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# Existence Results for $\psi$ -Caputo Fractional Differential Equations with Boundary Conditions

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**Abstract**

In this paper, we examine the existence and uniqueness of arrangements of a limit esteem problem for a fractional differential equation of order $\alpha \in (2, 3)$, including a general type of fractional derivative. initially, we demonstrate a comparability between the Cauchy problem and the Volterra condition. Then, at that point, two outcomes on the existence of arrangements are demonstrated, and we end for certain illustrative models.

**Keywords:** fractional differential equations, Cauchy problem,Volterra condition
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## 1. Introduction

Fractional differential conditions (FDEs) have obtained impressive importance since it has shifted applications in various applied sciences and designing [1-3]. There are different meanings of fractional calculus (FC) that created the FDEs whether in demonstrating or descript the memory precisely. The field of fragmentary differential conditions has been exposed to an escalated improvement of the hypothesis and

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applications in numerical phisics, finance, astronomy ,cosmology, combination model for low light upgrade, and in as of late applied the subjective examination in the hour of Coronavirus 19[5].

One of the notable subordinates with partial orders are Riemann-Liouville, Caputo, Weyl Hadamard and Grunwald-Letnikov, and so on, Partial administrators are assumed an indispensable part for the mathematical demonstrating and genuine applications.. As of late, some essential hypothesis for starting worth issues for fragmentary differential conditions including the Riemann-Liouville differential administrator of request $\alpha \in (0,1]$ as talked about by Lakshmikantham and , Vatsala [9,10,13]. Here we extend the work in [1] with mixed boundary conditions dependence on the lipschitz conditions. The motive of this paper is to explore the existence and uniqueness of solutions for a nonlinear partial differential equation (FDE) including the Caputo fractional derivative of a function u concerning another function $\psi$

$$\begin{cases} ({}^cD_{a+}^{\alpha,\psi}u)(t) = f\left(t, u(t), u'\left(t, u(t)\right)\right), \quad t \in [a,b] \\ u(a) = u_a \\ u'(a) = u_a^1 \\ u(b) = MI_{a+}^{\alpha,\psi}u(x) \end{cases} \tag{1}$$

Where $\alpha \in (2,3), u_a,\ u_a^1,\ M, x \in \mathbb{R}$ with $x \in (a,b], u \in C^2[a,b], ({}^cD_{a+}^{\alpha,\psi}u)(t)$ is the $\psi-$caputo fractional derivative of u [1]and consider the continuous function $f: [a,b] \times \mathbb{R} \to \mathbb{R}$.

If $f \in [a,b],\ f\left(t, u(t), u'\left(t, u(t)\right)\right) = 0$. Consider $\mathfrak{D}u(t) = u'(t, u(t))$

Then **(1)** changes to

$$\begin{cases} ({}^cD_{a+;\psi}^{\alpha,}u)(t) = f\left(t, u(t), \mathfrak{D}\mathrm{u}(\mathrm{t})\right), \quad t \in [a,b] \\ u(a) = u_a \\ u'(a) = u_a^1 \\ u(b) = MI_{a+;\psi}^{\alpha}u(x) \end{cases} \tag{2}$$

Here we postulate to the limitation.

$$K = \frac{2M}{\Gamma(\alpha+3)}\left(\psi(x) - \psi(a)\right)^{\alpha+2} - \left(\psi(b) - \psi(a)\right)^2 \quad \neq 0 \tag{3}$$

is satisfied

In this article we organized the basic needed results, definitions, lemmas and theorems . In section 3 we present the existence and uniqueness results using Banach and Leray Schauder fixed point theorems based on our system of equation (2). In section 4 we demonstrate one example based on our result.

## 2.Preliminaies

We presently review the a few documentations, definitions fundamental outcome that are utilized all through this article. By C(J,R),we mean the Banach space of all persistent capabilities from J to R with the norm $\|u\|_\infty = \sup\{|u(t)|: t \in J\}$. and an increasing differentiable function $\psi: [a, b] \to \mathbb{R} \ni: \psi'(t) \neq 0, \forall\, t \in [a, b]$

**Definition 2.1([11])** Let us defined the function $f: [0, T] \to \mathbb{R}$ be an integrable function. The $\psi$ –R-L necessary of request $\alpha > 0$ of f for $0 < t < T < +\infty$, is characterized by

$$I^{\alpha}_{0+;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\psi(t) - \psi(s)\right)^{\alpha-1} \psi'(s) f(s) ds \tag{5}$$

**Definition 2.2([6])** Considering $0 < t < T < +\infty$, the $\psi$ −R-L fractional derivative of order $\alpha > 0$ of a function $f \in AC^r_\psi([0, \mathrm{T}])$ is defined by

$$D^{\alpha}_{0+;\psi} f(t) = I^{r-\alpha}_{0+;\psi}\left(\omega^r_\psi f\right)(t) + \sum_{n=0}^{r-1} \frac{\left(\omega^n_\psi f\right)(0)}{\Gamma(n-\alpha+1)} \left(\psi(t) - \psi(0)\right)^{n-\alpha}$$

$$= \frac{1}{\Gamma(r-\alpha)} \int_0^t \left(\psi(t) - \psi(s)\right)^{r-\alpha-1} \psi'(s)\, \omega^r_\psi f(s) ds$$

$$+ \sum_{n=0}^{r-1} \frac{\left(\omega^n_\psi f\right)(0)}{\Gamma(n-\alpha+1)} \left(\psi(t) - \psi(0)\right)^{n-\alpha} \tag{6}$$

In the event that the basic exists, where $n = [\alpha] + 1$, $\Gamma(.)$ is the Euler's gamma capability characterized by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$

**Note:2.3** The fractional integral is satisfied the semigroup law, (i.e).,

$$I^{\alpha}_{a+;\psi} I^{\beta}_{a+;\psi} u(t) = I^{\alpha+\beta}_{a+;\psi} u(t), \quad \forall \alpha, \beta > 0.$$

the fractional derivative that we will bargain in our work is a Caputo type operator, and it is characterized by the articulation

$$({}^cD^{\alpha}_{a+;\psi} u)(t) = D^{\alpha}_{a+;\,\psi}\left[u(t) - \sum_{n=0}^{k-1} \frac{u^{[n]}_\psi(a)}{n!} \left(\psi(t) - \psi(a)\right)^n\right] \tag{7}$$

Where $u \in C^{n-1}[a, b], n = [\alpha] + 1$ for $\alpha \notin N, n = \alpha$ for $\alpha \in N$, and

$$u^{[n]}_\psi(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n u(t)$$

Now if $\alpha = h \in N$, we have $({}^{c}D^{\alpha}_{a+;\psi}u)(t) = u^{[\boldsymbol{h}]}_{\boldsymbol{\psi}}(\boldsymbol{t})$ And for $\alpha \notin N$, we have $({}^{c}D^{\alpha}_{a+;\psi}u)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \psi'(\xi)\big(\psi(t)-\psi(\xi)\big)^{n-\alpha-1} u^{[n]}_{\psi}(\xi)d\xi$

A significant function in fractional calculus is the Mittag-Leffler function, which is a speculation of the factorial function to real numbers, and it is given by

$$E_\alpha(Z) = \sum_{n=0}^{\infty} \frac{Z^n}{\Gamma(\alpha n + 1)}.$$

The power function derivative is given by

$$({}^{c}D^{\alpha}_{a+;\psi}\big(\psi(t)-\psi(a)\big)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\big(\psi(t)-\psi(a)\big)^{\gamma-\alpha-1}.$$

Thus the mittag leffler function is given by

$({}^{c}D^{\alpha}_{a+;\psi}E_\alpha\big(\delta\big(\psi(t)-\psi(a)\big)^{\alpha}\big)) = \delta E_\alpha\big(\delta\big(\psi(t)-\psi(a)\big)^{\alpha}\big)$n the accompanying theorem, gives the relation between fractional integrals and fractional derivatives.

**Theorem:2.4 [4]** Let $u: [a,b] \to \mathbb{R}$ be a function

a) If $u$ is continuous, then $({}^{c}D^{\alpha}_{a+;\psi}I^{\alpha}_{a+;\psi}u)(t) = u(t)$

b) If $u$ is of class $C^{n-1}$, then

$$I^{\alpha}_{a+;\psi}{}^{c}D^{\alpha}_{a+;\psi}u(t) = u(t) - \sum_{n=0}^{k-1}\frac{u^{[n]}_{\psi}(a)}{n!}\big(\psi(t)-\psi(a)\big)^{n}.$$

□

In [2.5] theorem is workout for a further class of function.

**Theorem:2.5** For $\gamma > 0$ and $\delta \in \mathbb{R} - \{0\}$, we have the subsequent couple of formulas

$$I^{\alpha}_{a+;\psi}\big(\psi(t)-\psi(a)\big)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\big(\psi(t)-\psi(a)\big)^{\gamma+\alpha-1}.$$

And $I^{\alpha}_{a+;\psi}E_\alpha\big(\delta\big(\psi(t)-\psi(a)\big)^{\alpha}\big) = \frac{1}{\delta}\Big(E_\alpha\big(\delta\big(\psi(t)-\psi(a)\big)^{\alpha}\big)-1\Big)$

**Proof**: For the first formula

$$I^{\alpha}_{a+;\psi}\big(\psi(t)-\psi(a)\big)^{\gamma-1} = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(\xi)\big(\psi(t)-\psi(\xi)\big)^{\alpha-1}\big(\psi(t)-\psi(\xi)\big)^{\gamma-1}d\xi$$

$$= \frac{1}{\Gamma(\alpha)}\big(\psi(t)-\psi(a)\big)^{\alpha-1}\int_a^t \psi'(\xi)\left(1-\frac{\psi(\xi)-\psi(a)}{\psi(t)-\psi(a)}\right)^{\alpha-1}\big(\psi(\xi)-\psi(a)\big)^{\gamma-1}d\xi$$

$$= \frac{1}{\Gamma(\alpha)}\big(\psi(t)-\psi(a)\big)^{\gamma+\alpha-1}\int_0^1 (1-p)^{\alpha-1}p^{\gamma-1}dp$$

$$= \frac{1}{\Gamma(\alpha)}\big(\psi(t)-\psi(a)\big)^{\gamma+\alpha-1}\frac{\Gamma(\gamma)\,\Gamma(\alpha)}{\Gamma(\alpha+\gamma)}$$

$$I^{\alpha}_{a+;\psi}\left(\psi(t)-\psi(a)\right)^{\gamma-1}=\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma+\alpha-1}.$$

Next,

$$I^{\alpha}_{a+;\psi}E_{\alpha}\left(\delta\left(\psi(t)-\psi(a)\right)^{\alpha}\right)=\sum_{n=0}^{\infty}\frac{\delta^n}{\Gamma(n\alpha+1)}I^{\alpha}_{a+;\psi}\left(\psi(t)-\psi(a)\right)^{n\alpha}$$

$$=\sum_{n=0}^{\infty}\frac{\delta^n}{\Gamma(n\alpha+\alpha+1)}I^{\alpha}_{a+;\psi}\left(\psi(t)-\psi(a)\right)^{n\alpha+\alpha}$$

$$=\frac{1}{\delta}\sum_{n=0}^{\infty}\frac{\delta^n}{\Gamma(n\alpha+1)}\left(\psi(t)-\psi(a)\right)^{n\alpha}$$

$$I^{\alpha}_{a+;\psi}E_{\alpha}\left(\delta\left(\psi(t)-\psi(a)\right)^{\alpha}\right)=\frac{1}{\delta}\left(E_{\alpha}\left(\delta\left(\psi(t)-\psi(a)\right)^{\alpha}\right)-1\right)$$

□

# 3. Existence and uniqueness

Our next outcome lays out an equality between issue (2) and Volterra integral equation.

**Theorem:3.1** The given system of equation is equivalent to the integral equation

$$u(t)=I^{\alpha}_{a+;\psi}f\left(t,u(t),\mathfrak{D}u(t)\right)+u_a+\frac{u_a^1}{\psi'(a)}\left(\psi(t)-\psi(a)\right)+\tau_u\left(\psi(t)-\psi(a)\right)^2,\quad t\in[a,b] \quad (\mathbf{6})$$

Where

$$\tau_u=\Big[I^{\alpha}_{a+;\psi}f\left(b,u(b),\mathfrak{D}u(b)\right)-MI^{2\alpha}_{a+;\psi}f\left(x,u(x),\mathfrak{D}u(x)\right)+u_a\Big(1-\frac{M}{\Gamma(1+\alpha)}\big(\psi(x)-$$
$$\psi a\alpha+ua1\psi'(a)\psi b-\psi a-M\,\Gamma 2+\alpha\psi x-\psi a\alpha+1/K \quad \mathbf{(7)}$$

**Proof:** By utilize the theorem 1 applying the fractional integral both sides on $({}^{c}D^{\alpha}_{a+;\psi}u)(t)=f(t,u(t),\mathfrak{D}u(t))$ and we attain the relation.

$$u(t)=I^{\alpha}_{a+;\psi}f\left(t,u(t),\mathfrak{D}u(t)\right)+\tau_0+\tau_1\left(\psi(t)-\psi(a)\right)+\tau_2\left(\psi(t)-\psi(a)\right)^2$$

Here $\tau_0,\tau_1,\tau_2$ are the constants.

Since $u(a)=u_a$ ; $u'(a)=u_a^1$ we deduce that $\tau_0=u_a$; $\tau_1=\frac{u_a^1}{\psi'(a)}$. Next we find the value of $\tau_2$

$$I^{\alpha}_{a+;\psi}u(t)=I^{\alpha}_{a+;\psi}\left(I^{\alpha}_{a+;\psi}f\left(t,u(t),\mathfrak{D}u(t)\right)+u_a+\frac{u_a^1}{\psi'(a)}\left(\psi(t)-\psi(a)\right)+\tau_2\left(\psi(t)-\psi(a)\right)^2\right)$$

$$= I_{a+;\psi}^{2\alpha} f\big(t,u(t),\mathfrak{D}u(t)\big) + \frac{u_a}{\Gamma(\alpha+1)}\big(\psi(t)-\psi(a)\big)^{\alpha} + \frac{u_a^1}{\psi'(a)\,\Gamma(\alpha+2)}\big(\psi(t)-\psi(a)\big)^{\alpha+1} + \frac{2\tau_2}{\Gamma(\alpha+3)}\big(\psi(t)-\psi(a)\big)^{\alpha+2}$$

Since $u(b) = MI_{a+;\psi}^{\alpha} u(x)$, we decide the value of $\tau_u$, in this manner finishing the initial segment of the proof. For the converse, if $u$ satisfies condition (5), applying the fractional derivative to the two sides of the equation and going to that

$$({}^{c}D_{a+;\psi}^{\alpha} u_a 0 = ({}^{c}D_{a+;\psi}^{\alpha}\big(\psi(t)-\psi(a)\big) = ({}^{c}D_{a+;\psi}^{\alpha}\big(\psi(t)-\psi(a)\big)^2 = 0$$

We attain that $({}^{c}D_{a+;\psi}^{\alpha} u(t) = f\big(t,u(t),\mathfrak{D}u(t)\big)$. □

**Note 3.2**: Now we define the functional $\mathbb{F}$ based on the theorem 3

$$\mathbb{F}(u)(t) = I_{a+;\psi}^{\alpha} f\big(t,u(t),\mathfrak{D}u(t)\big) + u_a + \frac{u_a^1}{\psi'(a)}\big(\psi(t)-\psi(a)\big) + \tau_u\big(\psi(t)-\psi(a)\big)^2 \quad (8)$$

From the equation (7) we attain the value of $\tau_u$.

**(A1)** Let us define a continuous function $f \in (C[a,b]\times\mathfrak{R}\times\mathfrak{R},\mathfrak{R})$ and $u \in C[a,b]$ and there exists a positive constants $\mathfrak{M}_1, \mathfrak{M}_2$ and such that

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \le \mathfrak{M}_1(\|u_1-u_2\| + \|v_1-v_2\|)$$

for each $u_1, u_2, v_1, v_2$ in $Y$, $\mathfrak{M}_2 = \max_{t\in R}\|f(t,0,0)\|$ and $\mathfrak{M} = \max\{\mathfrak{M}_1,\mathfrak{M}_2\}$. Let Y=C[R,X] be the set of continuous functions on R with in the Banach space X values.

**(A2)** Let $u' \in C[a,b]$ satisfy the Lipschitz condition. i.e., There exists a positive constants $\mathfrak{N}_1, \mathfrak{N}_2$ and N such that

$$\|\mathfrak{D}(t,u) - \mathfrak{D}(t,v)\| \le \mathfrak{N}_1(\|u-v\|),$$

for all u,v in Y. $\mathfrak{N}_2 = \max_{t\in D}\|\mathfrak{D}(t,0)\|$ and $\mathfrak{N} = \max\{\mathfrak{N}_1,\mathfrak{N}_2\}$.

**Lemma 3.3** If (A1) and (A2) are satisfied, then the estimate $\|\mathfrak{D}u(t))\| \le t(\mathfrak{N}_1\|u\| + \mathfrak{N}_2)$, $\|\mathfrak{D}u(t) - \mathfrak{D}v(t)\| \le \mathfrak{N}t\|u-v\|$ are satisfied for any $t\in R$ and $u,v \in Y$ and consider $\mathfrak{B} = \mathfrak{M} + \mathfrak{N}t$

**Theorem:3.4** If ∃ an integrable function $\mathfrak{B}: [a,b] \to \mathbb{R}_0^+$ ∋: from the assumption there exists then a real $\mathfrak{h} > 0$ ($\mathfrak{h} \le b-a$) then the given system (1) has a unique solution in $[a, a+\mathfrak{h}]$,

**Proof:** Let $\mathfrak{h} > 0$ be such that

$$\sup_{t\in[a,a+\mathfrak{h}]}\left\{I_{a+}^{\alpha}\mathfrak{B}(t)+[I_{a+}^{\alpha}\mathfrak{B}(b)+MI_{a+}^{2\alpha}\mathfrak{B}(x)]\frac{\left(\psi(t)-\psi(a)\right)^2}{K}\right\}<1 \tag{9}$$

And Y the set $Y=\{u\in C^2[a,a+\mathfrak{h}]:{}^{c}D_{a+;\psi}^{\alpha}\mathfrak{B}(t)$ exists and is cotinuous in $[a,a+\mathfrak{h}]$.

We prove that the function $\mathbb{F}$ is a contraction. Allow us to see that $\mathbb{F}$ is well defined. $(i.e).,\mathbb{F}(Y)\subseteq Y$. Given a function $u\in Y$, then we have the map $t\longmapsto\mathbb{F}(u)(t)$ is of class $C^2$ and the below function is continuous . $({}^{c}D_{a+;\psi}^{\alpha}\mathbb{F}(u))(t)=f(t,u(t),\mathfrak{D}u(t))$

Let $u_1,u_2\in Y$ be arbitrary, then

$$\begin{aligned}&\|\mathbb{F}(u_1)-\mathbb{F}(u_2)\|\\&\le\sup_{t\in[a,a+\mathfrak{h}]}\Bigg\{I_{a+;\psi}^{\alpha}\left|f\big(t,u_1(t),\mathfrak{D}u_1(t)\big)-f\big(t,u_2(t),\mathfrak{D}u_2(t)\big)\right|\\&\qquad+\left[I_{a+;\psi}^{\alpha}\left|f\big(t,u_1(b),\mathfrak{D}u_1(b)\big)-f\big(t,u_2(b),\mathfrak{D}u_2(b)\big)\right|\right.\\&\qquad+MI_{a+;\psi}^{2\alpha}\left|f\big(t,u_1(x),\mathfrak{D}u_1(x)\big)\right.\\&\qquad\left.-f\big(t,u_2(x),\mathfrak{D}u_2(x)\big)\right|\Big]\frac{\left(\psi(t)-\psi(a)\right)^2}{K}\Bigg\}\end{aligned}$$

$$\le\sup_{t\in[a,a+\mathfrak{h}]}\left\{I_{a+;\psi}^{\alpha}(\mathfrak{M}+\mathfrak{N}\mathrm{t})\|u_1-u_2\|+\left[I_{a+;\psi}^{\alpha}(\mathfrak{M}+\mathfrak{N}\mathrm{b})\|u_1-u_2\|+\right.\right.$$

*MIa+;ψ2α(𝔐+ 𝔑x)u1−u2ψt−ψa2K*

$$\le\sup_{t\in[a,a+\mathfrak{h}]}\left\{I_{a+;\psi}^{\alpha}\mathfrak{B}(t)\quad+\left[I_{a+;\psi}^{\alpha}\mathfrak{B}(b)+MI_{a+;\psi}^{2\alpha}\mathfrak{B}(x)\right]\frac{\left(\psi(t)-\psi(a)\right)^2}{K}\right\}\|u_1-u_2\|$$

Hence, $\mathbb{F}$ is a contraction, By using the Banach Fixed point theorem the given system of equation has unique solution. □

Review that a group of functions $\vartheta$ is equicontinuous if for each $\in>0,\exists$ some $\delta>0$ to such an extent that $d\big(u(t_1),u(t_2)\big)<\in$ for all $u\in\vartheta$ and all $t_1,t_2$ to such an extent that $d(t_1,t_2)<\delta$.

**Theorem:3.5** Suppose that ∃ the following assumption hold

**(A3)** $\exists$ *two continuous function* $H_1,H_2:[a,b]\to\mathbb{R}$ and a non-decreasing function $\Upsilon:\mathbb{R}_0^+\to\mathbb{R}_0^+$ ∋:, $\forall\, t\in[a,b]$ and $u\in\mathbb{R}$ we have

$$|f(t,u(t),\mathfrak{D}u(t))|\le H_1(t)\Upsilon(\mathfrak{B}|u|)+H_2(t)\le\mathfrak{B}H_1(t)\Upsilon(|u|)+H_2(t)$$

Then the real system $\mathbb{R}>0$ such that

$$\lambda_1 \mathfrak{B}\Upsilon(\mathbb{R}) + \lambda_2 + |u_a| + \frac{|u_a^1|}{\psi'(a)}\big(\psi(b)-\psi(a)\big) + \left| u_a\left[1 - \frac{M}{\Gamma(\alpha+1)}\big(\psi(x)-\psi(a)\big)^{\alpha}\right] + \frac{|u_a^1|}{\psi'(a)}\left[\psi(b)-\psi(a) - \frac{M}{\Gamma(\alpha+2)}\big(\psi(x)-\psi(a)\big)^{\alpha+1}\right]\right| \frac{\big(\psi(b)-\psi(a)\big)^2}{K}$$

$\leq \mathbb{R}$.

Where

$$\lambda_i = \sup\nolimits_{t\in[a,b]} I^{\alpha}_{a+;\psi}|H_i(t)| + \left(I^{\alpha}_{a+;\psi}|H_i(b)| + MI^{2\alpha}_{a+;\psi}|H_i(x)|\right)\frac{\big(\psi(b)-\psi(a)\big)^2}{K}, \quad i = 1,2$$

Then by the Cauchy problem (2) has atleast one solution in [a,b].

**Proof:** By the equation (8) $\mathbb{F}$ is given by, consider the closed ball

$$\mathbb{B}_{\mathbb{R}} = \{u \in C^2[a,b] : \ \|u\| < R\}$$

**Step 1:** $\mathbb{F}(\mathbb{B}_{\mathbb{R}}) \subseteq \mathbb{B}_{\mathbb{R}}$. Given $u \in \mathbb{B}_{\mathbb{R}}$ then

$$\|\mathbb{F}(x)\| \leq \sup_{t\in[a,b]}\left\{ I^{\alpha}_{a+;\psi}\left|f\big(t,u(t),\mathfrak{D}u(t)\big)\right| + |u_a| + \frac{|u_a^1|}{\psi'(a)}\big(\psi(t)-\psi(a)\big) + |\tau_u|\big(\psi(t)-\psi(a)\big)^2 \right\}$$

$$\leq \sup_{t\in[a,b]}\left\{ I^{\alpha}_{a+;\psi}|\mathfrak{B}H_1(t)\Upsilon(|u(t)|) + H_2(t)| + |u_a| + \frac{|u_a^1|}{\psi'(a)}\big(\psi(t)-\psi(a)\big) + \left[I^{\alpha}_{a+;\psi}|\mathfrak{B}H_1(b)\Upsilon(|u(b)|) + H_2(b)|\right] + MI^{2\alpha}_{a+;\psi}\mathfrak{B}H_1(x)\Upsilon(|u(x)|) + H_2(x) + \left| u_a\left[1 - \frac{M}{\Gamma(\alpha+1)}\big(\psi(x)-\psi(a)\big)^{\alpha}\right] + \frac{|u_a^1|}{\psi'(a)}\left[\psi(b)-\psi(a) - \frac{M}{\Gamma(\alpha+2)}\big(\psi(x)-\psi(a)\big)^{\alpha+1}\right]\right| \frac{\big(\psi(b)-\psi(a)\big)^2}{K} \right\}$$

$$\leq \lambda_1 \mathfrak{B}\Upsilon(\mathbb{R}) + \lambda_2 + |u_a| + \frac{|u_a^1|}{\psi'(a)}\big(\psi(b)-\psi(a)\big)$$
$$+\left|u_a\left[1-\frac{M}{\Gamma(\alpha+1)}\big(\psi(x)-\psi(a)\big)^{\alpha}\right]\right.$$
$$+\frac{|u_a^1|}{\psi'(a)}\Big[\psi(b)-\psi(a)$$
$$\left.-\frac{M}{\Gamma(\alpha+2)}\big(\psi(x)-\psi(a)\big)^{\alpha+1}\Big]\right|\frac{\big(\psi(b)-\psi(a)\big)^2}{K}$$
$$\leq \mathbb{R}.$$

**Step :2** $\mathbb{F}(\mathfrak{B}_{\mathbb{R}})$ is equicontinuous. Let $t_1, t_2 \in [a,b]$ with $t_1 > t_2$ and

$$L = \max_{(\mathrm{t,x})\in[\mathrm{a,b}]\times\mathfrak{B}_{\mathbb{R}}} \big|f\big(t,u(t),\mathfrak{D}u(t)\big)\big|$$

Thus $|\mathbb{F}(u)(t_1) - \mathbb{F}(u)(t_2)|$
$$\leq \frac{L}{\Gamma(\alpha+1)}\Big[\big(\psi(t_1)-\psi(a)\big)^{\alpha} - \big(\psi(t_2)-\psi(a)\big)^{\alpha}\Big] +$$
$$\frac{|u_a^1|}{\psi'(a)}\big(\psi(t_1)-\psi(t_2)\big) + |\tau_u|\Big[\big(\psi(t_1)-\psi(a)\big)^2 - \big(\psi(t_2)-\psi(a)\big)^2\Big]$$

As $t_1 \to t_2$ and which combines to zero. By the Ascoli-Arzela Hypothesis, $\mathbb{F}(\mathbb{B}_{\mathbb{R}})$ is contained in a compact set. At last, we presently demonstrate that there are no $u \in \partial\mathbb{B}_{\mathbb{R}}$ and $\rho \in (0,1)$ $\ni: u = \rho\mathbb{F}(u)$. For that reason, assume that such u and $\rho$ exist. For this situation,

$$\mathbb{R} = \|u\| < \|\mathbb{F}(u)\| \leq \mathbb{R}$$

Now we have seen previously, which show our case. All in all, by the Leray-Schauder alternative, we demonstrate the presence of a fixed point to $\mathbb{F}$. □

# 4. Example

In this section we discuss the problem which is related to the main result.Consider the fractional differential equation

$$\begin{cases} ({}^{c}D^{\alpha}_{a+;\psi}u)(t) = \dfrac{\mathrm{t}}{3\sqrt{\pi}}\sin\left(\mathrm{u(t)}+\mathrm{u}'(\mathrm{t})\right),, \qquad t \in [a,b] \\ u(0) = u_0 \\ u'(0) = u_0^1 \\ u(1) = MI^{2.5}_{0+;\psi}u(0.5) \end{cases}$$

Now consider Where $\alpha \in (2,3), u_a,\ u_a^1,\ M\ , x \in\ \mathbb{R}$. From [2] we consider the value of M. Let $u, v \in [0,\infty)$ $and\ \ t \in J, thus\ we\ have$

$$|\mathrm{f(t,u_1,v_1)} - \mathrm{f(t,u_2,v_2)}| \le \frac{\mathrm{t}}{3\sqrt{\pi}}(|\mathrm{u_1}-\mathrm{u_2}| + |\mathrm{v_1}-\mathrm{v_2}|),$$

$$\le \frac{1}{3\sqrt{\pi}}|\mathrm{u}-\mathrm{v}| \le \frac{1}{5}|\mathrm{u}-\mathrm{v}|$$

Hence from the assumption $\mathfrak{B}(t) = \frac{1}{5}, u, v \in [0,\infty)$, then we have by the theorem 4 we conclude that

$$I^{\alpha}_{a+}\mathfrak{B}(t) + [I^{\alpha}_{a+}\mathfrak{B}(b) + MI^{2\alpha}_{a+}\mathfrak{B}(x)]\frac{\left(\psi(t)-\psi(a)\right)^2}{K} < 1$$

Thus, the existence and uniqueness of the solution in the interval [0, 1].

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