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# Status Connectivity indices of Middle graph

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**Abstract**

Topological index is sometimes also known as graph theoretic index, is a numerical invariant of a graph, the topological indices are classified on degree and distance based concepts. The status $\sigma(u)$ of a vertex $u \in V(G)$ is defined as the sum of its distances between each other vertex in V(G) of the graph G. Ramane and Yalnaik defined the distance based topological indices such as first and second status connectivity indices. In this article bounds for the first and second status connectivity indices of middle graph of a graph are established and further status connectivity indices of middle graph of certain graphs are computed.

**Keywords**: Middle graph; Status connectivity indices; Sunlet graph; Tree.
**2020 AMS subject classifications**: 05C10, 05C38, 05C76, 05C05.

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# 1 Introduction

Let G={V, E} be a graph of order n and size m. The edge is formed by joining any two vertices. A vertex degree u denoted by deg(u) and is defined as the number of edges that are incident on it. The length of the shortest path connecting the any two vertices u and v in G is the distance between the vertices u and v and it is denoted by d(u, v). The eccentricity of a vertex u is defined as $e(u) = max\{d(u,v)|v \in V(G)\}$. The diameter of G is the maximum distance between any two vertices and it is denoted by diam(G).

Topological index also known as molecular descriptors, play a vital role in the fields including chemical engineering, material science, chemistry and so on. Topological indices are classified on degree and distance based concepts. Oldest degree based index were introduced by Gutman and Trinajstić [1] in 1972 and the oldest distance based index was introduced by Weiner [14] in 1947 while working on the paraffin boiling point. The Wiener index of graph G is defined as

$$W(G) = \sum_{uv \in E(G)} d(u,v)$$

The status [4] of a vertex $u \in V(G)$ is defined as the sum of its distances between each other vertex in V(G) and it is denoted by $\sigma(u)$.

$$i.e, \sigma(u) = \sum_{v \in V(G)} d(u,v)$$

Status connectivity indices were defined by Ramane and Yalnaik [11]

$$i.e, S_1(G) \quad = \quad \sum_{uv \in E(G)} [\sigma(u) + \sigma(v)]$$

and

$$S_2(G) \quad = \quad \sum_{uv \in E(G)} [\sigma(u)\sigma(v)]$$

$S_1(G)$ and $S_2(G)$ are the first and second status connectivity indices respectivity.
For details on status connectivity indices see [7, 10, 11, 12].

Hamada and Yoshimura introduced the middle graph M(G) concept [2]. M(G) is the graph whose vertex set is $V(G) \cup E(G)$ and is defined as, two vertices are connected in M(G) if and only if either they are adjacent edges in G or one vertex of G and the other is an edge incident with it and for related work on indices and graph valued functions can be found in [6, 8, 9]. In this article status connectivity indices for middle graph of connected graph are established. Throughout this paper we consider graph which are simple and connected. For undefined terms related to graph theory, we refer the book [5].

**Definition 1.1.** *[3] The friendship graph $F_n$ obtained by connecting n copies of triangles with a common vertex, which is a universal vertex for the graph.*

**Definition 1.2.** *[13] The cycle $C_n$ is converted into a graph called the n-sunlet graph $S_n$ by adding n - pendant edges to each vertices of $C_n$.*

The following remarks are immidiately used in theorem
Remark 1: $S_1(G) \leqslant S_1(M(G))$
Remark 2: $S_2(G) \leqslant S_2(M(G))$

Status connectivity index of the graph increase, when new edges or new vertices are added to the graph. by the definition of middle graph, edges are consider as vertices. So, number of vertices and edges are more than the graph $G$ hence, Status connectivity index of middle graph is more than the graph.
Remark 3: $S_1 M(K_{1,q}) \leqslant S_1(M(P_n)), n \geqslant 3$
Remark 4: $S_2 M(K_{1,q}) \leqslant 3\, S_2(M(P_n)), n \geqslant$

Status of the vertex is increase along with distance. If status of the vertex increase then, status connectivity index graph also increase. The middle graph of star graph has the $diam = 3$ but, in middle graph of path graph the diameter always $diam > 3$ so, status connectivity index of middle graph of star graph is always less than or equal to middle graph of path.

# 2 Bounds for status connectivity indices of middle graph

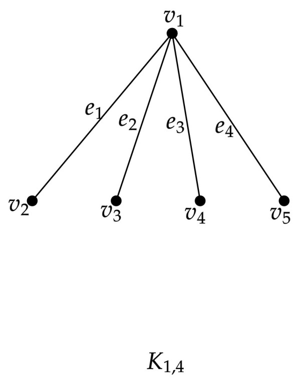
**Theorem 2.1.** *Let G be a connected graph of order $n \geq 3$ then,*

$$(n-1)[(14n-18)] + [(n^2-3n+2)](3n-4) \leqslant S_1(M(G)) \leqslant n(2n^3-3n^2+1) + 2nm(n-2)(n-1)$$

*and*

$$(n-1)[(24n^2-62n+40)] + \frac{1}{2}[n^2-3n+2](3n-4)^2 \leqslant S_2(M(G)) \leqslant n^2(n-1)^2[(n^2-1)+(nm-2m)]$$

*equality for lower bound holds, when G=$K_{1,q}$ where q=n-1 , $n > 1$ and equality for upper bound holds, when G= $K_n$.*



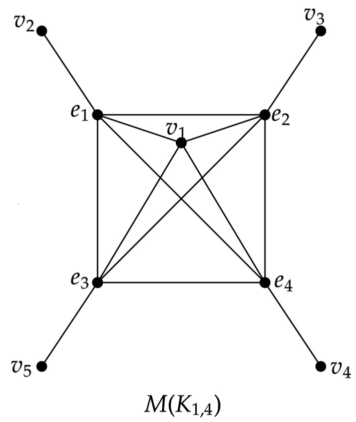




Figure 1: Star graph $K_{1,4}$ and their middle graph $M(K_{1,4})$.

***Proof*.** *For the equality part of the lower bound, let G= $K_{1,q}$ be the star graph with n vertices m edges. $M(K_{1,q})$ be the middle graph of star graph having n+m vertices and $\frac{m^2+3m}{2}$ edges, it can be verified from the figure 1, then,*

*Status of the vertices having different degree in $M(K_{1,q})$ is as follows*
*Vertex having degree 1,*

$$\begin{aligned}\sigma(u) &= \sum_{v\in V(M(K_{1,q}))} d(u,v)\\ &= 5n-7\end{aligned}$$

*Vertex having the degree n,*

$$\begin{aligned}\sigma(u) &= \sum_{v\in V(M(K_{1,q}))} d(u,v)\\ &= 3n-4\end{aligned}$$

*Vertex having the degree n-1,*

$$\begin{aligned}\sigma(u) &= \sum_{v\in V(M(K_{1,q}))} d(u,v)\\ &= 3n-3\end{aligned}$$

*The edge set of $M(K_{1,q})$ is partitioned as,*

$E_1 = (uv|[deg(u) = n, deg(v) = 1]),|E_1| = n-1.$
$E_2 = (uv|[deg(u) = n-1, deg(v) = n]),\ |E_2| = n-1.$

$E_3 = (uv|[deg(u) = n, deg(v) = n])$, $|E_3| = \frac{1}{2}[n^2 - 3n + 2]$.

*by definition,*

$$\begin{aligned}
S_1(M(K_{1,q})) &= \sum_{uv \in E(M(K_{1,q}))} [\sigma(u) + \sigma(v)] \\
&= \sum_{uv \in E_1} (5n - 7 + 3n - 4) + \sum_{uv \in E_2} (3n - 3 + 3n - 4) \\
&\quad + \sum_{uv \in E_3} (3n - 4 + 3n - 4) \\
&= (n-1)(8n-11) + (n-1)(6n-7) \\
&\quad + \frac{1}{2}[n^2 - 3n + 2](6n - 8) \\
&= (n-1)(14n-18) + (n^2 - 3n + 2)(3n - 4) \qquad (1)
\end{aligned}$$

*and*

$$\begin{aligned}
S_2(M(K_{1,q})) &= \sum_{uv \in E(M(K_{1,q}))} [\sigma(u)\sigma(v)] \\
&= \sum_{uv \in E_1(G)} (5n-7)(3n-4) + \sum_{uv \in E_2(G)} (3n-3)(3n-4) + \\
&\quad \sum_{uv \in E_3(G)} (3n-4)(3n-4) \\
&= (n-1)(15n^2 - 41n + 28) + (n-1)(9n^2 - 21n + 12) \\
&\quad + \frac{1}{2}[n^2 - 3n + 2](3n-4)^2 \\
&= (n-1)(24n^2 - 62n + 40) + \frac{1}{2}[(n^2 - 3n + 2) \\
&\quad (3n-4)^2]. \qquad (2)
\end{aligned}$$

*Equality For the upper bounds, let G=* $K_n$ *be the complete graph with n vertices and m edges.* $M(K_n)$ *be the middle graph of complete graph having n+m vertices and nm edges. Status of vertex of* $M(K_n)$ *which having degree n-1 is* $n^2 - 1$ *and status of the vertex of* $M(K_n)$*which having the degee 2(n-1) is* $n^2 - n$*. The edge set of* $M(K_n)$ *is partitioned as,*

$E_1 = (uv|deg(u) = n - 1, deg(v) = 2n - 2)$, $|E_1| = n(n - 1)$,
$E_2 = (uv|deg(u) = deg(v) = 2n - 2)$, $|E_2| = nm - 2m$.

*by definition,*

$$\begin{aligned} S_1(M(K_n)) &= \sum_{uv\in E(M(K_n))} [\sigma(u)+\sigma(v)] \\ &= \sum_{uv\in E_1}(n^2-1+n^2-n)+\sum_{uv\in E_2}(n^2-n)+(n^2-n) \\ &= n(n-1)(2n^2-n-1)+m(n-2)(2n(n-1)) \\ &= n(2n^3-3n^2+1)+2nm(n-2)(n-1) \qquad (3) \end{aligned}$$

$$\begin{aligned} S_2(M(K_n)) &= \sum_{uv\in E(M(K_n))} [\sigma(u)\sigma(v)] \\ &= \sum_{uv\in E_1}(n^2-1)(n^2-n)+\sum_{uv\in E_2}(n^2-n)(n^2-n) \\ &= n(n-1)((n^2-1)(n^2-n))+(nm-2m)n^2(n-1)^2 \\ &= n^2(n-1)^2[(n^2-1)+(nm-2m)]. \qquad (4) \end{aligned}$$

*For Lower bounds, G only has a minimum number of edges if it is a tree. M(G) has least number of edges if $G \cong T$. M(T) has minimum number of edges compared to every other graph M(G).*

*Status connectivity index is decreases when number of vertices or edges are deleted from the graph.*
*by Remark (3) and (4),*

$$S_1(M(G)) \geqslant S_1(M(K_{1,q}))$$
$$S_2(M(G)) \geqslant S_2(M(K_{1,q}))$$

*from equation (1)*

$$S_1(M(G)) \geqslant (n-1)[(14n-18)]+(n^2-3n+2)(3n-4) \qquad (5)$$

*from equation (2)*

$$S_2(M(G)) \geqslant (n-1)[(24n^2-62n+40)]+\frac{1}{2}(n^2-3n+2)(3n-4)^2 \qquad (6)$$

*For upper bounds, G has maximum number of edges if and only if G is complete graph, M(G) has maximum number of edges if and only if $G \cong K_n$. When additional edges or vertices are joined to the graph, the graph's status connectivity index extends and comparing to other middle graph of G, $M(K_n)$ has maximum number of edges.*

*Therefore,*

$$S_1(M(G)) \leqslant S_1(M(K_n))$$
$$S_2(M(G)) \leqslant S_2(M(K_n))$$

*from equation (3)*

$$S_1(M(G)) \leqslant n(2n^3 - 3n^2 + 1) + 2nm(n-2)(n-1) \tag{7}$$

*from equation (4)*

$$S_2(M(G)) \leqslant n^2(n-1)^2[(n^2-1) + (nm - 2m)] \tag{8}$$

*Hence, from the equation (5), (6), (7) and (8) the status connectivity indices of the middle graph will be,*

$$(n-1)[(14n-18)] + [(n^2-3n+2)](3n-4) \leqslant S_1(M(G)) \leqslant n(2n^3 - 3n^2 + 1) + 2nm(n-2)(n-1)$$

*and*

$$(n-1)[(24n^2 - 62n + 40)] + \frac{1}{2}[n^2 - 3n + 2](3n-4)^2 \leqslant S_2(M(G)) \leqslant n^2(n-1)^2[(n^2-1) + (nm - 2m)].$$

□

# 3 Status connectivity indices of middle graph of some class of graphs

**Theorem 3.1.** *Let $C_n$ be the cycle graph then,*

$$S_1(M(C_n)) = 2n(n^2 + 2n - 1) + m(n^2 + n)$$

*and*

$$S_2(M(C_n)) = \frac{n(n^2+3n-2)(n^2+n)}{2} + \frac{m(n^2+n)}{4}.$$

***Proof.*** *Let $C_n$ be the cycle graph with n vertices and m edges. $M(C_n)$ be the middle graph of the $C_n$ has 2n vertices, 2n+m edges.*
*Status of vertex of $M(C_n)$ having the degree 2 is*

$$\begin{aligned} \sigma(v) &= \sum_{u \in V(M(K_n))} d(u,v) \\ &= 1 + 2 + 3 + 4 + \cdots + (n-1) + (2n-1) \\ &= \frac{n^2 + 3n - 2}{2} \end{aligned}$$

*Status of the vertex of $M(C_n)$ having the degree 4 is*

$$\begin{aligned}\sigma(v) &= \sum_{u\in V(M(K_n))} d(u,v)\\ &= 1+2+3+\cdots+n\\ &= \frac{n^2+n}{2}\end{aligned}$$

*The edge set of $M(C_n)$ is partitioned as,*
$E_1 = (uv|deg(u) = 4, deg(v) = 2)$, $|E_1| = 2n$
$E_2 = (uv|deg(u) = deg(v) = 4)$ , $|E_2| = m$.

*by definition,*

$$\begin{aligned}S_1(M(C_n)) &= \sum_{uv\in E(M(C_n))} [\sigma(u)+\sigma(v)]\\ &= \sum_{uv\in E_1}\left(\frac{n^2+3n-2}{2}+\frac{n(n+1)}{2}\right)+\sum_{uv\in E_2}\left(\frac{n^2+n}{2}\right)+\left(\frac{n^2+n}{2}\right)\\ &= 2n(n^2+2n-1)+m(n^2+n).\end{aligned}$$

*and*

$$\begin{aligned}S_2(M(C_n)) &= \sum_{uv\in E(M(C_n))} [\sigma(u)\sigma(v)]\\ &= \sum_{uv\in E_1}\left(\frac{n^2+3n-2}{2}\right)\left(\frac{n^2+n)}{2}\right)+\sum_{uv\in E_2}\left(\frac{n^2+n}{2}\right)\left(\frac{n^2+n}{2}\right)\\ &= \frac{n(n^2+3n-2)(n^2+n)}{2}+\frac{m(n^2+n)}{4}.\end{aligned}$$

□

**Theorem 3.2.** *Let $K_{p,q}$ be the complete bipartite graph then,*

$$S_1(M(K_{p,q})) = 2m(n^2+3n-5)+m(\tfrac{n}{2}-1)(n^2+2n-4)$$

*and*

$$S_2(M(K_{p,q}) = \tfrac{m}{2}(n^2+4n+6)(n^2+2n-4)+\tfrac{m}{4}(\tfrac{n}{2}-1)(n^2+2n-4)^2.$$

***Proof.*** *Let $K_{p,q}$, $p = q$ with n=(p+q) vertices and m=(pq) edges and $M(K_{p,q})$ be the middle graph of complete bipartite graph. Status of vertex of $M(K_{p,q})$ having degree $\frac{n}{2}$ is $\frac{n^2+4n-6}{2}$ and status of the vertex of $M(K_{p,q})$ having the degree n is $\frac{n^2+2n-4}{2}$.*

*The edge set of $M(K_{p,q})$ partitioned as,*
$E_1 = (uv|deg(u) = \frac{n}{2}, deg(v) = n)$, $|E_1| = 2m$
$E_2 = (uv|deg(u) = deg(v) = n)$, $|E_2| = m(\frac{n}{2} - 1)$.

*by definition,*

$$\begin{aligned} S_1(M(K_{p,q})) &= \sum_{uv\in E((M(K_{p,q})))} [\sigma(u) + \sigma(v)] \\ &= \sum_{uv\in E_1} (\frac{n^2+4n-6}{2} + \frac{n^2+2n-4}{2}) + \sum_{uv\in E_2} (\frac{n^2+2n-4}{2} \\ &\quad + \frac{n^2+2n-4}{2}) \\ &= 2m(n^2+3n-5) + m(\frac{n}{2} - 1)(n^2+2n-4) \end{aligned}$$

*and*

$$\begin{aligned} S_2(M(K_{p,q})) &= \sum_{uv\in E((M(K_{p,q})))} [\sigma(u)\sigma(v)] \\ &= \sum_{uv\in E_1(G)} (\frac{n^2+4n-6}{2})(\frac{n^2+2n-4}{2}) + \sum_{uv\in E_2(G)} (\frac{n^2+2n-4}{2}) \\ &\quad (\frac{n^2+2n-4}{2}) \\ &= \frac{m}{2}(n^2+4n+6)(n^2+2n-4) + \frac{m}{4}(\frac{n}{2} - 1)(n^2+2n-4)^2. \end{aligned}$$

□

**Theorem 3.3.** *Let $F_n$ be the friendship graph where n is number of triangles then,*

$$S_1(M(F_n)) = 2n(16n^2 + 72n - 28)$$
*and*
$$S_2(M(F_n)) = 2n(64n^3 + 377n^2 - 300n + 57)$$

***Proof.*** *Let $M(F_n)$ be the middle graph of friendship graph, status of vertex of $M(F_n)$ which having degree 2 is $8 + 13(n - 1)$, status of the vertex of $M(F_n)$*

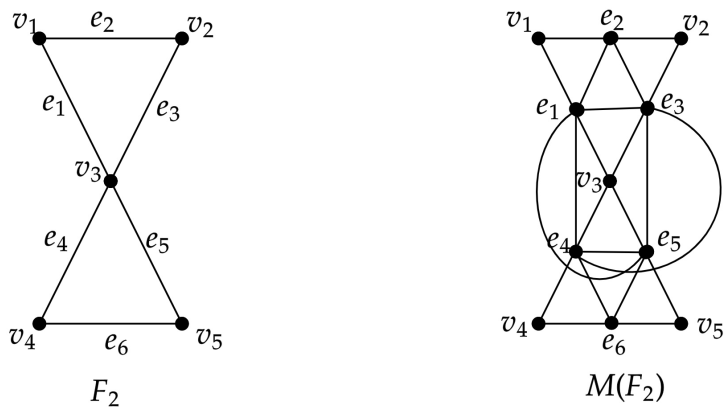



Figure 2: Friendship graph $F_2$ and their middle graph $M(F_2)$.

*having degree 2n is* $2n + 6n$*, status of vertex of* $M(F_n)$ *having degree 2n+2 is* $6 + 8(n-1)$ *and status of vertex of* $M(F_n)$ *having degree 4 is* $6 + 13(n-1)$*, it can be verified from the figure 2.*
*The edge set of* $M(F_n)$ *is partitioned as,*
$E_1 = (uv|deg(u) = 2, deg(v) = 4)$, $|E_1| = 2n$
$E_2 = (uv|deg(u) = 2, deg(v) = 2n+2)$, $|E_2| = 2n$
$E_3 = (uv|deg(u) = 4, deg(v) = 2n+2)$, $|E_3| = 2n$
$E_4 = (uv|deg(u) = deg(v) = 2n+2)$, $|E_4| = n(2n-1)$
$E_5 = (uv|deg(u) = 2n+2, deg(v) = 2n)$, $|E_5| = 2n$.

*by definition,*

$$
\begin{aligned}
& S_1(M(F_n)) \\
&= \sum_{uv\in E(M(F_n))} \sigma(u)+\sigma(v) \\
&= \sum_{uv\in E_1} (8+13(n-1)+6+13n-13) + \sum_{uv\in E_2} (8+13n-13)+(6+8n-8) \\
&\quad + \sum_{uv\in E_3} (6+13(n-1)+6+8(n-1)) + \sum_{uv\in E_4} (6+8n-8)+(6+8n-8) \\
&\quad + \sum_{uv\in E_5} (6+8(n-1)+8n) \\
&= 2n(26n-12)+2n(21n-7)+2n(21n-9)+n(2n-1)(16n-4) \\
&\quad +2n(16n-2) \\
&= 2n[8n^2+78n-29]
\end{aligned}
$$

*and*

$$
\begin{aligned}
&S_2(M(F_n)) \\
&= \sum_{uv\in E(M(F_n))} \sigma(u)\sigma(v) \\
&= \sum_{uv\in E_1}(8+13n-13)(6+13n-13)+\sum_{uv\in E_2}(8+33n-13)(6+8n-8) \\
&\quad +\sum_{uv\in E_3}(6+13(n-1))(6+8(n-1))+\sum_{uv\in E_4}(6+8n-8)(6+8n-8) \\
&\quad +\sum_{uv\in E_5}(6+8(n-1))(8n) \\
&= 2n(13n-5)(13n-7)+2n(13n-5)(8n-2)+2n(13n-7)(8n-2)+ \\
&\quad n(2n-1)(8n-2)^2+2n(6+8(n-1))(8n) \\
&= 2n[32n^3+409n^2-310n+58].
\end{aligned}
$$

□

Illustration 1: Let $M(F_2)$ be the middle graph of friendship graph $F_2$.
From the figure 2, $V(M(F_2))=\{v_1,v_2,v_3,v_4,v_5,e_1,e_2,e_3,e_4,e_5,e_6\}$.
From the definition of status of the vertex,

$$
\begin{aligned}
\sigma(v_1)=21=\sigma(v_2)=\sigma(v_5)=\sigma(v_5) \\
\sigma(e_2)=19=\sigma(e_6) \\
\sigma(e_1)=14=\sigma(e_3)=\sigma(e_4)=\sigma(e_5) \\
\sigma(v_3)=16
\end{aligned}
\tag{9}
$$

$$
S_1(M(F_2)) = \sum_{uv\in E(M(F_2))}\sigma(u)+\sigma(v)
$$

$$
\begin{aligned}
&= [\sigma(v_1)+\sigma(e_2)]+[\sigma(v_2)+\sigma(e_2)]+[\sigma(e_1)+\sigma(e_2)]+[\sigma(e_3)+\sigma(e_2)] \\
&\quad +[\sigma(e_1)+\sigma(e_3)]+[\sigma(e_1)+\sigma(e_4)]+[\sigma(e_1)+\sigma(e_5)]+[\sigma(e_3)+\sigma(e_5)] \\
&\quad +[\sigma(e_3)+\sigma(e_4)]+[\sigma(e_1)+\sigma(v_3)]+[\sigma(e_3)+\sigma(v_3)]+[\sigma(v_3)+\sigma(e_4)] \\
&\quad +[\sigma(v_3)+\sigma(e_5)]+[\sigma(e_4)+\sigma(e_5)]+[\sigma(e_4)+\sigma(e_6)]+[\sigma(e_5)+\sigma(e_6)] \\
&\quad +[\sigma(v_4)+\sigma(e_4)]+[\sigma(e_5)+\sigma(v_5)]+[\sigma(v_4)+\sigma(e_6)]+[\sigma(v_5)+\sigma(e_6)] \\
&\quad +[\sigma(v_1)+\sigma(e_1)]+[\sigma(v_2)+\sigma(e_3)].
\end{aligned}
$$

Substitute the values from the equation (9) we get,

$$S_1(M(F_2)) \;=\; 720 \tag{10}$$

From the Theorem 3.3
$S_1(M(F_n)) = 2n(16n^2 + 72n - 28)$,
Put $n = 2$

$$\begin{aligned} S_1(M(F_2)) &= 2(2)(16(2^2) + 72(2) - 28) \\ S_1(M(F_2)) &= 720. \end{aligned} \tag{11}$$

The equation (10) and (11) are equal.
Hence the result.

**Theorem 3.4.** *For $S_n$, $n \geqslant 3$ be the Sunlet graph having 2n vertices and 2n edges then,*

$$S_1(M(S_n)) = \begin{cases} n[14n^2 + 55n - 15] & \text{if n is even ,} \\ n[14n^2 + 55n - 16] & \text{if n is odd .} \end{cases}$$

$$S_2(M(S_n)) = \begin{cases} 7n^5 + 55n^4 + \frac{379}{4}n^3 - \frac{254}{4}n^2 + 10n & \text{if n is even,} \\ 7n^5 + 55n^4 + \frac{375}{4}n^3 - 68n^2 + \frac{45}{4}n & \text{if n is odd.} \end{cases}$$

***Proof.*** *$M(S_n)$ be the middle graph of sunlet graph has 4n vertices and 7n edges.*
*If n is even then, status of vertex of $M(S_n)$ which having degree 3 is $n^2 + \frac{9}{2}n - 1$, status of the vertex of $M(S_n)$ having the degree 1 is $n^2 + \frac{17}{2}n - 4$, status of vertex of $M(S_n)$ which having degree 4 is $n^2 + \frac{9}{2}n - 2$ and status of vertex of $M(S_n)$ which having degree 6 is $n^2 + \frac{5}{2}n$, it can be verified from the figure 3.*

*If n is odd then, the status of vertex which having degree 3 is $n^2 + \frac{9}{2}n - \frac{3}{2}$, status of the vertex of $M(S_n)$ which having the degree 1 is $n^2 + \frac{17}{2}n - \frac{9}{2}$, status of vertex which having degree 4 is $n^2 + \frac{9}{2}n - \frac{5}{2}$ and status of vertex of $M(S_n)$ which having degree 6 is $n^2 + \frac{5}{2}n + \frac{1}{2}$.*
*We partition edge set of $M(S_n)$ as,*
$E_1 = (uv|deg(u) = 6, deg(v) = 3), |E_1| = 2n$
$E_2 = (uv|deg(u) = 4, deg(v) = 3), |E_2| = n$
$E_3 = (uv|deg(u) = 1, deg(v) = 4), |E_3| = n$
$E_4 = (uv|deg(u) = 4, deg(v) = 6), |E_4| = 2n$
$E_5 = (uv|deg(u) = deg(v) = 6), |E_5| = n.$

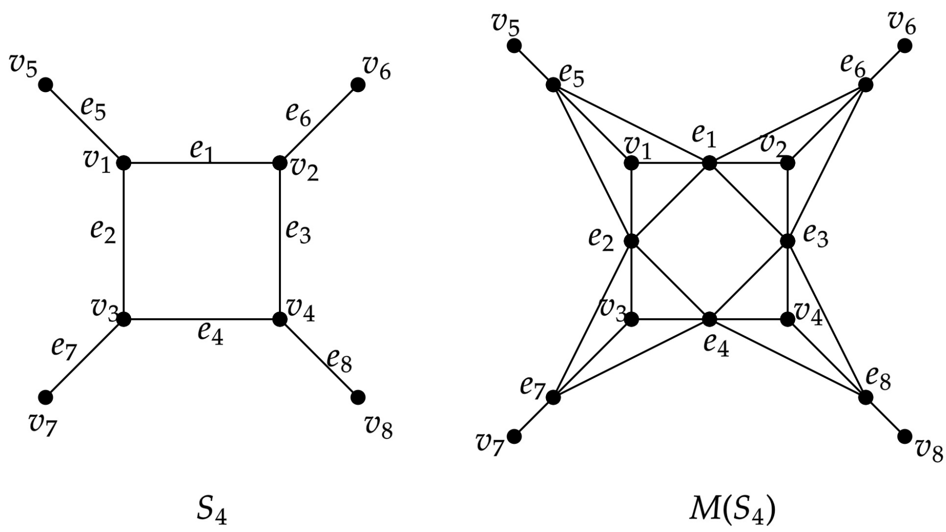



Figure 3: Sunlet graph $S_4$ and their middle graph $M(S_4)$.

*by definition,*

$S_1(M(S_n))$

$$\begin{aligned}
&= \sum_{uv\in E(M(S_n))} (\sigma(u)+\sigma(v))\\
&= \sum_{uv\in E_1}(\frac{2n^2+9n-2}{2}+\frac{2n^2+5n}{2})+\sum_{uv\in E_2}(\frac{2n^2+9n-2}{2}+\frac{2n^2+9n-4}{2})\\
&\quad+\sum_{uv\in E_3}(\frac{2n^2+9n-2}{2}+\frac{2n^2+17n-8}{2})+\sum_{uv\in E_4}(\frac{2n^2+9n-4}{2}+\frac{2n^2+5n}{2})\\
&\quad+\sum_{uv\in E_5}(\frac{2n^2+5n}{2}+\frac{2n^2+5n}{2})\\
&= 2n(2n^2+13n-6)+n(2n^2+9n-3)+n(2n^2+7n-1)+\\
&\quad 2n(2n^2+7n-2)+n(2n^2+5n)\\
&= n[14n^2+55n-15]
\end{aligned}$$

$$
\begin{aligned}
& S_1(M(S_n)) \\
= & \sum_{uv\in E(M(S_n))} \sigma(u)+\sigma(v) \\
= & \sum_{uv\in E_1} \frac{2n^2+9n-3}{2}+\frac{n^2+5n+1}{2}+\sum_{uv\in E_2}\frac{2n^2+9n-5}{2}+\frac{2n^2+9n-3}{2}+ \\
& \sum_{uv\in E_3} \frac{2n^2+9n-5}{2}+\frac{2n^2+17n-9}{2}+\sum_{uv\in E_4}\frac{2n^2+9n-5}{2}+\frac{2n^2+5n+1}{2}+ \\
& \sum_{uv\in E_5} \frac{2n^2+5n+1}{2}+\frac{2n^2+5n+1}{2} \\
= & 2n(2n^2+13n-7)+n(2n^2+9n-4)+n(2n^2+7n-1)+2n(2n^2+7n-2) \\
& +n(2n^2+5n+1) \\
= & n[14n^2+55n-16].
\end{aligned}
$$

*and*

$$
\begin{aligned}
& S_2(M(S_n)) \\
= & \sum_{uv\in E(M(S_n))} (\sigma(u)+\sigma(v)) \\
= & \sum_{uv\in E_1} \frac{(2n^2+9n-2)(2n^2+5n)}{4}+\sum_{uv\in E_2}\frac{(2n^2+9n-2)(2n^2+9n-4)}{4}+ \\
& \sum_{uv\in E_3} \frac{(2n^2+9n-2)(2n^2+17n-8)}{4}+\sum_{uv\in E_4}\frac{(2n^2+9n-4)(2n^2+5n)}{4}+ \\
& \sum_{uv\in E_5} \frac{(2n^2+5n)(2n^2+5n)}{4} \\
= & 2n\frac{(2n^2+9n-2)(2n^2+5n)}{4}+n\frac{(2n^2+9n-2)(2n^2+9n-4)}{4} \\
& +n\frac{(2n^2+9n-2)(2n^2+17n-8)}{4}+2n\frac{(2n^2+9n-4)(2n^2+5n)}{4} \\
& +n\frac{(2n^2+5n)(2n^2+5n)}{4} \\
= & 7n^5+55n^4+\frac{379}{4}n^3-\frac{254}{4}n^2+10n.
\end{aligned}
$$

$$\begin{aligned}
&S_2(M(S_n))\\
&= \sum_{uv\in E(M(S_n))} (\sigma(u)\sigma(v))\\
&= \sum_{uv\in E_1} \frac{(2n^2+9n-3)(2n^2+5n+1)}{4} + \sum_{uv\in E_2} \frac{(2n^2+9n-5)(2n^2+9n-3)}{4} +\\
&\quad \sum_{uv\in E_3} \frac{(2n^2+9n-5)(2n^2+17n-9)}{4} + \sum_{uv\in E_4} \frac{(2n^2+9n-5)(2n^2+5n+1)}{4}\\
&\quad + \sum_{uv\in E_5} \frac{(2n^2+5n+1)(2n^2+5n+1)}{4}\\
&= 2n(\frac{4n^2+14n-2}{4}) + n(\frac{4n^2+18n-8}{4}) + n(\frac{4n^2+26n-14}{4}) +\\
&\quad 2n(\frac{4n^2+14n-4}{4}) + n(\frac{4n^2+10n+2}{4})\\
&= 7n^5+55n^4+\frac{375}{4}n^3-68n^2+\frac{45}{4}n.
\end{aligned}$$

□

Illustration 2: Let $M(S_4)$ be the middle graph of Sunlet graph $S_4$. From the figure 3, $V(M(S_4)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$. From the definition of status of the vertex,

$$\begin{aligned}
\sigma(v_1) &= 33 = \sigma(v_2) = \sigma(v_3) = \sigma(v_4)\\
\sigma(v_5) &= 46 = \sigma(v_6) = \sigma(v_7) = \sigma(v_8)\\
\sigma(e_1) &= 26 = \sigma(e_2) = \sigma(e_3) = \sigma(e_4)\\
\sigma(e_5) &= 32 = \sigma(e_6) = \sigma(e_7) = \sigma(e_8) \qquad (12)
\end{aligned}$$

$$S_1(M(S_4)) = \sum_{uv\in E(M(S_4))} \sigma(u)+\sigma(v)$$

$$\begin{aligned}
= \; & [\sigma(v_5)+\sigma(e_5)] + [\sigma(e_5)+\sigma(e_1)] + [\sigma(e_5)+\sigma(v_1)] + [\sigma(e_5)+\sigma(e_2)]\\
& +[\sigma(v_6)+\sigma(e_6)] + [\sigma(e_6)+\sigma(e_1)] + [\sigma(e_6)+\sigma(v_2)] + [\sigma(e_3)+\sigma(e_6)]\\
& +[\sigma(v_7)+\sigma(e_7)] + [\sigma(e_7)+\sigma(v_3)] + [\sigma(e_7)+\sigma(e_2)] + [\sigma(e_7)+\sigma(e_4)]\\
& +[\sigma(v_8)+\sigma(e_8)] + [\sigma(e_8)+\sigma(e_4)] + [\sigma(e_8)+\sigma(v_4)] + [\sigma(e_8)+\sigma(e_3)]\\
& +[\sigma(v_1)+\sigma(e_1)] + [\sigma(e_1)+\sigma(v_2)] + [\sigma(v_2)+\sigma(e_3)] + [\sigma(e_3)+\sigma(v_4)]\\
& +[\sigma(v_4)+\sigma(e_4)] + [\sigma(e_4)+\sigma(v_3)] + [\sigma(v_3)+\sigma(e_2)] + [\sigma(e_2)+\sigma(v_1)]\\
& +[\sigma(e_1)+\sigma(e_2)] + [\sigma(e_1)+\sigma(e_3)] + [\sigma(e_3)+\sigma(e_4)] + [\sigma(e_4)+\sigma(e_2)].
\end{aligned}$$

Substitute the values from the equation (12) we get,

$$S_1(M(S_4)) \quad = \quad 1716 \tag{13}$$

From the Theorem 3.4

$S_1(M(S_4)) = n[14n^2 + 55n - 15]$

Put $n = 4$

$$\begin{aligned} S_1(M(S_4)) &= 4(14(4^2) + 55(4) - 15) \\ S_1(M(S_4)) &= 1716 \end{aligned} \tag{14}$$

The equation (13) and (14) are equal.
Hence the result.

**Theorem 3.5.** *Let T be the tree with n vertices and diam(T)=3 then,*

$$S_1(M(T)) = 25k^2 + 28k + 10 + \tfrac{k^2-2k}{4}(8k+4)$$

*and*

$$S_2(M(T)) = 52k^3 + 82k^2 + 50k + 12 + (k^2 - 2k)[4k^2 + 4k + 1].$$

***Proof.*** *T be the tree obtained by joining equal number of pendent edges to both end vertices of path $P_2$. $M(T)$ be the middle graph of tree contaning 4n vertices and 7n edges,* ***k*** *be the pendent vertices of the tree. Status of vertex which having degree 1 is $6k + 3$, status of the vertex which having the degree $\frac{k}{2} + 2$ is $4k + 2$ ,status of vertex which having degree $\frac{k}{2} + 1$ is $4k + 3$ and status of vertex which having degree k+2 is $3k + 2$.*
*The edge set of $M(T)$ is partitioned as,*
$E_1 = (uv|deg(u) = 1, deg(v) = \frac{k}{2} + 2)$, $|E_1| = k$
$E_2 = (uv|deg(u) = \frac{k}{2} + 2, deg(v) = \frac{k}{2} + 1)$, $|E_2| = k$
$E_3 = (uv|deg(u) = \frac{k}{2} + 1, deg(v) = k + 2)$, $|E_3| = 2$
$E_4 = (uv|deg(u) = \frac{k}{2} + 2, deg(v) = k + 2)$, $|E_4| = k$
$E_5 = (uv|deg(u) = deg(v) = \frac{k}{2} + 2)$, $|E_5| = \frac{k^2-2k}{4}$.

*by definition,*

$$
\begin{aligned}
&S_1(M(T))\\
&\quad= \sum_{uv\in E(M(T))} (\sigma(u)+\sigma(v))\\
&\quad= \sum_{uv\in E_1}(6k+3+4k+2)+\sum_{uv\in E_2}(4k+2+4k+3)+\sum_{uv\in E_3}(4k+3\\
&\qquad +3k+2)+\sum_{uv\in E_4}(4k+2+3k+2)+\sum_{uv\in E_5}(4k+2+4k+2)\\
&\quad= k(10k+5)+k(8k+5)+2(7k+5)+k(7k+4)+\frac{k^2-2k}{4}\\
&\qquad (8k+4)\\
&\quad= 25k^2+28k+10+\frac{k^2-2k}{4}(8k+4).
\end{aligned}
$$

*and*

$$
\begin{aligned}
&S_2(M(T))\\
&\quad= \sum_{uv\in E(M(T))} (\sigma(u)\sigma(v))\\
&\quad= \sum_{uv\in E_1}2(6k+3)(2k+1)+\sum_{uv\in E_2}2(2k+1)(4k+3)+\sum_{uv\in E_3}[(4k+3)\\
&\qquad (3k+2)]+\sum_{uv\in E_4}2(2k+1)(3k+2)+\sum_{uv\in E_5}(2(2k+1))(2(2k+1))\\
&\quad= k(6k+3)(4k+2)+k(4k+2)(4k+3)+2(4k+3)(3k+2)\\
&\qquad +k(4k+2)(3k+2)+\frac{k^2-2k}{4}[2(2k+1)]^2\\
&\quad= 52k^3+82k^2+50k+12+(k^2-2k)(4k^2+4k+1).
\end{aligned}
$$

□

# 4 Conclusions

In the present research work bounds for first and second status connectivity indices of middle graph are obtained. Futher, first and second status connectivity indices for middle graph of some class of graphs like cycle, complete bipartite, friendship graph, sunlet graph and tree of diameter 3 are established. For future research work, one can calculate the results on status connectivity indices of iterated middle graph of graph and other class of graphs including tree of diameter greater than 3.

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