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Small weak hyperfields in hadronic mechanics

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Abstract

It was in mid 90es when Professor R. M. Santilli realized, for the first time, that his innovating theories can be appropriate expressed by multi-valued systems. At that time the largest class of hyper-structures, the H_v-structures, based on the weak properties, were introduced and studied deeply. The extremely large number of weak hyper-structures defined on the same set can be reduced by demanding a lot of properties. Therefore, whenever the Santilli's Hadronic Mechanics need special elements satisfying a number of properties then the class of the H_v-structures is the appropriate one. The first H_v-structures in Hadronic Mechanics, introduced in 1996, are called e-hyper-structures. The main question in this theory proposed by Santilli is: What are the hyper-numbers? Hyper-numbers or H_v-numbers are called the elements of any H_v-field and they are important in representation theory, as well. In this presentation we give some constructions and properties on minimal H_v-fields.

Keywords: Lie-Santilli iso-theory, weak hyperstructures, H_v-fields.

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1. Introduction to H_v-structures

The hyper-structures were introduced by F.Marty in 1934 [13] when he first defined the *hypergoup* as a set equipped with an associative and reproductive hyper-operation. We deal with the largest class of hyper-structures called *H_v-structures*, introduced in 1990 [17], [29], [30]. These satisfy the *weak axioms* where the non-empty intersection replaces equality. Basic definitions [2], [4], [5], [6], [10], [11], [12], [21], [23], [24], [25], [26], [30], [31], [32], [34], [43], [51]:

A set H equipped with, at least, one *hyper-operation*: $\cdot: H\times H\to P(H)-\{\varnothing\}$, is called *hyper-structure*, we write $(H,\cdot)$. The $(\cdot)$ is *weak associative,* if $(xy)z\cap x(yz)\neq\varnothing$, $\forall x,y,z\in H$ and *weak commutative* if $xy\cap yx\neq\varnothing$, $\forall x,y\in H$. The $(H,\cdot)$ is an *H_v-semigroup* if it is weak associative, and it is an *H_v-group* if it is *reproductive* H_v-semigroup: $xH=Hx=H$, $\forall x\in H$.

Similarly, more complicated hyper-structures can be defined: $(R,+,\cdot)$ is *H_v-ring* if (+) and $(\cdot)$ are weak associative, the reproduction axiom is valid for (+) and $(\cdot)$ is *weak distributive* with respect to (+):

$$x(y+z)\cap(xy+xz)\neq\varnothing,\quad (x+y)z\cap(xz+yz)\neq\varnothing,\quad \forall x,y,z\in R.$$

Let $(R,+,\cdot)$ be H_v-ring, $(M,+)$ is a weak commutative H_v-group and there exists an external hyper-operation $\cdot: R\times M\to P(M)$: $(a,x)\to ax$ such that, $\forall a,b\in R$ and $\forall x,y\in M$, we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\quad (a+b)x\cap(ax+bx)\neq\varnothing,\quad (ab)x\cap a(bx)\neq\varnothing,$$

then M is an *H_v-module* over F. In the case that we have an H_v-field F, defined later, instead of the H_v-ring R, then the *H_v-vector space*, is obtained.

In an H_v-semigroup $(H,\cdot)$, the *powers* are defined by

$$h^1=\{h\},\quad h^2=h\cdot h\ \dots\quad h^n=h\circ h\circ\dots\circ h,$$

where $(\circ)$ is the *n-ary circle hope*: take the union of hyper-products n times, with all possible patterns of parentheses on them. An $(H,\cdot)$ is a *cyclic of period s*, if there is a *generator* h and the minimum natural s, such that

$$H = h^1\cup h^2\cup\dots\cup h^s.$$

Analogously, cyclicity for the infinite period is defined. If there are h and s, the minimum one, such that $H=h^s$, then we say that the $(H,\cdot)$, is a *single-power cyclic of period s*.

For more definitions and applications on H_v-structures one can see books and papers as [1], [2], [4], [5], [6], [11], [12], [22], [30], [32], [34], [38], [43], [49].

Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups on H. $(\cdot)$ is called *smaller* than $(*)$, if there is an

$$f\in Aut(H,*)\ \text{ such that }\ xy\subset f(x*y),\ \forall x,y\in H,$$

then we say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a classical structure then it the *basic structure,* and $(H,*)$ is called *H_b-structure*. *Minimal* is called an H_v-group if it contains no other H_v-group on the same set.

Theorem 1.1 (*The Little Theorem*). Greater hyper-operations than the ones which are weak associative or weak commutative, are also weak associative or weak commutative, respectively.

This Theorem leads to a partial order on H_v-structures and reveal the enormous number of defined on the same set. The problem of enumeration of hyper-structures, is complicate in H_v-structures because we have great numbers. The Little Theorem, transfer the problem into finding the minimal, up to isomorphisms, H_v-structures. For example, the number of H_v-groups with three elements, up to isomorphism, is 1.026.462, there are 7.926 abelian.

Interesting large classes of H_v-structures, are the following [25], [26], [30], [31], [41]:

Definition 1.2 An H_v-structure is called *very-thin* if all hyper-operations are operations except one, were all results are singletons except one, which is a set with more elements.

Definition 1.3 Let $(G,\cdot)$ groupoid, then $\forall P\subset G$, $P\neq\varnothing$, we define the *P-hyper-operations*:

$$\underline{P}\text{: } x\underline{P}y=(xP)y\cup x(Py),\ \forall x,y\in G,$$

$(G,\underline{P})$ is a *P-hyper-structure*. If $(G,\cdot)$ semigroup, then $x\underline{P}y=xPy$, $(G,\underline{P})$ is semi-hypergroup.

A generalization of P-hyper-operation, used in Lie Santilli's theory, is the following: Let $(G,\cdot)$ be abelian group and $P\subset G$, we define the *P_e-hyper-operation* $(\times_P)$ by

$$\begin{cases} x\times_P y = x\cdot P\cdot y=\{x\cdot h\cdot y \mid h\in P\} & \text{if } x\neq e \text{ and } c\neq e \\ x\cdot y & \text{if } x=e \text{ or } y=e \end{cases}$$

The hyper-structure $(G,\times_P)$ is an abelian H_v-group.

Hyper-products on any type of ordinary matrices can be defined [10], [21], [54], [55].

Definition 1.4 Let $A=(a_{ij})\in M_{m\times n}$, $m\times n$ matrix and $s,t\in N$, $1\leq s\leq m$, $1\leq t\leq n$. Denote $\underline{st}$ a map

$$\underline{st}\text{: } M_{m\times n}\to M_{s\times t}\text{: } A\to A\underline{st}=(\underline{a}_{ij}),$$

where $\underline{a}_{ij}=\{a_{i+\kappa s,j+\lambda t} \mid 1\leq i\leq s,\ 1\leq j\leq t,\ \kappa,\lambda\in N,\ i+\kappa s\leq m,\ j+\lambda t\leq n\}$.

(a) Let $A=(a_{ij})\in M_{m\times n}$, $B=(b_{ij})\in M_{u\times v}$, $s=\min(m,u)$, $t=\min(n,v)$. The *helix-sum*, is

$$\oplus\text{: } M_{m\times n}\times M_{u\times v}\to P(M_{s\times t})\text{: } (A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset M_{s\times t},$$

where $(\underline{a}_{ij})+(\underline{b}_{ij})=\{(c_{ij})=(a_{ij}+b_{ij}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}$.

(b) Let $A=(a_{ij})\in M_{m\times n}$, $B=(b_{ij})\in M_{u\times v}$, $s=\min(n,u)$. The *helix-product*, is

$$\otimes\text{: } M_{m\times n}\times M_{u\times v}\to P(M_{m\times v})\text{: } (A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset M_{m\times v},$$

where $(\underline{a}_{ij})\cdot(\underline{b}_{ij})=\{(c_{ij})=(\sum a_{it}b_{tj}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}$.

The helix-sum is commutative, the helix-product is weak associative.

The *helix-Lie Algebra* is defined as well. We study the matrices $M_{m\times n}$ where $m<n$, since we have analogous cases if $m>n$. Several classes appropriate in helix-operations are defined. For example: Let $A=(a_{ij})\in M_{m\times n}$ and $s,t\in N$, $1\leq s\leq m$, $1\leq t\leq n$. We call A *S-helix matrix of type $s\times t$*, if $A\underline{st}$ has diagonal entries elements. An S-helix A, is an *S_o-helix matrix* if, moreover, $a_{11}\ldots a_{mm}\neq 0$. The set of S-helix matrices $A=(a_{ij})\in M_{m\times n}$, is closed under the helix-product. It has $\underline{st}$ left scalar unit I_c. The S_0-helix matrices X have inverses X^{-1}: $I_c\in X\otimes X^{-1}\cap X^{-1}\otimes X$.

2. Fundamental relations

The main tool to study the hyper-structures is the, so called, *fundamental relation.* In 1970 M. Koskas [30] introduced in classical hypergroups the relation β and its transitive closure β^*. This relation connects the hyperstructures with the corresponding classical structures and is defined in H_v-groups as well. T. Vougiouklis [11], [27], [29], [30], [31], [52] introduced, and gave new proofs, the analogous relations γ^* and ε^*, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named the relations β^*, γ^* and ε^*, *fundamental* since they play crutial role mainly in representation theory.

Definition 2.1 The *fundamental relations* β^*, γ^*, and ε^* are defined in H_v-groups, H_v-rings, and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring, and vector spaces, respectively.

Let $(G,\cdot)$ be group and R a partition in G, then $(G/R,\cdot)$ is H_v-group, so $(G/R,\cdot)/\beta^*$ is a group, the *fundamental*. The classes of $(G/R,\cdot)/\beta^*$ are a union of the R-classes.

The main theorems, and a way to find the fundamental classes, are the following:

(1) Let $(H,\cdot)$ be H_v-group and denote U the set of finite products in H. Define the relation β in H setting $x\beta y$ if $\{x,y\}\subset u$, where $u\in U$. Then, β^* is transitive closure of β.

(2) Let $(R,+,\cdot)$ be H_v-ring, denote U the set of finite polynomials in R. Define the relation γ in R setting $x\gamma y$ if $\{x,y\}\subset u$, where $u\in U$. Then, γ^* is transitive closure of γ.

(3) Let $(M,+)$ be H_v-vector space over the H_v-field $(F,+,\cdot)$, denote U the set of expressions in M. Define ε in M setting $x\varepsilon y$ if $\{x,y\}\subset u$, $u\in U$. Then, ε^* is transitive closure of ε.

We present a proof for the second of the above theorems, for the H_v-rings, and the rest are analogous [11], [29], [30], [33], [39], [50], [52]:

Theorem 2.2 Let $(R,+,\cdot)$ be an H_v-ring. Denote by U the set of all finite polynomials of elements of R. We define the relation γ in R as follows:

$$x\gamma y \text{ if } \{x,y\}\subset u, \text{ where } u\in U.$$

Then, the relation γ^* is the transitive closure of γ.

Proof. Let $\underline{\gamma}$ the transitive closure of γ, denote $\underline{\gamma}(a)$ the class of a. First, we prove that the quotient $R/\underline{\gamma}$ is a ring.

In $R/\underline{\gamma}$ the sum ($\oplus$) and the product ($\otimes$) are defined in the usual manner:

$$\underline{\gamma}(a)\oplus\underline{\gamma}(b)=\{\underline{\gamma}(c):c\in\underline{\gamma}(a)+\underline{\gamma}(b)\},\quad \underline{\gamma}(a)\otimes\underline{\gamma}(b)=\{\underline{\gamma}(d):d\in\underline{\gamma}(a)\cdot\underline{\gamma}(b)\},\ \forall a,b\in R.$$

Take $a'\in\underline{\gamma}(a)$, $b'\in\underline{\gamma}(b)$. Then, we have

$$a'\underline{\gamma}a \text{ if } \exists x_1,\dots,x_{m+1} \text{ with } x_1=a',\ x_{m+1}=a \text{ and } u_1,\dots,u_m\in U\text{: } \{x_i,x_{i+1}\}\subset u_i,\ i=1,\dots,m,$$

$$b'\underline{\gamma}b \text{ if } \exists y_1,\dots,y_{n+1} \text{ with } y_1=b',\ y_{n+1}=b \text{ and } v_1,\dots,v_n\in U\text{: } \{y_j,y_{j+1}\}\subset v_j,\ j=1,\dots,n.$$

From the above, we obtain

$$\{x_i,x_{i+1}\}+y_1\subset u_i+v_1,\ i=1,\dots,m-1,\ x_{m+1}+\{y_j,y_{j+1}\}\subset u_m+v_j,\ j=1,\dots,n.$$

The sums $u_i+v_1=t_i$, $i=1,\dots,m-1$ and $u_m+v_j=t_{m+j-1}$, $j=1,\dots,n$, are polynomials, so, $t_k\in U$, $\forall k\in\{1,\dots,m+n-1\}$.

Now, pick up elements $z_1,\dots,z_{m+n}$ such that

$$z_i \in x_i + y_1,\ i=1,\dots,n \ \text{ and } \ z_{m+j} \in x_{m+1} + y_{j+1},\ j=1,\dots,n,$$

therefore, we obtain $\{z_k, z_{k+1}\} \subset t_k$, $k=1,\dots,m+n-1$.
Thus, every element $z_1 \in x_1+y_1=a'+b'$ is $\underline{\gamma}$ equivalent to every $z_{m+n} \in x_{m+1}+y_{n+1}=a+b$. Thus, $\underline{\gamma}(a) \oplus \underline{\gamma}(b)$ is a singleton so we can write $\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \underline{\gamma}(c)$, $\forall c \in \underline{\gamma}(a) + \underline{\gamma}(b)$.
In a similar way, we prove that $\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \underline{\gamma}(d)$, $\forall d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)$.
The weak associativity and the weak distributivity on R give that the associativity and the distributivity are valid for the quotient R/γ^*. Therefore, R/γ^* is a ring.

Let σ equivalence with R/σ ring. Then, $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \otimes \sigma(b)$ are singletons, so $\forall a,b \in R$, we have $\sigma(a) \oplus \sigma(b) = \sigma(c)$, $\forall c \in \sigma(a)+\sigma(b)$, and $\sigma(a) \otimes \sigma(b) = \sigma(d)$, $\forall d \in \sigma(a) \cdot \sigma(b)$. Thus, $\sigma(a) \oplus \sigma(b) = \sigma(a+b) = \sigma(A+B)$, $\sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B)$, $\forall a,b \in R$, $A \subset \sigma(a)$, $B \subset \sigma(b)$. By induction, extend it on sums and products. So $\forall u \in U$, we have $\sigma(x)=\sigma(u)$, $\forall x \in u$. So, $x \in \gamma(a)$ implies $x \in \sigma(a)$, $\forall x \in R$. But σ is transitively closed, thus: $x \in \underline{\gamma}(x)$ implies $x \in \sigma(a)$. That means that $\underline{\gamma}$ is the smallest equivalence in R such that $R/\underline{\gamma}$ is a ring, so, $\underline{\gamma} = \gamma^*$. ■

An element is called *single* if its fundamental class is singleton, so if $\beta^*(s)=\{s\}$.

Denote S_H the set of singles. If $S_H \neq \varnothing$, then we can answer to the hard problem, which is to find the fundamental classes. There is the following [11], [30], [42], [56]:

Theorem 2.3 Let $(H,\cdot)$ be H_v-group and $s \in S_H \neq \varnothing$. Let $a \in H$, take a $v \in H$ such that $s \in av$, then $\beta^*(a)=\{h \in H: hv=s\}$, the core of H is $\omega_H=\{u \in H: us=s\}=\{u \in H: su=s\}$. Moreover, $sx=\beta^*(sx)$ and $xs=\beta^*(xs)$, $\forall x \in H$.

The fundamental relations are used for general definitions.

Definition 2.4 An H_v-ring $(R,+,\cdot)$ is called *H_v-field* if R/γ^* is a field. In sequence the *H_v-vector space* is defined. The elements of an H_v-field are called *hyper-numbers* or *H_v-numbers*.

Definition 2.5 Let $(L,+)$ H_v-vector space on $(F,+,\cdot)$, $\varphi: F \to F/\gamma^*$, $\omega_F=\{x \in F: \varphi(x)=0\}$, 0 zero of F/γ^*. Let ω_L core of $\varphi': L \to L/\varepsilon^*$, 0 zero of L/ε^*. Take the *bracket hyper-operation*:

$$[\ ,\ ]: L \times L \to P(L): (x,y) \to [x,y],$$

then L is an *H_v-Lie algebra* over F if the following axioms are satisfied:

(L1) $[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1,y] + \lambda_2 [x_2,y]) \neq \varnothing$

$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x,y_1] + \lambda_2 [x,y_2]) \neq \varnothing$, $\forall x,x_1,x_2,y,y_1,y_2 \in L$, $\forall \lambda_1,\lambda_2 \in F$

(L2) $[x,x] \cap \omega_L \neq \varnothing$, $\forall x \in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]]) \cap \omega_L \neq \varnothing$, $\forall x,y \in L$.

Theorem 2.6 Let $(H,\cdot)$ be H_v-group, H/β^* fundamental group. If H/β^* is not commutative or not cyclic, then $(H,\cdot)$ is not weak commutative or cyclic, respectively.

Proof. Straightforward since if $(H,\cdot)$ is weak commutative or cyclic, then its fundamental group H/β^* is commutative or cyclic, respectively. ■

The *uniting elements* method, introduced by Corsini & Vougiouklis in 1989, [3], [30], is the following: Let G be structure and a no-valid property d, described by equations. Take the partition in G setting in the same class, all pairs of elements that cause the nonvalidity of d. The quotient by this partition *G*/d is an H_v-structure. Then, quotient out G/d by β^*, is a stricter structure $(G/d)/\beta^*$ for which the property d is valid.

Theorem 2.7 Let $(R,+,\cdot)$ be ring, $F=\{f_1,\dots, f_m, f_{m+1},\dots, f_{m+n}\}$ system of equations on R consisting of subsystems $F_m=\{f_1,\dots,f_m\}$ and $F_n=\{f_{m+1},\dots,f_{m+n}\}$. Let σ, σ_m the equivalence relations defined by uniting elements using F and F_m respectively, and σ_n the equivalence on F_n on the ring $R_m=(R/\sigma_m)/\gamma^*$. Then $(R/\sigma)/\gamma^* \cong (R_m/\sigma_n)/\gamma^*$.

Motivation for F. Marty to introduce the hypergroup was the quotient of a group by any not normal subgroup. Motivation to introduce the weak hyper-group, or H_v-group, is the quotient of a group by any partition of the group. Thus, we summarize:

Motivation for hyper-structures.

The quotient of a group by any invariant subgroup is a group.
The quotient of a group by any subgroup is a hypergroup.
The quotient of a group by any partition is an H_v-group.

3. Representation Theory. Small H_v-fields

The representations of H_v-groups, can be considered either by H_v-matrices [24], [29], [30], [36], [38], [46], [47], [49] or by generalized permutations [28].

Definition 3.1 *H_v-matrix* is called a matrix with entries elements of H_v-ring or H_v-field. The hyper-product of H_v-matrices $A=(a_{ij})$ and $B=(b_{ij})$, of type $m\times n$ and $n\times r$, respectively, is a set of $m\times r$ H_v-matrices, defined in a usual manner:

$$A\cdot B = (a_{ij})\cdot(b_{ij}) = \{C=(c_{ij}) \mid c_{ij}\in\oplus\Sigma a_{ik}\cdot b_{kj} \},$$

where $(\oplus)$ is the n-ary circle hyper-sum.

The representation problem by H_v-matrices is the following:

Definition 3.2 Let $(H,\cdot)$ be H_v-group, $(R,+,\cdot)$ H_v-ring and $M_R=\{(a_{ij}) \mid a_{ij}\in R\}$, then any

$$T:H\to M_R:\ h\to T(h) \ \text{ with } \ T(h_1h_2)\cap T(h_1)T(h_2)\neq\emptyset,\ \ \forall h_1,h_2\in H,$$

is called *H_v-matrix representation.* If the $T(h_1h_2)\subset T(h_1)T(h_2)$, is valid, then T is called *inclusion* representation, if the $T(h_1h_2)=T(h_1)T(h_2)$, is valid then T is called *good* representation. If T is one to one and good, then it is a *faithful* representation.

Theorem 3.3 A necessary condition in order to have an inclusion representation T of an H_v-group $(H,\cdot)$ by $n\times n$ H_v-matrices over the H_v-ring $(R,+,\cdot)$ is the following:
For all $\beta^*(x)$, $x\in H$ there are elements $a_{ij}\in H$, $i,j\in\{1,...,n\}$ such that

$$T(\beta^*(a)) \subset \{A=(a'_{ij}) \mid a'_{ij}\in\gamma^*(a_{ij}),\ i,j\in\{1,...,n\}\}.$$

Thus, every inclusion representation $T:H\to M_R$: $a\mapsto T(a)=(a_{ij})$ induces a homomorphic representation T^* of H/β^* over R/γ^* by setting $T^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall\beta^*(a)\in H/\beta^*$, where the element $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$.

The representation problem is simplified in cases such as if the H_v-rings have scalars 0 and 1, and they are used as singleton entries.

The representation problem by generalized permutations, is described as follows [28]:

Definitions 3.4 Let X be a set, then a map $f:X\rightarrow P(X)-\{\varnothing\}$, is a *generalized permutation* of X if it is reproductive: $\cup_{x\in X}f(x)=f(X)=X$. Denote by M_X the set of all generalized permutations on X. For an H_v-group $(X,\cdot)$ and $a\in X$, the generalized permutation f_a defined by $f_a(x)=ax$ is called *inner*. *Arrow* of f is any $(x,y)\in X^2$ with $y\in f(x)$. The $f_2\in M_X$ *contains* $f_1\in M_X$, if $f_1(x)\subset f_2(x)$, $\forall x\in X$, we write $f_1\subset f_2$. If, moreover, $f_1\neq f_2$, then f_1 is a *proper* sub-generalized permutation of f_2. An $f\in M_X$ is called *minimal* if it has no proper sub-generalized permutation. Denote $\underline{M}_X$ the set of all minimal generalized permutations of M_X. The *converse* $\underline{f}$ of a generalized permutation f is obtained by reversing arrows: $\underline{f}(x)=\{z\in X: f(z)\ni x\}$. We call *associated* to $f\in M_X$ the generalized permutation $f\circ\underline{f}$.

Theorem 3.5 Let $f\in M_X$, then $f\in\underline{M}_X$ if, and only if, the following condition is valid: if $a\neq b$ and $f(a)\cap f(b)\neq\varnothing$, then $f(a)=f(b)$ and f(a) is singleton. If $f\in\underline{M}_X$ then, $\underline{f}\in\underline{M}_X$. If $f\in\underline{M}_X$ then, $(f\circ\underline{f})(x)=\{y\in X:f(y)=f(x)\}$.

We summarize the representation theory of H_v-groups by H_v-matrices saying that the problem has two steps: First, find small H_v-fields and second, find appropriate H_v-matrices. We focus on small H_v-fields which can be applied in Hadronic Mechanics. These are the small non-degenerate H_v-fields, which in isotheory, are:

(a) very-thin minimal, (b) weak commutative, (c) 0 and 1, scalars.

Definition 3.6 We call *raised very-thin* the H_v-fields which are obtained enlarging one result of a ring adding one element, such that the fundamental structure is a field.

Combining the uniting elements procedure with the raise theory we can obtain stricter structures or hyperstructures [3], [35], [36], [44], [46], [50], [51]. So, raising operations or hyper-operations we can obtain more strict structures as we can see in the following.

Theorem 3.7 In the ring of integers $(Z,+,\cdot)$, fix an m>1. We raise in the product the result $0\cdot m$ by $0\otimes m=\{0,m\}$, the rest results are the same. Then $(Z,+,\otimes)$ becomes H_v-ring, with finite fundamental ring: $(Z,+,\otimes)/\gamma^*\cong(Z_m,+,\cdot)$.

If m=p, prime, then $(Z,+,\otimes)$ is raised very-thin H_v-field, with finite fundamental field.

Raising $a\cdot b$ of fixed $a,b\in Z-\{0,1\}$, setting $a\otimes b=\{a\cdot b,a\cdot b+m\}$, we have the same result, moreover, $(Z,+,\otimes)$ is raised very-thin H_v-field, where 0 and 1 are scalars.

Proof. Sums and products which contain more than one element are the ones which have $0\otimes m$. Adding to $0\otimes m$ the 1, several times, we have modm equivalence classes. On the other side, if add or multiply elements of the same class the results remain in one class, the class of the representatives. Thus, the γ^*-classes form a ring isomorphic to $(Z_m,+,\cdot)$. The rest of the proof is straightforward. Notice, that we can transfer the generalized raised case if we consider the expression $a\otimes b-a\cdot b=\{0,m\}$. ■

Theorem 3.8 In the ring $(Z_n,+,\cdot)$, with n=ms we raise in the product the result $0\cdot m$ by $0\otimes m=\{0,m\}$ and the rest results are the same. Then, $(Z_n,+,\otimes)/\gamma^*\cong(Z_m,+,\cdot)$.

If m=p, prime, then $(Z_n,+,\otimes)$ is raised very-thin H_v-field.

Raising $a\cdot b$ of fixed $a,b\in Z_n-\{0,1\}$, setting $a\otimes b=\{a\cdot b,a\cdot b+m\}$, we have the same result, $(Z_n,+,\otimes)$ is raised very-thin H_v-field, where 0 and 1, are scalars.

All H_v-fields obtained from a field raising any sum or product, are degenerate:

Theorem 3.9 In a field $(F,+,\cdot)$, we raise only the product of $a\cdot b$, by $a\otimes b=\{a\cdot b,c\}$, $c\neq a\cdot b$. Then we obtain the degenerate minimal very-thin H_v-field $(F,+,\otimes)/\gamma^*\cong\{0\}$.

Proof. Take any $x\in F-\{0\}$, then from $a\otimes b=\{ab,c\}$, we obtain $(a\otimes b)-ab=\{0,c-ab\}$ and then $(x(c-ab)^{-1})\otimes((a\otimes b)-ab)=\{0,x\}$ so, $0\gamma x$, $x\in F-\{0\}$ or, every x is in the fundamental class of 0. Thus, $(F,+,\otimes)/\gamma^*\cong\{0\}$. ■

Theorem 3.10 In a field $(F,+,\cdot)$, we raise only the sum of a+b, by $a\oplus b=\{a+b,c\}$, $c\neq a+b$. Then we obtain the degenerate, minimal very-thin, H_v-field $(F,\oplus,\cdot)/\gamma^*\cong\{0\}$.

Proof. Take any $x\in F-\{0\}$, then from $a\oplus b=\{a+b,c\}$, we obtain $(a\oplus b)-(a+b)=\{0,c-(a+b)\}$ and then $[x(c-(a+b))^{-1}]\cdot[(a\oplus b)-(a+b)]=\{0,x\}$, so, $0\gamma x$, $x\in F-\{0\}$ or, every x is in the fundamental class of 0. Thus, $(F,\oplus,\cdot)/\gamma^*\cong\{0\}$. ■

We present some ‘small’ H_v-fields.

Constructions 3.11 On rings $(\boldsymbol{Z}_4,+,\cdot)$ and $(\boldsymbol{Z}_6,+,\cdot)$ we define all multiplicative H_v-fields with non-degenerate fundamental field and, moreover they are,

(a) very-thin minimal, (b) weak commutative, (c) 0 and 1, scalars.

I. On $(\boldsymbol{Z}_4,+,\cdot)$ the isomorphic cases are: $2\otimes3=\{0,2\}$, $3\otimes2=\{0,2\}$.
Fundamental classes: $[0]=\{0,2\}$, $[1]=\{1,3\}$, and $(\boldsymbol{Z}_4,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot)$.

II. On $(\boldsymbol{Z}_6,+,\cdot)$, the isomorphic cases are:

(i) $2\otimes3=\{0,3\}$, $2\otimes4=\{2,5\}$, $3\otimes4=\{0,3\}$, $3\otimes5=\{0,3\}$, $4\otimes5=\{2,5\}$
Fundamental classes: $[0]=\{0,3\}$, $[1]=\{1,4\}$, $[2]=\{2,5\}$, and $(\boldsymbol{Z}_6,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_3,+,\cdot)$.

(ii) $2\otimes3=\{0,2\}$ or $2\otimes3=\{0,4\}$, $2\otimes4=\{0,2\}$ or $\{2,4\}$, $2\otimes5=\{0,4\}$ or $2\otimes5=\{2,4\}$, $3\otimes4=\{0,2\}$ or $3\otimes4=\{0,4\}$, $3\otimes5=\{3,5\}$, $4\otimes5=\{0,2\}$ or $\{2,4\}$.
Fundamental classes: $[0]=\{0,2,4\}$, $[1]=\{1,3,5\}$ and $(\boldsymbol{Z}_6,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot)$.

We can also define the analogous cases for the rings $(\boldsymbol{Z}_9,+,\cdot)$, and $(\boldsymbol{Z}_{10},+,\cdot)$.

We present examples on the topic [44], [47], [49], [50], [51]:

Example 3.12 Take the 2×2 upper triangular H_v-matrices on the above H_v-field $(\boldsymbol{Z}_4,+,\otimes)$ of the case that only $2\otimes3=\{0,2\}$ is a hyperproduct:

$$I=E_{11}+E_{22},\ a=E_{11}+E_{12}+E_{22},\ b=E_{11}+2E_{12}+E_{22},\ c=E_{11}+3E_{12}+E_{22},$$

$$d=E_{11}+3E_{22},\ e=E_{11}+E_{12}+3E_{22},\ f=E_{11}+2E_{12}+3E_{22},\ g=E_{11}+3E_{12}+3E_{22},$$

then, for $\boldsymbol{X}=\{I,a,b,c,d,e,f,g\}$, we obtain the following multiplicative table:

⊗	**I**	**a**	**b**	**c**	**d**	**e**	**f**	**g**
I	I	a	b	c	d	e	f	g
a	a	b	c	I	g	d	e	f
b	b	c	I	a	d,f	e,g	d,f	e,g
c	c	I	a	b	e	f	g	d
d	d	e	f	g	I	a	b	c
e	e	f	g	d	c	I	a	b
f	f	g	d	e	I,b	a,c	I,b	a,c
g	g	d	e	f	a	b	c	b

The $(\boldsymbol{X},\otimes)$ is weak commutative H_v-group with fundamental classes $\underline{I}=\{I,b\}$, $\underline{a}=\{a,c\}$, $\underline{d}=\{d,f\}$, $\underline{e}=\{e,g\}$, and fundamental group isomorphic to $(\boldsymbol{Z}_2\times\boldsymbol{Z}_2,+)$.

Example 3.13 Consider the 2×2 upper triangular H_v-matrices on the H_v-field $(\boldsymbol{Z}_4,+,\otimes)$ where only $3\otimes2=\{0,2\}$ is hyper-product:

$$a=E_{11}+E_{22},\ a_1=E_{11}+E_{12}+E_{22},\ a_2=E_{11}+2E_{12}+E_{22},\ a_3=E_{11}+3E_{12}+E_{22},$$

$$b=E_{11}+3E_{22},\ b_1=E_{11}+E_{12}+3E_{22},\ b_2=E_{11}+2E_{12}+3E_{22},\ b_3=E_{11}+3E_{12}+3E_{22},$$

$$c=3E_{11}+E_{22},\ c_1=3E_{11}+E_{12}+E_{22},\ c_2=3E_{11}+2E_{12}+E_{22},\ c_3=3E_{11}+3E_{12}+E_{22},$$

$$d=3E_{11}+3E_{22},\ d_1=3E_{11}+E_{12}+3E_{22},\ d_2=3E_{11}+2E_{12}+3E_{22},\ d_3=3E_{11}+3E_{12}+3E_{22},$$

then, for $\boldsymbol{X}=\{a,a_1,a_2,a_3,b,b_1,b_2,b_3,c,c_1,c_2,c_3,d,d_1,d_2,d_3\}$, we obtain the multiplicative table.

$\otimes$	**a**	**a_1**	**a_2**	**a_3**	**b**	**b_1**	**b_2**	**b_3**	**c**	**c_1**	**c_2**	**c_3**	**d**	**d_1**	**d_2**	**d_3**
a	a	a_1	a_2	a_3	b	b_1	b_2	b_3	c	c_1	c_2	c_3	d	d_1	d_2	d_3
a_1	a_1	a_2	a_3	a	b_3	b	b_1	b_2	c_1	c_2	c_3	c	d_3	d	d_1	d_2
a_2	a_2	a_3	a	a_1	b_2	b_3	b	b_1	c_2	c_3	c	c_1	d_2	d_3	d	d_1
a_3	a_3	a	a_1	a_2	b_1	b_2	b_3	b	c_3	c	c_1	c_2	d_1	d_2	d_3	d
b	b	b_1	b_2	b_3	a	a_1	a_2	a_3	d	d_1	d_2	d_3	c	c_1	c_2	c_3
b_1	b_1	b_2	b_3	b	a_3	a	a_1	a_2	d_1	d_2	d_3	d	c_3	c	c_1	c_2
b_2	b_2	b_3	b	b_1	a_2	a_3	a	a_1	d_2	d_3	d	d_1	c_2	c_3	c	c_1
b_3	b_3	b	b_1	b_2	a_1	a_2	a_3	a	d_3	d	d_1	d_2	c_1	c_2	c_3	c
c	c	c_3	c,c_2	c_1	d	d_3	d,d_2	d_1	a	a_3	a,a_2	a_1	b	b_3	b,b_2	b_1
c_1	c_1	c	c_1,c_3	c_2	d_3	d_2	d_1,d_3	d	a_1	a	a_1,a_3	a_2	b_3	b_2	b_1,b_3	b
c_2	c_2	c_1	c,c_2	c_3	d_2	d_1	d,d_2	d_3	a_2	a_1	a,a_2	a_3	b_2	b_1	b,b_2	b_3
c_3	c_3	c_2	c_1,c_3	c	d_1	d	d_1,d_3	d_2	a_3	a_2	a_1,a_3	a	b_1	b	b_1,b_3	b_2
d	d	d_3	d,d_2	d_1	c	c_3	c,c_2	c_1	b	b_3	b,b_2	b_1	a	a_3	a,a_2	a_1
d_1	d_1	d	d_1,d_3	d_2	c_3	c_2	c_1,c_3	c	b_1	b	b_1,b_3	b_2	a_3	a_2	a_1,a_3	a
d_2	d_2	d_1	d,d_2	d_3	c_2	c_1	c,c_2	c_3	b_2	b_1	b,b_2	b_3	a_2	a_1	a,a_2	a_3
d_3	d_3	d_2	d_1,d_3	d	c_1	c	c_1,c_3	c_2	b_3	b_2	b_1,b_3	b	a_1	a	a_1,a_3	a_2

$(\boldsymbol{X},\otimes)$ is weak commutative H_v-group, fundamental classes: $\underline{a}=\{a,a_2\}$, $\underline{a}_1=\{a_1,a_3\}$, $\underline{b}=\{b,b_2\}$, $\underline{b}_1=\{b_1,b_3\}$, $\underline{c}=\{c,c_2\}$, $\underline{c}_1=\{c_1,c_3\}$, $\underline{d}=\{d,d_2\}$, $\underline{d}_1=\{d_1,d_3\}$, with table as the above example. In $(\boldsymbol{X},\otimes)$ there is unit a, and every element has unique double inverse. The c_2 is right inverse to c and c_2, because $a\in cc_2$, $a\in c_2c_2$. The d_2 is right inverse to d and d_2, because $a\in dd_2$, $a\in d_2d_2$. $(\boldsymbol{X},\otimes)$ is not cyclic, since, from Theorem 2.6, $(\underline{\boldsymbol{X}},\underline{\otimes})$ is not cyclic.

4. Santilli's iso-theory and e-hyper-structures

The H_v-structures have applications in mathematics and other sciences, as well. These range from biomathematics -conchology, inheritance- and hadronic physics or on leptons, to mention but a few. The hyper-structure theory is related to fuzzy theory; consequently, hyper-structures can be widely applicable in linguistic, sociology, production and industry, too [1], [6], [7], [11], [37], [39], [52], [53], [56].

In '*The Santilli's theory 'invasion' in hyperstructures*' [40], there is a description on how Santilli's theories effect in hyper-structures and how new theories in Mathematics appeared by Santilli's pioneer research. In 1996 Santilli & Vougiouklis [19], point out that in physics the most interesting hyper-structures are the e-hyperstructures.

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit [11], [14], [15], [16], [17], [18], [19], [37], [45], [48].

Definition 4.1 According to Santilli's iso-theory on a field F=(F,+,·), *a general isofield* $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ is defined to be a field with elements $\hat{a}=a\times\hat{1}$, called *isonumbers*, where $a\in F$, and $\hat{1}$ is positive-defined outside F, with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with unit 0, and $\hat{\times}$ is a new multiplication

$$\hat{a}\ \hat{\times}\ \hat{b}:=\hat{a}\times\hat{T}\times\hat{b},\ \text{ with }\ \hat{1}=\hat{T}^{-1},\ \forall\hat{a},\hat{b}\in\hat{F} \qquad \text{(i)}$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of $\hat{F}$,

$$\hat{1}\ \hat{\times}\ \hat{a}=\hat{a}\times\hat{1}=\hat{a},\ \forall\hat{a}\in\hat{F} \qquad \text{(ii)}$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

In order to transfer this theory into the hyper-structure case we generalize only the new product $\hat{\times}$ from (i), replacing with a hyper-product including the old one. There is a general construction on this direction [9], [14], [15], [20], [33], [45], [46]:

Definition 4.2 *General enlargement*. On a field F and its isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ replace in the results of the iso-product

$$\hat{a}\ \hat{\times}\ \hat{b}=\hat{a}\times\hat{T}\times\hat{b},\ \text{ with }\ \hat{1}=\hat{T}^{-1}$$

the $\hat{T}$ by a set $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$, $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}$, containing $\hat{T}$, $\forall\hat{a}\ \hat{\times}\ \hat{b}$ for which $\hat{a},\hat{b}\notin\{\hat{0},\hat{1}\}$, $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}-\{\hat{0},\hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Therefore, the new iso-hyper-operation is

$$\hat{a}\ \hat{\times}\ \hat{b}=\hat{a}\times\hat{H}_{ab}\times\hat{b}=\hat{a}\times\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}\times\hat{b},\ \forall\hat{a},\hat{b}\in\hat{F} \qquad \text{(iii)}$$

$\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ becomes *very-thin isoH_v-field.* The elements of $\hat{F}$ are called *isoH_v-numbers* or *isonumbers*.

Important hyper-products are if only for few $(\hat{a},\hat{b})$ the result enlarged, even more, the extra elements $\hat{x}_i$, are only few. Thus, if there exists only one pair $(\hat{a},\hat{b})$ for which

$$\hat{a}\ \hat{\times}\ \hat{b}=\hat{a}\times\{\hat{T},\hat{x}\}\times\hat{b},\ \ \forall\hat{a},\hat{b}\in\hat{F}$$

and the rest are ordinary, then we have a *very-thin isoH_v-field*.

The assumption $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, and $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give that the isoH_v-field has a scalar absorbing $\hat{0}$, a scalar $\hat{1}$, and $\forall\hat{a}\in\hat{F}$, has one inverse.

The *isofields* corresponding to hyperstructures, introduced by Santilli & Vougiouklis in 1999, are called *e-hyperfields*. [9], [11], [14], [19], [33], [35], [40], [41], [42].

Definitions 4.3 A hyper-structure (H,·) is called *e-hyper-structure* if contains a unique scalar unit e, and every element x, has a, not unique, inverse x^{-1}: $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

A triad (F,+,·), with (+) operation, (·) hyper-operation, is called *e-hyper-field* if: (F,+) is abelian group; (·) weak associative; (·) weak distributive to (+); 0 absorbing: $0\cdot x=x\cdot 0=0$,

$\forall x\in F$; 1 scalar: $1\cdot x=x\cdot 1=x$, $\forall x\in F$; and $\forall x\in F$ there is unique inverse x^{-1}: $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$. The elements of F are called *e-hyper-numbers*. If $1=x\cdot x^{-1}=x^{-1}\cdot x$, then $(F,+,\cdot)$ is *strong*.

Definition 4.4 *The Main e-Construction.* Given a group $(G,\cdot)$, e unit, define in G, a large number of hyper-operations ($\otimes$) as follows:

$$x\otimes y=\{xy, g_1, g_2,\dots\},\ \forall x,y\in G\text{-}\{e\},\ \ g_1, g_2,\dots\in G\text{-}\{e\}$$

$g_1, g_2,\dots$ are not the same for all pairs (x,y). $(G,\otimes)$ is an H_v-group, it is an H_b-group which contains the $(G,\cdot)$. $(G,\otimes)$ is an e-hypergroup. Moreover, if for all x,y such that $xy=e$, we have $x\otimes y=xy$, then $(G,\otimes)$ becomes a strong e-hypergroup.

The proof is immediate since we enlarge the results of the group by putting elements from G applying the Little Theorem. Moreover, the unit e is a unique scalar and $\forall x\in G$, there is a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$. Finally, if we have $1=x\cdot x^{1}=x^{-1}\cdot x$ then the hyper-structure $(G,\otimes)$ is a strong e-hypergroup.

Remark that the main e-construction gives an extremely large number of e-hyper-operations.

Example 4.5 Consider the quaternion group $\boldsymbol{Q}=\{1,-1, i,-i, j,-j, k,-k\}$ with $i^2=j^2=k^2=-1$, $ij=k$, $jk=i$, $ki=j$. Denote $\underline{i}=\{i,-i\}$, $\underline{j}=\{j,-j\}$, $\underline{k}=\{k,-k\}$, we may define a large number $(*)$ hyper-operations by enlarging few products. For example, we set $(-1)*k=\underline{k}$, $k*i=\underline{j}$ and $i*j=\underline{k}$, then $(Q,*)$ is a strong e-hyper-group.

5. Applications in Lie-Santilli's admissibility

A new field in hyper-mathematics comes from Santilli's theory of admissibility. One can transfer Santilli's admissibility, either using ordinary matrices and a hyper-operation on them, or using hyper-matrices and ordinary operations. The Lie-Santilli admissibility in H_v-structures is defined as follows [1], [7], [8], [9], [11], [19], [20], [42], [43], [48]:

Definition 5.1 Let L be H_v-vector space on F, $\varphi:F\to F/\gamma^*$, canonical, $\omega_F=\{x\in F:\varphi(x)=0\}$, 0 zero of F/γ^*, ω_L core of $\varphi':L\to L/\varepsilon^*$, 0 zero of L/ε^*. Take any $R,S\subset L$, then, a *Lie-Santilli admissible hyperalgebra* is obtained by taking the *Lie-bracket* which is hyper-operation:

$$[\ ,\ ]_{RS}: L\times L\to P(L): [x,y]_{RS}= (xR)y-(yS)x= \{xry-ysx \mid r\in R, s\in S\}$$

Special cases: (a) Take only S, so, $[x,y]_S=xy-ySx$. (b) Take only R, so, $[x,y]_R=xRy-yx$.

On non square matrices we can define hyper-admissibility, as well:

Definition 5.2 Let $(L=M_{m\times n},+)$ be H_v-vector space of $m\times n$ H_v-matrices on the H_v-field $(F,+,\cdot)$; $\varphi:F\to F/\gamma^*$, canonical map; $\omega_F=\{x\in F:\varphi(x)=0\}$, 0 the zero of F/γ^*. Similarly, let ω_L core of $\varphi':L\to L/\varepsilon^*$ and 0 zero of L/ε^*. Take two subsets $R,S\subseteq L$, then, a *Santilli's Lie-admissible hyperalgebra* is obtained by taking the hyper-operation Lie bracket:

$$[\ ,\ ]_{RS}: L\times L\to P(L): [x,y]_{RS}=xR^ty-yS^tx.$$

Notice that $[x,y]_{RS}=xR^ty-yS^tx=\{xr^ty-ys^tx \mid r\in R \text{ and } s\in S\}$.

Special cases, but not degenerate, are the 'small' and 'strict' ones:

(a) $R=\{e\}$ then $[x,y]_{RS} = xy-yS^tx = \{xy-ys^tx \mid s\in S\}$

(b) $S=\{e\}$ then $[x,y]_{RS} = xR^ty–yx = \{xr^ty–yx \mid r\in R\}$
(c) $R=\{r_1,r_2\}$ and $S=\{s_1,s_2\}$ then

$$[x,y]_{RS}=xR^ty–yS^tx=\{xr_1{}^ty–ys_1{}^tx,\ xr_1{}^ty–ys_2{}^tx,\ xr_2{}^ty–ys_1{}^tx,\ xr_2{}^ty–ys_2{}^tx\}$$

Definition 5.3 *The P-hope*. Consider isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ with isonumbers $\hat{a}=a\times\hat{1}$, $a\in F$, and $\hat{1}$ is positive-defined element outside F, with operations $\hat{+}$, $\hat{\times}$, where $\hat{+}$ the sum with conventional unit 0, and $\hat{\times}$ the iso-product

$$\hat{a}\ \hat{\times}\ \hat{b} := \hat{a}\times\hat{T}\times\hat{b},\ \text{ with } \hat{1} = \hat{T}^{-1},\ \forall\hat{a},\hat{b}\in\hat{F}.$$

Take $\hat{P}=\{\hat{T},\hat{p}_1,\ldots,\hat{p}_s\}$, $\hat{p}_1,\ldots,\hat{p}_s\in\hat{F}-\{\hat{0},\hat{1}\}$, and define the isoP-$H_v$-field, $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times}_P)$, where the hyper-operation $\hat{\times}_P$ is defined by:

$$\hat{a}\ \hat{\times}_P\ \hat{b} := \begin{cases} \hat{a}\times\hat{P}^{\wedge}\times\hat{b} = \{\hat{a}\times\hat{h}^{\wedge}\times\hat{b} \mid \hat{h}^{\wedge}\in\hat{P}^{\wedge}\} & \text{if } \hat{a}\neq\hat{1} \text{ and } \hat{b}\neq\hat{1} \\ \hat{a}\times\hat{T}^{\wedge}\times\hat{b} & \text{if } \hat{a}=\hat{1} \text{ or } \hat{b}=\hat{1} \end{cases}$$

Special 'small' case: $\hat{P}=\{\hat{T},\hat{p}\}$, so $\hat{P}$ contains only one $\hat{p}$ except $\hat{T}$. Remark that we can take this small case on rings to obtain isoP-H_v-fields. We can see this in the following:
Example 5.4 For, $\hat{Z}_{10} = Z_{10}(\underline{\hat{a}},\hat{+},\hat{\times})$, and if we take $\hat{P}=\{\underline{\hat{2}},\underline{\hat{7}}\}$, then we have the table

$\hat{\times}$	$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{\mathbf{6}}}$	$\underline{\hat{\mathbf{7}}}$	$\underline{\hat{\mathbf{8}}}$	$\underline{\hat{\mathbf{9}}}$
$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$
$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$	$\underline{\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{9}}$
$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$
$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{0}}$	$\underline{\hat{3}}$	$\underline{\hat{2}}$	$\underline{\hat{3},\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{2},\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{4},\hat{9}}$
$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$
$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{0}}$	$\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$
$\underline{\hat{\mathbf{6}}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$
$\underline{\hat{\mathbf{7}}}$	$\underline{\hat{0}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{2},\hat{7}}$	$\underline{\hat{6}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{4}}$	$\underline{\hat{3},\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{1},\hat{6}}$
$\underline{\hat{\mathbf{8}}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$
$\underline{\hat{\mathbf{9}}}$	$\underline{\hat{0}}$	$\underline{\hat{9}}$	$\underline{\hat{6}}$	$\underline{\hat{4},\hat{9}}$	$\underline{\hat{2}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{8}}$	$\underline{\hat{1},\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2},\hat{7}}$

The fundamental classes are $(0)=\{\underline{\hat{0}},\underline{\hat{5}}\}$, $(1)=\{\underline{\hat{1}},\underline{\hat{6}}\}$, $(2)=\{\underline{\hat{2}},\underline{\hat{7}}\}$, $(3)=\{\underline{\hat{3}},\underline{\hat{8}}\}$, $(4)=\{\underline{\hat{4}},\underline{\hat{9}}\}$ and the multiplicative table of them is the following:

×	**(0)**	**(1)**	**(2)**	**(3)**	**(4)**
(0)	(0)	(0)	(0)	(0)	(0)
(1)	(0)	(1),(2)	(2),(4)	(3),(1)	(4),(3)
(2)	(0)	(2),(4)	(3)	(2)	(1)
(3)	(0)	(3),(1)	(2)	(3)	(4)
(4)	(0)	(4),(3)	(1)	(4)	(2)

Consequently, $\hat{Z}_{10} = Z_{10}(\underline{\hat{a}},\hat{+},\hat{\times})$, is an H_v-field.

All biological structures, including cells, viruses and living organisms, are irreversible over time because they are born, grow and die. Santilli introduced in 1994 [16], [17], [18], the representation of biological structures via multivalued, Lie-admissible formulations on a 3-dimensional Euclidean space. They have genounits, *hyperunits*, who correspond the product of generic non-singular quantities, *classical hyper-products*, are equally multivalued, yet defined in our 3-dimensional Euclidean space. The introduced *classical hyper-fields*, are sets of multi-valued elements, products and units which verify the axioms of numeric fields. The hyperstructure representation in biological structures was initiated by T. Vougiouklis in 1999 [35], [37], with the lifting of the classical hyper-structures to Vougiouklis H_v-structures which are formulated via hyper-operations which are weak associative, weak commutative and so on. The advantages of lifting the classical hyper-structures to Vougiouklis H_v-structures are several [7], [8], [9], [16], [20], [40], [41], [45], [48]. The main advantage is a large increase of the representational capabilities which is necessary for a representation of biological structures such as the DNA.

In the transition from Lie theory to Lie-admissible theory, we must specify an element S on the right and an element R to the left. In hyper-structure realization we can use as S and R, sets instead of elements. But in this case, we have hyper-operations of constant length and the living organisms are not the case. Therefore, Santilli & Vougiouklis suggest the use of the special case of the main e-construction to face the problem. Therefore, as the Living Organism Construction the Main e-Construction is applied.

6. Conclusions

When applied sciences ask from mathematics models to express a new theory, one has to search on the existing mathematics if there is an appropriate theory. If there is not any, then the mathematicians try to create a new one to obey the axioms which are required. The hyper-structures offer models in multivalued problems as the ones appeared in real life. The largest class of hyperstructures, called H_v-structures, can offer to Lie-Santilli theory some of the models needed, because they define a huge number of H_v-structures defined on the same set. The appropriate H_v-fields, obtained by the Main e-Construction, can answer to the basic question of Santilli's isotheory, which is: What are the hyper-numbers? The reason is that the Main e-Construction can enlarge every single result with the needed elements.

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