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A Nonlinear Generalizations of Dynamic Inequalities on Time Scales

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Abstract

Considering the distinctive generalization of the Gronwall-Bellman inequality, in this paper we innovates new dynamical inequalities on time scale. These inequalities will be acted a profound tools for studying qualitative properties such as uniqueness, existence, stability and asymptotic behavior of solutions of dynamic equations on time scale.

Keywords: Dynamic and integral equations, Young's Inequality, Time Scale calculus.

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1 Introduction

Inequalities play pivoted role in the development of mathematics and sciences. Most of the inequalities have been acting as an extensive aid for the study of the qualitative properties of solutions of differential and integral equations. Consequently over the years many such new inequalities have been revealed, which are determined by uncontested applications. For example, see (1; 2; 3; 5; 10; 11; 13; 17) and the references therein.

In the analysis of some dynamic differential and dynamic integral equations, existing inequalities are inadequate. Thus, it is necessary to have some new inequalities in order to study these equations.

In this paper, we have established nonlinear dynamic inequalities on time scales and their applications. We generalized some crucial inequalities by making use of well-known Young's inequality. Here we have used certain analytical techniques to develop nonlinear dynamic inequalities on time scales. Our work is refinements and generality of existing results. Few applications of our are given to find importance of our work. It is an interesting and important techniques study differential and integral equations. Section 2 includes basic definitions, results and preliminaries which are useful for our main results. In Section 3, we are establishing some generalizations and variants of the inequalities established by A. A. El-Deeb (6). In Section 4, we present some applications of our main results. In particular, we discuss the boundedness behavior of solutions of a nonlinear dynamic equation on time scales.

2 Preliminaries

In this section, we are presenting some basic concepts on time scales which are required to set up our main result. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+$ represents the set of positive real numbers, $\mathbb{Z}$ represents the set of integers. $\mathbb{T}$, time scale has a topology that it inherits from standard topology on real numbers $\mathbb{R}$ and $\mathbb{T}_0 = [t_0, \infty) \cap \mathbb{T}$.

Suppose $\phi : \mathbb{T} \to \mathbb{T}$ is defined as

$$\phi(t) := \inf \{x \in \mathbb{T} : x > t\}$$

and ϕ is called as forward jump operator and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as

$$\rho(t) := \sup \{x \in \mathbb{T} : x < t\}.$$

Let inf $\emptyset$ =sup $\mathbb{T}$ and sup $\emptyset$ =inf $\mathbb{T}$, where $\emptyset$ is the empty set. We say that a point $t \in \mathbb{T}$ is left dense if $\rho(t) = t$ and right dense if $\phi(t) = t$, where inf

$\mathbb{T} < t <$ sup $\mathbb{T}$. A function $h : \mathbb{T} \to \mathbb{R}$ is said to be a right dense continuous (rd-continuous) function if h is continuous at right dense points and at left dense points in $\mathbb{T}$, left-sided limit exist and is finite. $C_{rd}(\mathbb{T})$ denotes the set of all such rd-continuous functions.

Let $\mu(t)$ and $\nu(t)$ be forward and backward graininess function on a time scale and are defined as $\mu(t) := \phi(t) - t$ and $\nu(t) = t - \rho(t)$ respectively.

Given a time scale $\mathbb{T}$ we introduce sets $\mathbb{T}^\kappa, \mathbb{T}_\kappa$ and $\mathbb{T}^\kappa_\kappa$ as follows: If $\mathbb{T}$ has left scattered maximum t_1, then $\mathbb{T}^\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has right scattered minimum t_2, then $\mathbb{T}^\kappa = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.Finally $\mathbb{T}^\kappa_\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$. We will give the definition of the generalized exponential function and its derivatives. We say that $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1+\mu(t)p(t) \neq 0$, for all $t \in \mathbb{T}^\kappa$,$\mathfrak{R}$ be set of all regressive and rd-continuous functions. We define the set $\mathfrak{R}^+$ of all positive regressive elements of $\mathfrak{R}$ by $\mathfrak{R}^+ = \{p \in \mathfrak{R} : 1+\mu(t)\rho(t) > 0, \forall t \in \mathbb{T}\}$. If $p \in \mathfrak{R}$, then we define the exponential function by

where $\xi_h(z)$ is the cylinder transformation, which is defined by

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h}, & h \neq 0 \\ z, & h = 0. \end{cases}$$

Lemma 2.1. *[(12), Young's Inequality] If $x \geq 0, y \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then*

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}$$

with equality holds if and only if $x = y$.

Lemma 2.2. *Let $t_0 \in \mathbb{T}^\kappa$ and $k : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ be continuous at (t, t), where $t > t_0$ and $t \in \mathbb{T}^\kappa$. Let $k^\triangle(t, .)$ be rd-continuous on $[t_0, \sigma(t)]$. If for any $\epsilon > 0$, there exist a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$|[k(\sigma(t), \tau) - k(s, \tau)] - k^\triangle(t, \tau)[\sigma(t), s]| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$.

If $k^\triangle$ represent the derivative of k with respect to the first variable, then

$$f(t) = \int_{t_0}^{t} k(t, \tau) \triangle\tau,$$

yields

$$f^\triangle(t) = \int_{t_0}^{t} k^\triangle(t, \tau)\triangle\tau + k(\sigma(t), \tau).$$

Note 2.1. *If* $\mathbb{T}$=$\mathbb{R}$, *then*

$$\phi(t) = t, \mu(t) = 0, f^{\triangle}(t) = f^{'}(t), \int_a^b f(t)\triangle t = \int_a^b f(t)dt \tag{1}$$

and if $\mathbb{T} = \mathbb{Z}$, *then*

$$\phi(t) = t+1, \mu(t) = 1, f^{\triangle}(t) = \triangle f(t), \int_a^b f(t)\triangle t = \sum_{t=a}^{b-1} f(t)dt. \tag{2}$$

Lemma 2.3. *Let* $u, b \in C_{rd}$ *and* $a \in \mathfrak{R}^+$. *Then*

$$u^{\triangle}(t) \leq a(t)u(t) + b(t), t \geq t_0, t \in \mathbb{T}^k,$$

yields

$$u(t) \leq u(t_0)e_a(t, t_0) + \int_{t_0}^{t} e_a(t, \sigma(\tau))b(\tau)\triangle\tau, t \geq t_0, t \in \mathbb{T}^k.$$

Lemma 2.4. *If p,q,*$\in \mathfrak{R}$ *and* $a, b, c \in \mathbb{T}$*, then*

1. $e_p(t, t_0) = 1$ *and* $e_0(t, s) = 1$.
2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$.

If $\mathbb{T} = \mathbb{R}$*, then*

$$e_a(t, t_0) = exp\left(\int_{t_0}^{t} a(s)ds\right) \tag{3}$$

If $\mathbb{T} = \mathbb{Z}$*, then*

$$e_a(t, t_0) = \prod_{s=t_0}^{t-1} (1 + a(s)). \tag{4}$$

3 Main Results

In this section, we state and prove inequalities

Theorem 3.1. *Suppose that* $u(t), a(t), b(t), x(t), y(t) \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ *and* $h(t, s)$, $h^{\triangle}(t, s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}_+)$. *Let* l, m, n *be real numbers such that* $l > m > 0$, $l > n > 0$. *If* $u^l(t)$ *satisfies the inequality*

$$u^l(t) \leq a(t) + b(t)\int_{t_0}^{t} [x(s)u^m(s) + y(s)u^n(s) + h(t, s)]\triangle s, \tag{5}$$

for all $t \in \mathbb{T}_0$, then

$$u(t) \leq [a(t) + b(t) \int_{t_0}^{t} e_\mu(t, \phi(s))\rho(s)\triangle s]^{\frac{1}{l}}, \tag{6}$$

for all $t \in \mathbb{T}_0$, where μ and ρ are

$$\mu(t) = b(t)\left(\frac{m}{l}x(t) + \frac{n}{l}y(t)\right) \tag{7}$$

and

$$\begin{aligned}\rho(t) =& x(t)\left[\frac{m}{l}a(t) + \frac{l-m}{l}\right] + y(t)\left[\frac{l-n}{l} + \frac{n}{l}a(t)\right] + \\ & h(\phi(t), t) + \int_{t_0}^{t} h^{\triangle}(t,s)\triangle s.\end{aligned} \tag{8}$$

Proof. Define a function $\tau(t)$ as

$$\tau(t) = \int_{t_0}^{t} [x(s)u^m(s) + y(s)u^n(s) + h(t,s)]\triangle s, \tag{9}$$

for all t $\in \mathbb{T}_0$. Then we have $\tau(t) \geq 0$ non decreasing function on $\mathbb{T}_0$ with $\tau(t_0) = 0$ and (5) can be written as

$$u^l(t) \leq a(t) + b(t)\tau(t), \tag{10}$$

for all $t \in \mathbb{T}_0$. From (10) and using lemma 2.1, we find

$$u^m(t) \leq \frac{l-m}{l} + \frac{m}{l}a(t) + \frac{m}{l}b(t)\tau(t), \tag{11}$$

for all $t \in \mathbb{T}_0$. Correspondingly, we get

$$u^n(t) \leq \frac{l-n}{l} + \frac{n}{l}a(t) + \frac{n}{l}b(t)\tau(t), \tag{12}$$

for all $t \in \mathbb{T}_0$. Now differentiating (9), we find

$$\tau^{\triangle}(t) = x(t)u^m(t) + y(t)u^n(t) + h(\phi(t), t) + \int_{t_0}^{t} h^{\triangle}(t,s)\triangle s, \tag{13}$$

for all $t \in \mathbb{T}_0$.

Now applying lemma 2.2 and using inequalities (11), (12), we obtain

$$\begin{aligned}\tau^{\triangle}(t) \leq & x(t)\left[\frac{l-m}{l} + \frac{m}{l}a(t) + \frac{m}{l}b(t)\tau(t)\right] + \\ & y(t)\left[\frac{l-n}{l} + \frac{n}{l}a(t) + \frac{n}{l}b(t)\tau(t)\right] + h(\phi(t), t) + \int_{t_0}^{t} h^{\triangle}(t,s)\triangle s,\end{aligned}$$

$$\begin{aligned}\tau^{\triangle}(t) \leq & b(t)\left[\frac{m}{l}x(t) + \frac{n}{l}y(t)\right]\tau(t) + x(t)\left[\frac{m}{l}a(t) + \frac{l-m}{l}\right] + \\ & y(t)\left[\frac{l-n}{l} + \frac{n}{l}a(t)\right] + h(\sigma(t), t) + \int_{t_0}^{t} h^{\triangle}(t,s)\triangle s,\end{aligned} \tag{14}$$

$$\tau^{\triangle}(t) \leq \tau(t)\mu(t) + \nu(t). \tag{15}$$

for all $t \in \mathbb{T}_0$, where $\mu(t)$ and $\rho(t)$ are given by (3.3) and (3.4) respectively. Now a suitable application of lemma 2.3 to (15) with $\tau(t_0) = 0$, we have

$$\tau(t) \leq \int_{t_0}^{t} e_{\mu}(t, \phi(s))\rho(s)\triangle s. \tag{16}$$

Finally, (6) follows from (10) and (16). □

Remark 3.1. *If $\mathbb{T} = \mathbb{R}$ and $h(t,s) = 0$ in theorem 5 and using relation (1) and (3), then we obtain [(14), theorem 1 part(a_1)].*

Remark 3.2. *If $\mathbb{T} = \mathbb{Z}$ and $h(t,s) = 0$ in theorem 5 and using relation (2) and (4), then we obtain discrete analogue [(14), theorem 3 part(c_1)].*

Remark 3.3. *If $m = l$ and $n = 1$, then the theorem 5 reduces to the theorem 3.4 of (6).*

Corolary 3.1. *Let $u(t), x(t), y(t), b(t), h(t,s), h^{\triangle}(t,s), l, m, n$ be defined as same as in theorem 5 and $c \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ be a non decresing function on $\mathbb{T}_0$. If $u^l(t)$ satisfies the inequality*

$$u^l(t) \leq c^l(t) + b(t)\int_{t_0}^{t}[x(s)u^m(s) + y(s)u^n(s) + h(t,s)]\triangle s, \tag{17}$$

for all $t \in \mathbb{T}_0$, then

$$u(t) \leq c(t)[1 + b(t)\int_{t_0}^{t} e_{\mu_1}(t, \phi(s))\rho_1(s)\triangle s]^{\frac{1}{l}}, \tag{18}$$

for all $t \in \mathbb{T}_0$, *where* $\mu_1(t)$ *and* $\rho_1(t)$ *are defined as*

$$\mu_1(t) = b(t)\Big[\frac{m\, c^{m-l}(t)x(t)}{l} + \frac{nc^{n-l}(t)y(s)}{l}\Big] \tag{19}$$

and

$$\begin{aligned}\rho_1(t) =& c^{m-l}(t)x(t)\Big[\frac{m}{l} + \frac{m-l}{l}\Big] + c^{n-l}(t)y(t)\Big[\frac{l-n}{l} + \frac{n}{l}\Big] + \\ & h(\phi(t), t)c^{-l}(t) + \int_{t_0}^{t} h^{\triangle}(t,s)c^{-l}(s)\triangle s.\end{aligned} \tag{20}$$

Proof. Since $c \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ is non decreasing function, from (17) we find,

$$\begin{aligned}\Big[\frac{u(t)}{c(t)}\Big]^l \leq & 1 + b(t)\int_{t_0}^{t}\Big[c^{m-l}(t)x(s)\Big(\frac{u(t)}{c(t)}\Big)^m \\ & + y(s)c^{n-l}(s)\Big(\frac{u(t)}{c(t)}\Big)^n + c^{-l}(s)h(t,s)\Big]\triangle s,\end{aligned} \tag{21}$$

for all $t \in \mathbb{T}_0$. By the theorem 5, we deduce that

$$u(t) \leq c(t)[1 + b(t)\int_{t_0}^{t} e_{\mu_1}(t, \phi(s))\rho_1(s)\triangle s]^{\frac{1}{l}}, \tag{22}$$

where μ_1 and ρ_1 are given by (19) and (20). This completes the proof $\square$

Theorem 3.2. *Let* $u(t), x(t), y(t), a(t), b(t), h(t,s), h^{\triangle}(t,s), l, m, n$ *be defined as same as in theorem 5 and* $d \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$. *If* $u^l(t)$ *satisfies the inequality*

$$u^l(t) \leq a(t) + b(t)\int_{t_0}^{t} h(t,s)\big[x(s)u^m(s) + y(s)u^n(s) + d(s)\big]\triangle s, \tag{23}$$

for all $t \in \mathbb{T}_0$, *then*

$$u(t) \leq \big[a(t) + b(t)\int_{t_0}^{t} e_{\mu_2}(t, \phi(s))\rho_2(s)\triangle s\big]^{\frac{1}{l}}, \tag{24}$$

for all $t \in \mathbb{T}_0$ *where* μ_2 *and* ρ_2 *are defined as*

$$\begin{aligned}\mu_2(t) =& h(\phi(t), t)\Big[b(t)\Big(\frac{m}{l}x(t) + \frac{n}{l}y(t)\Big)\Big] + \\ & \int_{t_0}^{t} h^{\triangle}(t,s)b(s)\left(\frac{m}{l}x(s) + \frac{n}{l}y(s)\right)\triangle s\end{aligned} \tag{25}$$

and

$$\rho_2(t) = h(\phi(t),t)\Big[x(t)\Big(\frac{l-m}{l} + \frac{m}{l}a(t)\Big) + y(t)\Big(\frac{l-n}{l} + \frac{n}{l}a(t)\Big) + d(t)\Big] + \int_{t_0}^{t} k^{\triangle}(t,s)\left[x(s)\Big(\frac{l-m}{l} + \frac{m}{l}a(s)\Big) + y(s)\Big(\frac{l-n}{l} + \frac{n}{l}a(s)\Big) + d(s)\right]\triangle s, \tag{26}$$

for all $t \in \mathbb{T}_0$

Proof. Define a function $\tau_1(t)$ as

$$\tau_1(t) = \int_{t_0}^{t} h(t,s)\big[x(s)u^m(s) + y(s)u^n(s) + d(s)\big]\triangle s, \tag{27}$$

for all $t \in \mathbb{T}_0$. Then $\tau_1(t_0) = 0$ and $\tau_1(t) \geq 0$. Then as in the proof of theorem (5) from (23) we see that inequalities (11) and (12) holds for τ_1. On differentiating (27) and applying lemma 2.2. we obtain

$$\tau_1^{\triangle}(t) = h(\phi(t),t)\big[x(t)u^m(t) + y(t)u^n(t) + d(t)\big] + \int_{t_0}^{t} h^{\triangle}(t,s)\big[x(s)u^m(s) + y(s)u^r(s) + d(s)\big]\triangle s, \tag{28}$$

for all $t \in \mathbb{T}_0$. From lemma (2.2) and using (11) and (12) in (28), we obtain,

$$\begin{aligned}\tau_1^{\triangle}(t) \leq & h(\phi(t),t)\Big[x(t)\Big(\frac{l-m}{l} + \frac{m}{l}a(t) + \frac{m}{l}b(t)\tau_1(t)\Big) \\ & + y(t)\Big(\frac{l-n}{l} + \frac{n}{l}a(t) + \frac{n}{l}b(t)\tau_1(t)\Big) + d(t)\Big] + \\ & \int_{t_0}^{t} h^{\triangle}(t,s)\Big[x(s)\Big(\frac{l-m}{l} + \frac{m}{l}a(s) + \frac{m}{l}b(s)\tau_1(s)\Big) + \\ & y(s)\Big(\frac{l-n}{l} + \frac{n}{l}a(s) + \frac{n}{l}b(s)\tau_1(s)\Big) + d(s)\Big]\triangle s.\end{aligned} \tag{29}$$

The inequality (29) gives

$$
\begin{aligned}
{\tau_1}^{\triangle}(t) \leq & \Bigg[h(\phi(t),t)\Bigg[b(t)\Bigg(\frac{m}{l}x(t)+\frac{n}{l}y(t)\Bigg)\Bigg] \\
& + \int_{t_0}^{t} h^{\triangle}(t,s)b(s)\Bigg(\frac{m}{l}x(s)+\frac{n}{l}y(s)\Bigg)\triangle s\Bigg]\nu_1(t) \\
& + h(\phi(t),t)\Bigg[x(t)\Bigg(\frac{l-m}{l}+\frac{m}{l}a(t)\Bigg) \\
& + y(t)\Bigg(\frac{l-n}{l}+\frac{n}{l}a(t)\Bigg)+d(t)\Bigg]+ \\
& \int_{t_0}^{t} h^{\triangle}(t,s)\Bigg[x(s)\Bigg(\frac{l-m}{l}+\frac{m}{l}a(s)\Bigg)+ \\
& y(s)\Bigg(\frac{l-n}{l}+\frac{n}{l}a(s)\Bigg)+d(s)\Bigg]\triangle s.
\end{aligned}
\tag{30}
$$

$$
\tau_1^{\triangle}(t) \leq \mu_2(t)\tau_1(t)+\rho_2(t) \tag{31}
$$

for all $t \in \mathbb{T}_0$ where $\mu_2(t)$ and $\nu_2(t)$ are given by (25) and (26) respectively. By (31), lemma 2.3 and with $\tau_1(t_0)=0$, we obtain

$$
\tau_1(t) \leq \int_{t_0}^{t} e_{\mu_2}(t,\phi(s))\rho_2(s)\triangle s, \tag{32}
$$

for all $t \in \mathbb{T}_0$.
Making use of the (32) in $u^l(t) \leq a(t)+b(t)\tau_1(t)$, we get the inequality (24). □

Corolary 3.2. *Let $u(t), x(t), y(t), a(t), d(t), b(t), h(t,s), h^{\triangle}(t,s), l, m, n$ be defined as same as in theorem 6 and $c(t)$ be defined as same as in corollaty 5. If*

$$
u^l(t) \leq c^l(t)+b(t)\int_{t_0}^{t} h(t,s)[x(s)u^m(s)+y(s)u^n(s)+d(s)]\triangle s \tag{33}
$$

for all $t \in \mathbb{T}_0$, then

$$
u(t) \leq c(t)[1+b(t)\int_{t_0}^{t} e_{\mu_3}(t,\phi(s))\rho_3(s)\triangle s]^{\frac{1}{l}}, \tag{34}
$$

for all $t \in \mathbb{T}_0$, where μ_3 and ρ_3 are defined as

$$
\begin{aligned}
\mu_3(t) = & h(\phi(t),t)\Bigg[b(t)\Bigg(\frac{mc^{m-l}x(t)}{l}+\frac{nc^{n-l}y(t)}{l}\Bigg)\Bigg]+ \\
& \int_{t_0}^{t} h^{\triangle}(t,s)b(s)\Bigg(\frac{mc^{m-l}}{l}b(s)+\frac{nc^{n-l}}{l}y(s)\Bigg)\triangle s
\end{aligned}
\tag{35}
$$

and

$$\begin{aligned}\nu_3(t) =&h(\phi(t),t)\Big[b(t)c^{m-l}(t)\Big(\frac{l-m}{l}+\frac{m}{l}\Big)+\\ &y(t)c^{n-l}(t)\Big(\frac{l-n}{l}+\frac{n}{l}\Big)+d(t)\Big]+\\ &\int_{t_0}^{t}h^{\triangle}(t,s)\Big[x(s)c^{m-l}(t)\Big(\frac{l-m}{l}+\frac{m}{l}\Big)+\\ &y(s)c^{n-l}(t)\Big(\frac{l-n}{l}+\frac{n}{l}\Big)+d(s)c^{-l}(t)\Big]\triangle s,\end{aligned}$$

for all $t\in\mathbb{T}_0$

Proof. Since $c\in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$ is non decreasing function, from (17), we write

$$\begin{aligned}\left[\frac{u(t)}{c(t)}\right]^l \leq&1+b(t)h(t,s)\int_{t_0}^{t}\Big[c^{m-l}(t)b(s)\Big(\frac{u(t)}{c(t)}\Big)^m+\\ &y(s)c^{n-l}(s)\Big(\frac{u(t)}{c(t)}\Big)^n+c^{-l}(s)d(s)\Big]\triangle s,\end{aligned}\tag{36}$$

for all $t\in\mathbb{T}_0$. By theorem (6), we have

$$\frac{u(t)}{c(t)}\leq[1+b(t)\int_{t_0}^{t}e_{\mu_3}(t,\phi(s))\rho_3(s)\triangle s]^{\frac{1}{l}},\tag{37}$$

where, $\mu_3(t)$ and $\nu_3(t)$ are given by (35) and (**??**) respectively for all $t\in\mathbb{T}_0$. This completes the proof □

Theorem 3.3. *Let* $u(t),x(t),y(t),a(t),b(t),h(t,s),h^{\triangle}(t,s)$ *be defined as same as in theorem 5,* l,m *be constant such that* $l>m>0$ *and* $d\in C_{rd}(\mathbb{T}_0\times\mathbb{R}_+,\mathbb{R}_+)$*, such that*

$$0\leq d(t,\alpha)-d(t,\beta)\leq\delta(t,\beta)(\alpha-\beta)\tag{38}$$

for all $t\in\mathbb{T}_0$ *and* $\alpha\geq0,\beta\geq0$*, where* $\delta\in C_{rd}(\mathbb{T}_0\times\mathbb{R}_+,\mathbb{R}_+)$*. If*

$$u^l(t)\leq a(t)+b(t)\int_{t_0}^{t}\big[d(s,u^m(s))+h(t,s)\big]\triangle s,\tag{39}$$

for all $t\in\mathbb{T}_0$*, then*

$$u(t)\leq\big[a(t)+b(t)\int_{t_0}^{t}e_{\mu_3}(t,\phi(s)),\rho_3(s)\triangle s\big]^{\frac{1}{l}},\tag{40}$$

for all $t \in \mathbb{T}_0$, *where* $\mu_3(t)$ *and* $\rho_3(t)$ *are defined as*

$$\mu_3(t) = \delta\bigg(t, \frac{l-m}{l} + \frac{m}{l}a(t)\bigg)\frac{m}{l}b(t) \tag{41}$$

and

$$\rho_3(t) = d\bigg(t, \frac{l-m}{l} + \frac{m}{l}a(t)\bigg) + h(\phi(t), t) + \int_{t_0}^{t} h^{\triangle}(t,s)\triangle s. \tag{42}$$

Proof. Suppose that

$$\tau_2(t) = \int_{t_0}^{t} \big(d(s, u^m(s)) + h(t,s)\big)\triangle s, \tag{43}$$

for all $t \in \mathbb{T}_0$. Then from (39), we get

$$u(t) \le \big[a(t) + b(t)\tau_2(t)\big]^{\frac{1}{l}},$$

$$u^m(t) \le \big[a(t) + b(t)\tau_2(t)\big]^{\frac{m}{l}}(1)^{\frac{l-m}{l}}. \tag{44}$$

By lemma 2.1 and the inequality (44), we obtain

$$u^m(t) \le \bigg[\frac{l-m}{l} + \frac{m}{l}a(t) + \frac{m}{l}b(t)\tau_2(t)\bigg] \tag{45}$$

From (43), (45), (39) and lemma 2.2, we deduce that

$$\begin{aligned}\tau_2{}^{\triangle}(t) \le \delta\bigg(t, \frac{l-m}{l} + \frac{m}{l}a(t)\bigg)\frac{m}{l}b(t)\tau_2(t) +\\ d\bigg(t, \frac{l-m}{l} + \frac{m}{l}a(t)\bigg) + h(\phi(t), t) + \int_{t_0}^{t} h^{\triangle}(t,s)\triangle s,\end{aligned}$$

$$\tau_2{}^{\triangle}(t) \le \mu_3(t)\tau_2(t) + \rho_3(t), \tag{46}$$

for all $t \in \mathbb{T}_0$ where $\mu_3(t)$ and $\rho_3(t)$ are given by (41) and (42) respectively. From (46) and lemma 2.3 with $\tau_2(t_0) = 0$, we obtain

$$\tau_2(t) = \int_{t_0}^{t} e_{\mu_3}(t, \phi(s))\rho_3(s)\triangle s, \tag{47}$$

for all $t \in \mathbb{T}_0$. Using (47) in $u^l(t) \le a(t) + b(t)\tau_2(t)$, we obtain (40). $\square$

Theorem 3.4. *Let $u(t), a(t), b(t), h(t,s), h^{\triangle}(t,s)$ be defined as same as in theorem 5, l, m be constants such that $l > m > 0$ and $d \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}_+, \mathbb{R}_+)$ and $\lambda \in C(\mathbb{T}_0, \mathbb{R}_+)$ be a strictly increasing function with $\lambda(t_0) = 0$ such that*

$$0 \leq d(t,\alpha) - d(t,\beta) \leq \delta(t,\beta)\lambda^{-1}(\alpha - \beta), \tag{48}$$

for all $t \in \mathbb{T}_0$ and $\alpha, \beta \geq 0$ where $\delta \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}_+, \mathbb{R}_+)$ and

$$\lambda^{-1}(\alpha\beta) = \lambda^{-1}(\alpha)\lambda^{-1}(\beta), \tag{49}$$

for all $\alpha, \beta \in \mathbb{T}_0$ where λ^{-1} is the inverse of λ. If $u^l(t)$ satisfies inequality

$$u^l(t) \leq a(t) + b(t)\phi \int_{t_0}^{t} \big[d(s, u^m(s)) + h(t,s)\big] \triangle s, \tag{50}$$

for all $t \in \mathbb{T}_0$, then

$$u(t) \leq a(t) + b(t) \int_{t_0}^{t} e_{\mu_5}(t, \phi(s))\rho_5(s) \triangle s, \tag{51}$$

for all $t \in \mathbb{T}_0$, where $\mu_5(t)$ and $\rho_5(t)$ are defined as,

$$\mu_5(t) = \delta\bigg(t, \frac{l-m}{l} + \frac{m}{l}a(t)\bigg)\lambda^{-1}\bigg(\frac{nb(t)}{l}\bigg) \tag{52}$$

and

$$\rho_5(t) = d\bigg(t, \frac{l-m}{l} + \frac{m}{l}a(t)\bigg) + h(\phi(t), t) + \int_{t_0}^{t} h^{\triangle}(t,s) \triangle s. \tag{53}$$

Proof. Let

$$\tau_2(t) = \int_{t_0}^{t} \big[d(s, u^m(s)) + h(t,s)\big] \triangle s. \tag{54}$$

Then by lemma 2.1, we deduce that

$$u^l(t) \leq a(t) + b(t)\lambda(\tau_2(t)), \tag{55}$$

$$u^m(t) \leq \bigg[\frac{l-m}{l} + \frac{ma(t)}{l} + \frac{mb(t)}{l}\lambda(\tau_2(t))\bigg], \tag{56}$$

$$\tau_2{}^{\triangle}(t) = d(t, u^m(t)) + h(\phi(t), t) + \int_{t_0}^{t} h^{\triangle}(t, s)\triangle s, \tag{57}$$

$$\begin{aligned}\tau_2{}^{\triangle}(t) \leq & d\left(t, \frac{l-m}{l} + \frac{ma(t)}{l} + \frac{mb(t)}{l}\lambda(\tau_2(t))\right) + \\ & h(\phi(t), t) + \int_{t_0}^{t} h^{\triangle}(t, s)\triangle s,\end{aligned}$$

$$\begin{aligned}\tau_2{}^{\triangle}(t) \leq\ & d\left(t, \frac{l-m}{l} + \frac{ma(t)}{l} + \frac{mb(t)}{l}\lambda(\tau_2(t))\right) - \\ & d\left(t, \frac{l-m}{l} + \frac{ma(t)}{l}\right) + d\left(t, \frac{l-m}{l} + \frac{ma(t)}{l}\right) \\ & + h(\phi(t), t) + \int_{t_0}^{t} h^{\triangle}(t, s)\triangle s,\end{aligned}$$

$$\begin{aligned}\tau_2{}^{\triangle}(t) \leq\ & \delta\left(t, \frac{l-m}{l} + \frac{m}{l}a(t)\right)\lambda^{-1}\left(\frac{m}{l}b(t)\lambda(\tau_2(t))\right) + \\ & d\left(t, \frac{l-m}{l} + \frac{m}{l}a(t)\right) + h(\phi(t), t) + \int_{t_0}^{t} h^{\triangle}(t, s)\triangle s\end{aligned}$$

$$\begin{aligned}\tau_2{}^{\triangle}(t) \leq\ & \delta\left(t, \frac{l-m}{l} + \frac{m}{l}a(t)\right)\lambda^{-1}\left(\frac{m}{l}a(t)\right)\lambda(\tau_2(t)) + \\ & d\left(t, \frac{l-m}{l} + \frac{m}{l}a(t)\right) + h(\phi(t), t) + \int_{t_0}^{t} h^{\triangle}(t, s)\triangle s,\end{aligned}$$

$$\tau_2{}^{\triangle}(t) \leq \mu_5(t)\tau_2(t) + \rho_5(t), \tag{58}$$

for all $t \in \mathbb{T}_0$
where μ_5 and τ_5 are given by (52) and (53) respectively.
From the inequality (49)and lemma (2.3), we get

$$\tau_2(t) \leq \int_{t_0}^{t} e_{\mu_5(t)}(t, \phi(s))\rho_5(s)\triangle s, \tag{59}$$

for all $t \in \mathbb{T}_0$
The inequality(51) follows from (55) and (59). □

4 Applications

In this section, we present some applications of our main results. In particular, we discuss the boundedness behavior of solutions of a nonlinear dynamic equation on time scales.

Example 4.1. *Consider the following dynamic integral equation:*

$$u^l(t) = \Omega(t, \int_{t_0}^{t} \Gamma(s, u(s), h(t,s))\triangle s, u(t_0) = a(t_0) = 0, \tag{60}$$

for all $t \in \mathbb{T}_0$, where $\Omega \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}, \mathbb{R})$ and $\Gamma \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^2, \mathbb{R})$ satisfying

$$|\Omega(t,j)| \leq a(t) + b(t)|j|, \tag{61}$$

and

$$|\Gamma(t,f,g)| \leq d(s, |f^m|) + g, \tag{62}$$

where $a, b, d \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$

$$u(t) \leq [a(t) + b(t)\int_{t_0}^{t} e_{\mu_3}(t, \phi(s))\rho_3(s)\triangle s]^{\frac{1}{l}}, \tag{63}$$

for all $t \in \mathbb{T}_0$, where l, m, μ_3, ρ_3 are given by the theorem 3.3. From (60) and by the assumption (61) and (62), we have

$$|u^l(t)| = |\Omega(t, \int_{t_0}^{t} \Gamma(s, u(s), h(t,s))\triangle s|,$$
$$u^l(t) \leq \Omega(t, \int_{t_0}^{t} |\Gamma(s, u(s), h(t,s))\triangle s|),$$

$$|u^l(t)| \leq |a(t)| + |b(t)| \int_{t_0}^{t} [|d(s, u^m(s))| + h(t,s)]\triangle s, \tag{64}$$

for all $t \in \mathbb{T}_0$. By theorem 3.3 and (64) yields

$$u(t) \leq [a(t) + b(t)\int_{t_0}^{t} \mu_3(t, \phi(s))\rho_3(s)\triangle s]^{\frac{1}{l}}. \tag{65}$$

Note that each function of the right hand side of (65) is known. Therefore if t is fix, then solution $u(t)$ is bounded.

Example 4.2. *Consider the following dynamic equation on time scales:*

$$(j^l)^{\triangle}(t) = \Theta(t, j(t), h(t,s)), t \in \mathbb{T}_0, \tag{66}$$

with initial condition $j(t_0) = C^{\frac{1}{l}}$, *where* $\Theta \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^3, \mathbb{R})$.

Theorem 4.1. *Assume that*

$$|\Theta(t, j(t), h(t,s))| \leq h(t,s)[x(t)j^m(t) + y(t)j^n(t) + d(t)] \tag{67}$$

where $j, x, y, d \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+), h(t,s), h^{\triangle}(t,s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}_+)$. *If* j *is a solution of dynamic equation (66), then*

$$j(t) \leq [C + \int_{t_0}^{t} e_{\mu_2}(t, \phi(s))\rho_2(s)\triangle s]^{\frac{1}{l}}, \tag{68}$$

for all $t \in \mathbb{T}_0$, *where* l, m, n, μ_2 *and* ρ_2 *are given by theorem 3.2.*

Proof. Clearly, the solution j of dynamic equation (66) with initial condition $j(t_0) = C^{\frac{1}{l}}$, satisfies the equivalent dynamic integral equation on time scales

$$j^l(t) = C + \int_{t_0}^{t} \Theta\big(s, j(s), h(t,s)\big)\triangle s, \tag{69}$$

for all $t \in \mathbb{T}_0$ with initial condition $j(t_0) = C^{\frac{1}{l}}$. In fact from (69) and by using assumption (67), we have

$$|j^l(t)| = |C + \int_{t_0}^{t} \Theta\big(s, j(s), h(t,s)\big)\triangle s|,$$

$$|j^l(t)| \leq |C| + \int_{t_0}^{t} |\Theta\big(s, j(s), h(t,s)\big)\triangle s|,$$

$$|j^l(t)| \leq |C| + \int_{t_0}^{t} h(t,s)\big[x(s)|j^m(s)| + d(s)|j^n(s)| + d(s)\big]|, \tag{70}$$

with initial condition $j(t_0) = C^{\frac{1}{l}}$. Then a suitable application of Theorem 3.2 with $a(t) = C$ and $b(t) = 1$ to (70), gives (68). □

Example 4.3. *Consider the following dynamic integral equation on time scales:*

$$j^l(t) = \Phi\left(t, j(t), \int_{t_0}^{t} \Psi(s, j(s))\triangle s\right), t \in \mathbb{T}_0 \tag{71}$$

where $\Phi \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^2, \mathbb{R})$ *and* $\Psi \in C_{rd}(\mathbb{T}_0 \times \mathbb{R})$.

Theorem 4.2. *Assume that*

$$\left|\Phi(t)\left(t, j(t), \int_{t_0}^{t} \Psi(s, j(s))\triangle s\right)\right| \leq a(t) + b(t)\left|\int_{t_0}^{t} \Psi(s, j(s))\right|, \tag{72}$$

and

$$|\Psi(t, j(t))| \leq a(t)j^m(t) + y(t)j^n(t) + h(t,s), \tag{73}$$

where $j, x, y, a, b \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+), h(t,s), h^{\triangle}(t,s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}_+)$. *If j is solution of dynamic equation (71), then*

$$j(t) \leq \left[a(t) + b(t)\int_{t_0}^{t} e_{\mu}(t, \phi(s))\rho(s)\triangle s\right]^{\frac{1}{l}}, \tag{74}$$

for all $t \in \mathbb{T}_0$*, where* l, m, n, μ *and* $\rho(t)$ *are given by theorem 5.*

Proof. In fact from (71) and by using assumption (72) and (73) we have

$$|j^l(t)| = \left|\Phi\left(t, j(t), \int_{t_0}^{t} \Psi(s, j(s))\triangle s\right)\right|$$

$$|j^l(t)| \leq a(t) + b(t)\left|\int_{t_0}^{t} \psi(s, j(s))\triangle s\right|$$

$$|j^l(t)| \leq a(t) + b(t)\int_{t_0}^{t} |\psi(s, j(s))|\triangle s$$

$$|j^l(t)| \leq a(t) + b(t)\int_{t_0}^{t}\left[x(s)|j^m(s)| + y(s)|j^n(s)| + h(t,s)\right]\triangle s. \tag{75}$$

An application of theorem 5 to (75) we get (74). □

5 Conclusion

In this paper, we begin with review of nonlinear dynamic inequalities on time scales and their applications. We generalize nonlinear dynamic inequalities using Young's inequality and lemma 2.3. Our work includes several nonlinear dynamic inequalities on time scales . we have obtained new nonlinear dynamic inequalities. Our results are refinements and generality of some existing results in (6). Many existing results in the literature are reduced to the special case of our results when $m = n = 1$. Some applications to dynamic equations are provided. The findings of this study can be utilized in the analysis of dynamic equations. It is an interesting and new problems that upcoming researchers can develop similar inequalities for in their future research work, about differential and integral equations.

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