Ratio Mathematica Volume 51, 2024

Value distribution of meromorphic functions whose differential polynomials share a small function

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Abstract

In this article, we study the uniqueness of differential polynomials $\mathcal{P}(f) = f_1^p P(f_1)$ and $P[f]$ generated by meromorphic functions f and g respectively sharing a small function. Our results generalises the result due to Harina P. Waghamore and Husna V. [7].

Keywords: Differential Polynomial, Small Function, Meromorphic Function etc.,
2020 AMS subject classifications: Primary 30D35 [1]

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[1]Received on December 20, 2023. Accepted on March 1, 2024. Published on March 21, 2024. DOI:10.23755/rm.v51i0.1551. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Throughout this paper, we use the standard notations of the Nevanlinna Theory of meromorphic function as explained in (see [8, 9, 10]). Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, then we say that f and g share the value a counting multiplicities (CM) and if we do not consider the multiplicities, then f and g are said to share the value a ignoring multiplicites (IM). If $a = \infty$, then the zeros of $f - a$ means the poles of f. A meromorphic function $a = a(z)$ $(\not\equiv 0,\ \infty)$ is called a small function with respect to f provided that $T(r, a) = S(r, f)$ as $r \to \infty$, $r \notin E$, where E is a set of positive real numbers with finite Lebesgue measure. If $a = a(z)$ is a small function, then we say that f and g share a IM (resp. CM) according to $f - a$ and $g - a$ share 0 IM (resp. CM).

In 2008, Zhang and Lü [14] obtained the following result.

Theorem 1.1. *be (see [14]) Let $k,\ n$ be the positive integers, f be a non-constant meromorphic function, and a $(\not\equiv 0, \infty)$ be a meromorphic function satisfying $T(r, a) = o\{T(r, f)\}$, as $r \to \infty$. If f^n and $f^{(k)}$ share a IM and*

$$(2k + 6)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{2+k}(0, f) > 2k + 12 - n,$$

or f^n and $f^{(k)}$ share a CM and

$$(k + 3)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{2+k}(0, f) > k + 6 - n$$

then $f^n \equiv f^{(k)}$.

In the same paper T. Zhang and W. Lü [14] asked the following question:

Question 1. What will happen if f^n and $(f^{(k)})^m$ share a meromorphic function a $(\not\equiv 0, \infty)$ satisfying $T(r, a) = o\{T(r, f)\}$ as $r \to \infty$?

In 2013, S. S. Bhoosnurmath and Kabbur [15] proved the following result.

Theorem 1.2. *(see [15]) Let f be a non-constant meromorphic function and a $(\not\equiv 0, \infty)$ be a meromorphic function satisfying $T(r, a) = o\{T(r, f)\}$, as $r \to \infty$. Let $P[f]$ be a non-constant differential polynomial in f. If f and $P[f]$ share a IM and*

$$(2Q + 6)\Theta(\infty, f) + (2 + 3\underline{d}(P))\delta(0, f) > 2Q + 2\underline{d}(P) + \overline{d}(P) + 7,$$

or if f and $P[f]$ share a CM and

$$3\Theta(\infty, f) + (\underline{d}(P) + 1)\delta(0, f) > 4,$$

then $f \equiv P[f]$.

Banerjee and Majumder [16] considered the weighted sharing of $f^n$and $(f^m)^{(k)}$ and proved the following result.

Theorem 1.3. *(see [16]) Let f be a non-constant meromorphic function, $k,\ n,\ m \in \mathbb{N}$ and l be a non-negative integer. Suppose $a\ (\not\equiv 0, \infty)$ be a meromorphic function satisfying $T(r,a) = o\{T(r,f)\}$ as $r \to \infty$ such that f^n and $(f^m)^{(k)}$ share (a,l). If $l \geq 2$ and*

$$(k+3)\Theta(\infty,f) + (k+4)\Theta(0,f) > 2k+7-n,$$

or $l = 1$ and

$$\Big(k+\frac{7}{2}\Big)\Theta(\infty,f) + \Big(k+\frac{9}{2}\Big)\Theta(0,f) > 2k+8-n,$$

or $l = 0$ and

$$(2k+6)\Theta(\infty,f) + (2k+7)\Theta(0,f) > 4k+13-n,$$

then $f \equiv (f^m)^{(k)}$.

In 2015, Kuldeep S. Charak and Banarasi Lal [17] proved the following result.

Theorem 1.4. *(see [17]) Let f be a non-constant meromorphic function, n be a positive integer and $a\ (\not\equiv 0, \infty)$ be a meromorphic function satisfying $T(r,a) = o\{T(r,f)\}$ as $r \to \infty$. Let $P[f]$ be a non-constant differential polynomial in f. Suppose f^n and $P[f]$ share (a,l) such that any one of the following holds.*
(i) when $l \geq 2$ and

$$(Q+3)\Theta(\infty,f) + 2\Theta(0,f) + \overline{d}(P)\delta(0,f) > Q+5+2\overline{d}(P) - \underline{d}(P) - n,$$

(ii) when $l = 1$ and

$$\Big(Q+\frac{7}{2}\Big)\Theta(\infty,f) + \frac{5}{2}\Theta(0,f) + \overline{d}(P)\delta(0,f) > Q+6+2\overline{d}(P) - \underline{d}(P) - n,$$

(iii) when $l = 0$ and

$$(2Q+6)\Theta(\infty,f) + 4\Theta(0,f) + 2\overline{d}(P)\delta(0,f) > 2Q+10+4\overline{d}(P) - 2\underline{d}(P) - n.$$

Then $f^n \equiv P[f]$.

In 2017, H. P. Waghamore and Husna V. [7] proved the following.

Theorem 1.5. *(see [7]) Let $k\ (\geq 1),\ n\ (\geq 1),\ p\ (\geq 1)$ and $m\ (\geq 0)$ be integers and f and $f_1 = f - \omega_p$ be two non-constant meromorphic functions and $M[f]$ be a differential monomial of degree d_M and weight Γ_M and k is the highest derivative in $M[f]$. Let $\mathcal{P}(z) = a_{m+n}z^{m+n} + \ldots + a_n z^n + \ldots + a_0$, $a_{m+n} \neq 0$, be a polynomial in z of degree $m+n$ such that $\mathcal{P}(f) = f_1^p P(f_1)$. Also let $a(z)\ (\not\equiv 0, \infty)$ be a small function with respect to f. Suppose $\mathcal{P}(f) - a$ and $M[f] - a$ share $(0, l)$. If $l \geq 2$ and*

$$(3 + 2\lambda)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 2d_M\delta_{1+k}(0, f) > 2\Gamma_M + 2\mu_2 + 3 - p,$$

or $l = 1$

$$\left(\frac{7}{2} + 2\lambda\right)\Theta(\infty, f) + \frac{1}{2}\Theta(\omega_p, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 2d_M\delta_{1+k}(0, f) > 2\Gamma_M + \mu_2 + 4 + \frac{(m+n) - 3p}{2},$$

or $l = 0$ and

$$(6+3\lambda)\Theta(\infty, f) + 2\Theta(\omega_p, f) + \mu_2\delta_{\mu_2^*}(\omega_p, f) + 3d_M\delta_{1+k}(0, f) > 3\Gamma_M + \mu_2 + 8 + 2(m+n) - 3p$$

then $\mathcal{P}(f) \equiv M[f]$.

Question 2. What happens if differential monomial $M[f]$ is replaced by differential polynomial $P[f]$ in Theorem 1.5 ?

To answer the above question, we prove one Theorem which improves and generalizes Theorem 1.5.

2 Some Definitions

Definition 2.1. *For any constant a, we define the quantities*

$$\Theta(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

$$\delta(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N(r, a; f)}{T(r, f)}.$$

Definition 2.2. *(see [13]) We denote by $N_{k)}\left(r, \frac{1}{f-a}\right)$ the counting function for zeros of $f - a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ the corresponding one* *for which multiplicity is not counted. Let* $N_{(k}\Big(r, \frac{1}{f-a}\Big)$ *be the counting function for zeros of* $f-a$ *with multiplicity atleast* k *and* $\overline{N}_{(k}\Big(r, \frac{1}{f-a}\Big)$ *the corresponding one for which multiplicity is not counted. Set*

$$N_k\Big(r, \frac{1}{f-a}\Big) = \overline{N}\Big(r, \frac{1}{f-a}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{f-a}\Big) + \ldots + \overline{N}_{(k}\Big(r, \frac{1}{f-a}\Big).$$

Definition 2.3. *(see [19]) For two positive integers* $n,\ p$ *we define* $\mu_p = min\{n, p\}$ *and* $\mu_p^* = p + 1 - \mu_p$. *Then it is clear that*

$$N_p(r, 0; f^n) = \mu_p N_{\mu_p^*}(r, 0; f).$$

Definition 2.4. *(see [20]) Let* z_0 *be a zero of* $f-a$ *of multiplicity* p *and a zero of* $g-a$ *of multiplicity* q. *We denote by* $\overline{N}_L(r, a; f)$ *the counting function of those* a*-points of* f *and* g *where* $p > q \geq 1$, *by* $N_E^{1)}(r, a; f)$ *the counting function of those* a*-points of* f *and* g *where* $p = q = 1$ *and by* $\overline{N}_E^{(2}(r, a; f)$ *the counting function of those* a*-points of* f *and* g *where* $p = q \geq 2$, *each point in these counting functions is counted only once. In the same way we can define* $\overline{N}_L(r, a; g)$, $N_E^{1)}(r, a; g)$, $\overline{N}_E^{(2}(r, a; g)$.

Definition 2.5. *(see [21]) Let* k *be a non-negative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a, f)$ *the set of all* a*-points of* f, *where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and* $k+1$ *times if* $m > k$. *If* $E_k(a, f) = E_k(a, g)$, *we say that* $f,\ g$ *share the value* a *with weight* k.
The definition implies that if $f,\ g$ *share a value* a *with weight* k, *then* z_0 *is an* a*-point of* f *with multiplicity* m $(\leq k)$ *if and only if it is an* a*-point of* g *with multiplicity* m $(\leq k)$ *and* z_0 *is an* a*-point of* f *with multiplicity* m $(> k)$ *if and only if it is an* a*-point of* g *with multiplicity* n $(> k)$, *where* m *not necessarily not equal to* n. *We write* $f,\ g$ *share* (a, k) *to mean that* $f,\ g$ *share the value* a *with weight* k. *Clearly if* $f,\ g$ *share* (a, k) *then* $f,\ g$ *share* (a, p) *for all integers* p, $0 \leq p < k$. *Also we note that* $f,\ g$ *share a value* a *IM or CM if and only if* $f,\ g$ *share* $(a, 0)$ *or* (a, ∞) *respectively.*

Definition 2.6. *(see [1]) Suppose that* $f(z)$ *is a meromorphic function in the complex plane* $\mathbb{C}$ *and* $a(z)$ *is a small function of* $f(z)$.*Then the differential monomial is defined by* $M[f] = \prod_{j=0}^{k} (f^{(j)})^{n_{ij}}$ *where* $n_{i_0}, \cdots, n_{i_k}$ *are non-negative integers.*

Definition 2.7. *(see [1]) Suppose that* $f(z)$ *is a meromorphic function in the complex plane* $\mathbb{C}$ *and* $M_1(f), \cdots, M_m(f)$ *be the differential monomials in* $f(z)$ *of* *degree* $n_1, \cdots, n_m$ *respectively. Then* $P[f] = \sum_{j=0}^{m} M_j(f)$ *is said to be differential poynomial in* $f(z)$ *and* $n = max\{n_1, \cdots, n_m\}$ *the degree of* $P[f]$.

With the notion of weighted sharing of values Lahiri-Sarkar [11] improved the result of Zhang [12]. In [13] Zhang extended the result of Lahiri-Sarkar [11] and replaced the concept of value sharing by small function sharing.

3 Some Lemmas

Let $F,\ G$ be two non-constant meromorphic functions. We define ψ as meromorphic function which is defined as follows.

$$\psi = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{1}$$

Lemma 3.1. *(see [1]) Let* $p,\ n$ *be two positive integers. Then for* $\epsilon > 0$

$$N_p(r, 0; f^n) \leq (n - n\delta_p(0; f) + \epsilon)T(r, f).$$

Lemma 3.2. *(see [2]) Let* $P[f]$ *be a differential polynomial generated by non-constant meromorphic function* f*. Then*

$$N(r, \infty; P) \leq \overline{d}(P)N(r, \infty; f) + (\Gamma_P - \overline{d}(P))\overline{N}(r, \infty; f).$$

Lemma 3.3. *(see [3, 4]) Let* f *be a meromorphic function and* $P[f]$ *be a differential polynomial. Then*

$$m\Big(r, \frac{P[f]}{f^{\overline{d}(P)}}\Big) \leq (\overline{d}(P) - \underline{d}(P))m\Big(r, \frac{1}{f}\Big) + S(r, f).$$

Lemma 3.4. *(see [5, 6]) Let* $P[f]$ *be a differential polynomial generated by non-constant meromorphic function* f*. Then*

$$\begin{aligned} N\left(r, \infty; \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq\ & (\Gamma_P - \overline{d}(P))\overline{N}(r, \infty; f) + (\overline{d}(P) - \underline{d}(P))N(r, 0; f| \geq k+1) \\ & + Q\overline{N}(r, 0; f| \geq k+1) + \overline{d}(P)N(r, 0; f| \leq k) + S(r, f). \end{aligned}$$

Lemma 3.5. *(see [1]) For the differential polynomial* $P[f]$*,*

$$\begin{aligned} N(r, 0; P[f]) \leq\ & (\Gamma_P - \overline{d}(P))\overline{N}(r, \infty; f) + \underline{d}(P)\overline{N}(r, 0; f) \\ & + (\overline{d}(P) - \underline{d}(P))\left(m\Big(r, \frac{1}{f}\Big) + T(r, f)\right) + S(r, f). \end{aligned}$$

Lemma 3.6. *(see [1]) Let j and p be two positive integers satisfying $j \geq p+1$. Let $P[f]$ be a differential polynomial with $\Gamma_P > (k+1)\underline{d}(P) - (p+1)$. Then*

$$\overline{N}_{(j+\Gamma_P-\underline{d}(P)}(r,0;f^{\underline{d}(P)}) \leq \overline{N}_{(j}(r,0;P[f]).$$

Lemma 3.7. *(see [1]) Let j and p be two positive integers satisfying $j \geq p+1$. Let $P[f]$ homogeneous differential polynomial with $\Gamma_P > (k+1)\underline{d}(P) - (p+1)$. Then*

$$N_p(r,0;P[f]) \leq N_{p+\Gamma_P-\underline{d}(P)}(r,0;f^{\underline{d}(P)}) + (\Gamma_P - \underline{d}(P))\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 3.8. *(see [19]) Let F and G share $(1,l)$. Then*

$$\overline{N}_L(r,1;F) \leq \frac{1}{l+1}\overline{N}(r,\infty;F) + \frac{1}{l+1}\overline{N}(r,0;F) + S(r,F) \ \ if \ \ l \geq 1,$$

and

$$\overline{N}_L(r,1;F) \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + S(r,F) \ \ if \ \ l = 0.$$

Lemma 3.9. *(see [19]) Let f be non-constant meromorphic function and $a(z)$ be a small function of f. Also, let $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{M[f]}{a}$. If F and G shares $(1,\infty)$, then one of the following cases holds.*
(i) $T(r) = N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_L(r,\infty;F) + \overline{N}_L(r,\infty;G) + S(r)$,
(ii) $F \equiv G$,
(iii) $FG \equiv 1$,
where $T(r) = max\{T(r,F),\ T(r,G)\}$ and $S(r) = \{T(r)\}$, $r \in I$, I is set of infinite linear measure of $r \in (0,\infty)$.

Lemma 3.10. *Let f be a non-constant meromorphic function and $a(z)$ be a small function in f. Let us define $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{P[f]}{a}$. Then $FG \not\equiv 1$.*

Proof. On contrary, assume that $FG \equiv 1$ i.e., $P[f]f_1^p P(f_1) \equiv a^2$. Then

$$N(r,0;f| \geq k+1) = S(r,f)$$

Now applying Lemmas 3 and 4 and First fundamental Theorem of Nevanlinna, we get

$$
\begin{aligned}
(n+m+\overline{d}(P))T(r,f) &= T\Big(r, \frac{P[f]}{f^{\underline{d}(P)}}\Big) + S(r,f)\\
&\le (\overline{d}(P)-\underline{d}(P))\Big[T(r,f) - \{N(r,0;f|\le k)\\
&+ N(r,0;f|\ge k+1)\}\Big]\\
&+ (\Gamma_P - \overline{d}(P))\overline{N}(r,\infty;f) + (\overline{d}(P)-\underline{d}(P))N(r,0;f|\ge k+1)\\
&+ Q\overline{N}(r,0;f|\ge k+1) + \overline{d}(P)N(r,0;f|\le k) + S(r,f)\\
&\le (\overline{d}(P)-\underline{d}(P))T(r,f) + \underline{d}(P)N(r,0;f|\le k)\\
&+ (\Gamma_P - \overline{d}(P))\overline{N}(r,\infty;f) + S(r,f)\\
&\le \overline{d}(P)T(r,f) + (\Gamma_P - \overline{d}(P))\overline{N}(r,\infty;P[f]f_1^pP(f_1)) + S(r,f)\\
&\le \overline{d}(P)T(r,f) + (\Gamma_P - \overline{d}(P))\overline{N}(r,\infty;a^2) + S(r,f)\\
&\le \overline{d}(P)T(r,f) + S(r,f).
\end{aligned}
$$

which is a contradiction. □

4 Main Results

Theorem 4.1. *Let k (≥ 1), n (≥ 1), p (≥ 1) and m (≥ 0) be integers and f and $f_1 = f - \omega_p$ be two non-constant meromorphic functions and $P[f]$ be a homogeneous differential polynomial of degree $\overline{d}(P)$ and weight Γ_P such that $\Gamma_P > (k+1)\underline{d}(P) - 2$, where k is the highest derivative in $P[f]$. Let $\mathcal{P}(z) = a_{m+n}z^{m+n} + \ldots + a_nz^n + \ldots + a_0$, $a_{m+n} \ne 0$, be a polynomial in z of degree $m+n$ such that $\mathcal{P}(f) = f_1^pP(f_1)$. Also let $a(z)$ $(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose $\mathcal{P}(f) - a$ and $P[f] - a$ share $(0, l)$.*
(i) If $l \ge 2$ and

$$(\Gamma_P - \underline{d}(P)+3)\Theta(\infty;f) + \mu_2\delta_{\mu_2^*}(0;f) + \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0;f) > \Gamma_P + \mu_2 + 3 - n - m.$$

(ii) or $l = 1$ and

$$
\begin{aligned}
\Big(\Gamma_P - \underline{d}(P) + \frac{7}{2}\Big)\Theta(\infty;f) + \frac{1}{2}\Theta(0;f) + \mu_2\delta_{\mu_2^*}(0;f)&\\
+ \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0;f) &> \Gamma_P + \mu_2 + 4 - n - m.
\end{aligned}
$$

(iii) or $l = 0$ and

$$
\begin{aligned}
(2(\Gamma_P - \underline{d}(P)) + 6)\Theta(\infty;f) + 2\Theta(0;f) + \mu_2\delta_{\mu_2^*}(0;f) + \underline{d}(P)\delta_{1+\Gamma_P-\underline{d}(P)}(0;f)&\\
+ \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0;f) &\le 2\Gamma_P + \mu_2 + 8 - n - m,
\end{aligned}
$$

then $\mathcal{P}(f) = P[f]$.

Proof. Let $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{P[f]}{a}$. Then $F - 1 = \frac{f_1^p P(f_1) - a}{a}$ and $G - 1 = \frac{P[f]-a}{a}$. Since $\mathcal{P}(f)$ and $P[f]$ share (a, l), it follows that F and G share $(1, l)$ except the zeros and poles of $a(z)$. Let ψ be defined as in equation (1). We consider the following cases.

Case 1. When $\psi \not\equiv 0$. Then from (1) we have $m(r, \psi) = S(r, f)$. By Nevanlinna Second Fundamental Theorem we have,

$$\begin{aligned} T(r,F) + T(r,G) \leq{}& 2\overline{N}(r,f) + \overline{N}\Big(r, \frac{1}{F}\Big) + \overline{N}\Big(r, \frac{1}{F-1}\Big) \\ &+ \overline{N}\Big(r, \frac{1}{G}\Big) + \overline{N}\Big(r, \frac{1}{G-1}\Big) - N_0\Big(r, \frac{1}{F'}\Big) \\ &- N_0\Big(r, \frac{1}{G'}\Big) + S(r,f). \end{aligned} \tag{2}$$

where $N_0\Big(r, \frac{1}{F'}\Big)$ denotes the counting function of the zeros of F' which are not zeros of $F(F-1)$ and $N_0\Big(r, \frac{1}{G'}\Big)$ denotes the counting function of the zeros of G' which are not zeros of $G(G-1)$.

Subcase 1.1. When $l \geq 1$. Then from (1) we have

$$\begin{aligned} N_E^{1)}\Big(r, \frac{1}{F-1}\Big) &\leq N\Big(r, \frac{1}{\psi}\Big) + S(r,f) \\ &\leq T(r,\psi) + S(r,f) \\ &\leq N(r,\psi) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}_{(2}\Big(r, \frac{1}{F}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{G}\Big) \\ &+ \overline{N}_L\Big(r, \frac{1}{F-1}\Big) + \overline{N}_L\Big(r, \frac{1}{G-1}\Big) + N_0\Big(r, \frac{1}{F'}\Big) \\ &+ N_0\Big(r, \frac{1}{G'}\Big) + S(r,f). \end{aligned}$$

and so,

$$\begin{aligned}\overline{N}\Big(r,\frac{1}{F-1}\Big)+\overline{N}\Big(r,\frac{1}{G-1}\Big) &= N_E^{1)}\Big(r,\frac{1}{F-1}\Big)+N_E^{(2}\Big(r,\frac{1}{F-1}\Big)+\overline{N}_L\Big(r,\frac{1}{F-1}\Big)\\ &\quad+\overline{N}_L\Big(r,\frac{1}{G-1}\Big)+\overline{N}\Big(r,\frac{1}{G-1}\Big)+S(r,f)\\ &\le \overline{N}(r,f)+\overline{N}_{(2}\Big(r,\frac{1}{F}\Big)+\overline{N}_{(2}\Big(r,\frac{1}{G}\Big)\\ &\quad+2\overline{N}_L\Big(r,\frac{1}{F-1}\Big)+2\overline{N}_L\Big(r,\frac{1}{G-1}\Big)\\ &\quad+\overline{N}_E^{(2}\Big(r,\frac{1}{F-1}\Big)+\overline{N}\Big(r,\frac{1}{G-1}\Big)\\ &\quad+N_0\Big(r,\frac{1}{F'}\Big)+N_0\Big(r,\frac{1}{G'}\Big)+S(r,f).\end{aligned} \tag{3}$$

For $l \geq 2$

$$\begin{aligned}2\overline{N}_L\Big(r,\frac{1}{F-1}\Big)+2\overline{N}_L\Big(r,\frac{1}{G-1}\Big)+\overline{N}_E^{(2}\Big(r,\frac{1}{F-1}\Big)&\\ +\overline{N}\Big(r,\frac{1}{G-1}\Big)\le N\Big(r,\frac{1}{G-1}\Big)+S(r,f).&\end{aligned}$$

Thus from (3) we obtain

$$\begin{aligned}\overline{N}\Big(r,\frac{1}{F-1}\Big)+\overline{N}\Big(r,\frac{1}{G-1}\Big) &\le \overline{N}(r,f)+\overline{N}_{(2}\Big(r,\frac{1}{F}\Big)+\overline{N}_{(2}\Big(r,\frac{1}{G}\Big)\\ &\quad+N\Big(r,\frac{1}{G-1}\Big)+N_0\Big(r,\frac{1}{F'}\Big)\\ &\quad+N_0\Big(r,\frac{1}{G'}\Big)+S(r,f).\end{aligned} \tag{4}$$

From Lemma 3.7 and inequalities (2) and (4) we get

$$\begin{aligned}(n+m)T(r,f) &\le (\Gamma_P-\underline{d}(P)+3)\overline{N}(r,f)+\mu_2 N_{\mu_2^*}(r,\omega_p;f)+N_{2+\Gamma_P-\underline{d}(P)}(r,0;f^{\underline{d}(P)})\\ &\quad+S(r,f)\\ &\le \{(\Gamma_P-\underline{d}(P)+3)-(\Gamma_P-\underline{d}(P)+3)\Theta(\infty;f)+\mu_2-\mu_2\delta_{\mu_2^*}(0;f)\\ &\quad+\underline{d}(P)-\underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0;f)+\epsilon\}T(r,f)+S(r,f).\end{aligned}$$

i.e.,

$$(\Gamma_P-\underline{d}(P)+3)\Theta(\infty;f)+\mu_2\delta_{\mu_2^*}(0;f)+\underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0;f)\le\Gamma_P+\mu_2+3-n-m.$$

which violates to the condition (i) of Theorem 4.1.
Subcase 1.2. Next, consider the case when $l = 1$. First note that

$$\overline{N}_L\Big(r, \frac{1}{F-1}\Big) \leq \frac{1}{2} N\Big(r, \frac{1}{F'}\Big| F \neq 0\Big) \leq \frac{1}{2}\overline{N}(r, F) + \frac{1}{2}\overline{N}\Big(r, \frac{1}{F}\Big), \tag{5}$$

when $N\Big(r, \frac{1}{F'}\Big| F \neq 0\Big)$ denotes the zeros of F' that are not the zeros of F. From inequalities (1) and (5), we have

$$\begin{aligned} 2\overline{N}_L\Big(r, \frac{1}{F-1}\Big) &+ 2\overline{N}_L\Big(r, \frac{1}{F-1}\Big) + \overline{N}_E^{(2}\Big(r, \frac{1}{F-1}\Big) + \overline{N}\Big(r, \frac{1}{G-1}\Big) \\ &\leq N\Big(r, \frac{1}{G-1}\Big) + \overline{N}_L\Big(r, \frac{1}{F-1}\Big) + S(r, f) \\ &\leq N\Big(r, \frac{1}{G-1}\Big) + \frac{1}{2}\overline{N}(r, F) + \frac{1}{2}\overline{N}\Big(r, \frac{1}{F}\Big) + S(r, f). \end{aligned} \tag{6}$$

Thus, from inequalities (2) and (6)

$$\begin{aligned} \overline{N}\Big(r, \frac{1}{F-1}\Big) + \overline{N}\Big(r, \frac{1}{G-1}\Big) &\leq \overline{N}(r, f) + \overline{N}_{(2}\Big(r, \frac{1}{F}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{G}\Big) + \frac{1}{2}\overline{N}(r, F) \\ &+ \frac{1}{2}\overline{N}\Big(r, \frac{1}{F}\Big) + T(r, G) + N_0\Big(r, \frac{1}{F'}\Big) + N_0\Big(r, \frac{1}{G'}\Big) \\ &+ S(r, f). \end{aligned} \tag{7}$$

From Lemma 3.6, (2) and (7), we have

$$\begin{aligned} (n+m)T(r, f) &\leq \Big(\Gamma_P - \underline{d}(P) + \frac{7}{2}\Big)\overline{N}(r, f) + \frac{1}{2}\overline{N}\Big(r, \frac{1}{f}\Big) + \mu_2 N_{\mu_2^*}(r, 0; f) \\ &+ N_{2+\Gamma_P - \underline{d}(P)}(r, 0; f^{\underline{d}(P)}) + S(r, f) \\ &\leq \Big\{\Big(\Gamma_P - \underline{d}(P) + \frac{7}{2}\Big) - (\Gamma_P - \underline{d}(P) + \frac{7}{2})\Theta(\infty; f) + \frac{1}{2} - \frac{1}{2}\Theta(0; f) \\ &+ \mu_2 - \mu_2\delta_{\mu_2^*}(0; f) + \underline{d}(P) - \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0; f) + \epsilon\Big\}T(r, f) \\ &+ S(r, f). \end{aligned}$$

i.e.,

$$\begin{aligned} \Big(\Gamma_P - \underline{d}(P) + \frac{7}{2}\Big)\Theta(\infty; f) &+ \frac{1}{2}\Theta(0; f) + \mu_2\delta_{\mu_2^*}(0; f) \\ &+ \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0; f) \leq \Gamma_P + \mu_2 + 4 - n - m, \end{aligned} \tag{8}$$

which violates to the condition (ii) of Theorem 4.1.
Subcase 1.3. When $l = 0$. Then, we have

$$N_E^{1)}\Big(r, \frac{1}{F-1}\Big) = N_E^{1)}\Big(r, \frac{1}{G-1}\Big) + S(r,f),$$

$$\overline{N}_E^{(2}\Big(r, \frac{1}{F-1}\Big) = N_E^{(2}\Big(r, \frac{1}{G-1}\Big) + S(r,f),$$

and also from (1), we get

$$\begin{aligned}
\overline{N}\Big(r, \frac{1}{F-1}\Big) + \overline{N}\Big(r, \frac{1}{G-1}\Big) &\leq N_E^{1)}\Big(r, \frac{1}{F-1}\Big) + \overline{N}_E^{(2}\Big(r, \frac{1}{F-1}\Big) \\
&+ \overline{N}_L\Big(r, \frac{1}{F-1}\Big) + \overline{N}_L\Big(r, \frac{1}{G-1}\Big) \\
&+ \overline{N}\Big(r, \frac{1}{G-1}\Big) + S(r,f) \\
&\leq N_E^{1)}\Big(r, \frac{1}{F-1}\Big) + \overline{N}_L\Big(r, \frac{1}{F-1}\Big) + \overline{N}\Big(r, \frac{1}{G-1}\Big) \\
&+ S(r,f) \\
&\leq \overline{N}(r,F) + \overline{N}_{(2}\Big(r, \frac{1}{F}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{G}\Big) \\
&+ 2\overline{N}_L\Big(r, \frac{1}{F-1}\Big) + \overline{N}_L\Big(r, \frac{1}{G-1}\Big) + N\Big(r, \frac{1}{G-1}\Big) \\
&+ N_0\Big(r, \frac{1}{F'}\Big) + N_0\Big(r, \frac{1}{G'}\Big) + S(r,f).
\end{aligned} \tag{9}$$

From Lemma 3.8, (3) and (6) we get

$$\begin{aligned}
(n+m)T(r,f) &\leq (2(\Gamma_P - \underline{d}(P)) + 6)\overline{N}(r,f) + 2\overline{N}\Big(r, \frac{1}{f}\Big) + \mu_2 N_{\mu_2^*}(r,0;f) \\
&+ N_{2+\Gamma_P-\underline{d}(P)}(r,0;f^{\underline{d}(P)}) + N_{1+\Gamma_P-\underline{d}(P)}(r,0;f^{\underline{d}(P)}) + S(r,f) \\
&\leq \{(2(\Gamma_P - \underline{d}(P)) + 6) - (2(\Gamma_P - \underline{d}(P)) + 6)\Theta(\infty;f) \\
&+ 2 - 2\Theta(0;f) + \mu_2 - \mu_2\delta\mu_2^*(0;f) \\
&+ \underline{d}(P) - \underline{d}(P)\delta_{1+\Gamma_P-\underline{d}(P)}(0;f) + \underline{d}(P) \\
&- \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0;f) + \epsilon\}T(r,f) + S(r,f).
\end{aligned}$$

i.e.,

$$\begin{aligned}
(2(\Gamma_P - \underline{d}(P)) + 6)\Theta(\infty;f) + 2\Theta(0;f) + \mu_2\delta_{\mu_2^*}(0;f) + \underline{d}(P)\delta_{1+\Gamma_P-\underline{d}(P)}(0;f) \\
+ \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0;f) \leq 2\Gamma_P + \mu_2 + 8 - n - m^{\cdot}
\end{aligned}$$

which violates condition (iii) of Theorem 4.1.
Case 2. If $H \equiv 0$. On integration of (1), we get

$$\frac{1}{G-1} = \frac{A}{F-1} + B.$$

where $A\ (\neq 0)$ and B are complex constants . Clearly, F and G share $(1, \infty)$. Also, by construction of F and G, F and G share $(\infty, 0)$. So, by Lemma 3.6 and condition (2), we obtain

$$\begin{aligned}
&N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_L(r,F) + \overline{N}_L(r,G) + S(r)\\
&\leq \mu_2 N_{\mu_2^*}(r,0;f) + N_{2+\Gamma_P-\underline{d}(P)}(r,0;f^{\underline{d}(P)}) + (\Gamma_P - \underline{d}(P)+3)\overline{N}(r,f) + S(r)\\
&\leq \{(\Gamma_P + \mu_2 + 3) - (\Gamma_P - \underline{d}(P)+3)\Theta(\infty;f) - \mu_2\delta\mu_2^*(0;f)\\
&- \underline{d}(P)\delta_{2+\Gamma_P-\underline{d}(P)}(0;f) + \epsilon\}\\
&< T(r,F) + S(r),
\end{aligned}$$

where $\epsilon > 0$ is any small quantity. Hence from the case $l \geq 2$ and Lemma 3.9 does not hold. Again in view of Lemma 3.10, we get $FG \not\equiv 1$. Therefore $F \equiv G$ i.e., $\mathcal{P}(f) \equiv P[f]$. $\square$

5 Conclusion

In this research article, we have proved one result by using the concepts of differential polynomials. In this paper, we have replaced the differential monomial by differential polynomial and $\mathcal{P}(f) = f_1^p P(f_1)$ where f is a meromorphic function. Thus, the result have been improved and generalizes the previous result obtained in [7]. Finally, for further research we are posing a question What happens if $\mathcal{P}(f) = f_1^p P(f_1)$ is replaced with $\mathcal{P}(f) = [f_1^p P(f_1)]^{(k)}$ which shares "(0,1)" ?

Acknowledgements

The authors are grateful to the anonymous reviewer for their valuable comments and suggestions towards the improvement of this paper.

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