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Partial sharing of small functions with exact difference

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Abstract

We have proven certain uniqueness results related owing to the partial sharing of small functions of f with difference polynomials of the form $L(f) = b_k(z)f(z+kc) + \ldots + b_0(z)f(z)$, where b_i are small functions of f. As a consequence of this, we have deduced the conditions under which, f is identically equal to its exact difference operator $\Delta_c^k f$.
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1 Introduction

Two meromorphic functions f and g are said to share a value a CM (resp. IM) if whenever $f(z) = a$, then $g(z) = a$ with the same multiplicity (resp. ignoring multiplicity). The sets $E(a, f)$ and $\overline{E}(a, f)$ are also defined as the set of points z_0 where $f(z_0) = a$, z_0 being counted according to multiplicity and ignoring multiplicity respectively. The celebrated Nevanlinna's five value theorem which says that if for two meromorphic functions f and g, there are five distinct complex values a_i such that $\overline{E}(a_i, f) = \overline{E}(a_i, g)$, for $i = 1, \ldots, 5$, then they are identical. This is also true for small functions of f and g as well. Let us denote the set of all small functions of f by $\mathscr{S}(f)$. Also, if the function g is replaced by the derivative of f, then the sharing conditions are changed from five to two shared values; that is, if f and f' share two finite values CM, then $f = f'$ Gundersen [1983]. Several further modifications to these theorems have been done thereafter, replacing the derivative f' by n-th order derivatives of f. However, in this paper, we are interested in the difference analogues of the uniqueness theorems. For the preliminary ideas and definitions of the classical Nevanlinna theory, we refer the reader to Hayman [1964], Lo [1993]. The reader can also obtain a comprehensive idea on the uniqueness theory of meromorphic functions from Yang and Yi [2003].

Let c be a non-zero complex number. Then the exact differences are defined as

$$\Delta_c f(z) = f(z + c) - f(z) \quad \text{and} \quad \Delta_c^n f(z) = \Delta_c \left(\Delta_c^{n-1} f(z)\right).$$

If $c = 1$, then we denote $\Delta_c^n f$ by $\Delta^n f$. Recent times have witnessed increasing interest in the difference analogues of the classical Nevanlinna theory owing to its wide applications in the theory of difference equations. The inception of the theory is credited to Halburd and Korhonen Halburd and Korhonen [2006a]; and Chiang and Feng Chiang and Feng [2008], who independently determined estimates of the proximity function of $\dfrac{f(z+c)}{f(z)}$ for a finite ordered meromorphic function f, respectively in the years 2006 and 2008. Thereafter, several existing results have been looked upon in this new perspective culminating in the disclosure of new theories in multitudes and methodically developing the theory of complex difference equations which runs parallel to the theory of complex differential equations Laine [1993]. We refer the reader to Chiang and Feng [2008, 2009], Halburd and Korhonen [2006a,b] for preliminaries in this field.

In their celebrated work on the difference analogue of the Nevanlinna theory, Halburd and Korhonen have deduced a uniqueness theorem by taking into account the counting function for the c-separated pairs Halburd and Korhonen [2006b]. Further studies on uniqueness theorems, and more specifically, periodicity conditions, for meromorphic functions have been done henceforth Heittokangas et al. [2009, 2011]. Since the difference $\Delta_c f$ can be thought of as the difference analogue of the derivative of f, it is interesting to study the uniqueness results for values shared by f with its exact differences. We shall study a more generalised version of this by taking into account certain difference polynomials as stated later in the article. Significant contributions have been done in this field. Li et al. [2015] proved the following theorem for entire functions.

Theorem 1.1. *Li et al. [2015] Let $f(z)$ be a non-constant entire function of finite order, $c \in \mathbb{C} \setminus \{0\}$. If $E(a_i, f) = E(a_i, \Delta_c f)$ where a_i, $i = 1, 2, 3$ are distinct complex values in the extended complex plane, then $f(z) \equiv \Delta_c f(z)$.*

However, Li and Gao gave a stronger version of the above theorem as follows.

Theorem 1.2. *Li and Gao [2011] Let $f(z)$ be a non-constant entire function of finite order, $c \in \mathbb{C}$ and n be a positive integer. Suppose that $E(a_i, f) = E(a_i, \Delta_c^n f)$ for two distinct complex numbers a_1 and a_2 and one of the following cases is satisfied:*

1. *$a_1 a_2 = 0$;*
2. *$a_1 a_2 \neq 0$ and order of f $\rho(f) \notin \mathbb{N}$.*

Then $f(z) \equiv \Delta_c^n f(z)$.

The sharing condition was improved further in the following theorem.

Theorem 1.3. *Li and Gao [2011] Let $f(z)$ be a non-constant entire function of finite order $\rho(f)$, $c \in \mathbb{C}$ and n be a positive integer. If $E(a, f) = E(a, \Delta_c^n f)$ for some $a \in \mathbb{C}$, and for a finite value $b \neq a$, $f(z) - b$ and $\Delta_c^n f(z) - b$ have $\max\{1, [\rho(f)]\}$ distinct common zeros of multiplicity ≥ 2, then $f(z) \equiv \Delta_c^n f(z)$.*

The theorem has been extended to meromorphic conditions imposing further restrictions as in the following theorem by Gao et al.

Theorem 1.4. *Gao et al. [2019] Let $f(z)$ be a transcendental meromorphic function of hyper order less than 1 such that $E(a_i, f) = E(a_i, \Delta^n f)$ for three distinct periodic functions $a_i \in \mathscr{S}(f) \cup \{\infty\}$, $i = 1, 2, 3$ of period 1, the $\Delta^n f(z) \equiv f(z)$.*

Natural questions that can be posed are as follows:

1. Can we replace three functions by two functions?
2. Can we replace sharing of functions by partial sharing?
3. Can we replace the exact difference of f by any function of the form $b_k(z)f(z+k) + \ldots + b_0(z)f(z)$, where $b_i(z) \in \mathscr{S}(f)$ for $i = 1, \ldots, k$?

The first question has been answered by Gao et al. [2019]. They showed for any meromorphic solution f, of $\Delta^n f(z) - 1 = k(f(z) - 1)$, where $k \neq 0, 1$ is a constant, $E(1, f) = E(1, \Delta^n f)$ and $E(\infty, f) = E(\infty, \Delta^n f)$, but $\Delta^n f \not\equiv f$. In 2020, Chen and Xu have also discussed certain uniqueness results related to sharing of two values by entire functions with its differences Chen and Xu [2020]. We will however, examine further conditions that need to be imposed to ensure the equality, with the partial sharing condition defined as follows.

Definition 1.1. *Let f and g be two meromorphic functions and $a \in \mathscr{S}(f) \cap \mathscr{S}(f)$. Then f share a with g partial if $E(a, f) \subseteq E(a, g)$.*

In the year 2016, however, Charak et al. [2016] used a different definition for partial sharing where they considered $\overline{E}(a, f)$ instead of $E(a, f)$ and proved the following theorem.

Theorem 1.5. *Charak et al. [2016] Let f be a meromorphic function of hyper order < 1 and $c \in \mathbb{C} \setminus \{0\}$. Let $a_1, a_2, a_3 \in \mathscr{S}(f)$ be distinct periodic functions with period c such that f share a_1, a_2 and a_3 partially with $f(z + c)$ and also $\sum_{i=1}^{3} \delta(a_i, f) > 1$. Then $f(z) \equiv f(z + c)$.*

Thereafter, in 2018, Chen [2018] deduced a result replacing the shift $f(z + c)$ by $\Delta_c f(z)$.

Theorem 1.6. *Chen [2018] Let f be a non-constant meromorphic function of hyper order less than 1, and let c be a non-zero finite complex number such that $f(z + c) \not\equiv f(z)$. If $\Delta_c f(z) = f(z + c) - f(z)$, and $f(z)$ share the value 1 CM and satisfy*

$$E(0, f) \subset E(0, \Delta_c f) \quad \textit{and} \quad E(\infty, f) \supset E(\infty, \Delta_c f),$$

then $\Delta_c f \equiv f$.

In the next section, will deduce uniqueness results relating to partial sharing of small functions with a difference polynomial.

2 Main Results

Let us begin by defining the following function.

$$L(f) = b_k(z) f(z + kc) + \ldots + b_0(z) f(z), \tag{1}$$

where $b_0, \ldots, b_k \in \mathscr{S}(f)$, not all of which are equal to zero.

We have generalised Chen's result Chen [2018] in the following.

Theorem 2.1. *Let f be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$. Also let f and $L(f)$ satisfy the following conditions:*

1. *$E(a, f) \subseteq E(a, L(f))$, where $a \in \mathscr{S}(f)$ is a periodic function of period c;*
2. *$E(0, f) \subseteq E(0, L(f))$ and $E(\infty, f) \supseteq E(\infty, L(f))$;*

then $f \equiv L(f)$.

The following corollaries can be immediately deduced from the above theorem.

Corolary 2.1. *Let f be a transcendental entire function of hyper order $\rho_2(f) < 1$. Also, let $E(a, f) \subseteq E(a, L(f))$ and $E(0, f) \subseteq E(0, L(f))$, where $a \in \mathscr{S}(f)$. Then $f \equiv L(f)$.*

Corolary 2.2. *Let f be a function as mentioned in theorem 2.1 and let for any positive integer n, $L(f) = \Delta_c^n f$. If f and $L(f)$ satisfies all the conditions of theorem 2.1, then $f \equiv \Delta_c^n f$.*

The first condition of theorem 2.1 is essential and non-replaceable as proved by the example below.

Example 2.1. *Consider $f(z) = e^z$, and $L(f) = (z+1)f(z) - ef(z) = ze^{z+1}$. Then $E(0, f) \subseteq E(0, L(f))$ and $E(\infty, L(f)) = E(\infty, f)$. But, $f \not\equiv L(f)$.*

Henceforth, we have proved results using arbitrary functions $a_1, a_2, a_3 \in \mathscr{S}(f)$, periodic with period c.

Theorem 2.2. *Let f be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ such that $E(a_i, f) \subseteq E(a_i, L(f))$ for small periodic functions a_i of f of period c, $i = 1, 2, 3$. Also, if*

$$\limsup_{r \to \infty} \frac{N(r, f)}{T(r, f)} < \frac{1}{k+1},$$

then $f \equiv L(f)$.

We can deduce the following corollaries from the above theorem.

Corolary 2.3. *Let f be a transcendental meromorphic function of hyper order less than 1. Let $a_i \in \mathscr{S}(f)$, $i = 1, 2, 3$ are periodic functions with period c.*

1. *Let $L(f) \equiv f(z + c)$. If $E(a_i, f) \subseteq E(a_i, L(f))$, $i = 1, 2, 3$, and also*

   $$\limsup_{r \to \infty} \frac{N(r, f)}{T(r, f)} < \frac{1}{2},$$

   then $f(z) \equiv f(z + c)$.

2. *If $L(f) \equiv \Delta_c^k f(z)$, and all the conditions in the above theorem are satisfied, then $f(z) \equiv \Delta_c^k f(z)$.*

Corolary 2.4. *Let f be a transcendental entire function with hyper order less than 1 and a_i be defined as in the previous corollary. If $E(a_i, f) \subseteq E(a_i, L(f))$, $i = 1, 2, 3$, then $f \equiv L(f)$.*

We have further tried to strengthen the results replacing $E(a, f)$ by $\overline{E}(a, f)$.

Theorem 2.3. *If f is a transcendental meromorphic function of hyper order $\rho_2(f) < 1$, and for $a_i \in \mathscr{S}(f)$, such that $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $i = 1, 2, \ldots, k+4$, then, $f \equiv L(f)$.*

Putting $k = 1$, we can immediately deduce the following corollary.

Corolary 2.5. *If f be a function and we take $L(f) = \Delta_c f$ satisfying the conditions in theorem 2.3, such that f and $L(f)$ satisfy the condition $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $i = 1, 2, \ldots, 5$, then $f \equiv \Delta_c f$.*

Can we strengthen the sharing condition of the above theorem? We have deduced the following result in this context.

Theorem 2.4. *Let f be a transcendental entire function of hyper order $\rho_2(f) < 1$, such that $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $a_i \in \mathscr{S}(f)$, $i = 1, 2, 3, 4$. Then $f \equiv L(f)$.*

Can we prove this for meromorphic functions as well? We see that in case of entire functions, the quantity $N(r, f)$ is "small". Can we do something by making the quantity somewhat "small" in the meromorphic case too? We have deduced the following theorem thus.

Theorem 2.5. *Let f be a transcendental meromorphic function of hyper order less than 1, such that $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $a_i \in \mathscr{S}(f)$, $i = 1, 2, 3, 4$. Also, let*

$$\limsup_{r\to\infty} \frac{N(r, f)}{T(r, f)} < \frac{1}{k^2}.$$

Then $f \equiv L(f)$.

Theorem 2.4 can be deduced as a corollary of the above theorem. Here, we are inclined to make a conjecture that we can replace $N(r, f)$ by $\overline{N}(r, f)$. Under what conditions can we change the sharing condition from four functions to three functions?

Theorem 2.6. *Let k be a positive integer ≥ 3 and f be a non-constant meromorphic function of hyper order less than 1, such that $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $a_i \in \mathscr{S}(f)$, $i = 1, 2, 3$. Also let*

$$\limsup_{r\to\infty} \frac{N(r, f)}{T(r, f)} < \frac{1}{k^2},$$

alongwith the condition $\delta(a_4, f) > \dfrac{2}{k}$, holds, where $a_4 \in \mathscr{S}(f) \setminus \{a_1, a_2, a_3\}$. Then $f \equiv L(f)$.

From the proof of the above theorem, we can easily show that we can replace $\delta(a_4, f)$ by $\Theta(a_4, f)$ which is defined by

$$\Theta(a_4, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{f-a_4}\right)}{T(r, f)}.$$

What further conditions do we need to impose if f and $L(f)$ share only two small functions partially? Our next theorem points in that direction.

Theorem 2.7. *Let k be any positive integer ≥ 2 and f be a non-constant meromorphic function of hyper order less than 1, such that $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, L(f))$, for $a_i \in \mathscr{S}(f)$, for $i = 1, 2$. Also, let for some $a_3, a_4 \in \mathscr{S}(f) \setminus \{a_1, a_2\}$, be such that*

$$\limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{f-a_3}\right) + \overline{N}\left(r, \frac{1}{f-a_4}\right)}{T(r, f)} < \frac{k-1}{k}.$$

Further, if we assume that

$$\limsup_{r\to\infty} \frac{N(r, f)}{T(r, f)} < \frac{1}{k^2},$$

then, $f \equiv L(f)$.

From all the theorems we so far stated, we can easily deduce the condition $\Delta_c^k f(z) \equiv f(z)$, under the required conditions, by replacing $L(f)$ by the difference operator.

3 Lemmas

We state a few lemmas in this section, that we will need in the sequel.

Lemma 3.1. *Halburd et al. [2014] Let f be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, and $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f)$$

outside an exceptional set of finite logarithmic measure.

Lemma 3.2. *Let f be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, and $c \in \mathbb{C}$. Also, let $L(f)$ be as defined in equation (1). Then*

$$m\left(r, \frac{L(f)}{f}\right) = S(r,f)$$

outside an exceptional set of finite logarithmic measure.

Proof. Follows by using the principle of mathematical induction on 3.1. □

Lemma 3.3. *Halburd et al. [2014] Let $T : [0,\infty) \to [0,\infty)$ be a non-decreasing continuous function and let $s \in (0,\infty)$. If the hyper order ρ_2 of T is strictly less than one, and $\delta \in (0, 1-\rho_2)$, then*

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right),$$

where r runs to infinity outside of a set of finite logarithmic measure.

Lemma 3.4. *Let f be a meromorphic function of hyper order $\rho_2 < 1$ and a be any small periodic function of f of period c. Then*

$$m\left(r, \frac{L(f)}{f-a}\right) = S(r,f),$$

outside an exceptional set with finite logarithmic measure.

Proof. Since we have

$$m\left(r, \frac{L(f)}{f-a}\right) = m\left(r, \frac{L(f)-L(a)}{f-a}\right) + O(1) = S(r, f-a),$$

so using the properties of $T(r,f)$ and lemma 3.3, the lemma is proved. □

Lemma 3.5. *Let $f(z)$ be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$ and $c \in \mathbb{C}$ such that $L(f) \not\equiv 0$. Let $q \geq 2$ and $a_1, \ldots, a_q$ be q meromorphic periodic functions in $\mathscr{S}(f)$ of period c. Then*

$$\begin{aligned} m(r,f) + \sum_{i=1}^{q} m\left(r, \frac{1}{f-a_i}\right) \quad &\leq \quad 2T(r,f) - 2N(r,f) + N(r, L(f)) \quad (2) \\ &\quad -N\left(r, \frac{1}{L(f)}\right) + S(r,f), \end{aligned}$$

for all r outside an exceptional set of finite logarithmic measure.

Proof. Let $P(f)=(f-a_1)(f-a_2)\dots(f-a_q)$ which, on rewriting, gives

$$\frac{1}{P(f)}=\sum_{i=1}^{q}\frac{c_i}{f-a_i},$$

where $c_k\in\mathscr{S}(f)$ are periodic functions with period c. By previous lemma, we obtain

$$m\left(r,\frac{L(f)}{P(f)}\right)=m\left(r,L(f)\sum_{i=1}^{q}\frac{c_i}{f-a_i}\right)=S(r,f).$$

Thus,

$$m\left(r,\frac{1}{P(f)}\right)\le m\left(r,\frac{1}{L(f)}\right)+S(r,f).$$

The preceding equation, first fundamental theorem and further properties of $T(r,f)$ gives

$$T(r,L(f)) \;\ge\; \sum_{i=1}^{q}m\left(r,\frac{1}{f-a_i}\right)+N\left(r,\frac{1}{L(f)}\right)+S(r,f).$$

Finally, we will have,

$$\begin{aligned}m(r,f)+\sum_{i=1}^{q}m\left(r,\frac{1}{f-a_i}\right) &\le T(r,f)-N(r,f)+T(r,L(f))\\ &\quad -N\left(r,\frac{1}{L(f)}\right)+S(r,f)\\ &\le 2T(r,f)-2N(r,f)+N(r,L(f))\\ &\quad -N\left(r,\frac{1}{L(f)}\right)+S(r,f).\end{aligned}$$

□

Remark 3.1. *The equation* (2) *can be rewritten as follows*

$$\begin{aligned}(q-1)T(r,f) &\le N(r,f)+\sum_{i=1}^{q}N\left(r,\frac{1}{f-a_i}\right)-2N(r,f)+N(r,L(f))\\ &\quad -N\left(r,\frac{1}{L(f)}\right)+S(r,f).\end{aligned}$$

We also state the second fundamental theorem for small functions, due to Yamanoi.

Lemma 3.6. *Yamanoi [2004] Let f be a non-constant meromorphic function and $a_i\in\mathscr{S}(f)$ be distinct for $i=1,2,\dots,n$. Then for any $\epsilon>0$,*

$$(k-2-\epsilon)T(r,f)\le\sum_{i=1}^{n}\overline{N}\left(r,\frac{1}{f-a_i}\right)+S(r,f).$$

4 Proof of Main Results

Proof of theorem 2.1

Proof. Since $E(a,f) \subseteq E(a,L(f))$, and $E(0,f) \subseteq E(0,L(f))$, so,

$$N\left(r,\frac{1}{f-a}\right) \leq N\left(r,\frac{1}{L(f)-a}\right) \quad \text{and} \quad N\left(r,\frac{1}{f}\right) \leq N\left(r,\frac{1}{L(f)}\right). \tag{3}$$

Also, since $E(\infty,L(f)) \subseteq E(\infty,f)$, so $N(r,L(f)) \leq N(r,f)$. Now,

$$\begin{aligned} T(r,f) &= T\left(r,\frac{1}{f}\right)+O(1) \\ &= m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)+O(1) \\ &\leq m\left(r,\frac{L(f)}{f}\right)+m\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(f)}\right)+O(1) \\ &= T(r,L(f))+S(r,f). \end{aligned} \tag{4}$$

Also,

$$T(r,L(f)) \leq \sum_{i=0}^{k} T(r,f(z+ic))+S(r,f) = (k+1)T(r,f)+S(r,f). \tag{5}$$

Thus, we have, from equations (4) and (5), $S(r,L(f)) = S(r,f) = S(r)$, say.

Now, let us take

$$g = \frac{L(f)}{f}.$$

Then, by the given assumptions, we can say that g is an entire function of hyper order less than 1.

By lemma 3.2, we get

$$T(r,g) = S(r).$$

If possible,suppose that $L(f) \not\equiv f$. We show that

$$N\left(r,\frac{1}{f-a}\right) = S(r).$$

Since $E(a,f) \subseteq E(a,L(f))$, so,

$$\overline{N}\left(r,\frac{1}{f-a}\right) \leq N\left(r,\frac{1}{g-1}\right) \leq S(r). \tag{6}$$

Next, we estimate $N_{(2}\left(r, \dfrac{1}{f-a}\right)$, the counting function for the number of a-points of f with multiplicity greater than or equal to 2.

Also, we have

$$\frac{g'}{g} = \frac{[L(f)]'}{L(f)} - \frac{f'}{f}.$$

Let z_0 be an a-point of f of multiplicity p greater than or equal to 2. Then it is also so of $L(f)$. By the above equation, we can see that z_0 is a zero of $\dfrac{g'}{g}$ of multiplicity $\geq p-1$. Combining, if g is non-constant, we get

$$\begin{aligned}
N_{(2}\left(r, \frac{1}{f-a}\right) \leq 2N\left(r, \frac{g}{g'}\right) &\leq 2T\left(r, \frac{g}{g'}\right) \\
&= 2T\left(r, \frac{g'}{g}\right) + O(1) \\
&\leq 6T(r,g) + S(r,g) \\
&= S(r,g) = S(r). \qquad (7)
\end{aligned}$$

Thus, from equations (6) and (7), we get

$$N\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + N_{(2}\left(r, \frac{1}{f-a}\right) = S(r).$$

Also, if g is constant and $L(f) \not\equiv f$, then $a(z)$ is not assumed by f and hence,

$$N\left(r, \frac{1}{f-a}\right) \leq S(r).$$

By remark 3.1, and equations (3), we get,

$$\begin{aligned}
T(r,f) &\leq N(r,f) + N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f}\right) - 2N(r,f) \\
&\quad + N(r, L(f)) - N\left(r, \frac{1}{L(f)}\right) + S(r) \\
&= N\left(r, \frac{1}{f-a}\right) + (N(r, L(f)) - N(r,f)) \\
&\quad + \left(N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{L(f)}\right)\right) + S(r) \\
&\leq S(r),
\end{aligned}$$

which is a contradiction to the fact that f is transcendental. Thus, $L(f) \equiv f$. $\square$

Proof of theorem 2.2

Proof. Since

$$\limsup_{r\to\infty}\frac{N(r,f)}{T(r,f)}<\frac{1}{k+1},$$

so for $\epsilon>0$, we have, for large r,

$$N(r,f)\leq\frac{1-2\epsilon}{k+1}T(r,f).$$

If possible, let $f\not\equiv L(f)$. Since $E(a_i,f)\subseteq E(a_i,L(f))$, we have

$$N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_2}\right)+N\left(r,\frac{1}{f-a_3}\right)\leq N\left(r,\frac{1}{L(f)-f}\right). \quad (8)$$

Now, by first main theorem, elementary Nevanlinna theory, and lemma 3.3,

$$\begin{aligned}
N\left(r,\frac{1}{L(f)-f}\right) &\leq T(r,L(f)-f)+O(1)\\
&\leq m(r,L(f)-f)+(k+1)N(r+k|c|,f)+S(r,f)\\
&= m\left(r,f\left(\frac{L(f)}{f}-1\right)\right)+(k+1)N(r,f)+S(r,f)\\
&\leq T(r,f)+kN(r,f)+S(r,f),
\end{aligned}$$

outside an exceptional set of finite logarithmic measure. Thus, by equation (8), we get,

$$\sum_{i=1}^{3}N\left(r,\frac{1}{f-a_i}\right)\leq T(r,f)+kN(r,f)+S(r,f).$$

Finally, by second main theorem, it follows that

$$\begin{aligned}
(2-\epsilon)T(r,f) &\leq N(r,f)+\sum_{i=1}^{3}N\left(r,\frac{1}{f-a_i}\right)+S(r,f)\\
&\leq \frac{1-2\epsilon}{k+1}T(r,f)+T(r,f)+kN(r,f)+S(r,f)\\
&\leq (1-2\epsilon)T(r,f)+T(r,f)+S(r,f)\\
\text{or,}\qquad \epsilon T(r,f) &\leq S(r,f),
\end{aligned}$$

which is a contradiction. Hence $f\equiv L(f)$. □

Proof of theorem 2.3

Proof. If $f \not\equiv L(f)$, then by the given assumption,

$$\begin{aligned}\sum_{i=1}^{k+4} \overline{N}\left(r, \frac{1}{f-a_i}\right) &\leq N\left(r, \frac{1}{L(f)-f}\right) \\ &\leq T(r, L(f)-f) + O(1) \\ &\leq (k+1)T(r,f) + O(1).\end{aligned}$$

Finally, by the second main theorem for small functions, we have, for $0 < \epsilon < 1$,

$$\begin{aligned}(k+2-\epsilon)T(r,f) &\leq \sum_{i=1}^{k+4} \overline{N}\left(r, \frac{1}{f-a_i}\right) + S(r,f) \\ &\leq (k+1)T(r,f) + S(r,f),\end{aligned}$$

which contradicts our assumption that f is a transcendental meromorphic function. Hence the result. □

Proof of theorem 2.4

Proof. If $f \not\equiv L(f)$, then we have

$$\sum_{i=1}^{4} \overline{N}\left(r, \frac{1}{f-a_i}\right) \leq N\left(r, \frac{1}{L(f)-f}\right). \tag{9}$$

Also, since f is an entire function of hyper order less than 1, we have

$$N\left(r, \frac{1}{L(f)-f}\right) \leq T(r,f) + S(r,f). \tag{10}$$

Finally, by lemma 3.6, and equations (9) and (10), we have, for $0 < \epsilon < 1$,

$$\begin{aligned}(2-\epsilon)T(r,f) &\leq \sum_{i=1}^{4} \overline{N}\left(r, \frac{1}{f-a_i}\right) + S(r,f) \\ &\leq T(r,f) + S(r,f),\end{aligned}$$

which is a contradiction. Hence the result. □

Proof of theorem 2.5

Proof. For $\epsilon > 0$ and large r, we have

$$N(r,f) \leq \frac{1-\epsilon}{k^2} T(r,f). \tag{11}$$

Also, see that by the sharing assumptions and lemma 3.3, we have,

$$\begin{aligned}\sum_{i=1}^{4} \overline{N}\left(r, \frac{1}{f-a_i}\right) &\leq N\left(r, \frac{1}{L(f)-f}\right) \\ &\leq T(r,f) + kN(r,f) + S(r,f).\end{aligned}$$

Using this theorem, we have, by lemma 3.6 and $0 \leq \epsilon < 1$,

$$\begin{aligned}(1-\epsilon)T(r,f) &\leq kN(r,f) + S(r,f) \\ &\leq \frac{k-k\epsilon}{k^2} T(r,f) + S(r,f),\end{aligned}$$

which is a contradiction. Hence the result. □

Proof of theorem 2.6

Proof. We have, for large r and $\epsilon > 0$,

$$\frac{N(r,f)}{T(r,f)} \leq \frac{1-\epsilon}{k^2}.$$

By the assumption, if $f \not\equiv L(f)$, then we have by previous computations,

$$\sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{f-a_i}\right) + \leq T(r,f) + kN(r,f) + S(r,f).$$

Thus, by the second main theorem, for the previously assumed $\epsilon > 0$, we have,

$$\begin{aligned}(2-\epsilon)T(r,f) &\leq \sum_{i=1}^{4} \overline{N}\left(r, \frac{1}{f-a_i}\right) + S(r,f) \\ &\leq kN(r,f) + T(r,f) + N\left(r, \frac{1}{f-a_4}\right) + S(r,f) \\ \text{or,}\quad (1-\epsilon)T(r,f) &\leq kN(r,f) + N\left(r, \frac{1}{f-a_4}\right) + S(r,f) \\ \text{or,}\quad (1-\epsilon) &\leq \frac{1-\epsilon}{k} + 1 - \delta(a_4,f) \\ \text{or,}\quad \delta(a_4,f) &\leq \frac{1}{k} + \frac{k-1}{k}\epsilon.\end{aligned}$$

If we choose ϵ such that

$$\frac{1}{k}+\frac{k-1}{k}\epsilon<\frac{2}{k}\Rightarrow\epsilon<\frac{1}{k-1},$$

then we arrive at a contradiction to our assumption that $\delta(a_4,f)>\frac{2}{k}$. Thus, the result. $\square$

Proof of theorem 2.7

Proof. Let $f\not\equiv L(f)$. Then by previous computations, we have,

$$\sum_{i=1}^{2}\overline{N}\left(r,\frac{1}{f-a_i}\right)\leq T(r,f)+kN(r,f)+S(r,f).$$

Thus, by second main theorem, we have, for $\epsilon>0$,

$$\begin{aligned}(2-\epsilon)T(r,f) &\leq \sum_{i=1}^{4}\overline{N}\left(r,\frac{1}{f-a_i}\right)+S(r,f)\\ &\leq kN(r,f)+T(r,f)+\overline{N}\left(r,\frac{1}{f-a_3}\right)\\ &\quad +\overline{N}\left(r,\frac{1}{f-a_4}\right)+S(r,f)\\ \text{or,}\qquad (1-\epsilon) &\leq k\frac{N(r,f)}{T(r,f)}+\frac{\overline{N}\left(r,\frac{1}{f-a_3}\right)+\overline{N}\left(r,\frac{1}{f-a_4}\right)}{T(r,f)}+S(r,f).\end{aligned}$$

Using limit supremum on both sides of the above equation, as $r\to\infty$ we get,

$$\begin{aligned}\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{f-a_3}\right)+\overline{N}\left(r,\frac{1}{f-a_4}\right)}{T(r,f)} &\geq 1-\frac{1}{k}-\epsilon\\ &= \frac{k-1}{k}-\epsilon.\end{aligned}$$

Since $\epsilon>0$ is arbitrary, so we arrive at a contradiction. Hence the result. $\square$

5 Results and Discussions

Our primary objective in this paper has been to deduce the uniqueness theorems related to the exact differences of meromorphic functions. In this process, we have proven various results on the uniqueness of meromorphic functions with partially shared values with the difference polynomial given in (1), from which, we have shown, how many results done earlier by other authors can be improved substantially. We started with theorem 2.1 which has specific sharing conditions for a transcendental meromorphic function. Afterwards, from theorem 2.2 onwards, it can be seen that given the partial sharing condition, the "smallness" of the function $N(r, f)$ is somewhat necessary to ensure the equality of f with $L(f)$. It is needless saying that the results can be easily proven for entire functions without this additional condition on $N(r, f)$ since in that case, $N(r, f) = S(r, f)$. On changing the sharing conditions further, i,e., by replacing $E(a, f)$ by $\overline{E}(a, f)$, or by reducing the number of functions partially shared, we have been able to stick to the same equality. We have excluded the case when f shares one small function with $L(f)$, which we presume can be done using the similar tools that we have used in proving the previous theorems. It is to be noted that as we have kept reducing the number of functions shared, the counting function $N\left(r, \dfrac{1}{f-a}\right)$, or the reduced counting function $\overline{N}\left(r, \dfrac{1}{f-a}\right)$ becomes necessary to be "small" with respect to the characteristic function $T(r, f)$. It would be interesting to delve further into the topic and explore the possibilities of replacing the set $E(a, f)$ by $E_{p)}(a, f)$ (resp. $E_{(p}(a, f)$), which is the set of a-points of f having multiplicity p atmost (resp. at least). At this juncture, we are inclined to make a mention of the set $\tilde{E}_c(a, f)$, which is the number of a-points of f ignoring the c-separated a-pairs. It would also be worthy to study the case when $E(a, f)$ is replaced by $\tilde{E}_c(a, f)$. Further, in case of theorems 2.2, and thereafter, it can also be checked whether the limit supremum condition can be replaced by limit infimum. Finally, we would like to point that all discussions in this paper have been done for functions of "much higher" growth, which is measured by using the hyper order. But, the case of the functions of "much lower" growth may also be studied under the same light.

6 Open Problems

In this article, we have mainly concentrated on meromorphic functions of hyper order $\rho_2(f) < 1$. The case of $\rho_2(f) \geq 1$ is open. Also, the case of functions of "much lower" growth may also be studied under the same light. A possible measure of growth for such functions was developed by Chern in the year 2005 when he first came up with the idea of functions with logarithmic order of growth Chern [2006]. The topic itself is still largely unexplored. However, from the results deduced by Chern in Chern [2006], we conjecture that even stronger uniqueness results can be obtained for such functions.

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