

Fixed Point Results for (ψ, ϕ) -Contractive Mapping in $G_{\mathcal{F}}$ -Metric Space

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Abstract

The main goal of this work is to present $G_{\mathcal{F}}$ -metric space, a new generalization of G -metric space. A comparison between the classes of G -metric spaces, GP -metric spaces, G_b -metric spaces, generalized G_b -metric spaces, and G^* -metric spaces and the class of $G_{\mathcal{F}}$ -metric spaces is also presented. We examine a few fundamental aspects of this newly defined abstract space. Proving the Banach contraction principle and the fixed point result for (ψ, ϕ) -contractive mapping in the context of $G_{\mathcal{F}}$ -metric spaces is the paper's secondary goal.

Keywords: Fixed point, G -metric space, G_b -metric space, $\mathcal{F}$ -metric space, Contractive mapping.

2020 AMS subject classifications: 47H10; 54H25.¹

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¹Received on November 2, 2023. Accepted on January 15, 2024. Published on January 31, 2024. DOI:10.23755/rm.v51i0.1455. ISSN: 1592-7415. eISSN: 2282-8214. ©Kapil Jain et al. This paper is published under the CC-BY licence agreement.

1 Introduction

Gahler (1) proposed the idea of 2-metric space, which is an extension of the famous concept of metric space (X, d) . Various writers have demonstrated that there is no relation between the two functions. For example, Ha *et al.* (2) demonstrate that the 2-metric does not necessarily need to be continuous. Dhage (3) introduced the concept of D -metric space, a new class of generalized metric space, in 1992. Most of the assertions about the basic topological structure of D -metric space were later proved inappropriate by Mustafa and Sims (4), Naidu *et al.* ((5; 6)). Therefore, Mustafa and Sims (7) created a more suitable concept, known as G -metric space. (7) Let $\mathcal{A}$ be a non-empty set and $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a function satisfying:

- (G1) $G(\zeta, \eta, \vartheta) = 0$ if $\zeta = \eta = \vartheta$;
- (G2) $0 < G(\zeta, \zeta, \eta)$, for all $\zeta, \eta \in \mathcal{A}$ with $\zeta \neq \eta$;
- (G3) $G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta)$, for all $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\vartheta \neq \eta$;
- (G4) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in all three variables);
- (G5) $G(\zeta, \eta, \vartheta) \leq G(\zeta, a, a) + G(a, \eta, \vartheta)$, for all $\zeta, \eta, \vartheta, a \in \mathcal{A}$, (rectangle inequality).

The pair $(\mathcal{A}, G)$ is a G -metric space, and the function G is referred to as a *generalized metric* or a G -metric on $\mathcal{A}$. Assume that the set of real numbers is $\mathcal{A}$, define $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ as

$$G(\zeta, \eta, \vartheta) = |\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|, \text{ for all } \zeta, \eta, \vartheta \in \mathcal{A}.$$

G is therefore a G -metric on $\mathcal{A}$.

(8; 9; 10; 11; 12; 13; 15; 14; 16) has more results and more information in G -metric spaces. As a generalisation of partial metric space (17) and G -metric space, Zand and Nezhad (18) presented GP -metric space in 2011. (18) Let $\mathcal{A}$ be a non-empty set. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a function such that the following conditions hold:

- (G_p1) $\zeta = \eta = \vartheta$ if $G(\zeta, \eta, \vartheta) = G(\zeta, \zeta, \zeta) = G(\eta, \eta, \eta) = G(\vartheta, \vartheta, \vartheta)$;
- (G_p2) $G(\zeta, \zeta, \zeta) \leq G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta)$, for all $\zeta, \eta, \vartheta \in \mathcal{A}$;
- (G_p3) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in all three variables);
- (G_p4) $G(\zeta, \eta, \vartheta) \leq G(\zeta, a, a) + G(a, \eta, \vartheta) - G(a, a, a)$, for all $\zeta, \eta, \vartheta, a \in \mathcal{A}$.

Then, the function G is called a GP -metric on $\mathcal{A}$, and the pair $(\mathcal{A}, G)$ is a GP -metric space.

Later, in 2013, Parvaneh *et al.* (19) discovered that (G_p2) makes GP -metric spaces symmetric. Because those G -metric spaces are nonsymmetric, GP -metric spaces do not generalize them (see Example 1, (7)). Parvaneh *et al.* (19) redefined GP -metric space in light of this by modifying the inequality (G_p2) to read as follows:

$$(G_p2') \quad G(\zeta, \zeta, \zeta) \leq G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta), \text{ for all } \zeta, \eta, \vartheta \in \mathcal{A} \text{ with } \eta \neq \vartheta.$$

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(18) Let $\mathcal{A} = [0, \infty)$ and define a map $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ by setting

$$G(\zeta, \eta, \vartheta) = \max\{\zeta, \eta, \vartheta\}, \quad \text{for all } \zeta, \eta, \vartheta \in \mathcal{A}.$$

Consequently, $(\mathcal{A}, G)$ is a GP -metric space but not a G -metric space since $G(1, 1, 1) = 1 \neq 0$, i.e., $(G1)$ does not hold.

Further details about GP -metric spaces are provided in papers (20; 21; 22; 23; 24; 25; 26; 27; 28). By merging the ideas of G -metric spaces and b -metric spaces(30), Aghajani *et al.* introduced the notion of G_b -metric spaces in (29) as follows: (29) Let $s \geq 1$ be a real number and let $\mathcal{A}$ be a non-empty set. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a function such that:

- (G_b1) $G(\zeta, \eta, \vartheta) = 0$ if $\zeta = \eta = \vartheta$;
- (G_b2) $0 < G(\zeta, \zeta, \eta)$, for all $\zeta, \eta \in \mathcal{A}$ with $\zeta \neq \eta$;
- (G_b3) $G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta)$, for all $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\vartheta \neq \eta$;
- (G_b4) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in all three variables);
- (G_b5) $G(\zeta, \eta, \vartheta) \leq s[G(\zeta, a, a) + G(a, \eta, \vartheta)]$, for all $\zeta, \eta, \vartheta, a \in \mathcal{A}$.

Then, on $\mathcal{A}$, the function G is referred to as a G_b -metric or a *generalized b -metric*, and the pair $(\mathcal{A}, G)$ is a G_b -metric space or a *generalized b -metric space*. A G -metric space is a G_b -metric space with $s = 1$, but the opposite is not true in general. (29) Let $\mathcal{A} = \mathbb{R}$ represent the set of real numbers. Define $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ as follows:

$$G(\zeta, \eta, \vartheta) = \frac{1}{9}(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|)^2, \quad \text{for all } \zeta, \eta, \vartheta \in \mathcal{A}.$$

Hence, on $\mathcal{A}$, G is a G_b -metric but not a G -metric. Numerous researchers demonstrated different findings in G_b -metric spaces; refer to (31; 32; 33; 34; 35; 36; 37; 38; 39; 40; 41; 42). In (29), the authors proposed the idea of G_b -metric space. The term G_b -metric space was also used by Jain and Kaur in (43), although it referred to a different abstract space. Jain *et al.* (44) renamed this abstract space as ‘generalized G_b -metric space’, and its definition is as follows: (44) Let $\mathcal{A}$ be a non-empty set and $s \geq 1$ be a real number. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a function satisfying:

- (gG_b1) $G(\zeta, \eta, \vartheta) = 0$ if $\zeta = \eta = \vartheta$;
- (gG_b2) $0 < G(\zeta, \zeta, \eta)$, for all $\zeta, \eta \in \mathcal{A}$ with $\zeta \neq \eta$;
- (gG_b3) $G(\zeta, \zeta, \eta) \leq s G(\zeta, \eta, \vartheta)$, for all $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\vartheta \neq \eta$;
- (gG_b4) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in all three variables);
- (gG_b5) $G(\zeta, \eta, \vartheta) \leq s[G(\zeta, a, a) + G(a, \eta, \vartheta)]$, for all $\zeta, \eta, \vartheta, a \in \mathcal{A}$.

The pair $(\mathcal{A}, G)$ is a *generalized G_b -metric space*, and the function G is referred to as a *generalized G_b -metric* on $\mathcal{A}$. The following example shows that while it is evident that every G_b -metric space is a generalized G_b -metric space, the converse is not true: (44) For every $\zeta, \eta, \vartheta \in \mathbb{R}$, define a mapping $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$

as follows:

$$G(\zeta, \eta, \vartheta) = |\zeta - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2.$$

In that case, $(\mathbb{R}, G)$ is not a G_b -metric space, but it is a generalized G_b -metric space with $s = 2$. To calculate $G(\zeta, \eta, \vartheta) = |1 - 3|^2 + |3 - 2|^2 + |2 - 1|^2 = 6$ and $G(\zeta, \eta, \eta) = 2|1 - 3|^2 = 8$, let $\zeta = 1$, $\eta = 3$, and $\vartheta = 2$. Cosequently, $G(\zeta, \eta, \eta) \not\leq G(\zeta, \eta, \vartheta)$, that is, (G_b3) , is not true.

After that, Jain *et al.* (44) introduced G^* -metric space to generalize GP -metric space and generalized G_b -metric space. (44) Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty]$ be a mapping, where $\mathcal{A}$ is a non-empty set. If there is an $\alpha > 0$ such that for every $\zeta, \eta, \vartheta \in \mathcal{A}$, the following conditions hold, then we say that G is a G^* -metric on $\mathcal{A}$:

- (Gg1) $G(\zeta, \eta, \vartheta) = 0$ implies $\zeta = \eta = \vartheta$;
 - (Gg2) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in its variables);
 - (Gg3) if $\{\zeta_n\} \in C_{\mathcal{A}}(G, \zeta)$, then $G(\zeta, \eta, \vartheta) \leq \alpha \left(\limsup_{n \rightarrow \infty} G(\zeta_n, \eta, \vartheta) + G(\zeta, \zeta, \zeta) \right)$,
- where $C_{\mathcal{A}}(G, \zeta) = \left\{ \{\zeta_n\} \subset \mathcal{A} \mid \lim_{n, m \rightarrow \infty} G(\zeta_n, \zeta_m, \zeta) = G(\zeta, \zeta, \zeta) < \infty \right\}$.

The pair $(\mathcal{A}, G)$ in this instance is referred to as a G^* -metric space with constant α .

(44) Assume that $\mathcal{A} = \mathcal{B} \cup \{0\}$, where $\mathcal{B} = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty]$ be a mapping defined so that G satisfies (Gg2) and

$$G(\zeta, \eta, \vartheta) = \begin{cases} \zeta + \eta + \vartheta, & \text{if atleast one of } \zeta, \eta, \vartheta \text{ is } 0; \text{ or} \\ & \text{if } \zeta = \frac{1}{n}, \eta = \frac{1}{n+m}, \vartheta = \frac{1}{n+l}, \text{ where } n, m, l \geq 5; \\ 5, & \text{otherwise.} \end{cases}$$

Consequently, $(\mathcal{A}, G)$ is a G^* -metric space with constant α . However, $G(0.5, 0.5, 0.5) = 5 \neq 0$, $(\mathcal{A}, G)$ is not a generalized G_b -metric space. Also, $(\mathcal{A}, G)$ is not a GP -metric space as for $\zeta = \frac{1}{10}$ and $\eta = \frac{1}{5}$, $G(\zeta, \zeta, \zeta) = 5 \not\leq \frac{2}{5} = G(\zeta, \zeta, \eta)$, that is, (G_p2') is not true.

Meanwhile, in 2018, Jleli and Samet (45) established an exciting generalization of metric space as follows.

Let $\mathcal{F}$ be the set of functions $f : (0, \infty) \rightarrow \mathbb{R}$ satisfying the following conditions:

(F1) f is non-decreasing, i.e., $0 < s < t$ implies $f(s) \leq f(t)$.

(F2) For every sequence $\{t_n\}$ in $(0, \infty)$, we have

$$\lim_{n \rightarrow \infty} t_n = 0 \text{ if and only if } \lim_{n \rightarrow \infty} f(t_n) = -\infty.$$

(45) Let $\mathcal{A}$ be a non-empty set and let $D : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that

(D1) $(\zeta, \eta) \in \mathcal{A} \times \mathcal{A}$, $D(\zeta, \eta) = 0$ if and only if $\zeta = \eta$.

(D2) $D(\zeta, \eta) = D(\eta, \zeta)$, for all $(\zeta, \eta) \in \mathcal{A} \times \mathcal{A}$.

(D3) For every $(\zeta, \eta) \in \mathcal{A} \times \mathcal{A}$, for every $n \in \mathbb{N}$, $n \geq 2$, and for every $\{u_1, u_2, \dots, u_n\} \subset \mathcal{A}$ with $(u_1, u_n) = (\zeta, \eta)$, we have

$$D(\zeta, \eta) > 0 \text{ implies } f(D(\zeta, \eta)) \leq f\left(\sum_{i=1}^{n-1} D(u_i, u_{i+1})\right) + \alpha.$$

Then, the function D is said to be an $\mathcal{F}$ -metric on $\mathcal{A}$, and the pair $(\mathcal{A}, D)$ is said to be an $\mathcal{F}$ -metric space.

We refer to (46; 48; 49; 47; 50) for more details on $\mathcal{F}$ -metric spaces. Now, motivated to the work done in (45), we define a new generalization of G -metric space as in the following section.

2 $G_{\mathcal{F}}$ -Metric Space

Let $\mathcal{A}$ be a non-empty set. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a mapping. Let there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that

($G_{\mathcal{F}1}$) $G(\zeta, \eta, \vartheta) = 0$ if and only if $\zeta = \eta = \vartheta$.

($G_{\mathcal{F}2}$) $f(G(\zeta, \zeta, \eta)) \leq f(G(\zeta, \eta, \vartheta)) + \alpha$, for all $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\vartheta \neq \eta, \zeta \neq \eta$.

($G_{\mathcal{F}3}$) $G(\zeta, \eta, \vartheta) = G(\zeta, \vartheta, \eta) = G(\eta, \vartheta, \zeta) = \dots$ (symmetric in all three variables).

($G_{\mathcal{F}4}$) For every $(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$, for every $n \in \mathbb{N}$, $n \geq 3$, and every $\{a_1, a_2, \dots, a_{n-1}\} \subset \mathcal{A}$ with $a_1 = \zeta$,

$$G(\zeta, \eta, \vartheta) > 0 \text{ implies } f(G(\zeta, \eta, \vartheta)) \leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) + \alpha.$$

Then, the function G is called a $G_{\mathcal{F}}$ -metric on $\mathcal{A}$, and the pair $(\mathcal{A}, G)$ is said to be a $G_{\mathcal{F}}$ -metric space.

2.1 Examples

Every G -metric space is a $G_{\mathcal{F}}$ -metric space. Let $(\mathcal{A}, G)$ be a G -metric space. Then, G is a $G_{\mathcal{F}}$ metric on $\mathcal{A}$, as ($G_{\mathcal{F}1}$) and ($G_{\mathcal{F}3}$) can be obtained from ($G1$), ($G2$), ($G3$) and ($G4$). Also, with $\alpha = 0$ and $f(t) = \frac{-1}{t^2}$, ($G_{\mathcal{F}2}$) and ($G_{\mathcal{F}4}$) are satisfied using ($G3$) and ($G5$).

Now, we construct an example of a $G_{\mathcal{F}}$ -metric space which is a G_b -metric space as well, but not a G -metric space. Let $\mathcal{A} = \{a, b, c\}$ and define $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ as follows:

$G(a, a, a) = G(b, b, b) = G(c, c, c) = 0$, $G(a, a, b) = G(a, b, b) = 1$, $G(a, a, c) = G(a, c, c) = 1.2$, $G(b, b, c) = G(b, c, c) = 1.3$, $G(a, b, c) = 3.3$, and assume that ($G_{\mathcal{F}3}$) holds. Then G is a $G_{\mathcal{F}}$ -metric on $\mathcal{A}$ with $f(t) = \ln(t)$, $t > 0$, and

$\alpha = \ln(1.5)$. Also, G is a G_b -metric on $\mathcal{A}$ with $s = 1.5$, but G is not a G -metric on $\mathcal{A}$ as $G(a, b, c) = 3.3 \not\leq 2.3 = G(a, b, b) + G(b, b, c)$.

See another example of a $G_{\mathcal{F}}$ -metric space which is a generalized G_b -metric space as well, but not a G_b -metric space. Let $\mathcal{A} = \{1, 2, 3, \dots, l-2\} \cup \{l - \frac{1}{n} \mid n \in \mathbb{N}\}$, where l be a fixed natural number such that $l \geq 5$ and $\mathcal{B} = \{1, 2, 3\}$. Consider a mapping $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ defined by

$$G(\zeta, \eta, \vartheta) = \begin{cases} |\zeta - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2, & \text{if } (\zeta, \eta, \vartheta) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B}, \\ |\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|, & \text{otherwise.} \end{cases}$$

Then, $(G_{\mathcal{F}}1)$ and $(G_{\mathcal{F}}3)$ are satisfied for mapping G .

Let $(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$ such that $G(\zeta, \eta, \vartheta) > 0$ and for $n \geq 3$, $\{a_1, a_2, \dots, a_{n-1}\} \subset \mathcal{A}$ with $a_1 = \zeta$.

Let $P = \{i = 1, 2, 3, \dots, n-2 \mid a_i, a_{i+1} \in \mathcal{B}\}$ and $Q = \{1, 2, 3, \dots, n-2\} - P$.

Case 1: If $(\zeta, \eta, \vartheta) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B}$ and $a_{n-1} \in \mathcal{B}$, we have

$$\begin{aligned} & G(\zeta, \eta, \vartheta) \\ &= |\zeta - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2 \\ &\leq 2|\zeta - \eta| + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2 \\ &\leq 2 \left(\sum_{i \in P} |a_i - a_{i+1}| + \sum_{i \in Q} |a_i - a_{i+1}| \right) + 2|a_{n-1} - \eta| + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2 \\ &\leq 2 \left(\sum_{i \in P} |a_i - a_{i+1}|^2 + \sum_{i \in Q} |a_i - a_{i+1}| \right) + 2|a_{n-1} - \eta|^2 + |\eta - \vartheta|^2 + 2|\vartheta - a_{n-1}|^2 \\ &\quad + 2|a_{n-1} - \zeta|^2 \\ &\leq 2 \left(2 \sum_{i \in P} |a_i - a_{i+1}|^2 + 2 \sum_{i \in Q} |a_i - a_{i+1}| \right) + 2|a_{n-1} - \eta|^2 + 2|\eta - \vartheta|^2 + 2|\vartheta - a_{n-1}|^2 \\ &\quad + 2|a_{n-1} - \zeta|^2 \\ &\leq 10 \left(\left(2 \sum_{i \in P} |a_i - a_{i+1}|^2 + 2 \sum_{i \in Q} |a_i - a_{i+1}| \right) + |a_{n-1} - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - a_{n-1}|^2 \right) \\ &= 10 \left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \right). \end{aligned}$$

Case 2: If $(\zeta, \eta, \vartheta) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B}$ and $a_{n-1} \notin \mathcal{B}$, we have

$$\begin{aligned}
 G(\zeta, \eta, \vartheta) &= |\zeta - \eta|^2 + |\eta - \vartheta|^2 + |\vartheta - \zeta|^2 \\
 &\leq 2|\zeta - \eta| + 2|\eta - \vartheta| + 2|\vartheta - \zeta| \\
 &\leq 2 \left(\sum_{i \in P} |a_i - a_{i+1}| + \sum_{i \in Q} |a_i - a_{i+1}| \right) + 2|a_{n-1} - \eta| + 2|\eta - \vartheta| + 2|\vartheta - a_{n-1}| \\
 &\quad + 2|a_{n-1} - \zeta| \\
 &\leq 2 \left(2 \sum_{i \in P} |a_i - a_{i+1}|^2 + 2 \sum_{i \in Q} |a_i - a_{i+1}| \right) + 2|a_{n-1} - \eta| + 2|\eta - \vartheta| + 2|\vartheta - a_{n-1}| \\
 &\quad + 2|a_{n-1} - \zeta| \\
 &\leq 2l \left(\left(2 \sum_{i \in P} |a_i - a_{i+1}|^2 + 2 \sum_{i \in Q} |a_i - a_{i+1}| \right) + |a_{n-1} - \eta| + |\eta - \vartheta| + |\vartheta - a_{n-1}| \right) \\
 &= 2l \left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \right).
 \end{aligned}$$

Case 3: If $(\zeta, \eta, \vartheta) \notin \mathcal{B} \times \mathcal{B} \times \mathcal{B}$, then using Case 1 and Case 2, we have

$$\begin{aligned}
 G(\zeta, \eta, \vartheta) &= |\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta| \\
 &\leq l \left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \right).
 \end{aligned}$$

Thus, combining all cases, we have

$$G(\zeta, \eta, \vartheta) > 0 \text{ implies } G(\zeta, \eta, \vartheta) \leq 2l \left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \right).$$

Therefore, by taking $f(t) = \ln(t)$, $t > 0$, and $\alpha = \ln(2l)$, $(G_{\mathcal{F}}4)$ and $(G_{\mathcal{F}}2)$ are obtained. Hence, G is a $G_{\mathcal{F}}$ -metric on $\mathcal{A}$. Also, G is a generalized G_b -metric on $\mathcal{A}$ with $s = 2l$. But G is not a G_b -metric on $\mathcal{A}$ as $G(1, 1, 3) = 8 \not\leq 6 = G(1, 2, 3)$, i.e., (G_b3) does not hold.

In the following example, we see the existence of a $G_{\mathcal{F}}$ -metric and G^* -metric space, which is not a generalized G_b -metric space. Let $\mathcal{A} = \mathbb{R}$ and $\mathfrak{B} = [0, 1] \times [0, 1] \times [0, 1] - \{(\zeta, \zeta, \zeta) \mid \zeta \in [0, 1]\}$ and $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a mapping defined by

$$G(\zeta, \eta, \vartheta) = \begin{cases} 0, & \text{if } \zeta = \eta = \vartheta, \\ \frac{1}{3}(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|), & \text{if } (\zeta, \eta, \vartheta) \in \mathfrak{B}, \\ 2(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|), & \text{otherwise.} \end{cases}$$

Then, G satisfies $(G_{\mathcal{F}1})$ and $(G_{\mathcal{F}3})$ directly from the definition of G .

Let $f(t) = \frac{-1}{\sqrt{t}}$, $t > 0$, and $\alpha = 1$. Then, for any $(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$ such that $G(\zeta, \eta, \vartheta) > 0$ and for $n \geq 3$, $\{a_1, a_2, \dots, a_{n-1}\} \subset \mathcal{A}$ with $a_1 = \zeta$, we consider the following cases:

Case 1: If $(\zeta, \eta, \vartheta) \in \mathfrak{B}$, then

$$\begin{aligned} G(\zeta, \eta, \vartheta) &= \frac{1}{3}(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|) \\ &\leq \sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \end{aligned}$$

which implies that

$$\begin{aligned} f(G(\zeta, \eta, \vartheta)) &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) \\ &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) + \alpha. \end{aligned}$$

Case 2 : If $(\zeta, \eta, \vartheta) \notin \mathfrak{B}$, then

$$\begin{aligned} &f(G(\zeta, \eta, \vartheta)) - f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) - \alpha \\ &= \frac{-1}{\sqrt{2(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|)}} + \frac{1}{\sqrt{\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)}} - 1 \\ &\leq \frac{-1}{\sqrt{2(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|)}} + 1 - 1 \leq 0. \end{aligned}$$

Thus, $(G_{\mathcal{F}4})$ holds.

Also, for $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\zeta \neq \eta, \eta \neq \vartheta$, we consider the following two cases:

Case 1: If $(\zeta, \zeta, \eta) \in \mathfrak{B}$, then

$$G(\zeta, \zeta, \eta) = \frac{1}{3}(2|\zeta - \eta|) \leq G(\zeta, \eta, \vartheta),$$

which implies that

$$f(G(\zeta, \zeta, \eta)) \leq f(G(\zeta, \eta, \vartheta)) \leq f(G(\zeta, \eta, \vartheta)) + \alpha.$$

Case 2: If $(\zeta, \zeta, \eta) \notin \mathfrak{B}$, then

$$f(G(\zeta, \zeta, \eta)) = \frac{-1}{\sqrt{2(2|\zeta - \eta|)}} \leq \frac{-1}{\sqrt{2(|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|)}} \leq f(G(\zeta, \eta, \vartheta)) + \alpha.$$

Fixed Point Results for (ψ, ϕ) -Contractive Mapping in $G_{\mathcal{F}}$ -Metric Space

Thus, $(G_{\mathcal{F}}2)$ holds; hence, G is a $G_{\mathcal{F}}$ -metric on $\mathcal{A}$. But G is not a generalized G_b -metric on $\mathcal{A}$ as for any $s \geq 1$, and for $n \in \mathbb{N}$,

$$\begin{aligned} 2^{4n} &= G(2n+1, 1, 1) \\ &\leq s(G(2n+1, n+1, n+1) + G(n+1, 1, 1)) \\ &= s(2^{2n} + 2^{2n}), \end{aligned}$$

which gives $2^{2n-1} \leq s$, therefore, by taking $n \rightarrow \infty$, we have a contradiction.

The next example assures the existence of a $G_{\mathcal{F}}$ -metric space which is not a G^* -metric space. Let $\mathcal{B} = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $\mathcal{A} = \mathcal{B} \cup \mathbb{N} \cup \{0\}$. Let $G : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ be a mapping defined by

$$G(\zeta, \eta, \vartheta) = \begin{cases} 0, & \text{if } \zeta = \eta = \vartheta; \\ |\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|, & \text{if } \zeta, \eta, \vartheta \in \mathcal{B}; \text{ or if atleast one of } \zeta, \eta, \vartheta \text{ is } 0; \\ 1, & \text{otherwise.} \end{cases}$$

First, we prove that G is not a G^* -metric. Suppose that G is a G^* -metric with constant $\beta > 0$.

Let $\zeta = 0$ and $\zeta_n = \frac{1}{n}$, then $\{\zeta_n\} \in C_{\mathcal{A}}(G, \zeta)$. Therefore, for $\eta, \vartheta \in \mathbb{N}$, using $(Gg3)$, we get

$$G(\zeta, \eta, \vartheta) \leq \beta \left(\limsup_{n \rightarrow \infty} G(\zeta_n, \eta, \vartheta) + G(\zeta, \zeta, \zeta) \right),$$

that is,

$$|\eta| + |\eta - \vartheta| + |\vartheta| \leq \beta(1 + 0) = \beta.$$

Taking limit $\eta, \vartheta \rightarrow \infty$, we have a contradiction. Thus, $(\mathcal{A}, G)$ is not a G^* -metric space.

We now prove that $(\mathcal{A}, G)$ is a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, where $f(t) = \frac{-1}{t}$ and $\alpha = 1$. $(G_{\mathcal{F}}1)$ and $(G_{\mathcal{F}}3)$ hold obviously. Now, for $(G_{\mathcal{F}}2)$, let $\zeta, \eta, \vartheta \in \mathcal{A}$ with $\zeta \neq \eta, \eta \neq \vartheta$. Then consider the following two cases:

Case 1: If $\zeta, \eta, \vartheta \in \mathcal{B} \cup \{0\}$, then $G(\zeta, \zeta, \eta) \leq G(\zeta, \eta, \vartheta)$. Therefore,

$$f(G(\zeta, \zeta, \eta)) \leq f(G(\zeta, \eta, \vartheta)) \leq f(G(\zeta, \eta, \vartheta)) + \alpha.$$

Case 2 : If at least one of $\zeta, \eta, \vartheta \in \mathbb{N}$, then $G(\zeta, \eta, \vartheta) \geq 1$. Thus,

$$\begin{aligned} &f(G(\zeta, \zeta, \eta)) - f(G(\zeta, \eta, \vartheta)) - \alpha \\ &= \frac{-1}{G(\zeta, \zeta, \eta)} + \frac{1}{G(\zeta, \eta, \vartheta)} - 1 \\ &\leq \frac{-1}{G(\zeta, \zeta, \eta)} + 1 - 1 < 0. \end{aligned}$$

Thus, $(G_{\mathcal{F}2})$ holds.

Now, for $(G_{\mathcal{F}4})$ let $\zeta, \eta, \vartheta \in \mathcal{A}$ such that $G(\zeta, \eta, \vartheta) > 0$ and for $n \geq 3$, $\{a_1, a_2, \dots, a_{n-1}\} \subset \mathcal{A}$ with $a_1 = \zeta$. Consider the following two cases:

Case 1: If $\zeta, \eta, \vartheta \in \mathcal{B} \cup \{0\}$, then

$$G(\zeta, \eta, \vartheta) \leq \sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta),$$

therefore,

$$\begin{aligned} f(G(\zeta, \eta, \vartheta)) &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) \\ &\leq f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) + \alpha. \end{aligned}$$

Case 2 : If at least one of $\zeta, \eta, \vartheta \in \mathbb{N}$, then $\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta) \geq$

1. Thus,

$$\begin{aligned} &f(G(\zeta, \eta, \vartheta)) - f\left(\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)\right) - \alpha \\ &= \frac{-1}{G(\zeta, \eta, \vartheta)} + \frac{1}{\sum_{i=1}^{n-2} G(a_i, a_{i+1}, a_{i+1}) + G(a_{n-1}, \eta, \vartheta)} - 1 \\ &\leq \frac{-1}{G(\zeta, \eta, \vartheta)} + 1 - 1 < 0. \end{aligned}$$

Thus, $(G_{\mathcal{F}4})$ holds. Hence, $(\mathcal{A}, G)$ is $G_{\mathcal{F}}$ -metric space.

Abstract spaces in Example 1, Example 1, Example 1, and Example 1 are not $G_{\mathcal{F}}$ -metric spaces. A relation among the classes of abstract spaces discussed so far is described in the following diagram (Figure 1):

Figure 1: A comparison among various classes of abstract space.

2.2 Some Basic Concepts

This section defines a few fundamental ideas and examines their characteristics within the context of $G_{\mathcal{F}}$ -metric space. Let A be a subset of $G_{\mathcal{F}}$ -metric space $(\mathcal{A}, G)$. Then A is said to be a $G_{\mathcal{F}}$ -open set if for every $\zeta \in A$, there exists $r > 0$ such that $B(\zeta, r) \subseteq A$, where

$$B(\zeta, r) = \{\eta \in \mathcal{A} \mid G(\zeta, \eta, \eta) < r\}.$$

Let $\tau_{G_{\mathcal{F}}}$ be the collection of all such open sets, then $\tau_{G_{\mathcal{F}}}$ is a topology on $\mathcal{A}$.

Let $\{\zeta_n\}$ be a sequence of points in $G_{\mathcal{F}}$ -metric space $(\mathcal{A}, G)$. Then sequence $\{\zeta_n\}$ is said to be $G_{\mathcal{F}}$ -convergent to $\zeta_0 \in \mathcal{A}$ if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(\zeta_n, \zeta_m, \zeta_0) < \epsilon$, for all $n, m \geq n_0$.

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, then

$$f(G(\zeta, \eta, \eta)) \leq f(2G(\zeta, \zeta, \eta)) + \alpha, \text{ for all } \zeta, \eta \in \mathcal{A} \text{ with } \zeta \neq \eta.$$

Proof. For $\zeta, \eta \in \mathcal{A}$ with $\zeta \neq \eta$, using $(G_{\mathcal{F}}1)$, $(G_{\mathcal{F}}4)$ and $(G_{\mathcal{F}}3)$, we have,

$$f(G(\zeta, \eta, \eta)) \leq f(G(\eta, \zeta, \zeta) + G(\zeta, \zeta, \eta)) + \alpha = f(2G(\zeta, \zeta, \eta)) + \alpha.$$

□

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then, the following statements are equivalent.

- (I) $G(\zeta_n, \zeta_n, \zeta_0) \rightarrow 0$ as $n \rightarrow \infty$.
- (II) $G(\zeta_n, \zeta_0, \zeta_0) \rightarrow 0$ as $n \rightarrow \infty$.
- (III) $G(\zeta_n, \zeta_m, \zeta_0) \rightarrow 0$ as $n, m \rightarrow \infty$.

Proof. First, we prove (I) implies (II); for this, let $G(\zeta_n, \zeta_n, \zeta_0) \rightarrow 0$ as $n \rightarrow \infty$. If for infinitely many n , $\zeta_n \neq \zeta_0$, then for those n , using Proposition 2.2, we have

$$f(G(\zeta_n, \zeta_0, \zeta_0)) \leq f(2G(\zeta_n, \zeta_n, \zeta_0)) + \alpha.$$

Taking $n \rightarrow \infty$ on right-hand side and using $(\mathcal{F}2)$, we arrive at $G(\zeta_n, \zeta_0, \zeta_0) \rightarrow 0$. and for rest of n , $G(\zeta_n, \zeta_0, \zeta_0) = 0$, thus in overall $G(\zeta_n, \zeta_0, \zeta_0) \rightarrow 0$.

Now we prove (II) implies (III); for this, let $G(\zeta_n, \zeta_0, \zeta_0) \rightarrow 0$ as $n \rightarrow \infty$.

If for infinitely many m and n , $G(\zeta_n, \zeta_m, \zeta_0) \neq 0$, then for those (m, n) , using $(G_{\mathcal{F}}4)$, we have

$$f(G(\zeta_n, \zeta_m, \zeta_0)) \leq f(G(\zeta_n, \zeta_0, \zeta_0) + G(\zeta_0, \zeta_0, \zeta_m)) + \alpha.$$

Taking $n, m \rightarrow \infty$ on right-hand side and using $(\mathcal{F}2)$, we arrive at $G(\zeta_n, \zeta_m, \zeta_0) \rightarrow 0$,

and for rest of n, m , $G(\zeta_n, \zeta_m, \zeta_0) = 0$, thus in overall $G(\zeta_n, \zeta_m, \zeta_0) \rightarrow 0$ as $n, m \rightarrow \infty$.

Also, (III) implies (I) obviously. □

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then a sequence $\{\zeta_n\}$ in $\mathcal{A}$ which is $G_{\mathcal{F}}$ -convergent, converges to a unique element in $\mathcal{A}$.

Proof. Let, if possible that sequence $\{\zeta_n\}$ converges to ζ and η in $\mathcal{A}$ with $\zeta \neq \eta$, then, by $(G_{\mathcal{F}}4)$

$$f(G(\zeta, \eta, \eta)) \leq f(G(\zeta, \zeta_n, \zeta_n) + G(\zeta_n, \eta, \eta)) + \alpha,$$

taking $n \rightarrow \infty$ on right-hand side and using $(\mathcal{F}2)$, we arrive at a contradiction. $\square$

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space. Then a sequence $\{\zeta_n\}$ in $\mathcal{A}$ is said to be $G_{\mathcal{F}}$ -Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(\zeta_n, \zeta_m, \zeta_l) < \epsilon$, for all $n, m, l \geq n_0$.

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then, the following statements are equivalent.

(I) Sequence $\{\zeta_n\}$ in $\mathcal{A}$ is $G_{\mathcal{F}}$ -Cauchy sequence.

(II) $G(\zeta_n, \zeta_m, \zeta_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$.

(III) $G(\zeta_n, \zeta_m, \zeta_m) \rightarrow 0$ as $n, m \rightarrow \infty$.

Proof. (I) and (II) are equivalent using Definition 2.2.

Now (II) implies (III) obviously, so we prove (III) implies (II). If for infinitely many n, m and l , $G(\zeta_n, \zeta_m, \zeta_l) \neq 0$, then for those (n, m, l) , using $(G_{\mathcal{F}}4)$, we have

$$f(G(\zeta_n, \zeta_m, \zeta_l)) \leq f(G(\zeta_n, \zeta_m, \zeta_m) + G(\zeta_m, \zeta_m, \zeta_l)) + \alpha.$$

Taking $n, m, l \rightarrow \infty$ on the right-hand side and using $(\mathcal{F}2)$, we arrive at $G(\zeta_n, \zeta_m, \zeta_l) \rightarrow 0$,

and for rest of n, m, l , $G(\zeta_n, \zeta_m, \zeta_l) = 0$, thus in overall $G(\zeta_n, \zeta_m, \zeta_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$. $\square$

Every $G_{\mathcal{F}}$ -convergent sequence is a $G_{\mathcal{F}}$ -Cauchy sequence.

In Example 2.1, sequence $\{l - \frac{1}{n}\}$ is $G_{\mathcal{F}}$ -Cauchy sequence but not a $G_{\mathcal{F}}$ -convergent sequence.

A $G_{\mathcal{F}}$ -metric space is said to be $G_{\mathcal{F}}$ -complete if every $G_{\mathcal{F}}$ -Cauchy sequence is a $G_{\mathcal{F}}$ -convergent sequence.

In Example 2.1, $(\mathcal{A}, G)$ is a $G_{\mathcal{F}}$ -complete metric space as every $G_{\mathcal{F}}$ -Cauchy sequence is a $G_{\mathcal{F}}$ -convergent sequence.

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space and $\mathcal{B} \subseteq \mathcal{A}$. Then *closure* of $\mathcal{B}$ is denoted by $\overline{\mathcal{B}}$ and defined by

$$\overline{\mathcal{B}} = \{\zeta \in \mathcal{A} \mid \zeta \in \mathcal{B} \text{ or } B(\zeta, r) \cap \mathcal{B} \text{ is infinite set for every } r > 0\}.$$

Fixed Point Results for (ψ, ϕ) -Contractive Mapping in $G_{\mathcal{F}}$ -Metric Space

The following propositions are needed in the main results of this paper. Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and f is a continuous function. If $\{\zeta_n\}$ is $G_{\mathcal{F}}$ -convergent to ζ with $\zeta \neq b$ or $\zeta \neq c$, where b and c are real constants, then

$$f(G(\zeta, b, c)) - \alpha \leq \liminf_{n \rightarrow \infty} f(G(\zeta_n, b, c)) \leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, b, c)) \leq f(G(\zeta, b, c)) + \alpha.$$

Proof. Without loss of generality, consider $\zeta \neq b$, so except first finitely many n , $\zeta_n \neq b$. Now using $(G_{\mathcal{F}}4)$, we have

$$\begin{aligned} f(G(\zeta, b, c)) - \alpha &\leq \liminf_{n \rightarrow \infty} f(G(\zeta, \zeta_n, \zeta_n) + G(\zeta_n, b, c)) \\ &= \liminf_{n \rightarrow \infty} f(G(\zeta_n, b, c)) \\ &\leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, b, c)) \\ &\leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, \zeta, \zeta) + G(\zeta, b, c)) + \alpha \\ &= \limsup_{n \rightarrow \infty} f(G(\zeta, b, c)) + \alpha \\ &= f(G(\zeta, b, c)) + \alpha. \end{aligned}$$

□

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, and f is a continuous function. If $\{\zeta_n\}$ and $\{\eta_n\}$ are $G_{\mathcal{F}}$ -convergent to ζ and η , respectively, and c is a real constant such that $c \neq \zeta$ or $c \neq \eta$, then

$$f(G(\zeta, \eta, c)) - 2\alpha \leq \liminf_{n \rightarrow \infty} f(G(\zeta_n, \eta_n, c)) \leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, \eta_n, c)) \leq f(G(\zeta, \eta, c)) + 2\alpha.$$

Proof. Without loss of generality, assume that $c \neq \zeta$; therefore, except first

finitely many n , $c \neq \zeta_n$. Now using $(G_{\mathcal{F}}4)$, we have

$$\begin{aligned}
 f(G(\zeta, \eta, c)) - 2\alpha &\leq \liminf_{n \rightarrow \infty} f(G(\zeta, \zeta_n, \zeta_n) + G(\zeta_n, \eta, c)) - \alpha \\
 &= \liminf_{n \rightarrow \infty} f(G(\zeta_n, \eta, c)) - \alpha \\
 &\leq \liminf_{n \rightarrow \infty} f(G(\eta, \eta_n, \eta_n) + G(\eta_n, \zeta_n, c)) \\
 &= \liminf_{n \rightarrow \infty} f(G(\zeta_n, \eta_n, c)) \\
 &\leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, \eta_n, c)) \\
 &\leq \limsup_{n \rightarrow \infty} f(G(\zeta_n, \zeta, \zeta) + G(\zeta, \eta_n, c)) + \alpha \\
 &= \limsup_{n \rightarrow \infty} f(G(\zeta, \eta_n, c)) + \alpha \\
 &\leq \limsup_{n \rightarrow \infty} f(G(\eta_n, \eta, \eta) + G(\eta, \zeta, c)) + 2\alpha \\
 &= \limsup_{n \rightarrow \infty} f(G(\eta, \zeta, c)) + 2\alpha. \\
 &= f(G(\zeta, \eta, c)) + 2\alpha.
 \end{aligned}$$

□

3 Fixed Point Results in $G_{\mathcal{F}}$ -Metric Space

This section deals with some fixed point results in the context of $G_{\mathcal{F}}$ -metric space. Our first result is the following theorem. Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -complete metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and $T : \mathcal{A} \rightarrow \mathcal{A}$ be a mapping such that there exists a non-empty subset $\mathcal{B}$ of $\mathcal{A}$ with $T(\mathcal{B}) \subseteq \mathcal{B}$ and

$$G(T\zeta, T\eta, T\vartheta) \leq \lambda G(\zeta, \eta, \vartheta), \quad \text{for all } \zeta, \eta, \vartheta \in \overline{\mathcal{B}},$$

where $\lambda \in [0, 1)$. Then, T has a fixed point in $\mathcal{A}$. Moreover, if the fixed point belongs to $\mathcal{B}$, then T has a unique fixed point in $\mathcal{B}$.

Proof. Let $\zeta_0 \in \mathcal{B}$ be arbitrary. Define a sequence $\{\zeta_n\}$ in $\mathcal{A}$ by $\zeta_n = T\zeta_{n-1}$, for all $n \in \mathbb{N}$. If $\zeta_n = \zeta_{n+1}$ for some $n \in \mathbb{N}$, then ζ_n is a fixed point of T . So let $\zeta_n \neq \zeta_{n+1}$, for every $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$, we have

$$G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \leq \lambda G(\zeta_{n-1}, \zeta_n, \zeta_n).$$

Now, an easy induction gives that

$$G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \leq \lambda^n G(\zeta_0, \zeta_1, \zeta_1).$$

Fixed Point Results for (ψ, ϕ) -Contractive Mapping in $G_{\mathcal{F}}$ -Metric Space

Let $n, m \in \mathbb{N}$ with $m > n$; then, we have

$$\sum_{i=n}^{m-1} G(\zeta_i, \zeta_{i+1}, \zeta_{i+1}) \leq \frac{\lambda^n}{1-\lambda} G(\zeta_0, \zeta_1, \zeta_1).$$

If for infinitely many pairs (m, n) with $m > n$, $G(\zeta_n, \zeta_m, \zeta_m) \neq 0$, then for these m, n using $(G_{\mathcal{F}4})$, we have

$$\begin{aligned} f(G(\zeta_n, \zeta_m, \zeta_m)) &\leq f\left(\sum_{i=n}^{m-1} G(\zeta_i, \zeta_{i+1}, \zeta_{i+1})\right) + \alpha \\ &\leq f\left(\frac{\lambda^n}{1-\lambda} G(\zeta_0, \zeta_1, \zeta_1)\right) + \alpha, \end{aligned}$$

taking $n, m \rightarrow \infty$ and using $(\mathcal{F}2)$, we get $G(\zeta_n, \zeta_m, \zeta_m) \rightarrow 0$.

Also, for rest of m, n with $m > n$, $G(\zeta_n, \zeta_m, \zeta_m) = 0$. Thus in overall,

$$G(\zeta_n, \zeta_m, \zeta_m) \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Thus, $\{\zeta_n\}$ is a $G_{\mathcal{F}}$ -Cauchy sequence in $\mathcal{A}$, but $(\mathcal{A}, G)$ is a $G_{\mathcal{F}}$ -complete metric space, therefore $\{\zeta_n\}$ is $G_{\mathcal{F}}$ -convergent to some $\zeta' \in \mathcal{A}$. Suppose, if possible $G(T\zeta', \zeta', \zeta') > 0$, then using $(G_{\mathcal{F}4})$, we have

$$\begin{aligned} f(G(T\zeta', \zeta', \zeta')) &\leq f(G(T\zeta', T\zeta_n, T\zeta_n) + G(T\zeta_n, \zeta', \zeta')) + \alpha \\ &\leq f(\lambda G(\zeta', \zeta_n, \zeta_n) + G(\zeta_{n+1}, \zeta', \zeta')) + \alpha. \end{aligned}$$

Taking $n \rightarrow \infty$ on the right-hand side and using $(\mathcal{F}2)$, we arrive at a contradiction.

Thus, $G(T\zeta', \zeta', \zeta') = 0$ and which implies that $T\zeta' = \zeta'$.

Now, if $\zeta' \in \mathcal{B}$ and $\eta \in \mathcal{B}$ be another fixed point of T , then

$$G(\zeta', \eta, \eta) = G(T\zeta', T\eta, T\eta) \leq \lambda G(\zeta', \eta, \eta)$$

which implies that $G(\zeta', \eta, \eta) = 0$ as $\lambda \in [0, 1)$. Therefore, $\zeta' = \eta$, that is, ζ' is a unique fixed point of T in $\mathcal{B}$. □

Consider $G_{\mathcal{F}}$ -metric space $(\mathcal{A}, G)$ as in Example 2.1, which is a $G_{\mathcal{F}}$ -complete metric space. Define a mapping $T : \mathcal{A} \rightarrow \mathcal{A}$ as $T\zeta = \frac{\zeta(\zeta+1)}{4}$, for all $\zeta \in \mathcal{A}$.

Now, for $\mathcal{B} = [0, 1]$, we see that $T(\mathcal{B}) \subseteq \mathcal{B}$, and for $\zeta, \eta, \vartheta \in \mathcal{B}$, we have

$$\begin{aligned} &G(T\zeta, T\eta, T\vartheta) \\ &= \frac{1}{3} \left(\left| \frac{\zeta(\zeta+1)}{4} - \frac{\eta(\eta+1)}{4} \right| + \left| \frac{\eta(\eta+1)}{4} - \frac{\vartheta(\vartheta+1)}{4} \right| + \left| \frac{\vartheta(\vartheta+1)}{4} - \frac{\zeta(\zeta+1)}{4} \right| \right) \\ &\leq \frac{1}{3} \times \frac{3}{4} (|\zeta - \eta| + |\eta - \vartheta| + |\vartheta - \zeta|) \\ &= \frac{3}{4} G(\zeta, \eta, \vartheta). \end{aligned}$$

Thus, the hypothesis of Theorem 3 is satisfied. And we notice that T has two fixed points, 0 and 3. Also, 0 is the only fixed point of T in $\mathcal{B}$.

Consider $G_{\mathcal{F}}$ -metric space $(\mathcal{A}, G)$ as in Example 2.1, which is a $G_{\mathcal{F}}$ -complete metric space. Define a mapping $T : \mathcal{A} \rightarrow \mathcal{A}$ as $T\zeta = \frac{\zeta}{2}$, for all $\zeta \in \mathcal{A}$. Then for $\mathcal{B} = [0, 1]$, we can easily see that hypothesis of Theorem 3 is satisfied. Also, we notice that T has a unique fixed point in $\mathcal{A}$.

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -complete metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and $T : \mathcal{A} \rightarrow \mathcal{A}$ be a mapping such that

$$G(T\zeta, T\eta, T\vartheta) \leq \lambda G(\zeta, \eta, \vartheta), \quad \text{for all } \zeta, \eta, \vartheta \in \mathcal{A},$$

where $\lambda \in [0, 1)$. Then, T has a unique fixed point in $\mathcal{A}$.

Proof. Take $\mathcal{B} = \mathcal{A}$ in Theorem 3. □

In the following result, we find the unique fixed point for (ψ, ϕ) -contractive mapping (see detail of (ψ, ϕ) -contractive mapping in (51; 52; 14; 53; 29; 32; 54)) in the setting of $G_{\mathcal{F}}$ -complete metric space.

Let $(\mathcal{A}, G)$ be a $G_{\mathcal{F}}$ -complete metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that f is a continuous function. Let $T : \mathcal{A} \rightarrow \mathcal{A}$ be a mapping such that

$$\psi(f(M(\zeta, \eta, \vartheta)) + 4\alpha) \leq \psi(f(G(T\zeta, T\eta, T\vartheta))) - \phi(M(\zeta, \eta, \vartheta)), \quad (1)$$

for all $(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A} - \{(\zeta, \eta, \vartheta) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A} \mid T\zeta = T\eta = T\vartheta\}$, where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous non-decreasing function and $\phi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi-continuous function with $\phi^{-1}(0) = \{0\}$ and

$$M(\zeta, \eta, \vartheta) = \max\{G(\zeta, \eta, \vartheta), G(\zeta, T\zeta, T\eta), G(\eta, T\eta, T\vartheta), G(\vartheta, T\vartheta, T\zeta)\}.$$

Then, T has a unique fixed point.

Proof. Let $\zeta_0 \in \mathcal{A}$ be arbitrary. Define a sequence $\{\zeta_n\}$ in $\mathcal{A}$ by $\zeta_{n+1} = T\zeta_n$, $n = 0, 1, 2, \dots$.

If $\zeta_n = \zeta_{n+1}$, then ζ_n is a fixed point of T .

Now, assume that $\zeta_n \neq \zeta_{n+1}$, for all n . Let $\theta_n = f(G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}))$, $n = 1, 2, 3, \dots$.

Now,

$$\begin{aligned} \psi(f(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}))) &\leq \psi(f(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) + 4\alpha) \\ &\leq \psi(f(G(T\zeta_n, T\zeta_{n+1}, T\zeta_{n+2}))) - \phi(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) \\ &\leq \psi(\theta_{n+1}) - \phi(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})), \end{aligned}$$

where

$$\begin{aligned} M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) &= \max\{G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}), G(\zeta_n, T\zeta_n, T\zeta_{n+1}), \\ &\quad G(\zeta_{n+1}, T\zeta_{n+1}, T\zeta_{n+2}), G(\zeta_{n+2}, T\zeta_{n+2}, T\zeta_n)\} \\ &= \max\{G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}), G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}), \\ &\quad G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}), G(\zeta_{n+2}, \zeta_{n+3}, \zeta_{n+1})\} \end{aligned}$$

and, therefore,

$$\begin{aligned} f(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) &= \max\{f(G(\zeta_n, \zeta_{n+1}, \zeta_{n+2})), f(G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}))\} \\ &= \max\{\theta_n, \theta_{n+1}\}. \end{aligned}$$

If $\theta_n < \theta_{n+1}$ for some n , then we have $\psi(\theta_{n+1}) \leq \psi(\theta_{n+1}) - \phi(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}))$, which gives that $M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) = 0$, a contradiction, therefore $\theta_{n+1} \leq \theta_n$ for all n . Thus, $\{\theta_n\}$ is a non-increasing sequence. Suppose that $\{\theta_n\}$ is bounded below; then there exists a real θ such that $\lim_{n \rightarrow \infty} \theta_n = \theta$. Now,

$$\psi(\theta_n) \leq \psi(\theta_{n+1}) - \phi(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}))$$

taking limit supremum on both sides, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \psi(\theta_n) &\leq \limsup_{n \rightarrow \infty} \psi(\theta_{n+1}) - \liminf_{n \rightarrow \infty} \phi(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) \\ \text{i.e.,} \quad \psi(\theta) &\leq \psi(\theta) - \phi\left(\liminf_{n \rightarrow \infty} M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})\right) \end{aligned}$$

which gives that $\liminf_{n \rightarrow \infty} M(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) = 0$, therefore, by $(\mathcal{F}2)$

$\liminf_{n \rightarrow \infty} f(M(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) = -\infty$, i.e., $\lim_{n \rightarrow \infty} \theta_n = -\infty$. Hence by $(\mathcal{F}2)$,

$$\lim_{n \rightarrow \infty} G(\zeta_n, \zeta_{n+1}, \zeta_{n+2}) = 0. \quad (2)$$

Since $\zeta_n \neq \zeta_{n+1}$ for every n , therefore, by $(G_{\mathcal{F}}2)$,

$$f(G(\zeta_n, \zeta_n, \zeta_{n+1})) \leq f(G(\zeta_n, \zeta_{n+1}, \zeta_{n+2})) + \alpha.$$

So, by using (2) and $(\mathcal{F}2)$, we have

$$\lim_{n \rightarrow \infty} G(\zeta_n, \zeta_n, \zeta_{n+1}) = 0. \quad (3)$$

Also, using Proposition 2.2, we have

$$\lim_{n \rightarrow \infty} G(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) = 0. \quad (4)$$

Next, we prove that $\{\zeta_n\}$ is a $G_{\mathcal{F}}$ -Cauchy sequence. Suppose not, then there exists $\epsilon > 0$ such that we can find subsequences $\{\zeta_{m_k}\}$ and $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that m_k is the smallest index for which $m_k > n_k > k$ and

$$G(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k}) \geq \epsilon \quad (5)$$

this means that

$$G(\zeta_{n_k}, \zeta_{m_k-1}, \zeta_{m_k-1}) < \epsilon. \quad (6)$$

Now further, consider only those k for which left-hand side of (6) is greater than 0, and clearly, such k exists infinitely many.

Now,

$$\begin{aligned} & \psi(f(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) + 4\alpha) \\ &= \psi(f(G(T\zeta_{n_k}, T\zeta_{m_k-2}, T\zeta_{m_k-1}))) - \phi(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) \\ &\leq \psi(f(G(\zeta_{n_k+1}, \zeta_{m_k-1}, \zeta_{m_k}))) - \phi(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})), \end{aligned} \quad (7)$$

where

$$\begin{aligned} M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}) &= \max\{G(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}), G(\zeta_{n_k}, T\zeta_{n_k}, T\zeta_{m_k-2}), \\ & \quad G(\zeta_{m_k-2}, T\zeta_{m_k-2}, T\zeta_{m_k-1}), G(\zeta_{m_k-1}, T\zeta_{m_k-1}, T\zeta_{n_k})\} \\ &= \max\{G(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}), G(\zeta_{n_k}, \zeta_{n_k+1}, \zeta_{m_k-1}), \\ & \quad G(\zeta_{m_k-2}, \zeta_{m_k-1}, \zeta_{m_k}), G(\zeta_{m_k-1}, \zeta_{m_k}, \zeta_{n_k+1})\}. \end{aligned} \quad (8)$$

Also, using $(G_{\mathcal{F}4})$, (3), and (6), we have

$$\begin{aligned} & \limsup_{k \rightarrow \infty} f(G(\zeta_{m_k-1}, \zeta_{m_k}, \zeta_{n_k+1})) \\ &\leq \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k+1}, \zeta_{n_k}, \zeta_{n_k}) + G(\zeta_{n_k}, \zeta_{m_k-1}, \zeta_{m_k-1}) + G(\zeta_{m_k-1}, \zeta_{m_k-1}, \zeta_{m_k})) + \alpha \\ &\leq \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k}, \zeta_{m_k-1}, \zeta_{m_k-1})) + \alpha \\ &\leq f(\epsilon) + \alpha. \end{aligned} \quad (9)$$

Now, using (5), $(G_{\mathcal{F}4})$, and $(G_{\mathcal{F}2})$, we have

$$\begin{aligned} f(\epsilon) &\leq \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k}, \zeta_{m_k}, \zeta_{m_k})) \\ &\leq \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k}, \zeta_{n_k+1}, \zeta_{n_k+1}) + G(\zeta_{n_k+1}, \zeta_{m_k}, \zeta_{m_k})) + \alpha \\ &= \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k+1}, \zeta_{m_k}, \zeta_{m_k})) + \alpha \\ &\leq \limsup_{k \rightarrow \infty} f(G(\zeta_{n_k+1}, \zeta_{m_k}, \zeta_{m_k-1})) + 2\alpha. \end{aligned} \quad (10)$$

Now, using (10), (7), and (9), we have

$$\begin{aligned}
 \psi(f(\epsilon) + 2\alpha) &\leq \psi \left(\limsup_{k \rightarrow \infty} f(G(\zeta_{n_k+1}, \zeta_{m_k-1}, \zeta_{m_k})) + 2\alpha + 2\alpha \right) \\
 &\leq \psi \left(\limsup_{k \rightarrow \infty} f(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) + 4\alpha \right) \\
 &\leq \psi \left(\limsup_{k \rightarrow \infty} f(G(\zeta_{n_k+1}, \zeta_{m_k-1}, \zeta_{m_k})) \right) - \liminf_{k \rightarrow \infty} \phi(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) \\
 &\leq \psi(f(\epsilon) + \alpha) - \phi \left(\liminf_{k \rightarrow \infty} M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}) \right) \\
 &\leq \psi(f(\epsilon) + 2\alpha) - \phi \left(\liminf_{k \rightarrow \infty} M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}) \right)
 \end{aligned}$$

This gives

$$\liminf_{k \rightarrow \infty} M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1}) = 0,$$

therefore, we have

$$\liminf_{k \rightarrow \infty} f(M(\zeta_{n_k}, \zeta_{m_k-2}, \zeta_{m_k-1})) = -\infty,$$

which gives a contradiction in view of (8) and (10). Thus $\{\zeta_n\}$ is a $G_{\mathcal{F}}$ -Cauchy sequence in $(\mathcal{A}, G)$, but $(\mathcal{A}, G)$ is $G_{\mathcal{F}}$ -complete. Therefore, there exists $b \in \mathcal{A}$ such that $\lim_{n \rightarrow \infty} \zeta_n = b$.

Next, we prove that b is a fixed point of T . Suppose that $Tb \neq b$, then

$$\begin{aligned}
 &\psi(f(M(b, \zeta_{n+1}, \zeta_{n+2})) + 4\alpha) \\
 &\leq \psi(f(G(Tb, T\zeta_{n+1}, T\zeta_{n+2}))) - \phi(M(b, \zeta_{n+1}, \zeta_{n+2})) \\
 &= \psi(f(G(Tb, \zeta_{n+2}, \zeta_{n+3}))) - \phi(M(b, \zeta_{n+1}, \zeta_{n+2})), \tag{11}
 \end{aligned}$$

where

$$\begin{aligned}
 M(b, \zeta_{n+1}, \zeta_{n+2}) &= \max\{G(b, \zeta_{n+1}, \zeta_{n+2}), G(b, Tb, T\zeta_{n+1}), \\
 &\quad G(\zeta_{n+1}, T\zeta_{n+1}, T\zeta_{n+2}), G(\zeta_{n+2}, T\zeta_{n+2}, Tb)\} \\
 &= \max\{G(b, \zeta_{n+1}, \zeta_{n+2}), G(b, Tb, \zeta_{n+2}), \\
 &\quad G(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}), G(\zeta_{n+2}, \zeta_{n+3}, Tb)\}.
 \end{aligned}$$

Thus, by Proposition 2.2 and Proposition 2.2, we have

$$\begin{aligned}
 f(G(Tb, b, b)) - 2\alpha &\leq \liminf_{n \rightarrow \infty} f(M(b, \zeta_{n+1}, \zeta_{n+2})) \\
 &\leq \limsup_{n \rightarrow \infty} f(M(b, \zeta_{n+1}, \zeta_{n+2})) \\
 &\leq f(G(Tb, b, b)) + 2\alpha. \tag{12}
 \end{aligned}$$

Now, using (11) and (12), we have

$$\begin{aligned}
 \psi(f(G(Tb, b, b)) + 2\alpha) &\leq \psi\left(\limsup_{n \rightarrow \infty} f(M(b, \zeta_{n+1}, \zeta_{n+2})) + 4\alpha\right) \\
 &\leq \psi\left(\limsup_{n \rightarrow \infty} f(G(Tb, \zeta_{n+2}, \zeta_{n+3}))\right) - \liminf_{n \rightarrow \infty} \phi(M(b, \zeta_{n+1}, \zeta_{n+2})) \\
 &\leq \psi(f(G(Tb, b, b)) + 2\alpha) - \phi\left(\liminf_{n \rightarrow \infty} M(b, \zeta_{n+1}, \zeta_{n+2})\right).
 \end{aligned}$$

It gives that

$$\liminf_{n \rightarrow \infty} M(b, \zeta_{n+1}, \zeta_{n+2}) = 0,$$

therefore, we have

$$\liminf_{n \rightarrow \infty} f(M(b, \zeta_{n+1}, \zeta_{n+2})) = -\infty,$$

which gives a contradiction in (12); therefore, $Tb = b$. Next we prove that the fixed point of T is unique. For this, let c be another fixed point of T such that $c \neq b$. Then

$$\begin{aligned}
 M(b, b, c) &= \max\{G(b, b, c), G(b, Tb, Tb), G(b, Tb, Tc), G(c, Tc, Tb)\} \\
 &= \max\{G(b, b, c), G(b, b, b), G(b, b, c), G(c, c, b)\} \\
 &= \max\{G(b, b, c), G(c, c, b)\} \\
 &= M(c, c, b).
 \end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned}
 \psi(f(G(b, b, c))) &\leq \psi(f(M(b, b, c)) + 4\alpha) \\
 &\leq \psi(f(G(Tb, Tb, Tc))) - \phi(M(b, b, c)) \\
 &= \psi(f(G(b, b, c))) - \phi(M(b, b, c)).
 \end{aligned} \tag{14}$$

It gives that $M(b, b, c) = 0$, and hence $G(b, b, c) = G(c, c, b) = 0$. Thus $b = c$. $\square$

4 Conclusion

With the aid of $\mathcal{F}$ -metric space, we have introduced a new generalization of G -metric space, which we call $G_{\mathcal{F}}$ -metric space. We have also shown a comparison between $G_{\mathcal{F}}$ -metric space and several abstract spaces found in literature. This newly defined abstract space is also studied in terms of some fundamental concepts. In the framework of $G_{\mathcal{F}}$ -metric space, we have demonstrated the Banach Contraction Principle and the fixed point result for (ψ, ϕ) -contractive mapping. In

this newly defined abstract space, fixed point results for different mappings existing in the literature and for some new mappings can be studied.

Author Contributions: All the authors have equally contributed in the planning, execution, and analysis of the study.

Acknowledgements: Authors are very thankful to the reviewers for their valuable comments and suggestions.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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