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Unicity Results Concerning of difference monomials of $\mathcal{L}-$function and a meromorphic function

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Abstract

In this paper, we study the value distribution of $\mathcal{L}$-function in the extend Selberg class and a non-constant transcendental meromorphic f function with finitely many zeros of finite order, sharing a polynomial with its difference monomial. Meromorphic functions and L-functions are central to Nevanlinna theory, a branch of complex analysis focusing on the distribution of zeros and poles of analytic functions. The Meromorphic functions are unique due to their combination of meromorphicity and analyticity. They are defined on the entire complex plane except for isolated poles in the complex analysis. On the other hand, L-functions, arising notably in number theory and automorphic forms, exhibit unique properties related to their zeros and poles. One classic example of a prime number distribution is the Riemann ζ-function. In this paper, we analyse the uniqueness results between a non-constant meromorphic function f having finitely many zeros and L-function, when their difference monomial share a non-zero polynomial p. Our results extends and improves the earlier results of Harina P. Waghamore and Manjunatha B E[6].

Keywords: Difference product; uniqueness; Meromorphic function; $L-$function.
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1 Introduction

Nevanlinna theory can provide tools for proving uniqueness theorems for analytic functions. By analyzing the behavior of meromorphic functions and their singularities, one can often deduce properties that lead to uniqueness results. For instance, understanding the growth and distribution of values of a meromorphic function can help establish uniqueness within a certain class of functions.
Let $\mathsf{f}(z)$ and $\mathsf{g}(z)$ be two non-constant meromorphic functions defined in $\mathbb{C}$. We assume that the reader is familiar with the basic results and standard notations of Nevanlinna value distribution theory (see [7],[8],[23],[24],[25]). For some $a \in \mathbb{C} \cup \{\infty\}$, the zeros of $\mathsf{f} - a$ and $\mathsf{g} - a$ have same multiplicities, we say that f and g share the value a CM(counting multiplicities), otherwise if we do not consider the multiplicities, we say that f and g share the value a IM(ignoring multiplicities). If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, then f and g share ∞ CM. While $N\left(r, f\right)$ denotes the counting function of poles of f whose multiplicities are taken into account(respectively $\overline{N}(r, f)$ denotes the reduced counting function when multiplicities are ignored). Also we have $N(r, a; f) = N\left(r, \frac{1}{f-a}\right)$, denotes the counting function of a points of $f(z)$ whose multiplicities are counted(respectively $\overline{N}(r, a; f)$ denotes for the reduced counting function when multiplicities are ignored). In general, the notation $N\left(r, a; f| = 1\right)$ denotes the counting function of simple a-points of f and the notation $N\left(r, a; f| \geq 2\right)$ denotes the counting function of those of a-points of f whose multiplicities are atleast 2(respectively, $\overline{N}\left(r, a; f| + 1\right)$ and $\overline{N}\left(r, a; f| \geq 2\right)$ denotes the reduced counting functions).
We denote by $T(r, \mathsf{f})$ the Nevanlinna characteristic function of f plays a key role in the value distribution theory. we have $T\left(r, f\right) = m\left(r, f\right) + N\left(r, f\right)$, which clearly shows that $T\left(r, f\right)$ is non negative. $S(r, \mathsf{f})$ is small quantity defined by $S(r, \mathsf{f}) = o\{T(r, \mathsf{f})\}$ as $(r \to \infty, r \notin E)$ where E denote any set of positive real numbers having a finite linear measure.

Let $\mathcal{L}$ be an L-function is defined as a meromorphic function in the complex plane $\mathbb{C}$ that is connected to one or more mathematical properties. A $\mathcal{L}$-series is a Dirichlet series that typically converges on a half-plane and has the potential to analytically continued as an $\mathcal{L}$-function.We first distinguish between the $\mathcal{L}$-function, which is the function in the complex plane that is its analytic continuation, and the $\mathcal{L}$-series, which is an infinite series representation (such as the Dirichlet series for the Riemann zeta function)(see.[20],[21]). Generally, the constructions begin with an $\mathcal{L}$-series that is initially specified as a Dirichlet series, and subsequently expanded into an Euler product that is indexed by prime numbers. The $\mathcal{L}$-function so defined analytically to the rest of the complex plane $\mathbb{C}$ atmost one pole. The Selberg class is an axiomatic definition of a class of $\mathcal{L}$-functions. $\mathcal{L}$, consists essentially of those Dirichlet series that satisfy the Riemann hypothesis as well as the Riemann zeta function, $F(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

2 Definitions and Theorems

Let us recall the following standard definitions of Nevnalinna theory.

Definition 2.1. *[20, 21] The Selberg class S consists of the functions $\mathcal{F}$ satisfying the following axioms:*

1. *(Dirichlet series) $\mathcal{F}(s) = \sum_{n=1}^{\infty} a(n)/n^s$ where $s \in \mathbb{C}$, absolutely convergent for $\sigma > 1$.*
2. *(Analytic continuation) There exists a non-negative integer k such that $(s-1)^k \mathcal{L}(s)$ is an entire function of finite order.*
3. *(Functional equation) There exist an integer $k \geq 0$, positive real numbers Q, λ_j, complex numbers v_j with $Re\, v_j \geq 0$ and ω with $|\omega|= 1$, such that $\wedge_{\mathcal{L}}(s)$ defined by*
$$\wedge_{\mathcal{L}}(s) = \mathcal{L}(s) Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + v_j) = \gamma(s)\mathcal{F}(s),$$
satisfies the functional equation $\wedge_{\mathcal{L}}(s) = \overline{\omega \wedge_{\mathcal{L}} (1-\overline{s})}$. We would call the function $\gamma(s)$ the γ factor.
4. *(Ramanujan conjecture) For every $\epsilon > 0, a(n) = O(n^{\epsilon})$.*
5. *(Euler product hypothesis) L can be written over prime as*
$$\mathcal{L}(s) = \prod_{p} exp\left(\sum_{k=1}^{\infty} b(p^k)/p^{ks}\right)$$
with suitable coefficients $b(^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$ where the product is taken over all prime numbers p.
The degree $d_{\mathcal{L}}$ of an $\mathcal{L}$ is defined to be
$$d_{\mathcal{L}} = 2\sum_{j=1}^{K} \lambda_j,$$
where λ_j and K are respectively the positive real number and the positive integer as in axiom (iii) above.

Definition 2.2. *[9, 10, 11] Let $k \in \mathbb{N}\cup\{0\}\cup\{\infty\}$. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;\mathsf{h})$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;\mathsf{f}) = E_k(a;\mathsf{g})$, we say that f, g share the value a with weight k.*

Definition 2.3. *[4] Let k be a positive integer for any constant a in the complex plane $\mathbb{C}$.*
We denote

1. *By $N_{k)}\left(r, \frac{1}{\mathsf{f}-a}\right)$ the counting function of a-points of $\mathsf{f}(z)$ with multiplicity $\leq k$.*
2. *By $N_{(k}\left(r, \frac{1}{\mathsf{f}-a}\right)$ the counting function of a-points of $\mathsf{f}(z)$ with multiplicity $\leq k$. Similarly, the reduced counting functions $\overline{N}_{k)}\left(r, \frac{1}{\mathsf{f}-a}\right)$ and $\overline{N}_{(k}\left(r, \frac{1}{\mathsf{f}-a}\right)$ are defined.*

Definition 2.4. *[6] By a difference product, we mean a difference monomial and its shifts, that is an expression of the type $\prod\limits_{j=1}^{s} \mathsf{f}(z+c_j)^{\mu_j}$.*

In 2011, K. Liu, X. L. Liu, and T. B. Cao [12] studied the uniqueness of the difference monomials and obtained the following results.

Theorem 2.1. *[12] Let f and g be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C}\setminus\{0\}$ and $n \in \mathbb{N}$. If $n \geq 14$, $\mathsf{f}^n(z)\mathsf{f}(z+c)$ and $\mathsf{g}^n(z)\mathsf{g}(z+c)$ share 1 CM, then $\mathsf{f} \equiv t\mathsf{g}$ or $\mathsf{fg} \equiv t$, where $t^{n+1} = 1$.*

Theorem 2.2. *[12] Let f and g be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C}\setminus\{0\}$ and $n \in \mathbb{N}$. If $n \geq 26$, $\mathsf{f}^n(z)\mathsf{f}(z+c)$ and $\mathsf{g}^n(z)\mathsf{g}(z+c)$ share 1 IM, then $\mathsf{f} \equiv t\mathsf{g}$ or $\mathsf{fg} \equiv t$, where $t^{n+1} = 1$.*

In 2015, Y.Liu, J. P. Wang, and F. H. Liu [13] improved Theorems A and B and obtained the following results.

Theorem 2.3. *[13] Let $c \in \mathbb{C}\setminus\{0\}$ and let f and g be two transcendental meromorphic functions with finite order, and $n\,(\geq 14)$, $k\,(\geq 3)$ be two positive integers. If $E_k(1, \mathsf{f}^n(z)\mathsf{f}(z+c)) = E_k(1, \mathsf{g}^n(z)\mathsf{g}(z+c))$, then $\mathsf{f} \equiv t_1\mathsf{g}$ or $\mathsf{fg} \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem 2.4. *[13] Let $c \in \mathbb{C}\setminus\{0\}$ and let f and g be two transcendental meromorphic functions with finite order, and $n\,(\geq 16)$ be a positive integer. If $E_2(1, f^n(z)f(z+c)) = E_2(1, g^n(z)g(z+c))$, then $f \equiv t_1g$ or $fg \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

Theorem 2.5. *[13] Let $c \in \mathbb{C} \setminus \{0\}$ and let f and h be two transcendental meromorphic functions with finite order, and $n\,(\geq 22)$ be a positive integer. If $E_1(1, \mathsf{f}^n(z)\mathsf{f}(z+c)) = E_1(1, \mathsf{g}^n(z)\mathsf{g}(z+c))$, then $\mathsf{f} \equiv t_1\mathsf{g}$ or $\mathsf{fg} \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In 2017, Sujoy Majumder [14] replaced the sharing value 1 by a nonzero polynomial $p(z)$ in 2.3, 2.4, and 2.5 and obtained the following results.

Theorem 2.6. *[14] Let f and g be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 14$. Let $p\,(\not\equiv 0)$ be a polynomial such that deg$(p) < (n-1)/2$. If $\mathsf{f}^n(z)\mathsf{f}(z+c) - p(z)$ and $\mathsf{g}^n(z)\mathsf{g}(z+c) - p(z)$ share $(0,2)$, then one of the following two cases holds:*

(1) $\mathsf{f} \equiv t\mathsf{g}$ *for some constant t such that $t^{n+1} = 1$,*

(2) $\mathsf{fg} \equiv t$, *where $p(z)$ reduces to a nonzero constant c and t is a constant such that $t^{n+1} = c^2$.*

Theorem 2.7. *[14] Let f and g be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 16$. Let $p\,(\not\equiv 0)$ be a polynomial such that deg$(p) < (n-1)/2$. Suppose $\mathsf{f}^n(z)\mathsf{f}(z+c) - p(z)$ and $\mathsf{g}^n(z)\mathsf{g}(z+c) - p(z)$ share $(0,1)$. Then the conclusion of Theorem 2.6 holds.*

Theorem 2.8. *[14] Let f and g be two transcendental meromorphic functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ be such that $n \geq 26$. Let $p\,(\not\equiv 0)$ be a polynomial such that deg$(p) < (n-1)/2$. Suppose $\mathsf{f}^n(z)\mathsf{f}(z+c) - p(z)$ and $\mathsf{g}^n(z)\mathsf{g}(z+c) - p(z)$ share $(0,0)$. Then conclusion of Theorem 2.6 holds.*

In 2023, Harina P. Waghamore and Manjunatha B.E [6] replaced the shift by a difference monomial in 2.6, 2.7, and 2.8 and obtained the following results.

Theorem 2.9. *[6] Let f and g be two transcendental meromorphic function of finite order and n be a positive integer such that $n \geq 2s + 2\sigma + 6$. Suppose that $c_j \in \mathbb{C} \setminus \{0\}$ for $j = \{1, 2, 3, ...s\}$. Let $\mathsf{f}^n \prod_{j=1}^{s} \mathsf{f}(z+c_j)^{\mu_j} - p(z)$ and $\mathsf{g}^n \prod_{j=1}^{s} \mathsf{g}(z+c_j)^{\mu_j} - p(z)$ share $(0,2)$ where p be a nonzero polynomial such that $2deg\,(p(z)) < n - \sigma$, then one of the following two cases holds:*

(1) $f\mathsf{f} \equiv t\mathsf{g}$ *for some constant t such that $t^{n+\sigma} = 1$,*

(2) $\mathsf{f}(z)\mathsf{g}(z) \equiv t$, *where $p(z)$ reduces to a nonzero constant c and t is a constant such that $t^{n+\sigma} = c^2$.*

Theorem 2.10. *[6] Let* f *and* g *be two transcendental meromorphic function of finite order and* n *be a positive integer such that* $n > \frac{5s+2\sigma+11}{2}$. *Suppose that* $c_j \in \mathbb{C}\setminus\{0\}$ *for* $j = \{1,2,3,...s\}$. *Let* $\mathsf{f}^n \prod_{j=1}^{s} \mathsf{f}(z+c_j)^{\mu_j} - p(z)$ *and* $\mathsf{g}^n \prod_{j=1}^{s} \mathsf{g}(z+c_j)^{\mu_j} - p(z)$ *share* $(0,1)$ *where* p *be a nonzero polynomial such that* $2deg\,(p(z)) < n-\sigma$, *then the conclusion of Theorem 3.1 holds.*

Theorem 2.11. *[6] Let* f *and* g *be two transcendental meromorphic function of finite order and* n *be a positive integer such that* $n \geq 5s+\sigma+9$. *Suppose that* $c_j \in \mathbb{C}\setminus\{0\}$ *for* $j = \{1,2,3,...s\}$. *Let* $\mathsf{f}^n \prod_{j=1}^{s} \mathsf{f}(z+c_j)^{\mu_j} - p(z)$ *and* $\mathsf{g}^n \prod_{j=1}^{s} \mathsf{g}(z+c_j)^{\mu_j} - p(z)$ *share* $(0,0)$ *where* p *be a nonzero polynomial such that* $2deg\,(p(z)) < n-\sigma$, *then the conclusion of Theorem 3.1 holds.*

It is quite natural to ask the following question.

Question 1. When two finite order meromorphic functions f and g share a nonzero polynomial, what can be said about their uniqueness if we consider the difference monomial and its shift for an L-function instead of the difference polynomial? We have obtained the following results of Harina P. Waghamore and B. E. Manjunath [6] which improves and generalizes Theorems 2.9, 2.10 and 2.11.

3 Main Results

Following are the main results of our paper.

Theorem 3.1. *Let* f *be non-constant transcendental meromorphic function having finitely many zeros and* $\mathcal{L}$ *be an* L*-function of finite order and* n *be a positive integer such that* $n \geq 2s+2\sigma+5$. *Suppose that* $c_j \in \mathbb{C}\setminus\{0\}$ *for* $j = \{1,2,3,...s\}$. *Let* $\mathsf{f}^n \prod_{j=1}^{s} \mathsf{f}(z+c_j)^{\mu_j} - p(z)$ *and* $\mathcal{L}^n \prod_{j=1}^{s} \mathcal{L}(z+c_j)^{\mu_j} - p(z)$ *share* $(0,2)$ *where* p *be a nonzero polynomial such that* $2deg\,(p(z)) < n-\sigma$, *then one of the following two cases holds:*

1. $\mathsf{f} \equiv t\mathcal{L}$ *for some constant* t *such that* $t^{n+\sigma} = 1$,
2. $\mathsf{f}(z)\mathcal{L}(z) \equiv t$, *where* $p(z)$ *reduces to a nonzero constant* c *and* t *is a constant such that* $t^{n+\sigma} = c^2$.

Theorem 3.2. *Let* f *be anon-constant transcendental meromorphic function having finitely many zeros and* $\mathcal{L}$ *be an* L *function of finite order and* n *be a positive integer such that* $n > \frac{3s+4\sigma+9}{2}$. *Suppose that* $c_j \in \mathbb{C}\setminus\{0\}$ *for* $j = \{1,2,3,...s\}$. *Let* $\mathsf{f}^n \prod_{j=1}^{s} \mathsf{f}(z+c_j)^{\mu_j} - p(z)$ *and* $\mathcal{L}^n \prod_{j=1}^{s} \mathcal{L}(z+c_j)^{\mu_j} - p(z)$ *share* $(0,1)$ *where* p

be a nonzero polynomial such that $2deg\,(p(z)) < n-\sigma$, then the conclusion of Theorem 3.1 holds.

Theorem 3.3. *Let f be a non-constant transcendental meromorphic function having finitely many zeros and $\mathcal{L}$ be an L function of finite order and n be a positive integer such that $n \geq 4s+\sigma+7$. Suppose that $c_j \in \mathbb{C}\setminus\{0\}$ for $j=\{1,2,3,...s\}$. Let $\mathsf{f}^n\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}-p(z)$ and $\mathcal{L}^n\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}-p(z)$ share $(0,0)$ where p be a nonzero polynomial such that $2deg\,(p(z)) < n-\sigma$, then the conclusion of Theorem 3.1 holds.*

Example 3.1. *Let $f = exp^{(z)}$ and $\mathcal{L} = exp^{(-z)}$. By considering the values for $n > 9, \sigma = 1, s = 1$ and c_j, μ_j are complex constants. Then one can easily verify that $\mathsf{f}^n\prod_{j=1}^{s}\mathsf{g}(z+c_j)^{\mu_j}$ and $\mathcal{L}^n\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}$, where f and $\mathcal{L}$ satisfies the algebraic equations $R(f,\mathcal{L}) = 0$ with the conclusion of Theorem 3.1 holds.*

Example 3.2. *Let $f = exp^{(z)}$ and $\mathcal{L} = t\,exp^{(z)}$. By considering the values for $n > 9, \sigma = 1, s = 1$ and c_j, μ_j are complex constants. Then one can easily verify that $\mathsf{f}^n\prod_{j=1}^{s}\mathsf{g}(z+c_j)^{\mu_j}$ and $\mathcal{L}^n\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}$, where f and $\mathcal{L}$ satisfies the algebraic equations $R(f,\mathscr{L}) = 0$ with the conclusion of Theorem 3.1 holds.*

4 Remarks

Theorem 3.1 - 3.3 is an improvement of Theorem 2.8 - 2.9 respctively. Taking $s = 1$ we get $\sigma = \sum_{j=1}^{s}\mu_j = 1$, $\mathsf{f}^n\prod_{j=1}^{s}\mathsf{g}(z+c_j)^{\mu_j}$ and $\mathcal{L}^n\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}$ reduces to $\mathsf{f}^n\mathsf{f}(z+c)$ and $\mathsf{g}^n\mathsf{g}(z+c)$, then

(1) In Theorem 3.1 we obtain $n \geq 8$ which is an improvement of Theorem 2.9,

(2) In Theorem 3.2 we obtain $n > 8$ which is an improvement of Theorem 2.10,

(3) In Theorem 3.3 we obtain $n \geq 13$ which is an improvement of Theorem 2.11.

5 Lemmas

In this section, we present few lemmas needed in the sequel. Let $\mathcal{F}$,and L be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. We denote by $\mathcal{H}$ the function

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{L''}{L'} - \frac{2L'}{L-1}\right). \tag{1}$$

Lemma 5.1. *[5] Let* f *be a meromorphic function of finite order* ρ*, and let* $c \in \mathbb{C} \setminus \{0\}$ *be fixed. Then for each* $\epsilon > 0$*, we have*

$$m\left(r, \frac{\mathsf{f}(z+c)}{\mathsf{f}(z)}\right) + m\left(r, \frac{\mathsf{f}(z)}{\mathsf{f}(z+c)}\right) = O(r^{\rho-1+\epsilon}) = S(r, \mathsf{f}).$$

The following lemma has few modifications of the original version [Corollary 2.5 of [4]]

Lemma 5.2. *[22] Let* f *be a non-constant meromorphic function and let* $a_n(z)$ $(\not\equiv 0), a_{n-1}(z), ..., a_0(z)$ *be meromorphic functions such that* $T(r, a_i(z)) = S(r, \mathsf{f})$ *for* $i = 0, 1, 2, ..., n$*. Then*

$$T(r, a_n\mathsf{f}^n + a_{n-1}\mathsf{f}^{n-1} + ... + a_1\mathsf{f} + a_0) = nT(r, \mathsf{f}) + S(r, \mathsf{f}).$$

Lemma 5.3. *[8] Let* f *be a non-constant meromorphic function of finite order and* $c \in \mathbb{C}$*. Then*

$$N\left(r, 0; \mathsf{f}(z+c)\right) \leq N\left(r, 0; \mathsf{f}(z)\right) + S(r, \mathsf{f}), \quad N(r, \infty; \mathsf{f}(z+c)) \leq N(r, \infty; \mathsf{f}) + S(r, \mathsf{f}),$$

$$\overline{N}\left(r, 0; \mathsf{f}(z+c)\right) \leq \overline{N}\left(r, 0; \mathsf{f}(z)\right) + S(r, \mathsf{f}), \quad \overline{N}(r, \infty; \mathsf{f}(z+c)) \leq \overline{N}(r, \infty; \mathsf{f}) + S(r, \mathsf{f}).$$

Lemma 5.4. *[5] Let* f *be a transcendental meromorphic function of finite order,* $c \in \mathbb{C} \setminus \{0\}$ *be fixed. Then*

$$T(r, \mathsf{f}(z+c)) = T(r, \mathsf{f}) + S(r, \mathsf{f}).$$

Lemma 5.5. *[22] Let* f *and* g *be two non-constant meromorphic functions. Then*

$$N\left(r, \infty; \frac{\mathsf{f}}{\mathsf{g}}\right) - N\left(r, \infty; \frac{\mathsf{g}}{\mathsf{f}}\right) = N(r, \infty; \mathsf{f}) + N(r, 0; \mathsf{g}) - N(r, \infty; \mathsf{g}) - N(r, 0; \mathsf{f}).$$

Lemma 5.6. *[6] Let* f *be a transcendental meromorphic function of finite order and let*
$F = \mathsf{f}^n \prod\limits_{j=1}^{s} \left(\mathsf{f}(z+c_j)\right)^{\mu_j}$*, where* n *is positive integer. Then*

$$(n-\sigma)T\left(r, \mathsf{f}\right) \leq T(r, \mathcal{F}) + S(r, \mathsf{f}).$$

Lemma 5.7. *Let* f *be non-constant transcendental meromorphic function having finitely many zeros and* $\mathcal{L}$ *be an* $L-$*function of finite order,* $c \in \mathbb{C} \setminus \{0\}$ *and* $n \in \mathbb{N}$ *such that* $n > \sigma$*. Let* p *be a nonzero polynomial such that* $2deg\,(p) < n - \sigma$*. Then*

(1) if $deg(p) \geq 1$*, then* $\mathsf{f}^n \prod\limits_{j=1}^{s} \left(\mathsf{f}(z+c_j)\right)^{\mu_j} .\mathcal{L}^n \prod\limits_{j=1}^{s} \left(\mathcal{L}(z+c_j)\right)^{\mu_j} \not\equiv p^2(z)$

(2) if $p(z)=c\in\mathbb{C}\backslash\{0\}$, then the relation $\mathsf{f}^n\prod_{j=1}^{s}(\mathsf{f}(z+c_j))^{\mu_j}\,\mathcal{L}^n\prod_{j=1}^{s}(\mathcal{L}(z+c_j))^{\mu_j}\equiv p^2(z)$*, always implies that $\mathsf{f}.\mathcal{L}=t$, where t is a constant such that $t^{n+\sigma}=c^2$.*

Proof. Suppose

$$\left(\mathsf{f}^n\prod_{j=1}^{s}(\mathsf{f}(z+c_j))^{\mu_j}\right)\left(\mathcal{L}^n\prod_{j=1}^{s}(\mathcal{L}(z+c_j))^{\mu_j}\right)\equiv p^2(z). \tag{2}$$

Let $\mathsf{h}_1=\mathsf{f}\mathcal{L}$ then by (2), we have

$$\mathsf{h}_1^2(z)=\frac{p^2(z)}{\prod_{j=1}^{s}\mathsf{h}_1(z+c_j)^{\mu_j}}. \tag{3}$$

We now consider the following two cases,
Case 1. Suppose h_1 is a transcendental meromorphic function. Now by Lemma 5.1, 5.2 and 5.3, we get

$$\begin{aligned}
nT\left(r,\mathsf{h}_1\right)&=T\left(r,\mathsf{h}_1^n\right)+S\left(r,\mathsf{h}_1\right),\\
&=T\left(r,\frac{p^2(z)}{\prod_{j=1}^{s}\mathsf{h}_1(z+c_j)^{\mu_j}}\right)+S\left(r,\mathsf{h}_1\right),\\
&\leq N\left(r,0;\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}\right)+N\left(r,0;\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}\right)+S\left(r,h_1\right),\\
&\leq\sigma\left[T(r,\mathsf{f})+T(r,\mathcal{L})\right]+S\left(r,\mathsf{h}_1\right),
\end{aligned}$$

from $\mathsf{h}_1=\mathsf{f}\mathcal{L}$, which implies $T(r,\mathsf{h}_1)=T(r,\mathsf{f})+T(r,\mathcal{L})$, we get

$$n\left[T(r,\mathsf{f})+T(r,\mathcal{L})\right]\leq\sigma\left[T(r,\mathsf{f})+T(r,\mathcal{L})\right]+S\left(r,\mathsf{h}_1\right),$$

which is a contradiction.
Case 2. Suppose h_1 is a rational function. Let

$$\mathsf{h}_1=\frac{\mathsf{h}_2}{\mathsf{h}_3} \tag{4}$$

where h_2, h_3 are two non-zero relatively prime polynomials. By (3), we have

$$T(r,\mathsf{h}_1)=max\{deg(\mathsf{h}_2),deg(\mathsf{h}_3)\}\log r+O(1). \tag{5}$$

Now by (3) - (5), we have

$$max\{deg(\mathsf{h}_2), deg(\mathsf{h}_3)\}\log r = T\left(r, \mathsf{h}_1^n\right) + O(1),$$
$$\leq \sigma\left[T(r,\mathsf{f}) + T(r,\mathcal{L})\right] + 2T(r,p) + O(1),$$

$$max\{deg(h_2), deg(h_3)\}\log r \leq \sigma max\{deg(h_2),\ deg(h_3)\}\log r + 2T(r,p) + O(1). \tag{6}$$

We see that $max\{deg(h_2), deg(h_3)\} \geq 1$. Now by (6), we deduce that $(n-\sigma) \leq 2deg\left(p(z)\right)$. Which contradicts our assumption that $2deg\left(p(z)\right) < (n-\sigma)$. Hence h_1 must be a non-zero constant.
Let

$$\mathsf{h}_1 = t \in \mathbb{C}\setminus\{0\}. \tag{7}$$

Now when $deg(p) \geq 1$ by (3) and (7), we arrive at a contradiction.
In this case we have $\left(\mathsf{f}^n \prod_{j=1}^{s}\left(\mathsf{f}(z+c_j)\right)^{\mu_j}\right)\left(\mathcal{L}^n \prod_{j=1}^{s}\left(\mathcal{L}(z+c_j)\right)^{\mu_j}\right) \not\equiv p^2(z)$.
Suppose $p(z) = c \in \mathbb{C}\setminus\{0\}$. So by (2), we see that $\mathsf{h}_1^{n+\sigma} \equiv c^2$.
By (7), we get $t^{n+\sigma} \equiv c^2$. This completes the proof. □

Lemma 5.8. *Let* f *be non-constant transcendental meromorphic function having finitely many zeros and* $\mathcal{L}$ *be an* L *function of finite order,* $c \in \mathbb{C}\setminus\{0\}$ *be finite complex constant such that* $\mathsf{f}(z+c_j) \neq \mathsf{f}(z)$ *for* $i = 0, 1, 2, ..., s$. *and* $\mathcal{L}(z+c_j) \neq \mathcal{L}(z)$ *and let* n *be an integer such that* $n \geq (s+1)+\sigma$. *Let* $\mathcal{F}(z) = \frac{\mathsf{f}^n \prod_{j=1}^{s}(\mathsf{f}(z+c_j))^{\mu_j}}{p(z)}$ *and* $L(z) = \frac{\mathcal{L}^n \prod_{j=1}^{s}(\mathcal{L}(z+c_j))^{\mu_j}}{p(z)}$, *where* $p(z)$ *is nonzero polynomial. If* $\mathcal{H} \equiv 0$ *then either*

(1) $\mathsf{f}^n \prod_{j=1}^{s}\left(\mathsf{f}(z+c_j)\right)^{\mu_j} \mathcal{L}^n \prod_{j=1}^{s}\left(\mathcal{L}(z+c_j)\right)^{\mu_j} \equiv p^2(z)$,

where $\mathsf{f}^n \prod_{j=1}^{s}\left(\mathsf{f}(z+c_j)\right)^{\mu_j} - p(z)$ *and* $\mathcal{L}^n \prod_{j=1}^{s}\left(\mathcal{L}(z+c_j)\right)^{\mu_j} - p(z)$ *share* 0 *CM,*

(2) $\mathsf{f}(z) = t\mathcal{L}(z)$ *for a constant* t *with* $t^{n+\sigma} = 1$.

Proof: From (1) we have,

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{L''}{L} - \frac{2L'}{L-1}\right).$$

Since $\mathcal{H} \equiv 0$, by integration, we get

$$\frac{1}{\mathcal{F}-1} = \frac{BL + A - B}{L-1} \tag{8}$$

where $A \neq 0$, B are constant . From (7) it is clear that $\mathcal{F}$ and L share $(1,\infty)$. We now consider the following cases.
Case 1. Let $B \neq 0$ and $A \neq B$, if $B=-1$ then from (8), we have $\mathcal{F}=\frac{-A}{L-A-1}$
Therefore $\overline{N}\left(r,\frac{1}{L-(A+1)}\right)=\overline{N}\left(r,\mathcal{F}\right)\leq N\left(r,\frac{1}{p}\right)\leq T(r,p)=S(r,\mathcal{L})$.
So in view of Lemma 5.6 and the second fundamental theorem , we get

$$\begin{aligned}(n-\sigma)T(r,\mathcal{L}) &\leq T\left(r,\mathcal{L}^n\prod_{j=1}^{s}(\mathcal{L}(z+c_j))^{\mu_j}\right)+S(r,\mathcal{L}),\\ &\leq T(r,\mathcal{L})+S(r,\mathsf{g}),\\ &\leq \overline{N}\left(r,\infty;\mathcal{L}\right)+\overline{N}\left(r,0;\mathcal{L}\right)+\overline{N}\left(r,(A+1);\mathcal{L}\right)+S(r,\mathcal{L}),\\ &\leq (s+1)T(r,\mathcal{L})+S(r,\mathcal{L})\end{aligned}$$

which contradicts with $n>(s+1)+\sigma$.
If $B \neq 1$, from (8) we get that

$$\mathcal{F}-\left(1+\frac{1}{B}\right)=\frac{-A}{B^2\left(L+\frac{A-B}{B}\right)}.$$

So $\overline{N}\left(r,\frac{B-A}{B};L\right)=\overline{N}(r,\infty;\mathcal{F})\leq S(r,\mathcal{L})$. Using Lemma 5.6 and the same argument as used in the case when $B=-1$ we can get a contradiction.
Case 2. Let $B \neq 0$ and $A=B$. If $B=-1$, then from (8), we have

$$\mathsf{f}^n\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}\mathcal{L}^n\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}=p^2(z),$$

when $\left[\mathsf{f}^n\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}\right]-p(z)$ and $\left[\mathcal{L}^n\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}\right]-p(z)$ share 0 CM.
If $B \neq 1$, from (8) we have

$$\frac{1}{\mathcal{F}}=\frac{BL}{(1+B)L-1}.$$

Therefore $\overline{N}\left(r,(1+B);L\right)=\overline{N}(r,\infty;\mathcal{F})$. So in view of Lemma 5.8 and the second fundamental theorem, we get

$$\begin{aligned}(n-\sigma)T(r,\mathcal{L}) &\leq T(r,L)+S(r,L),\\ &\leq \overline{N}\left(r,\infty;L\right)+\overline{N}\left(r,0;L\right)+\overline{N}\left(r,(1+B);L\right)+S(r,L),\\ &\leq (s+1)T(r,\mathcal{L})+S(r,\mathsf{f})+S(r,\mathcal{L}),\end{aligned}$$

so for $r \in I$, we have

$$\left(n-\sigma-(s+1)\right)T(r,\mathcal{L})\leq S(r,\mathcal{L}).$$

Which is contradiction, since $n > (s+1)+\sigma$.
Case 3. Let $B = 0$. From (8) we obtain

$$\mathcal{F} = \frac{L + A - 1}{A} \tag{9}$$

if $A \neq 1$, then from (9), we obtain $\overline{N}\left(r,(1-A);L\right) = \overline{N}\left(r,0;\mathcal{F}\right)$.
We can similarly deduce a contradiction as in case 2. Therefore $A = 1$ and from (9), we obtain , $\mathcal{F}(z) \equiv L(z)$.
That is

$$\mathsf{f}^n \prod_{j=1}^{s} \mathsf{f}(z+c_j)^{\mu_j} \equiv \mathcal{L}^n \prod_{j=1}^{s} \mathcal{L}(z+c_j)^{\mu_j}. \tag{10}$$

Let $\mathsf{h} = \frac{\mathsf{f}}{\mathcal{L}}$ and then substituting $\mathsf{f} = \mathsf{h}\,\mathcal{L}$ in (10), we deduce

$$\mathsf{h}^n(z) = \frac{1}{\prod\limits_{j=1}^{s} \mathsf{h}(z+c_j)^{\mu_j}} \tag{11}$$

$$\mathsf{h}^{n+1} = \frac{\mathsf{f}}{\mathcal{L}\prod\limits_{j=1}^{s} \mathsf{h}(z+c_j)^{\mu_j}},$$

if h is not a constant, then we have

$$\begin{aligned}(n+1)T(r,\mathsf{h}) \leq\; & T\left(r, \frac{\mathsf{f}}{\prod\limits_{j=1}^{s} \mathsf{f}(z+c_j)^{\mu_j}}\right) + T\left(r, \frac{\prod\limits_{j=1}^{s} \mathcal{L}(z+c_j)^{\mu_j}}{\mathcal{L}}\right) \\ & + S(r,\mathsf{f}) + S(r,\mathcal{L}), \\ & \leq (1+\sigma)\left[T(r,\mathsf{f}) + T(r,\mathcal{L})\right] + S(r,\mathsf{f}) + S(r,\mathcal{L}),\end{aligned}$$

we obtain

$$(n-\sigma)\left[T(r,\mathsf{f}) + T(r,\mathcal{L})\right] \leq S(r,\mathsf{f}) + S(r,\mathcal{L}),$$

which is impossible. Therefore h is constant, then substituting $\mathsf{f} = \mathsf{h}\,\mathcal{L}$ in (10), we have $\mathsf{h}^{n+\sigma} \equiv 1$. Therefore $\mathsf{f} = t\mathcal{L}$ where t is a constant with $t^{n+\sigma} = 1$.

Lemma 5.9. *[1] If* f, g *be two non-constant meromorphic functions such that they share*$(1,1)$. *Then*

$$2\overline{N}_L(r,1;\mathsf{f}) + 2\overline{N}_L(r,1;\mathsf{g}) + \overline{N}_E^{(2}(r,1;\mathsf{f}) - \overline{N}_{\mathsf{f}>2}(r,1;\mathsf{g}) \leq N(r,1;\mathsf{g}) - \overline{N}(r,1;\mathsf{g}).$$

Lemma 5.10. *[3] Let* f *and* g *share* $(1,1)$. *Then*

$$\overline{N}_{\mathsf{f}>2}(r,1;\mathsf{g}) \leq \frac{1}{2}\overline{N}(r,0;\mathsf{f}) + \frac{1}{2}\overline{N}(r,\infty;\mathsf{f}) - \frac{1}{2}N_0(r,0;\mathsf{f}^{'}) + S(r,\mathsf{g}),$$

where $N_0\,(r,0;\mathsf{f}')$ *is the counting function of the zeros of* f' *which are not the zero of* $\mathsf{f}(\mathsf{f}-1)$.

Lemma 5.11. *[3] Let* f, g *share* $(1,0)$. *Then*

(i) $\overline{N}_{\mathsf{f}>1}(r,1;\mathsf{g}) \leq \overline{N}(r,0;\mathsf{f}) + \overline{N}(r,\infty;\mathsf{f}) - N_0(r,0;\mathsf{f}^{'}) + S(r,\mathsf{f})$

(ii) $\overline{N}_{\mathsf{g}>1}(r,1;\mathsf{f}) \leq \overline{N}(r,0;\mathsf{g}) + \overline{N}(r,\infty;\mathsf{g}) - N_0(r,0;\mathsf{g}^{'}) + S(r,\mathsf{g})$.

Lemma 5.12. *[25] Let* f, g *share* $(1,0)$. *Then*

$$\overline{N}_{\mathscr{L}}(r,1;\mathsf{f}) \leq \overline{N}(r,0;\mathsf{f}) + \overline{N}(r,\infty;\mathsf{f}) + S(r,\mathsf{f}).$$

Lemma 5.13. *[3] Let* f *and* g *be two non-constant meromorphic functions sharing* $(1,0)$. *Then*

$$\overline{N}_{\mathscr{L}}(r,1;\mathsf{f}) + 2\overline{N}_{\mathscr{L}}(r,1;\mathsf{g}) + \overline{N}^{(2}_{E}(r,1;\mathsf{f}) - \overline{N}_{\mathsf{f}>1}(r,1;\mathsf{g}) - \overline{N}_{\mathsf{g}>1}(r,1;\mathsf{f}) \leq N(r,1;\mathsf{g}) - \overline{N}(r,1;\mathsf{g}).$$

Lemma 5.14. *[10, 19] If* $N(r,0;\mathsf{f}^{(k)}|\mathsf{f}\neq 0)$ *denotes the counting function of those zeros of* $\mathsf{f}^{(k)}(z)$ *which are not the zeros of* $\mathsf{f}(z)$, *where a zero of* $\mathsf{f}^{(k)}(z)$ *is counted according to its multiplicity, then*

$$N(r,0;\mathsf{f}^{(k)}|\mathsf{f}\neq 0) \leq k\overline{N}(r,\infty;\mathsf{f}) + N(r,0;\mathsf{f}|<k) + k\overline{N}(r,0;\mathsf{f}|\geq k) + S(r,\mathsf{f}).$$

Lemma 5.15. *[18, 20, 21] Let* $\mathcal{L}$ *be an L-function with degree* q. *Then* $T\,(r,L) = \frac{q}{\pi} r logr + O(r)$.

Lemma 5.16. *[2, 15, 16, 17] Let* $\mathcal{L}$ *be an L-function. Then* $N\,(r,\infty) = S\,(r,\mathcal{L}) = O(logr)$.

Lemma 5.17. *Let* f *be a non-constant meromorphic function and* L *be an L-function. If* f *and* L *share* $(\infty,0)$, *then* $\overline{N}\,(r,\infty;\mathsf{f}) = S\,(r,\mathcal{L}) = O(logr)$.

Proof:

Since f and $\mathcal{L}$ share $(\infty,0)$, therefore by lemma 5.16, we have $\overline{N}\,(r,\infty;\mathsf{f}) = \overline{N}\,(r,\infty;\mathcal{L}) = S\,(r,\mathcal{L}) = O(logr)$. This completes the proof.

6 Proof of main Theorems

6.1 Proof of Theorem 3.1

Let $\mathcal{F} = \frac{\mathsf{f}^n \prod\limits_{j=1}^{s} \mathsf{f}(z+c_j)^{\mu_j}}{p(z)}$ and $L = \frac{\mathcal{L}^n \prod\limits_{j=1}^{s} \mathcal{L}(z+c_j)^{\mu_j}}{p(z)}$. It follows that $\mathcal{F}$ and L share $(1,2)$ except the zeors of $p(z)$.
Case 1. Let $\mathcal{H} \not\equiv 0$ from (1), we obtain

$$N(r,\infty;\mathcal{H}) \leq \overline{N}_*(r,1;\mathcal{F},L) + \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;L| \geq 2) + \overline{N}_0(r,0;\mathcal{F}^{'}) + \overline{N}_0(r,0;L^{'}). \tag{12}$$

$\overline{N}_0(r,0;\mathcal{F}^{'})$ is the reduced counting function of those zero's of $\mathcal{F}^{'}$ which are not the zero's of $\mathcal{F}(\mathcal{F}-1)$, and $\overline{N}_0(r,0;\mathcal{F}^{'})$ is similarly defined.
Let z_0 be a simple zero of $F-1$ such that $p(z_0) \neq 0$. Then z_0 is a simple zero of $L-1$ and a zero of H. So

$$N(r,1;\mathcal{F}| = 1) \leq N(r,0;L) \leq N(r,\infty;\mathcal{H}) + S(r,\mathsf{f}) + S(r,\mathcal{L}). \tag{13}$$

From (12) and (13)

$$\overline{N}(r,1;\mathcal{F}) \leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;L| \geq 2) + \overline{N}_*(r,1;F,L) + \overline{N}(r,1;\mathcal{F}| \geq 2) + \overline{N}_0(r,0;\mathcal{F}^{'}) + \overline{N}_0(r,0;L^{'}) + S(r,\mathsf{f}) + S(r,L). \tag{14}$$

Now in the view of Lemma 5.14, we get

$$\begin{aligned}\overline{N}_0(r,0;L^{'}) + \overline{N}(r,1;\mathcal{F}| \geq 2) + \overline{N}_*(r,1;\mathcal{F},L) &\leq \overline{N}_0(r,0;L^{'}) + \overline{N}(r,1;\mathcal{F}| \geq 2) \\ &\quad + \overline{N}(r,1;\mathcal{F}| \geq 3) \\ &\leq N(r,0;L^{'}|L \neq 0) \leq \overline{N}(r,0;L) + S(r,L).\end{aligned} \tag{15}$$

Using equations (14), (15) and Lemma 5.6, we get from second fundamental theorem that

$$\begin{aligned}(n-\sigma)T(r,\mathsf{f}) &\leq T(r,\mathcal{F}) + S(r,\mathsf{f}), \\ &\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) - N_0(r,0;\mathcal{F}^{'}) + S(r,\mathsf{f}), \\ &\leq \overline{N}(r,\infty;\mathcal{F}) + N_2(r,0;\mathcal{F}) + N_2(r,0;L) + S(r,\mathsf{f}) + S(r,L), \\ &\leq \overline{N}(r,\infty;\mathsf{f}) + \overline{N}\left(r,\infty;\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}\right) + 2\overline{N}(r,0;\mathsf{f}) \\ &\quad + N\left(r,0;\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}\right) + N_2\left(r,0;\mathcal{L}^n\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}\right) \\ &\quad + S(r,f) + S(r,\mathcal{L}), \\ &\leq (2+\sigma)\,T(r,\mathsf{f}) + (2+s)T(r,\mathcal{L}) + S(r,\mathsf{f}) + S(r,\mathcal{L}).\end{aligned} \tag{16}$$

Similarly, we can obtain

$$(n-\sigma)T(r,\mathcal{L}) \leq (2+\sigma)T(r,\mathcal{L}) + (2+s)T(r,\mathsf{f}) + S(r,\mathsf{f}) + S(r,\mathcal{L}). \quad (17)$$

Combining (16) and (17) we have

$$(n-2\sigma-s-4)\left[T(r,\mathsf{f}) + T(r,\mathcal{L})\right] \leq S(r,\mathsf{f}) + S(r,\mathcal{L}). \quad (18)$$

Since $n > 2\sigma + s + 4$, equation (18) leads to contradiction.
Case 2. Let $\mathcal{H} \equiv 0$. Then the theorem follows from Lemmas 5.7 and 5.8. This completes the proof.

6.2 Proof of Theorem 3.2

Let $\mathcal{F}(z) = \frac{\mathsf{f}^n(z)\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}}{p(z)}$ and $L(z) = \frac{\mathcal{L}^n(z)\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}}{p(z)}$. Then $\mathcal{F}$ and L share (1,1) except for the zeros of $p(z)$. We now consider the following two cases.
Case 1. $\mathcal{H} \not\equiv 0$.
Using Lemmas 5.6, 5.8, 5.9 and equations (12) and (13), we get

$$\begin{aligned}
&\overline{N}(r,1;\mathcal{F}) \leq N(r,1;\mathcal{F}|=1) + \overline{N}_L(r,1;\mathcal{F}) + \overline{N}_{\mathscr{L}}(r,1;L) + \overline{N}_E^{(2}(r,1;F),\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;L|\geq 2) + \overline{N}_*(r,1;\mathcal{F},L) + \overline{N}_{\mathscr{L}}(r,1;\mathcal{F})\\
&+ \overline{N}_{\mathscr{L}}(r,1;L) + \overline{N}_E^{(2}(r,1;\mathcal{F}) + \overline{N}_0(r,0;\mathcal{F}^{'}) + \overline{N}_0(r,0;L^{'}) + S(r,\mathsf{f}) + S(r,L),\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;L|\geq 2) + 2\overline{N}_{\mathscr{L}}(r,1;\mathcal{F}) + 2\overline{N}_{\mathscr{L}}(r,1;L)\\
&+ \overline{N}_E^{(2}(r,1;\mathcal{F}) + \overline{N}_0(r,0;\mathcal{F}^{'}) + \overline{N}_0(r,0;L^{'}) + S(r,\mathsf{f}) + S(r,L),\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;L|\geq 2) + \overline{N}_{\mathcal{F}>2}(r,1;L) + N(r,1;L) - \overline{N}(r,1;L)\\
&+ \overline{N}_0(r,0;\mathcal{F}^{'}) + \overline{N}_0(r,0;L^{'}) + S(r,\mathsf{f}) + S(r,L),\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \frac{1}{2}\overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;L|\geq 2) + N(r,0;L^{'}|L\neq 0)\\
&+ \overline{N}_0(r,0;\mathcal{F}^{'}) + S(r,\mathsf{f}) + S(r,L),\\
&\overline{N}(r,1;\mathcal{F}) \leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \frac{1}{2}\overline{N}(r,0;\mathcal{F}) + N_2(r,0;L) + \overline{N}_0(r,0;\mathcal{F}^{'})\\
&+ S(r,\mathsf{f}) + S(r,L). \qquad (19)
\end{aligned}$$

Hence by using second fundamental theorem, (19), Lemmas 5.1 and 5.6 we get

$$
\begin{aligned}
&(n-\sigma)T(r,\mathsf{f}) \leq T(r,\mathcal{F}) + S(r,\mathsf{f}),\\
&\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) - N_0(r,0;\mathcal{F}') + S(r,\mathsf{f}),\\
&\leq \overline{N}(r,\infty;\mathcal{F}) + \frac{1}{2}\overline{N}(r,0;\mathcal{F}) + N_2(r,0;\mathcal{F}) + N_2(r,0;L) + S(r,\mathsf{f}) + S(r,L),\\
&\leq \overline{N}(r,\infty;\mathsf{f}) + \overline{N}(r,\infty;\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}) + \frac{1}{2}\overline{N}(r,0;\mathsf{f}^n\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j})\\
&+ 2\overline{N}(r,0;\mathsf{f}) + N(r,0;\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}) + N_2\left(r,0;\mathcal{L}^n\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}\right)\\
&+ S(r,\mathsf{f}) + S(r,\mathcal{L}),\\
&(n-\sigma)T(r,\mathsf{f}) \leq \left(\frac{5+s+2\sigma}{2}\right)T(r,\mathsf{f}) + (2+s)T(r,\mathcal{L}) + S(r,\mathsf{f}) + S(r,\mathcal{L}).
\end{aligned} \tag{20}
$$

Similarly, we obtain

$$
(n-\sigma)T(r,\mathcal{L}) \leq \left(\frac{5+s+2\sigma}{2}\right)T(r,\mathcal{L}) + (2+s)T(r,\mathsf{f}) + S(r,\mathcal{L}) + S(r,\mathsf{f}). \tag{21}
$$

Combining (20) and (21), we get

$$
\left(n - \frac{3s+4\sigma+9}{2}\right)[T(r,f) + T(r,\mathcal{L})] \leq S(r,f) + S(r,\mathcal{L}). \tag{22}
$$

Since $n > \frac{3s+4\sigma+9}{2}$, equation (22) leads to contradiction.

Case 2. Let $\mathcal{H} \equiv 0$. Then the theorem follows from Lemmas 5.7 and 5.8. This completes the proof.

6.3 Proof of Theorem 3.3

Let $\mathcal{F}(z) = \frac{\mathsf{f}^n(z)\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}}{p(z)}$ and $L(z) = \frac{\mathcal{L}^n(z)\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}}{p(z)}$. Then $\mathcal{F}$ and L share $(1,0)$ except for the zeros of $p(z)$.
In this case equation (13) changes to

$$
N_E^{1)}(r,1;\mathcal{F}) \leq N(r,0;\mathcal{H}) \leq N(r,\infty;\mathcal{H}) + S(r,\mathcal{F}) + S(r,L). \tag{23}
$$

Using Lemmas 5.11 - 5.14 and equations (13) and (23), we get

$$\overline{N}(r,1;\mathcal{F}) \leq N_E^{1)}(r,1;\mathcal{F}) + \overline{N}_{\mathscr{L}}(r,1;\mathcal{F}) + \overline{N}_{\mathscr{L}}(r,1;L) + \overline{N}_E^{(2}(r,1;\mathcal{F})$$
$$\leq \overline{N}_*(r,1;\mathcal{F},L) + \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;L|\geq 2) + \overline{N}_{\mathscr{L}}(r,1;\mathcal{F})$$
$$+ \overline{N}_{\mathscr{L}}(r,1;L) + \overline{N}_E^{(2}(r,1;\mathcal{F}) + \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;L') + S(r,\mathsf{f}) + S(r,L),$$
$$\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;L|\geq 2) + 2\overline{N}_{\mathscr{L}}(r,1;\mathcal{F}) + 2\overline{N}_{\mathscr{L}}(r,1;L) + \overline{N}_E^{(2}(r,1;\mathcal{F})$$
$$+ \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;L') + S(r,\mathsf{f}) + S(r,L),$$
$$\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;L|\geq 2) + \overline{N}_{\mathcal{F}>1}(r,1;L) + \overline{N}_{\mathscr{L}>1}(r,1;\mathcal{F}) + \overline{N}_{\mathscr{L}}(r,1;\mathcal{F})$$
$$+ N(r,1;L) - \overline{N}(r,1;L) + \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;L') + S(r,\mathsf{f}) + S(r,L),$$
$$\leq N_2(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + N_2(r,0;L) + N(r,0;L'|L\neq 0) + \overline{N}_0(r,0;L') + S(r,\mathsf{f})$$
$$+ S(r,L),$$
$$\overline{N}(r,1;\mathcal{F}) \leq N_2(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + N_2(r,0;L) + \overline{N}(r,0;L) + \overline{N}_0(r,0;\mathcal{F}')$$
$$+ S(r,\mathsf{f}) + S(r,L). \tag{24}$$

Hence using (24), Lemmas 5.1 and 5.6, we obtain from second fundamental theorem that

$$(n-\sigma)T(r,\mathsf{f}) \leq T(r,\mathcal{F}) + S(r,\mathsf{f}),$$
$$\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) - N_0(r,0;\mathcal{F}') + S(r,\mathsf{f}),$$
$$\leq \overline{N}(r,\infty;\mathcal{F}) + 2N_2(r,0;\mathcal{F}) + N_2(r,0;L) + \overline{N}(r,0;L) + S(r,\mathsf{f}) + S(r,\mathcal{L}),$$
$$\leq \overline{N}(r,\infty;\mathsf{f}) + \overline{N}\left(r,\infty;\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}\right) + 4\overline{N}(r,0;\mathsf{f})$$
$$+ 2N\left(r,0;\prod_{j=1}^{s}\mathsf{f}(z+c_j)^{\mu_j}\right) + N_2\left(r,0;\mathcal{L}^n\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}\right)$$
$$+ \overline{N}\left(r,0;\mathcal{L}^n\prod_{j=1}^{s}\mathcal{L}(z+c_j)^{\mu_j}\right) + S(r,\mathsf{f}) + S(r,\mathcal{L}),$$
$$(n-\sigma)T(r,\mathsf{f}) \leq (4+2s)T(r,\mathsf{f}) + (3+2s)T(r,\mathcal{L}) + S(r,\mathsf{f}) + S(r,\mathcal{L}). \tag{25}$$

Similarly, we obtain

$$(n-\sigma)T(r,\mathcal{L}) \leq (4+2s)T(r,\mathcal{L}) + (3+2s)T(r,\mathsf{f}) + S(r,\mathcal{L}) + S(r,\mathsf{f}). \tag{26}$$

Combining (25) and (26), we get

$$(n-\sigma-4s-7)\left[T(r,\mathsf{f}) + T(r,\mathcal{L})\right] \leq S(r,\mathsf{f}) + S(r,\mathcal{L}). \tag{27}$$

Since $n > 4s+\sigma+7$, equation(27) leads to contradiction.

Case 2. Let $\mathcal{H} \equiv 0$. Then the theorem follows from Lemmas 5.7 and 5.8. This completes the proof.

7 Discussion and Conclusions

Nevanlinna theory has applications in various areas of mathematics, including number theory, differential equations, algebraic geometry, and mathematical physics. It provides powerful tools for understanding the intricate behavior of meromorphic functions and their connections to other mathematical structures. Also Applications of L-functions in various areas of mathematics often involve understanding their analytic properties, which can be studied using techniques from Nevanlinna theory.

In this article, with the help of the Nevanlinna theory and L-functions, we have studied the uniqueness results of two difference monomials generated by two finite order non-constant meromorphic function and L function when they share a non-zero polynomial and obtain some results which extend and improve some earlier results of Harina P. Waghamore and Manjunatha B.E.

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