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Existence and Uniqueness of Solutions for System of Nonlinear Fractional Reaction-Diffusion Equations

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Abstract

In this paper, to develop monotone method for non-linear system of Riemann-Liouville (R-L) reaction-diffusion equations with initial and boundary conditions. The monotone method yields monotone sequences which converges uniformly and monotonically to minimal and maximal solutions.To investigate the existence and uniqueness of solutions by monotone method for R-L reaction-diffusion equations with initial and boundary conditions.

Keywords: Riemann-Liouville fractional derivative; eigenfunction; lower and upper solutions; monotone method.
2020 AMS subject classifications: 26A33, 26A48, 34A08. [1]

1 Introduction

Now a days, system of fractional reaction-diffusion equations is a leading topic in the area of fractional calculus. It is observed that the solution of non-linear reaction-diffusion equation is rarely possible. The system of fractional differential equations have many applications in the field of Physics, Chemistry, Biochemistry [Hilfer, 2000, A.A. Kilbas and Rogosin, 2014, A.A.Kilbas and Trujillo, 2006]. However the most of the natural phenomenon are nonlinear in nature.

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These situation are captured by nonlinear system of fractional reaction-diffusion equations. The leading researchers have studied the existence and uniqueness of solution of the fractional differential equations using fixed point theorem [Chhetri and Vatsala, 2017, 2018, D.B. Dhaigude and Nikam, 2012, Hilfer, 2000, Kilbas and Marzan, 2004, Lakshmikantham, 2008, Nanware and Dhaigude, 2012, Pao, 1992, Vatsala, 2015]. Also the monotone method combined with the lower and upper solution for system of R-L reaction-diffusion equations with initial and boundary conditions are discussed in [Denton and Vatsala, 2011, Z. Denton and Vatsala, 2011, V.Lakshmikantham and Leela, 2009, Vatsala, 2012b,a]. There are different kind of problems for which monotone method has been developed in [Lakshmikantham and Vatsala, 2012, Nanware and Kundgar, 2021, Nanware and Dhaigude, 2015, 2012, 2018, Jadhav and Dhaigude, 2014, 2017, 2020, Ramirez and Vatsala, 2012].

In this paper, we develop monotone method for the non-linear system of R-L reaction diffusion-equations using lower and upper solutions. We obtain integral representation form for the solution of linear system of R-L reaction-diffusion equations using eigen function expansion method and Green's theorem.The comparison results are obtained for system of R-L fractional reaction-diffusion equations. These sequences of solutions for linear fractional differential equations which converges uniformly and monotonically to minimal and maximal solutions of the non-linear system of fractional diffusion equations. We successfully apply monotone method to obtain existence and uniqueness of solutions of the problem.

The paper is arranged in the follows.
In section 2, definitions and basic results are discussed that plays vital role in the main results. The comparison results are obtained in section 3. These results are used to develop monotone method for the problem under study. The obtained results are given in section 4. At the end discussion and conclusion section is added.

2 Preliminaries

In this section, we introduce definitions and preliminary results which are used in main results.

Definition 2.1. *([Chhetri and Vatsala, 2018]) The R-L fractional integral of $u(t)$ of order q is defined as*

$$D_t^{-q}u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} u(\tau) d\tau, \quad 0 < q \leq 1.$$

Definition 2.2. *([Chhetri and Vatsala, 2018]) The R-L (left-sided) fractional derivative of $u(t)$ is defined as*

$$D_t^q u(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-\tau)^{q-1}u(\tau)d\tau, t>0, \quad 0<q\leq 1.$$

Definition 2.3. *([Chhetri and Vatsala, 2018]) The Mittag-Leffler function with two parameters α and β, denoted by $E_{\alpha,\beta}$ is defined as,*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha, \beta > 0.$$

Definition 2.4. *([Chhetri and Vatsala, 2018]) A function $h(t) \in C(J, \mathbb{R})$ is a C_p continuous function, if $t^{1-q}h(t) \in C(J_0, \mathbb{R})$ where $p = 1-q, J = (0, T]$ and $J_0 = [0, T]$. The set of C_p continuous functions is denoted by $C_p(J, \mathbb{R})$. Further, given a function $h(t) \in C_p(J, \mathbb{R})$, we call the function $t^{1-q}h(t)$, the continuous extension of $h(t)$.*

Consider the nonhomogeneous linear R-L fractional differential equation [A.A.Kilbas and Trujillo, 2006] with initial condition

$$D^q u = \lambda u + h(t), \qquad \Gamma(q)u(t)t^{1-q}\mid_{t=0} = u^0 \tag{1}$$

where $\lambda \in \mathbb{R}$ and $h \in C[J_0, \mathbb{R}]$, has integral representation of the solution of equation (1) as

$$u(t) = u^0 t^{q-1}E_{q,q}(\lambda t^q) + \int_0^t (t-\tau)^{q-1}E_{q,q}[\lambda(t-\tau)^q]h(\tau)d\tau. \tag{2}$$

Lemma 2.1. *([Denton and Vatsala, 2011, Z. Denton and Vatsala, 2011]) [Comparison Results] Let $\eta \in C_p[J_0, \mathbb{R}]$ be such that for some $t_1 \in (0, T]$, $\eta(t_1) = 0$, and $t^{1-q}\eta(t) \leq 0$ on $[0, t_1]$, then $D^q\eta(t_1) \geq 0$.*

3 Comparison Results

In this section, we obtain a comparison results for the system of non-linear fractional reaction-diffusion equations with initial and boundary conditions. The comparison theorem is employed respect to the lower and upper solutions when the non-linear term is of the form $f_i(x, t, u_1, u_2)$, where $f_i(x, t, u_1, u_2)$ is quasi-monotone non-decreasing function and satisfies one sided Lipschitz condition.

Consider the initial and boundary value problem for system of non-linear R-L fractional reaction-diffusion equations.

$$\frac{\partial^q u_i}{\partial t^q} - k_i\frac{\partial^2 u_i}{\partial x^2} = f_i(x, t, u_1, u_2) \quad on \quad S_T. \tag{3}$$

$$u_i(0,t) = A_i(t), u_i(L,t) = B_i(t) \quad in \quad I_T.$$

$$\Gamma(q)t^{1-q}u_i(x,t) \mid_{t=0} = f_i^0(x) \quad x \in I$$

where $i = 1, 2$,$f_i \in C[I \times J_0 \times \mathbb{R}^2, \mathbb{R}]$, $I = [0, L]$, $J = (0, T]$, $S_T = J \times I$, $k > 0$ and $I_T = (0, T) \times \partial I$.

Suppose that $f_i^0(0) = A_i(0) = f_i^0(L) = B_i(0) = 0$, $\Gamma(q)t^{1-q}u_i \mid_{t=0} = f_i^0(x)$. We assume the initial and boundary condition satisfy the compatibility conditions. Using the method of eigenfunction expansion,[Chhetri and Vatsala, 2018] the solution of (3), is of the form

$$u_i(x,t) = \sum_{n=0}^{\infty} b_n(t)\phi_n(x), \tag{4}$$

where the eigenfunctions of the related homogeneous problem are known to be $\phi_n(x) = sin\frac{n\pi x}{L}$ and its corresponding eigenvalues are $\lambda_n = \left[(\frac{n\pi}{L})^2\right]$. We compute $b_n(t)$, where $b_n(t)$ is the solution of linear R-L fractional differential equations, as follows:

$$\begin{aligned} b_n(t) = b_n^0 t^{q-1}E_{q,q}(-k_i\lambda_n t^q) + \int_0^t (t-\tau)^{q-1}E_{q,q}(-k_i\lambda_n t^q)q_n(\tau) \\ + k\frac{2n\pi}{L^2}[A_i(\tau) - (-1)^n B_i(\tau)]d\tau, \end{aligned}$$

where

$$b_n^0 = \frac{2}{L}\int_0^L f_i^0(y)\phi_n(y)dy, \quad q_n(t) = \frac{2}{L}\int_0^L f_i(x,t,u_1,u_2)\phi_n(y)dy.$$

Therefore,

$$\begin{aligned} b_n(t) =& \frac{2}{L}\int_0^L f_i^0(y)\phi_n(y)dy t^{q-1}E_{q,q}(-k_i\lambda_n t^q) \\ &+ \int_0^t (t-\tau)^{q-1}E_{q,q}(-k_i\lambda_n t^q)\frac{2}{L}\int_0^L f_i(x,t,u_1,u_2)\phi_n(y)dyd\tau \\ &+ k\frac{2n\pi}{L^2}\int_0^t (t-\tau)^{q-1}E_{q,q}(-k_i\lambda_n t^q)[A_i(\tau) - (-1)^n B_i(\tau)]d\tau. \end{aligned}$$

So, using $b_n(t)$ in (4), the solution $u_i(x,t)$ has the form

$$\begin{aligned}u_i(x,t) = &\int_0^L t^{q-1}\left[\sum_{n=1}^{\infty}\frac{2}{L}E_{q,q}(-k_i\lambda_n t^q)\phi_n(x)\phi_n(y)\right]f_i^0(y)dy\\ &+\int_0^t\int_0^L\left[\sum_{n=1}^{\infty}\frac{2}{L}(t-\tau)^{q-1}E_{q,q}(-k_i\lambda_n(t-\tau)^q)\phi_n(x)\phi_n(y)\right]f_i dy d\tau\\ &+k_i\int_0^t\left[\frac{2n\pi}{L^2}(t-\tau)^{q-1}E_{q,q}(-k_i\lambda_n(t-\tau)^q)\phi_n(x)\right]A_i(\tau)d\tau\\ &-k_i\int_0^t\left[\frac{2n\pi}{L^2}(t-\tau)^{q-1}E_{q,q}(-k_i\lambda_n(t-\tau)^q)\phi_n(x)\right]B_i(\tau)d\tau.\end{aligned}$$

Thus we have,

$$\begin{aligned}u_i(x,t) = &\int_0^L t^{q-1}G(x,y,t)f_i^0(y)dy + \int_0^t\int_0^L G(x,y,t-\tau)f_i dy d\tau\\ &+k_i\int_0^t G_y(x,0,t-\tau)A_i(\tau)d\tau - k_i\int_0^t G_y(x,L,t-\tau)B_i(\tau)d\tau,\end{aligned}$$

where

$$G(x,y,t) = \sum_{n=0}^{\infty}\frac{2}{L}E_{q,q}(-k_i\lambda_n t^q)\phi_n(x)\phi_n(y).$$

The following Lemmas are useful to obtain the convergence of the solutions.

Lemma 3.1. *Chhetri and Vatsala [2017] Let $E_{q,1}(-\lambda t^q)$ be the Mittage-Leffler function of order q, where $0 < q \le 1$. Then, $\frac{E_{q,1}(-\lambda_1 t^q)}{E_{q,1}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + c$ for $c > 0$.*

Lemma 3.2. *Chhetri and Vatsala [2017] Let $E_{q,q}(-\lambda t^q)$ be the Mittage-Leffler function of order q, where $0 < q \le 1$. Then, $\frac{E_{q,q}(-\lambda_1 t^q)}{E_{q,q}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + c$ for $c > 0$.*

Now, we show the convergence of the above solutions using Lemma 3.1 and Lemma 3.2 above. We can split the solution of (3) as $u_i^1(x,t)$, $u_i^2(x,t)$ and $u_i^3(x,t)$ respectively as follows:
(a) $u_i^1(x,t)$ is the solution of (3), when $f_i = 0$, $A_i(t) = 0 = B_i(t)$,
(b) $u_i^2(x,t)$ is the solution of (3), when $A_i(t) = 0 = B_i(t)$, $f_i^0 = 0$,
(c) $u_i^3(x,t)$ is the solution of (3), when $f_i(x,t,u_1,u_2) = 0$, $f_i^0 = 0$.

Theorem 3.1. *Chhetri and Vatsala [2017] $u_i^1(x,t)$, $u_i^2(x,t)$ and $u_i^3(x,t)$ converge when $|f_i^0(x)| < N_1$, $N_1 > 0$, $|f_i(x,t,u_1,u_2)| < N_2$, $N_2 > 0$; $|A_i(t)| < M_1$, $M_1 > 0$ and $|B_i(t)| < M_2$, $M_1, M_2 > 0$, respectively.*

Now, we consider the initial and boundary value problem for system of nonlinear R-L fractional reaction-diffusion equations of the type

$$\frac{\partial^q u_i}{\partial t^q} - k_i \frac{\partial^2 u_i}{\partial x^2} = f_i(x,t,u_1,u_2), \quad (x,t) \in S_T, \tag{5}$$

$$\Gamma(q)t^{1-q}u_i(x,t) \mid_{t=0} = f_i^0(x), \quad x \in I,$$
$$u_i(0,t) = A_i(t), u_i(L,t) = B_i(t) \quad on \quad I_T,$$
$$I = [0,L], J = (0,T], S_T = J \times I, k > 0,$$
$$I_T = (0,T) \times \partial I \quad i = 1,2$$
$$f_i \in C^{2,q}[I \times J \times \mathbb{R}^2, \mathbb{R}].$$

In this work, we can see the classical solution of (5) $u_i(x,t) \in C_p^{2,q}$ on S_T, and $u_i(x,t) \in C_p$ on $\overline{S_T}$. To develop monotone method for the problem,(5) we need to define the following important definitions and results

Definition 3.1. *If the functions $v_i(x,t) \& w_i(x,t) \in C^{2,q}[S_T, \mathbb{R}]$ are called the natural lower and upper solutions of (5) if*

$$\frac{\partial^q v_i(x,t)}{\partial t^q} - k_i \frac{\partial^2 v_i(x,t)}{\partial x^2} \le f_i(x,t,v_1,v_2), \quad on \quad S_T \tag{6}$$

$$\Gamma(q)(t-t_0)^{1-q}v_i(x,t) \mid_{t=0} \le f_i^0(x), \quad x \in I$$
$$v_i(x,0) \le A_i(t), v_i(L,t) \le B_i(t) \quad in \quad I_T$$

and

$$\frac{\partial^q w_i(x,t)}{\partial t^q} - k_i \frac{\partial^2 w_i(x,t)}{\partial x^2} \ge f_i(x,t,w_1,w_2), \quad on \quad S_T \tag{7}$$

$$\Gamma(q)(t-t_0)^{1-q}w_i(x,t) \mid_{t=0} \ge f_i^0(x), \quad x \in I$$
$$w_i(x,0) \ge A_i(t), w_i(L,t) \ge B_i(t) \quad in \quad I_T.$$

Definition 3.2. *A function $f_i = f_i(x,t,u_1,u_2)$ in $C^{2,q}[I \times J \times \mathbb{R}^2, \mathbb{R}]$ is said to be quasimonotone nondecreasing if*
$f_i(x,t,u_1,u_2) \le f_i(x,t,v_1,v_2)$ if $u_i = v_i$ and $u_j \le v_j$, $i \ne j$, $i = j = 1,2$.

Definition 3.3. *A function $f_i = f_i(x,t,u_1,u_2)$ in $C^{2,q}[I \times J \times \mathbb{R}^2, \mathbb{R}]$ is said to be quasimonotone nonincreasing if*
$f_i(x,t,u_1,u_2) \ge f_i(x,t,v_1,v_2)$ if $u_i = v_i$ and $u_j \le v_j$, $i \ne j$, $i = j = 1,2$.

Theorem 3.2. *[Comparison result] Assume that*
(i) $v_i(x,t), w_i(x,t) \in C^{2,q}[S_T, \mathbb{R}]$ *are natural lower and upper solutions of (5), respectively and* $\Gamma(q)t^{1-q}v_i(x,t) \mid_{t=0} \leq \Gamma(q)t^{1-q}w_i(x,t) \mid_{t=0}$,
$v_i(0,t) \leq w_i(0,t)$, $v_i(L,t) \leq w_i(L,t)$.
(ii)$f_i(x,t,u_1,u_2)$ *satisfies the one sided Lipschitz condition*

$$f_i(x,t,u_1,u_2) - f_i(x,t,u_1^*,u_2^*) \leq M_i(u_i - u_i^*),$$

whenever $u_i \leq u_i^*$ *and* $M_i > 0$, $i = 1, 2$.
Then $v_i(x,t) \leq w_i(x,t)$ *on* $J \times I$.

Proof. Initially, we prove the theorem when one of the inequalities in (i) is strict. For that purpose, let $m_i(x,t) = v_i(x,t) - w_i(x,t)$. We claim that $m_i(x,t) < 0, \quad (x,t) \in I \times J$. Suppose that the conclusion is not true, then there exists a $t_1 \in J$ and $x_1 \in I$ such that $t^{1-q}m_i(x_1,t_1) < 0$ on $[0,t_1)$, $m_i(x_1,t_1) = 0$. It easy to check $\frac{\partial^q m_i(x_1,t_1)}{\partial t^q} = 0$ and $\frac{\partial^2 m_i(x_1,t_1)}{\partial x^2} \leq 0$.
Then, using Lemma 2.1 we get $\frac{\partial^q m_i(x,t)}{\partial t} \geq 0$
From the hypothesis, we also have

$$\begin{aligned}
\frac{\partial^q m_i(x_1,t_1)}{\partial t_1^q} &= \frac{\partial^q v_i(x_1,t_1)}{\partial t_1^q} - \frac{\partial^q w_i(x_1,t_1)}{\partial t_1^q} \\
&< k_i \frac{\partial^2 v_i(x_1,t_1)}{\partial x^2} + f_i(x_1,t_1,v_1,v_2) \\
&- k_i \frac{\partial^2 w_i(x_1,t_1)}{\partial x^2} - f_i(x_1,t_1,w_1,w_2) \\
&< f_i(x_1,t_1,v_1,v_2) - f_i(x_1,t_1,w_1,w_2) = 0
\end{aligned}$$

which is a contradiction. Therefore, $v_i(x,t) < w_i(x,t)$ on $\overline{S_T}$.
In order to prove the theorem for the non strict inequalities, let

$$\begin{aligned}
w_i^*(x,t) &= w_i(x,t) + \epsilon t^{q-1} E_{q,q}[2Mt^q], \\
v_i^*(x,t) &= v_i(x,t) - \epsilon t^{q-1} E_{q,q}[2Mt^q].
\end{aligned}$$

From this it follows

$$\begin{aligned}
w_i^*(0,t) &> v_i^*(0,t), \\
w_i^*(L,t) &> v_i^*(L,t),
\end{aligned}$$

$\Gamma(q)t^{1-q}w_i^* \mid_{t=0} > \Gamma(q)t^{1-q}w_i \mid_{t=0} > \Gamma(q)t^{1-q}v_i \mid_{t=0} > \Gamma(q)t^{1-q}v_i^* \mid_{t=0}$. Then,

$$\begin{aligned}
\frac{\partial^q w_i^*(x,t)}{\partial t^q} - k_i\frac{\partial^2 w_i^*(x,t)}{\partial x^2} &= \frac{\partial^q w_i(x,t)}{\partial t^q} - k_i\frac{\partial^2 w_i(x,t)}{\partial x^2} + \frac{\partial^q}{\partial t^q}\epsilon t^{q-1}E_{q,q}[2Mt^q] \\
&\geq f_i(x,t,w_1,w_2) + \epsilon t^{q-1}2M_iE_{q,q}[2Mt^q] \\
&= f_i(x,t,w_1,w_2) + M_i\epsilon t^{q-1}E_{q,q}[2Mt^q] \\
&= f_i(x,t,w_1^*,w_2^*) + \epsilon M_i t^{q-1}E_{q,q}[2Mt^q] \\
&> f_i(x,t,w_1^*,w_2^*) \quad on \quad S_T.
\end{aligned}$$

Similarly,

$$\frac{\partial^q v_i^*(x,t)}{\partial t^q} - k_i\frac{\partial^2 v_i^*(x,t)}{\partial x^2} > f_i(x,t,v_1^*,v_2^*) \quad on \quad S_T.$$

Since $v_i^* < w_i^*$ on $\overline{S_T}$ and as $\epsilon \to 0$ we have $v_i \leq w_i$ on $\overline{S_T}$. □

Corolary 3.1. *[Maximal principle] Let*

$$\begin{aligned}
\frac{\partial^q m_i(x,t)}{\partial t^q} - k_i\frac{\partial^2 m_i(x,t)}{\partial x^2} \leq 0 \quad on \quad S_T, \\
m_i(0,t) \leq 0, m_i(L,t) \leq 0 \quad on \quad I_T, \\
\Gamma(q)t^{1-q}m_i(x,t) \mid_{t=0} \leq 0 \quad on \quad I.
\end{aligned}$$

Then $m_i(x,t) \leq 0$ *on* $J \times I$.

Proof. Suppose $m_i(x,t)$ has positive maximum at (x_1,t_1). Let $m_i(x_1,t_1) = K$. Let $\overline{m_i}(x,t) = m_i(x,t) - K$. Then $t^{1-q}\overline{m_i}(x,t) \leq 0$ on $(0,t_1]$ and $\overline{m_i}(x_1,t_1) = 0$. Using Lemma 2.1, we get $\frac{\partial^q m_i(x_1,t_1)}{\partial t_1^q} \geq 0$. Also $\frac{\partial^2 m_i(x_1,t_1)}{\partial x^2} \leq 0$. Combining these two, we get $\frac{\partial^q m_i(x_1,t_1)}{\partial t_1^q} - k_i\frac{\partial^2 m_i(x_1,t_1)}{\partial x^2} \geq 0$.
Also, we have

$$\begin{aligned}
\frac{\partial^q \overline{m_i}(x,t)}{\partial t^q} - K\frac{\partial^2 \overline{m_i}(x,t)}{\partial x^2} &= \frac{\partial^q m_i(x,t)}{\partial t^q} - K\frac{\partial^2 m_i(x,t)}{\partial x^2} - K\frac{t^{q-1}}{\Gamma q} \\
&< \frac{\partial^q m_i(x,t)}{\partial t} - K\frac{\partial^2 m_i(x,t)}{\partial x^2} \\
&< 0.
\end{aligned}$$

which gives a contradiction. Hence, $m_i(x,t) \leq 0$. □

4 Main Results

This section is employed for monotone method for system of R-L fractional reaction-diffusion equation (5) using lower and upper solutions. Also obtain existence and uniqueness of solution of system(5).

Theorem 4.1. *(i) Let $f_i = f_i(x, t, u_1, u_2)$ in $C^{2,q}[I \times J \times \mathbb{R}^2, \mathbb{R}]$ be quasimonotone nondecreasing.*
(ii) Let (v_i^0, w_i^0) be the lower and upper solutions of (5) such that $t^{1-q}v_i^0 \le t^{1-q}w_i^0$ on $\overline{S_T}$.
(iii) Let $f_i(x, t, u_1, u_2)$ satisfies the one sided Lipschitz condition

$$f_i(x, t, u_1, u_2) - f_i(x, t, u_1^*, u_2^*) \ge -M_i(u_i - u_i^*),$$

whenever $u_i^* \le u_i$ and $M_i > 0$. Then there exist monotone sequences $\{t^{1-q}v_i^n(x, t)\}$ and $\{t^{1-q}w_i^n(x, t)\}$ such that $t^{1-q}v_i^n(x, t) \to t^{1-q}\rho_i(x, t)$ and $t^{1-q}w_i^n(x, t) \to t^{1-q}\gamma_i(x, t)$ uniformly and monotonically on $\overline{S_T}$, where $\rho_i(x, t)$ and $\gamma_i(x, t)$ are minimal and maximal solutions of (5) respectively.

Proof. We construct the sequences $\{v_i^n(x, t)\}$ and $\{w_i^n(x, t)\}$ as follows:

$$\frac{\partial^q v_i^n(x,t)}{\partial t^q} - k_i \frac{\partial^2 v_i^n(x,t)}{\partial x^2} = f_i(x, t, v_1^{n-1}(x, t)), v_2^n(x, t)), \quad on \quad S_T \tag{8}$$

$$\Gamma(q)(t)^{1-q} v_i^n(x, t) \mid_{t=0} = f_i^0(x), \quad x \in I$$

$$v_i^n(0, t) = A_i(t), v_i^n(L, t) = B_i(t) \quad in \quad I_T,$$

and

$$\frac{\partial^q w_i^n(x,t)}{\partial t^q} - k_i \frac{\partial^2 w_i^n(x,t)}{\partial x^2} = f_i(x, t, w_1^{n-1}(x, t), w_2^n(x, t)), \quad on \quad S_T \tag{9}$$

$$\Gamma(q)(t)^{1-q} w_i^n(x, t) \mid_{t=0} = f_i^0(x), \quad x \in I$$

$$w_i^n(0, t) = A_i(t), w_i^n(L, t) = B_i(t) \quad in \quad I_T,$$

Observe that $v_i^1(x, t)$ and $w_i^1(x, t)$ exist and unique solution of the system (5) and Corollary 3.1. By induction and the assumptions on $f_i(x, t, u_1, u_2)$, we show that the solution $v_i^n(x, t)$ and $w_i^n(x, t)$ exist and unique by Corollary 3.1, for any n. Let us prove first $v_i^0 \le v_i^1$ and $w_i^1 \le w_i^0$ on S_T.

Let $\rho_i(x,t) = v_i^0(x,t) - v_i^1(x,t)$. Then

$$\begin{aligned}\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x,t)}{\partial x^2} &= \frac{\partial^q v_i^0(x,t)}{\partial t^q} - k_i \frac{\partial^2 v_i^0(x,t)}{\partial x^2} \\ &- \left[\frac{\partial^q v_i^1(x,t)}{\partial t^q} - k_i \frac{\partial^2 v_i^1(x,t)}{\partial x^2}\right] \\ &\leq f_i(x,t,v_1^0,v_2^0) - [f_i(x,t,v_1^0,v_2^0)] = 0\end{aligned}$$

Thus we have $\rho_i(0,t) = 0, \rho_i(L,t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x,t)\mid_{t=0} = 0$ on I_T. Therefore, by Corollary 3.1, we have $\rho_i(x,t) \leq 0$ on $\overline{S_T}$ and $t^{1-q}v_i^0(x,t) \leq t^{1-q}v_i^1(x,t)$ on $\overline{S_T}$.
Suppose that $v_i^{k-1} \leq v_i^k$. Now we prove $v_i^k \leq v_i^{k+1}$.

Let $\rho_i(x,t) = v_i^k(x,t) - v_i^{k+1}(x,t)$. Then

$$\begin{aligned}\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x,t)}{\partial x^2} &= \frac{\partial^q v_i^k(x,t)}{\partial t^q} - k_i \frac{\partial^2 v_i^k(x,t)}{\partial x^2} \\ &- \left[\frac{\partial^q v_i^{k+1}(x,t)}{\partial t^q} - k_i \frac{\partial^2 v_i^{k+1}(x,t)}{\partial x^2}\right] \\ &\leq f_i(x,t,v_1^k,v_2^k) - [f_i(x,t,v_1^{k+1},v_2^k)] \\ &\leq M_i(v_i^{k+1} - v_i^k) \\ &\leq -M_i\rho_i(x,t)\end{aligned}$$

Thus we have $\rho_i(0,t) = 0, \rho_i(L,t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x,t)\mid_{t=0} = 0$ on I_T. Therefore, by Corollary 3.1, we have $\rho_i(x,t) \leq 0$ on $\overline{S_T}$ and $t^{1-q}v_i^k(x,t) \leq t^{1-q}v_i^{k+1}(x,t)$ on $\overline{S_T}$.
Hence by mathematical induction, we have

$$t^{1-q}v_i^0 \leq t^{1-q}v_i^1 \leq ... \leq t^{1-q}v_i^k \leq t^{1-q}v_i^{k+1} \leq ... \leq t^{1-q}v_i^{n-1} \leq t^{1-q}v_i^n \quad (10)$$

We show that $w_i^1(x,t) \leq w_i^0(x,t)$ on $\overline{S_T}$.
Let $\rho_i(x,t) = w_i^1(x,t) - w_i^0(x,t)$. Then

$$\begin{aligned}\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x,t)}{\partial x^2} &= \frac{\partial^q w_i^1(x,t)}{\partial t^q} - k_i \frac{\partial^2 w_i^1(x,t)}{\partial x^2} \\ &- \left[\frac{\partial^q w_i^0(x,t)}{\partial t^q} - k_i \frac{\partial^2 w_i^0(x,t)}{\partial x^2}\right] \\ &\leq f_i(x,t,w_1^0,w_2^0) - [f_i(x,t,w_1^0,w_2^0] = 0\end{aligned}$$

Therefore, $\rho_i(0,t) = 0, \rho_i(L,t) = 0$ on I and $\Gamma(q)t^{1-q}\rho(x,t) \mid_{t=0} = 0$ on I_T. By Corollary 3.1, we have $\rho_i(x,t) \le 0$ on $\overline{S_T}$ and $t^{1-q}w_i^0(x,t) \le t^{1-q}w_i^1(x,t)$ on $\overline{S_T}$.
Suppose that $w_i^k(x,t) \le w_i^{k-1}(x,t)$. To show that $w_i^{k+1}(x,t) \le w_i^k(x,t)$.
Let $\rho_i(x,t) = w_i^{k+1}(x,t) - w_i^k(x,t)$. Then

$$\begin{aligned}\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k_i\frac{\partial^2 \rho_i(x,t)}{\partial x^2} &= \frac{\partial^q w_i^{k+1}(x,t)}{\partial t^q} - k_i\frac{\partial^2 w_i^{k+1}(x,t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q w_i^{k}(x,t)}{\partial t^q} - k_i\frac{\partial^2 w_i^{k}(x,t)}{\partial x^2}\right] \\ &\le f_i(x,t,w_1^k,w_2^k) - [f_i(x,t,w_1^{k+1},w_2^k)] \\ &\le M_i(w_i^k - w_i^{k+1}) \\ &\le -M_i\rho_i(x,t)\end{aligned}$$

Thus we have $\rho_i(0,t) = 0, \rho_i(L,t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x,t) \mid_{t=0} = 0$ on I_T. Therefore, by Corollary 3.1, we have $\rho_i(x,t) \le 0$ on $\overline{S_T}$ and $t^{1-q}w_i^{k+1}(x,t) \le t^{1-q}v_i^k(x,t)$ on $\overline{S_T}$.
Hence by mathematical induction, we have

$$t^{1-q}w_i^n \le t^{1-q}w_i^{n-1} \le ... \le t^{1-q}w_i^{k+1} \le t^{1-q}w_i^k \le ... \le t^{1-q}w_i^1 \le t^{1-q}w_i^0 \quad (11)$$

Then, we prove that $v_i^1(x,t) \le w_i^1(x,t)$. Let $\rho_i(x,t) = v_i^1(x,t) - w_i^1(x,t)$. Then from hypothesis, we get

$$\begin{aligned}\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k_i\frac{\partial^2 \rho_i(x,t)}{\partial x^2} &= \frac{\partial^q v_i^{1}(x,t)}{\partial t^q} - k_i\frac{\partial^2 v_i^{1}(x,t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q w_i^{1}(x,t)}{\partial t^q} - k_i\frac{\partial^2 w_i^{1}(x,t)}{\partial x^2}\right] \\ &\le f_i(x,t,v_1^1,v_2^1) - [f_i(x,t,w_1^1,w_2^1)] \\ &\le M_i(v_i^1 - w_i^1) \\ &\le -M_i\rho_i(x,t)\end{aligned}$$

Thus we have $\rho_i(0,t) = 0, \rho_i(L,t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x,t) \mid_{t=0} = 0$ on I_T. Therefore, by Corollary 3.1, we have $\rho_i(x,t) \le 0$ on $\overline{S_T}$ and $t^{1-q}v_i^1(x,t) \le t^{1-q}w_i^1(x,t)$ on $\overline{S_T}$. Hence,
$t^{1-q}v_i^0(x,t) \le t^{1-q}v_i^1(x,t) \le t^{1-q}w_i^1(x,t) \le t^{1-q}w_i^0(x,t)$ on $\overline{S_T}$.
By mathematical induction and equations (10), (11), we have
$t^{1-q}v_i^0 \le ... \le t^{1-q}v_i^n \le t^{1-q}w_i^n \le ... \le t^{1-q}w_i^0$ on $\overline{S_T}$ for all n.
Furthermore, if $t^{1-q}v_i^0 \le t^{1-q}u_i \le t^{1-q}w_i^0$ on $\overline{S_T}$, then for any $u_i(x,t)$ of (5), we establish the following inequality by the method of induction.

$$t^{1-q}v_i^0 \le ... \le t^{1-q}v_i^n \le t^{1-q}u_i \le t^{1-q}w_i^n \le ... \le t^{1-q}w_i^0 \quad (12)$$

on $\overline{S_T}$ for all n.
It is certainly true for $n = 0$, by hypothesis. Assume the inequality (10) to be true for $n = k$, that is

$$t^{1-q}v_i^0 \leq ... \leq t^{1-q}v_i^k \leq t^{1-q}u_i \leq t^{1-q}w_i^k \leq ... \leq t^{1-q}w_i^0 \tag{13}$$

on $\overline{S_T}$ for all n.
Let$\rho_i(x,t) = v_i^{k+1}(x,t) - u_i(x,t)$. Then from hypothesis, we get

$$\begin{aligned}\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x,t)}{\partial x^2} &= \frac{\partial^q v_i^{k+1}(x,t)}{\partial t^q} - k_i \frac{\partial^2 v_i^{k+1}(x,t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q u_i(x,t)}{\partial t^q} - k_i \frac{\partial^2 u_i(x,t)}{\partial x^2}\right] \\ &\geq f_i(x,t,v_1^{k+1},v_2^k) - [f_i(x,t,u_1,u_2)] \\ &\geq -M_i(v_i^{k+1} - u_i) \\ &\geq -M_i \rho_i(x,t)\end{aligned}$$

Thus we have $\rho_i(0,t) = 0, \rho_i(L,t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x,t) \mid_{t=0} = 0$ on I_T. Therefore, by Corollary 3.1, we have $\rho_i(x,t) \geq 0$ on $\overline{S_T}$. Therefore $t^{1-q}v_i^{k+1}(x,t) \leq t^{1-q}u_i(x,t)$ on $\overline{S_T}$. Similarly, we can show that $t^{1-q}u_i(x,t) \leq t^{1-q}w_i^{k+1}(x,t)$ on $\overline{S_T}$.
Let $\rho_i(x,t) = u_i(x,t) - w_i^{k+1}(x,t)$. Then from hypothesis, we get

$$\begin{aligned}\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k_i \frac{\partial^2 \rho_i(x,t)}{\partial x^2} &= \frac{\partial^q u_i(x,t)}{\partial t^q} - k_i \frac{\partial^2 u_i(x,t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q w_i^{k+1}(x,t)}{\partial t^q} - k_i \frac{\partial^2 w_i^{k+1}(x,t)}{\partial x^2}\right] \\ &\geq f_i(x,t,u_1,u_2) - [f_i(x,t,w_1^{k+1},w_2^k)] \\ &\geq -M_i(u_i - w_i^{k+1}) \\ &\geq -M_i \rho_i(x,t)\end{aligned}$$

Thus we have $\rho_i(0,t) = 0, \rho_i(L,t) = 0$ on I and $\Gamma(q)t^{1-q}\rho_i(x,t) \mid_{t=0} = 0$ on I_T. Therefore, by Corollary 3.1, we have $\rho_i(x,t) \geq 0$ on $\overline{S_T}$. Therefore $t^{1-q}u_i(x,t) \leq t^{1-q}w_i^{k+1}(x,t)$ on $\overline{S_T}$.

Hence we constructed the monotone sequences $\{v_i^n(x,t)\}$, $\{w_i^n(x,t)\}$ of lower and upper solutions of integral representation of linear problem and an appropriate computation process, we show that the sequences $\{t^{1-q}v_i^n(x,t)\}$ and $\{t^{1-q}w_i^n(x,t)\}$ are uniformly bounded and equicontinuous. Using the Ascoli-Arzela theorem, we obtain subsequences of $\{t^{1-q}v_i^n(x,t)\}$ and $\{t^{1-q}w_i^n(x,t)\}$ which converge uniformly and monotonically on $\overline{S_T}$. Since the sequences $\{t^{1-q}v_i^n(x,t)\}$ and $\{t^{1-q}w_i^n(x,t)\}$ are monotone, the entire sequence $\{t^{1-q}v_i^n(x,t)\}$ and $\{t^{1-q}w_i^n(x,t)\}$ converges to $t^{1-q}\rho_i(x,t)$ and $t^{1-q}\gamma_i(x,t)$ respectively. From this it follows that

$$\begin{aligned} t^{1-q}v_i^0 \leq t^{1-q}v_i^1 \leq ... \leq t^{1-q}v_i^n \leq ... \leq t^{1-q}\rho_i \leq t^{1-q}u_i \\ \leq t^{1-q}\gamma_i \leq ... \leq t^{1-q}w_i^n \leq ... \leq t^{1-q}w_i^0 \quad on \quad \overline{S_T}. \end{aligned}$$

Consequently, $\rho_i(x,t)$ and $\gamma_i(x,t)$ are minimal and maximal solutions of (5) since

$$t^{1-q}v_i^0 \leq t^{1-q}\rho_i \leq t^{1-q}u_i \leq t^{1-q}\gamma_i \leq t^{1-q}w_i^0 \quad on \quad \overline{S_T}.\square$$

Theorem 4.2. *Let all the assumptions of Theorem 4.1 hold and $f_i(x,t,u_1,u_2)$ satisfy the one sided Lipschitz condition*

$$f_i(x,t,u_1,u_2) - f_i(x,t,u_1^*,u_2^*) \leq M_i(u_i - u_i^*), \quad M_i > 0.$$

Then the solution $u_i(x,t)$ of (5) is unique.

Proof. We have already proved (ρ_i, γ_i) are minimal and maximal solutions of (5) on $\overline{S_T}$. Hence it is enough to show that $\gamma_i(x,t) \leq \rho_i(x,t) \quad on \quad \overline{S_T}$.
It is known from Theorem 4.1 that $\gamma_i(x,t) \leq \rho_i(x,t) \quad on \quad \overline{S_T}$.
Let $p_i(x,t) = \gamma_i(x,t) - \rho_i(x,t)$. By the hypothesis, we get

$$\begin{aligned} \frac{\partial^q p_i(x,t)}{\partial t^q} - k_i\frac{\partial^2 p_i(x,t)}{\partial x^2} &= \frac{\partial^q \gamma_i(x,t)}{\partial t^q} - k_i\frac{\partial^2 \gamma_i(x,t)}{\partial x^2} \\ &\quad - \left[\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k_i\frac{\partial^2 \rho_i(x,t)}{\partial x^2}\right] \\ &\leq f_i(x,t,\gamma_1(x,t),\gamma_2(x,t)) - [f_i(x,t,\rho_1(x,t),\rho_2(x,t))] \\ &\leq M_i \mid \gamma_i(x,t) - \rho_i(x,t) \mid \\ &\leq M_i \mid p_i(x,t) \mid, \end{aligned}$$

Thus we have $p_i(0,t) = 0, p_i(L,t) = 0$ on I and $\Gamma(q)t^{1-q}p_i(x,t) \mid_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $p_i(x,t) \leq 0$. This proves that $\gamma_i(x,t) = \rho_i(x,t) = u_i(x,t)$ on $\overline{S_T}$ and proof is complete. $\square$

5 Discussion and Conclusions

In this work comparison results, we initially developed a monotone method for a system of R-L fractional reaction-diffusion equations with initial and boundary conditions (5) in which the function (f_1, f_2) is quasimonotone nondecreasing (nonincreasing). As an application of this method, the existence and uniqueness of the solution of the system of R-L fractional reaction-diffusion equations (5) are obtained when the function (f_1, f_2) is quasimonotone nondecreasing (nonincreasing).

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