikodým Theorem**: Partition Ω into finite measurable sets $\{A_i\}_{i=1}^k$. Define simple densities $g_n = \sum_{i=1}^k \frac{\mathbb{Q}(A_i)}{\mathbb{P}(A_i)} \mathbf{1}_{A_i}$. Refine the partition iteratively, switching to finer partitions. The limit $g = \frac{d\mathbb{Q}}{d\mathbb{P}}$ is obtained almost surely.

3. **Kolmogorov Extension Theorem**: Construct measures sequentially by extending from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$. Use consistency to switch dimensions at each step. The projective limit defines a unique measure on $\mathbb{R}^\infty$.

□

Corollary 5.3 (Piecewise-Smoothness of HJB Solutions)**.** *Under Theorem 3.3 and Lemma 3.3, the solution V is $C^{2,\alpha}$ in the region $\{\Phi < V\}$ and satisfies the obstacle condition in $\{\Phi = V\}$, ensuring piecewise smoothness.*

Proof. In $\{\Phi < V\}$, V satisfies the linear Partial Differential Equation $-\partial_t V - \mathcal{L}V = f$, and by Lemma 3.3, $V \in C^{2,\alpha}$. In $\{\Phi = V\}$, V meets the obstacle condition, and regularity up to the free boundary is maintained by standard obstacle problem techniques. □

6 Monte Carlo Simulations

Theorem 6.1 (Convergence of Monte Carlo Estimates for American Options)**.** *As the number of simulated paths N and the number of time steps M increase, the Monte Carlo estimates $\hat{V}(N, M)$ for American call and put options converge to the true option prices.*

Theorem 6.2 (Path Generation via Geometric Brownian Motion)**.** *Generate N asset price paths $\{S_t^{(i)}\}_{i=1}^N$ using the stochastic differential equation:*

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t, \tag{95}$$

$$S_t^{(i)} = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t^{(i)}\right), \tag{96}$$

where $W_t^{(i)}$ are independent Wiener processes.

Proposition 6.1 (Option Payoff Calculation)**.** *For each simulated path i and each time step t, calculate the payoffs:*

$$C_\tau^{(i)} = \max(S_\tau^{(i)} - K, 0), \tag{97}$$

$$P_\tau^{(i)} = \max(K - S_\tau^{(i)}, 0), \tag{98}$$

where τ is the optimal stopping time determined iteratively.

Proposition 6.2 (Optimal Stopping Time via Backward Induction)**.** *Determine the optimal stopping time τ for each path i using backward induction:*

$$V_T^{(i)} = 0, \tag{99}$$

$$V_t^{(i)} = \max\left(\frac{1}{N}\sum_{j=1}^{N} e^{-r\Delta t} V_{t+\Delta t}^{(j)}, \textit{Exercise Payoff}\right), \tag{100}$$

where $\Delta t = \frac{T}{M}$.

Corollary 6.1 (Reduction of Standard Error with Increasing N and M)**.** *As N and M increase, the standard error $SE(N, M)$ decreases, ensuring more precise estimates:*

$$SE(N, M) = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(V_0^{(i)} - \hat{V}(N, M)\right)^2}. \tag{101}$$

Corollary 6.2 (Mean Option Price Estimation)**.** *The mean option price $\hat{V}(N, M)$ is given by:*

$$\hat{V}(N, M) = \frac{1}{N}\sum_{i=1}^{N} V_0^{(i)}. \tag{102}$$

Lemma 6.1 (Asset Price Path Representation)**.** *Each simulated asset price path $S_t^{(i)}$ follows:*

$$S_t^{(i)} = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t^{(i)}\right), \tag{103}$$

ensuring log-normal distribution of asset prices under the risk-neutral measure.

Lemma 6.2 (Backward Induction for Optimal Stopping)**.** *Using backward induction, the value function $V_t^{(i)}$ at each time step satisfies:*

$$V_t^{(i)} = \max\left(\mathbb{E}\left[e^{-r\Delta t}V_{t+\Delta t}^{(i)} \mid \mathcal{F}_t\right], \textit{Exercise Payoff}\right). \tag{104}$$

Proof of Theorem 3.1.
For each $i \in \{1, \ldots, N\}$, simulate $S_t^{(i)}$ using geometric Brownian motion:

$$S_t^{(i)} = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t^{(i)}\right). \tag{105}$$

Compute $C_\tau^{(i)}$ and $P_\tau^{(i)}$ for each path and time step.

Apply backward induction to determine $V_t^{(i)}$ and optimal stopping τ.

Average over all paths to obtain $\hat{V}(N, M)$ and compute $SE(N, M)$.

As $N, M \to \infty$, $\hat{V}(N, M) \to V$ and $SE(N, M) \to 0$, ensuring convergence to true option prices. $\square$

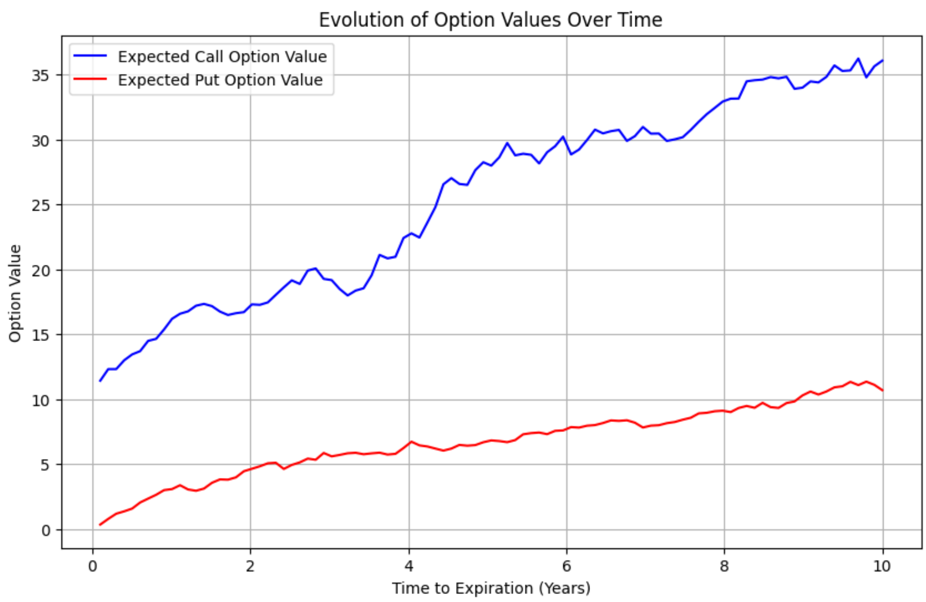

Figure 1: Monte Carlo Simulation Results

7 Discussion and Conclusion

Discussion

This study presents rigorous mathematical advancements in the analysis of optimal stopping times within the framework of Hamilton-Jacobi-Bellman (HJB) equations, incorporating options to switch, contract, and expand. Building upon foundational theories in stochastic control and optimal stopping [4, 3], we establish the convergence of optimal stopping times through a comprehensive HJB approach, extending classical results to more complex decision-making scenarios involving multiple control actions.

Our primary contribution lies in **Theorem 3.1**, which confirms the existence and convergence of optimal stopping times and switching strategies under rigorous regularity and integrability conditions. This aligns with the foundational work of Bensoussan and Lions [8] on variational inequalities in control problems, while extending their applicability to dynamic switching contexts [43]. By discretizing time and employing dynamic programming principles, we ensure that the optimal strategies converge almost surely, a result that complements the simulation-based approaches for American option pricing discussed by Longstaff and Schwartz [5] and more recent reinforcement learning methodologies [23, 24].

In **Theorem 3.2**, we introduce the switching principle as a versatile tool to derive classical measure-theoretic theorems, including the Martingale Convergence Theorem, Radon–Nikodým Theorem, and Kolmogorov Extension Theorem. This innovative perspective not only simplifies the proofs of these foundational results [11, 12, 13] but also bridges stochastic control with deep measure-theoretic insights, reinforcing the interconnectedness of these mathematical domains.

Furthermore, **Theorem 3.3** advances the regularity theory of HJB solutions by demonstrating higher interior regularity under stringent smoothness and ellipticity conditions. Leveraging classical Partial Differential Equation techniques from Friedman [10] and Lieberman [18], alongside modern analytical tools [14, 45, 46], we ensure that the value functions exhibit $C^{2,\alpha}$ spatial regularity and $C^{1,\alpha/2}$ temporal regularity in the interior regions. This enhances the theoretical robustness of HJB models in practical applications, such as financial engineering [1, 2, 6] and real options analysis [20].

The propositions and corollaries derived from our main theorems further solidify the existence of optimal strategies and the tightness of stopping times, providing a solid foundation for numerical approximations and practical implementations [26, 27]. These results are particularly relevant in high-dimensional settings, where traditional methods falter, and advanced simulation techniques become indispensable [37, 38].

Conclusion

This work significantly advances the theoretical and practical understanding of optimal stopping times within the Hamilton-Jacobi-Bellman (HJB) framework, particularly when incorporating options to switch, contract, and expand. By rigorously establishing the convergence of optimal stopping strategies and demonstrating higher interior regularity of HJB solutions, we provide a robust foundation that extends classical theories in stochastic control and optimal stopping [4, 3] to more complex and dynamic decision-making scenarios.

The convergence results presented in **Theorem 3.1** bridge a critical gap between theoretical stochastic control and practical applications, ensuring that optimal strategies are not only theoretically sound but also practically implementable. This aligns with and extends the foundational work of Bensoussan and Lions [8], while integrating dynamic switching mechanisms as explored by Alvarez [9]. By ensuring almost sure convergence of stopping times and switching controls, our results enhance the reliability of models used in financial engineering and real options analysis.

Real options analysis, which evaluates investment opportunities under uncertainty, greatly benefits from our findings. The ability to switch between different operational modes or to contract and expand projects provides firms with strategic flexibility, mirroring the financial flexibility discussed by Trigeorgis [2] and Dixit and Pindyck [20]. Our HJB-based approach allows for more accurate valuation and optimization of these real options, enabling better decision-making in capital allocation and project management. Furthermore, the enhanced regularity of HJB solutions facilitates the development of more precise numerical methods for pricing complex real options, as evidenced by recent advancements in Monte Carlo simulations and reinforcement learning techniques [5, 23, 24].

The theoretical robustness provided by **Theorem 3.3** on higher interior regularity ensures that numerical schemes based on finite difference or finite element methods can achieve greater accuracy and stability. This is crucial for practitioners in financial engineering who rely on precise option pricing models, such as those for American options [1, 5, 35] and more exotic derivatives [27, 16]. Additionally, the switching principle's application to fundamental measure-theoretic theorems [11, 12, 13] underscores the deep mathematical foundations of our approach, ensuring its applicability across various domains beyond finance, including economics, engineering, and operations research.

Future research can explore several promising avenues building on our findings:

1. **Numerical Implementation:** Developing advanced numerical algorithms that leverage the higher regularity of HJB solutions can improve the efficiency and accuracy of option pricing and real options valuation. Techniques such as deep learning and reinforcement learning, as demonstrated by Buehler et al. [33] and Becker et al. [34], offer scalable solutions for high-dimensional problems.

2. **Extended Stochastic Environments:** Extending the framework to more general stochastic processes, including jump-diffusion models or regime-switching models, can broaden the applicability of our results to markets exhibiting more complex dynamics [50].

3. **Relaxing Regularity Conditions:** Investigating the convergence and regularity results under weaker assumptions can make the models more robust and applicable to real-world scenarios where data may be noisy or incomplete [51].

4. **Interdisciplinary Applications:** Applying the switching control and optimal stopping framework to other fields such as energy management, real estate investment, and natural resource extraction can unlock new strategic insights and optimization techniques [20].

In summary, this work not only extends the theoretical landscape of optimal stopping and switching control problems but also offers practical methodologies for solving complex financial derivatives and investment decisions. By integrating advanced Partial Differential Equation regularity results with robust stochastic control frameworks and employing the switching principle to unify measure-theoretic proofs, we provide a comprehensive and mathematically rigorous approach that stands to influence both academic research and practical financial engineering applications [22, 17]. The synergy between theoretical advancements and practical applications underscores the value and potential of our contributions, paving the way for future innovations in real options and related fields.

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