The modified divisor graph of finite commutative ring

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Abstract

Let R be a finite commutative ring with unity. we have introduced a divisor graph of ring which is denoted by $\mathbb{D}[R]$. It is a undirected simple graph with vertex set $V = R - \{0, 1\}$ and two distinct vertices $u, v \in V$ are adjacent if and only if $\exists w \in R$ such that $v = uw$ or $u = vw$. Clearly 0 and unit elements are adjacent to all elements of ring. Thus, we adopt the idea of Anderson and Livingston and removed the zero and unit vertices from the vertex set. A new graph is called modified divisor graph noted by symbol $\mathbb{D}_0[R]$. In this study, we have focused on the structural properties of $\mathbb{D}_0[Z_n]$. Moreover, we have determined the diameter, girth, clique number,vertex connectivity and edge connectivity of the $\mathbb{D}_0[Z_n]$ for every natural number n. The purpose to study this graph is to find relationship between ring theoretic properties of R and graph theoretic properties of $\mathbb{D}_0[R]$.

Keywords: graph; eulerian graph; connected graph; clique number; etc...
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1 Introduction

The study of zero divisor graphs was initiated by I. Beck [1998]. He mostly interested in coloring the graph. Later on D. F.Anderson and P. S.Livingston [1999] simplified definition of zero divisor graph by omitting 0 from the graph of I. Beck. D. F.Anderson and P. S.Livingston [1999] studied properties of zero divisor graph

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more clearly and find relations between ring-theoretic properties and graph theoretic ones. Anderson and Livingston have given surprising results on zero divisor graph $\Gamma[R]$. They have study structural properties and graph theoretic properties of graph $\Gamma[R]$. Many of researcher motivated by their work and attracted in this area of research. See, Redmond and P [2002], Akbari and Mohammadian [2004], Redmond [2003]. The study of vertex and edge connectivity of zero divisor graph is done in B.S. Reddy [2020]. Associating graph to an algebraic structure become popular research topic since the last 2 decades. The graph of various algebraic structure and their properties can be found in Mathil et al. [2022], B.S. Reddy [2020], Das [2016], S. Akabari [2014], R. A. Muneshwar [2021], M. S. Rahman [2005], Khandare et al. [2022].
The concept of divisor graph introduced in Khandare et al. [2022]. The divisor graph $\mathbb{D}[R]$ is graph with vertex set, $V = \{r \in R \mid r \neq 0, r \neq 1\}$ and two vertices $x, y \in V$ are adjacent if and only if $\exists z \in R$ such that $y = zx \text{ or } x = zy$. In Khandare et al. [2022], unit vertices are adjacent with all other element of ring. So we adopt the idea of Anderson and Livingston and omitted unit elements and non zero divisors of ring from vertex set. Clearly the new graph is a induced subgraph of divisor graph $\mathbb{D}[R]$. we have denoted new graph as $\mathbb{D}_0[R]$. The section 2 includes the definition of divisor graph, modified divisor graph and some useful notation. In section 3, the graphical structure and graph theoretic properties like complete graph, connected, bipartite graph, planar graph is discussed. The section 4 contain diameter and girth of the graph $\mathbb{D}_0[Z_n]$. The section 5 includes condition for graph $\mathbb{D}_0[Z_n]$ to be an Eulerian and Hamiltonian. In section 6, a clique number is calculated and provides a short proof to Beck's conjecture. Also in the section 7, we derive formula to determine vertex and edge connectivity $\mathbb{D}_0[Z_n]$ for natural n .

2 Definition and Notation

Definition 2.1. divisor graph of ring R. *Khandare et al. [2022]*
Let R be finite commutative ring, then the graph $\mathbb{D}[R]$ is divisor graph of R, where $V = \{u \in R \mid u \neq 0, 1\}$ and for $u, v \in V, u \sim v$ if and only if $\exists$ w $\in R$ such that $u = wv$ or $v = wu$.

Definition 2.2. modified divisor graph of R
Let R be finite commutative ring, then the graph $\mathbb{D}_0[R]$ is modified divisor graph of R, where $V = \{u \mid u \in Z^*(R)\}$ and for $u, v \in V, u \sim v$ if and only if $\exists$ $w \in R$ such that $u = wv$ or $v = wu$.

Notation: Here, we discuss about the structure of the $\mathbb{D}_0[Z_n]$. Let $d_1, d_2, \cdots, d_k$ be the proper divisors of n. For $1 \leq i \leq k$, consider the following sets,

$$A_{d_i} = \{x \in Z_n \mid gcd(x, n) = d_i\}$$

Further, note that these sets $A_{d_1}, A_{d_2}, \cdots, A_{d_k}$ forms a partition of the vertex set of the graph $\mathbb{D}_0[Z_n]$. Thus $V(\mathbb{D}_0[Z_n]) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_k}$. Reader also note that, we use another partition for $Z^*[R]$ in Das [2016] such that if $n = p_1p_2\cdots p_m$, $Z^*[R] = \cup W_{d_i}$, $W_{d_i} = \{k(d_i) | k \in R^*, k \notin \langle d_j \rangle, d_i \neq d_j\}$, $W_{d_i} \cap W_{d_j} = \phi$ and if $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$ then $Z^*[R] = \cup T_{d_i}$, where $T_{d_i} = \{k \cdot d_i \mid k \in R^*, k \notin \langle d_j \rangle, d_i^s \neq d_j, s \in N\}$ but $T_{d_i} \cap T_{d_j} \neq \phi$. It is clear that, $W_{d_i} = A_{d_i}$, $T_{d_i} \neq A_{d_i}$

, $A_{d_i} \subseteq T_{d_i}$ and $Z^*[R] = \cup T_{d_i}$ is not partition of set. Therefore we consider the partition, $Z^*[R] = \cup A_{d_i}$. It needs to be noted that non-trivial divisor of n that is neither 1 nor n.

3 Modified Divisor Graph of Z_n

In order to illustrate the difference between divisor graph and modified divisor graph, we will discuss the following figures.

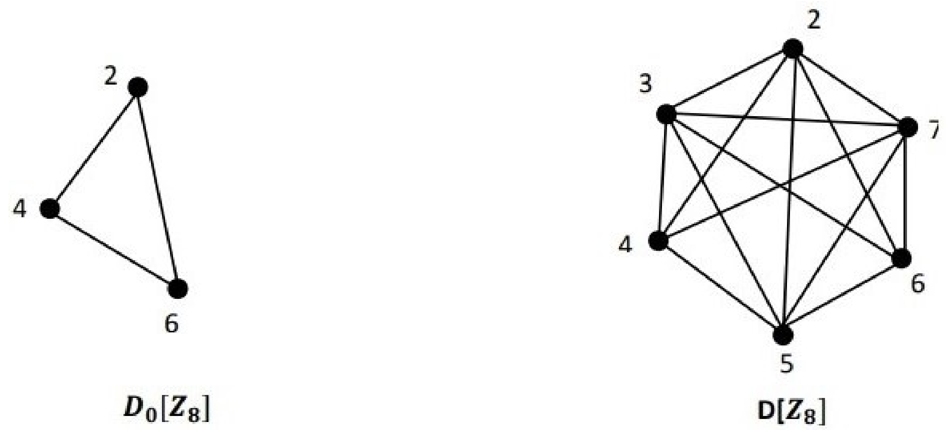

Figure 1: $Modified\ Divisor\ Graph\ and\ Divisor\ Graph\ of\ Z_8$

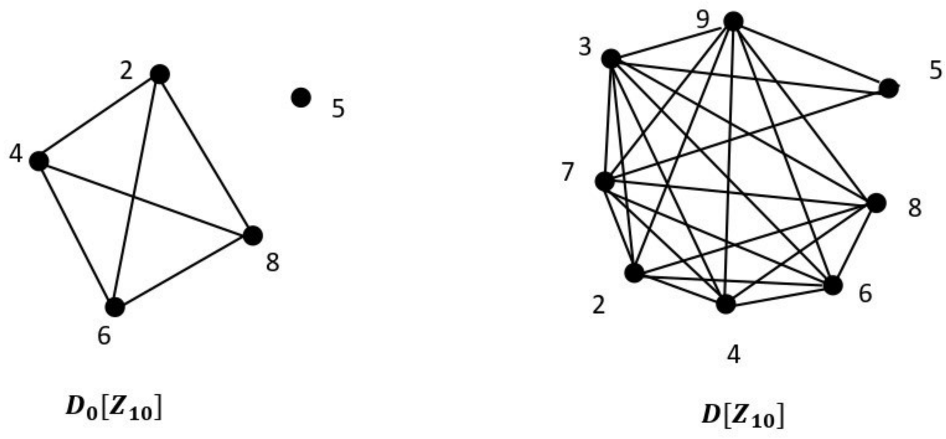

Figure 2: $Modified\ Divisor\ Graph\ and\ Divisor\ Graph\ of\ Z_{10}$

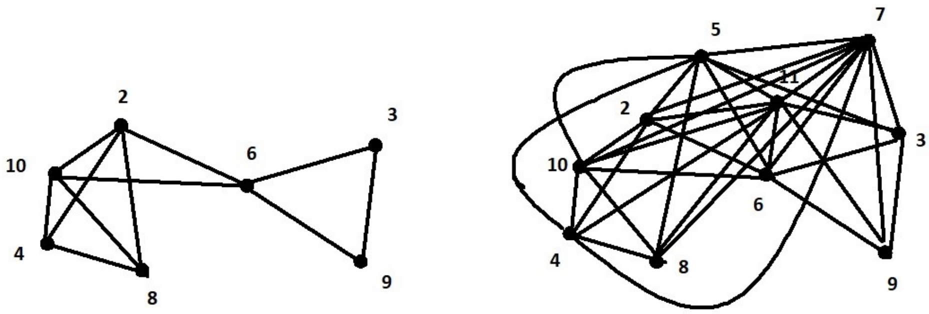

Figure 3: $Modified\ Divisor\ Graph\ and\ Divisor\ Graph\ of\ Z_{12}$

Remark 3.1. *Mathil et al. [2022] If n be positive integer then $|A_{d_j}| = \phi(\frac{n}{d_j})$, for each proper divisor d_j of n.*

Proposition 3.1. *If $n = p_1p_2\cdots p_m$ then $|A_d| = (p_{(i+1)}-1)(p_{(i+2)}-1)\cdots(p_m-1)$ for divisor $d = p_1p_2\cdots p_i, 3 \leq i \leq m$ of n.*

Proof: Let $n = p_1p_2\cdots p_m$ and any divisor d, $d = p_1p_2\cdots p_i, 3 \leq i \leq m$ Given set $A_{d_j} = \{x \in Z_n \mid gcd(x,n) = d_j\}$ and $|A_{d_j}| = \phi(\frac{n}{d_j})$. Therefore,

$$|A_d| \;=\; \phi(\frac{n}{d}) = \phi(p_ip_{(i+1)\cdots}p_m) = (p_{(i+1)}-1)(p_{(i+2)}-1)\cdots(p_m-1)$$

Proposition 3.2. *If $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}, p_1 < p_2 < \ldots < p_m$ and for divisor d, $d = p_1p_2\cdots p_i, 3 \leq i \leq m$ of n*

$$|A_d| \;=\; \phi(p_3^{k_3}p_4^{k_4}\cdots p_m^{k_m}).$$

$$|T_d| \;=\; (p_1^{k_1-1}p_2^{k_2-1}\cdots\cdots p_m^{k_m-1})(p_1-1)(p_2-1)\cdots(p_{(i-1)}-1)(p_{(i+1)}-1)(p_m-1).$$

Proof: Let $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$ and d be non trivial divisor of n. We have, $A_d = \{x \in Z_n \mid gcd(x,n) = d\}$ and $T_d = \{k{\cdot}d \mid k \in R^*, k \notin \langle d\rangle, d_i^s \neq d, s \in N\}$. To find $|A_d|, |T_d|$ consider simply $d = p_1^{k_1}p_2^{k_2}$. Thus,

$$|A_d| \;=\; \phi(\frac{n}{d}) = \phi(p_3^{k_3}p_4^{k_4}\cdots p_m^{k_m}).$$

Now suppose that $d = p_1$, then

$$\begin{aligned}
T_{p_1} &= \{k \cdot p_1 \mid k \in R^*, k \notin \langle d_j\rangle, p_1^s \neq d_j, s \in N\} \\
\therefore |T_{p_1}| &= \left[\frac{n}{p_1}\right] - \left[\frac{n}{p_1p_2}\right] - \left[\frac{n}{p_1p_3}\right]\cdots - \left[\frac{n}{p_1p_m}\right] + \left[\frac{n}{p_1p_2p_3}\right] + \left[\frac{n}{p_1p_2p_4}\right] + \cdots \\
&+ \left[\frac{n}{p_1p_2p_m}\right] + \left[\frac{n}{p_1p_3p_4}\right] + \left[\frac{n}{p_1p_3p_m}\right]\cdots + \cdots - \left[\frac{n}{p_1p_2p_3p_4}\right] + \cdots \\
&+ (-1)^{(n-1)}\left[\frac{n}{p_1p_2p_3\cdots p_m}\right]
\end{aligned}$$

After simplifying, we will obtain,

$$|T_{p_1}| \;=\; (p_1^{k_1-1}p_2^{k_2-1}\cdots\cdots p_m^{k_m-1})(p_2-1)(p_3-1)\cdots((p_m-1).$$

Hence, if $d = p_1^{k_1}p_2^{k_2}$

$$|T_d| \;=\; (p_3^{k_3-1}p_4^{k_4-1}\cdots\cdots p_m^{k_m-1})(p_3-1)(p_4-1)\cdots(p_m-1).$$

After the above discussion, we can now provide an important result of this article.

Theorem 3.1. *Let R be a ring of integer modulo n and d_i, d_j be distinct non trivial divisors of n. if $u_i, u_i' \in A_{d_i}, v_j, v_j' \in A_{d_j}$ then following statements hold.*
(i) $u_i \sim u_i'$ and $v_j \sim v_j'$.
(ii) $u_i \sim v_j$ if and only if $d_i \mid d_j$ or $d_j \mid d_i$,
(iii) $u_i \nsim v_j$ if and only if $d_i \nmid d_j$ and $d_j \nmid d_i$

Proof: Let R be ring of residue modulo n and d_i, d_j be distinct non trivial divisors of n. we say $u \mid v$, when $\exists w \in R$ such that, $v = wu$.
(i) Let $u_i, u_i' \in A_{d_i}$ then $gcd(u_i, n) = d_i$ and $gcd(u_i', n) = d_i$ i.e.$u_i = r_i \cdot d_i, u_i' = r_i' \cdot d_i$, for some $r_i, r_i' \in R$. Consider the following cases.
Case I: If r_i is multiple of r_i' then statement hold.
Case II: If either r_i or r_i' be unit element of ring then by theorem 3.6, Khandare et al. [2022], As every unit element is adjacent to other element of ring. Therefore $u_i \sim u_i'$. In the similar way, we obtain $v_j \sim v_j'$.
(ii) Let $u_i \in A_{d_i}, v_j \in A_{d_j}$. Suppose that $d_i \mid d_j$ i.e. $d_j = r \cdot d_i$ for some $r \in R$ and hence $v_j = r_j d_j = r_j \cdot r \cdot d_i$. This shows that $u_i \sim v_j$.
(iii) Let $u_i \in A_{d_i}, v_j \in A_{d_j}$. Suppose that $d_i \nmid d_j$ and $d_j \nmid d_i$. Assume that, $u_i \sim v_j$ then $\exists\ r \in R^*$ such that $v_j = r \cdot u_i = r \cdot r_i \cdot d_i$ i.e. $v_j \in A_{d_i}$. Which is contradiction to the set A_d. Hence $u_i \nsim v_j$.

Corolary 3.1. *The modified divisor graph $\mathbb{D}_0[R]$ is an induced subgraph of divisor graph $\mathbb{D}[R]$.*

Proof: Theorem statement holds using definitions 2.1 and 2.2.

Corolary 3.2. *If I be proper ideal of R then $\mathbb{D}_0[I]$ is proper subgraph of $\mathbb{D}_0[R]$.*

Proof: The definition 2.2 provides the proof.

Theorem 3.2. *A graph $\mathbb{D}_0[Z_n]$ is null if and only if $n = 1$ or $n = p$, for prime p.*

Proof: Since $V(\mathbb{D}_0[Z_n]) = Z^*[R] = \phi$ when $n = 1$ and $n = p$. Hence the result is hold.

Theorem 3.3. *If $n = p^k$ where $k > 1$ then modified divisor graph then $\mathbb{D}_0[Z_n]$ is complete. Moreover, $\mathbb{D}_0[Z_{p^k}] \cong K_{p^{k-1}-1}$.*

Proof: Let $R = Z_n$ and $n = p^k, k > 1$ where p be prime. Then $|V| = n - \phi(n) - 1 = p^{k-1} - 1$. Here $p, p^2, p^3, \cdots p^k$ are proper divisors of n. Therefore $V = \cup A_{d_i} = A_p \cup A_{p^2} \cup A_{p^3} \cup \cdots \cup A_{p^{k-1}}$. Also divisor $p \mid p^2 \mid p^3 \cdots \mid p^{k-1}$ Hence by theorem 3.4, each vertex of the graph is adjacent to other vertex. Thus $\mathbb{D}_0[Z_{p^k}] \cong K_{p^{k-1}-1}$. Conversely assume $\mathbb{D}_0[Z_{p^k}]$ is a complete graph and $n \neq p^k$. So n may be expressed as a product of power of distinct primes. say($n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_m^{k_m}$). Clearly $\exists p_1, p_2 \in V$ such that $p_1 \nmid p_2$. Thus we conclude, $\mathbb{D}_0[Z_{p^k}]$ is a complete graph if and only if $n = p^k$.

Theorem 3.4. *The modified divisor graph $\mathbb{D}_0[Z_n]$ is disconnected if and only if $n = pq$ for prime p and q. Moreover, it is a disjoint union of two connected components.*

Proof: Let R be a ring of residues modulo n and $n = p \cdot q$, where p and q are primes. As $p, q \in V$ and $p \nmid q$, Hence $\mathbb{D}_0[Z_n]$ is a disconnected graph. Conversely let $\mathbb{D}_0[Z_n]$ be disconnected. We claim that $n = p \cdot q$. Assume that, $n \neq pq$ i.e. n can be characterized in the following ways.
case I: If $n = p^k, k > 1$. Then from theorem 3.4, the result is true.
case II: If $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_m^{k_m}$. With the given disjoint partition of vertex set $V = \cup A_{d_i}$, where $d_i \mid n$ and $A_{d_i} = \{x \in Z_n \mid gcd(x, n) = d_i\}$. Let d_1, d_2 be any two proper divisor of n and $u_1 \in A_{d_1}, u_2 \in A_{d_2}$. If $gcd(d_1, d_2) = 1$ then $\exists d_3 = d_1 d_2$ such that $d_1 \mid d_3$ and $d_2 \mid d_3$. Thus, using theorem 3.4, we obtain $u \in A_{d_3}$ such that $u_1 \sim u_3 \sim u_2$. If $gcd(d_1, d_2) = d_4 > 1$ then again we can get the path $u_1 \sim u_4 \sim u_2$. Thus, a path is exist between any pair (u, v) of V. Hence $\mathbb{D}_0[Z_n]$ is disconnected graph if and only if $n = p \cdot q$.

Theorem 3.5. *If $n = pq$ then $\mathbb{D}_0[Z_n] \cong \bar{\Gamma}(Z_n)$.*

Proof: Let $R = Z_n, n = pq$. We have $V(\Gamma(R)) = V(\bar{\Gamma}(R)) = V(\mathbb{D}_0[R]) = Z^*(R)$. Since $u \sim v$ in $\Gamma(R)$ if and only if $uv = 0$. Let $u, v \in V$, such that, $uv = 0$. We claim that, $u \nsim v$ in $\mathbb{D}_0[R]$. Clearly u is multiple of p and v multiple of q or vice versa. Using theorem 3.1, $u \nsim v$ in $\mathbb{D}_0[R]$. Therefore $\mathbb{D}_0[R] \cong \bar{\Gamma}(R)$.

Theorem 3.6. *If R is ring of integers modulo n then $\mathbb{D}_0[R]$ is bipartite if and only if n is 1 or p or 9.*

Proof: Let R be a ring of integers modulo n. When n is 1 or p then there is nothing to prove as the graph is null. If $n = 9$ then the graph is isomorphic to K_2, which is bipartite graph.
To prove converse part, suppose that graph $\mathbb{D}_0[R]$ is bipartite. i.e. graph without cycle. Clearly using theorem 3.1, we obtain the cycle of length 3 for $n \geq 6, n \neq 1, p, 4$. Thus $\mathbb{D}_0[R]$ is bipartite if and only if n is 1 or p or 9.

Theorem 3.7. *If $n \geq 14, n \neq p, pq, 15$ then $\mathbb{D}_0[R]$ is not planar and planar if $n \leq 15, n \neq 14, pq$.*

Proof: The number of non-zero zero divisor in Z_n are $n - \phi(n) - 1$. The Kuratowski's theorem,West [2001] says that graph is not planar if and only if it contains K_5 *or* $K_{3,3}$. Clearly for $n \leq 15, n \neq 14, n \neq pq$, $\mathbb{D}_0[Z_n]$ does not contains K_5. When $n = 14$, an ideal of order 7 is exists in the ring. Hence by theorem 3.4, $\mathbb{D}_0[Z_7]$ contains K_5. Thus $\mathbb{D}_0[Z_n]$ is planar if $n \leq 15$ unless $n = 14$.

Theorem 3.8. *The graph $\mathbb{D}_0[Z_n]$ is regular if and only if $n = p^k$*

Proof: The theorem 3.8 provides the proof.

4 Diameter and Girth :

In this section, diameter and girth of modified divisor graph $\mathbb{D}_0[Z_n]$ are investigated.

Theorem 4.1. *The diameter of the graph $\mathbb{D}_0[Z_n]$ is less than or equals to 4.*

Proof: Since $Diam(G) = Sup\{d(u,v) : u, v$ are vertices of graph.
Case I: If $n = 1$ *or* $n = pq$ *or* $n = p$ then the graph $\mathbb{D}_0[Z_n]$ is disconnected graph for or null. Hence $Diam(\mathbb{D}_0[Z_n]) = \infty$.
Case II: If $n = p^k, k > 1$, $\mathbb{D}_0[Z_n]$ is complete graph. Thus $Diam(\mathbb{D}_0[Z_n]) = 1$.
Case III: If $n \neq p^k$ for positive integer k. Then by theorem 3.4, if $n = p^2q$ then there is path $p^2 \sim p \sim pq \sim q$ of length 3. Also when $n = p^2q^2$ then there exists path $p^2 \sim p \sim pq \sim q \sim q^2$ of length 4. i.e. $Diam(\mathbb{D}_0[R]) = 4$. In general $n = p_1^{k_1}p_2^{k_2}p_3^{k_3}$. Clearly, $p_1 \mid p_1^{(k_1)} \nmid p_1^{(k_1-i)}p_2^{(k_2-j)} \nmid p_2^{k_2} \mid p_2$, for $i < k_1, j < k_2$. Thus, a path can be found between vertices $p_1^{(k_1)} \sim p_1 \sim p_1^{(k_1-i)}p_2^{k_2-j} \sim p_2 \sim p_2^{(k_2)}$ of length 4. Thus, diameter of the modified divisor graph $\mathbb{D}_0[Z_n]$ is less than or equals to 4.

Theorem 4.2. *If R is ring of integer modulo n then*

$$girth(\mathbb{D}_0[Z_n]) = \begin{cases} \infty & \text{if } n < 8, n = 9 \text{ or } n = p \\ 3 & \text{if } n \geq 8, n \neq 9 \text{ or } n \neq p \end{cases}$$

Proof: Let R be ring of integer modulo n. We prove this in following cases.
Case I: Let $n < 8$ *or* $n = 9$ *or* $n = p$ then $\mid V(\mathbb{D}_0[R]) \mid < 3$ and hence girth is ∞.
Case II: Let $n \geq 8$. Since $\mathbb{D}_0[Z_{p^k}]$ is complete graph of size $p^{k-1} - 1$. hence for $n = p^k$, we get smallest cycle of length 3 unless $n = 9$. Let $n \neq p^k, n \geq 8$ then there exists proper divisor $d \mid n$, such that $|A_{d_i}| \geq 3$. Therefore by theorem 3.4, we conclude that girth of $\mathbb{D}_0[Z_n]$ is 3.

5 Eulerian and Hamiltonian of $\mathbb{D}_0[Z_n]$:

In this section, the condition of graph to be Eulerian and Hamiltonian of $\mathbb{D}_0[Z_n]$ is discussed. Let we recall following propositions.

Proposition 5.1. *A connected graph is Eulerian if and only if the degree of each vertex is even.*

Proposition 5.2. *The number of zero divisor in ring of integer modulo n are*

$$n - \phi(n) = p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_m^{\alpha_m-1}[p_1p_2\cdots p_m - (p_1-1)(p_2-1)\cdots(p_m-1)]$$

Theorem 5.1. *The graph $\mathbb{D}_0[Z_n]$ is Eulerian if and only if n is an even number.*

Proof: Let the following cases.
Case I: Let $n = p^k, k > 1$. Then using theorem 3.3, the graph $\mathbb{D}_0[Z_n]$ is complete of size $p^{k-1}-1$. The degree of each vertex becomes $p^{k-1}-2$. The number $p^{k-1}-2$ is even only if p is an even prime. The proposition 5.1 provides the proof.
Case II: Let $n \neq p^k$ i.e. $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$ be prime factorization of n.
If n is odd. Then we get the prime vertex $p_2 \in V$ such that $d(p_2) = \left[\frac{n}{p_2}\right] - 2 = odd - 2 = odd$. Thus, the graph $\mathbb{D}_0[Z_n]$ is Eulerian if and only if n is even.

Theorem 5.2. *The graph $\mathbb{D}_0[Z_n]$ is Hamiltonian graph if and only if $n = p^k$, for positive integer k.*

Proof: Suppose that $n = p^k, k > 1$ then using Theorem 3.3, $d(u) = p^{k-1} - 2, u \in V$. Now,

$$\begin{aligned}
p^{k-1} &\geq p^{k-1} \\
p^{k-1} - 1 &> p^{k-1} - 1 \\
p^{k-1} - 2 &\geq \frac{p^{k-1} - 1}{2} \\
\therefore d(u) \geq \frac{p^{k-1} - 1}{2} &= |V|, \forall\, u \in V
\end{aligned}$$

Hence, using M. S. Rahman [2005], the graph $\mathbb{D}_0[Z_n]$ is Hamiltonian. In the other hand, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ is a prime factorisation of n. Assume that $\mathbb{D}_0[Z_n]$ is Hamiltonian graph. Let $p_1, p_2 \in V$ be any pair of non adjacent vertices. Without loss of generality, we suppose that, then $d(p_1) = p_1^{(k_1-1)} p_2^{k_2} - 2, d(p_2) = p_1^{k_1} p_2^{(k_2-1)} - 2$ and

$$\begin{aligned}
d(p_1) + d(p_2) &= p_1^{(k_1-1)} p_2^{k_2} - 2 + p_1^{k_1} p_2^{(k_2-1)} - 2 \\
&= p_1^{(k_1-1)} p_2^{(k_2-1)} (p_2 + p_1) - 4 \\
&< p_1^{(k_1-1)} p_2^{(k_2-1)} (p_2 + p_1) - 1 \\
&< |V|
\end{aligned}$$

Now using M. S. Rahman [2005] , notice that, the assumption is wrong. Hence $\mathbb{D}_0[Z_n]$ is Hamiltonian graph if and only if $n = p^k$ for positive integer k.

6 Clique number

In this section, we provide formula to determine clique number of modified divisor graph $\mathbb{D}_0[Z_n]$.

Theorem 6.1. *The clique number of $\mathbb{D}_0[Z_n]$ is,*

$$cl(\mathbb{D}_0[Z_n]) = \begin{cases} \infty & \text{if } n < 4 \text{ or } n = p \\ p^{(k-1)} - 1 & \text{if } n = p^k, k > 1 \\ \sum_{i=1}^{k-1} \prod_{k=i+1}^{m} (p_k - 1) & \text{if } n = p_1 p_2 \cdots p_k \\ \sum_{i=1}^{m} \prod_{j=i}^{m} p_j^{k_j - 1} (p_{j+1} - 1) & \text{if} n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_m^{k_m} \end{cases}$$

Proof: Clearly $\mathbb{D}_0[Z_n]$ is null graph when $n = 1 \text{ or } n = p$. Therefore $cl(\mathbb{D}_0[Z_n]) = \infty$. Now, we consider following cases for the rest of values of n.
Case I: If $n = p^k, k > 1$ then graph $\mathbb{D}_0[Z_n]$ is complete graph of size $p^{(k-1)} - 1$. Therefore clique with maximum size is $p^{(k-1)} - 1$.
Case II: Let $n = p_1 p_2 p_3 \cdots p_m, m \geq 2$. With the given partition $V = \cup A_{d_i}$, $A_{d_i} = \{x \in Z_n \mid gcd(x, n) = d_i\}$ and $|A_{d_i}| = \phi(\frac{n}{d_i}), 1 \leq i \leq k,$. It is given that $A_{d_i} \leftrightarrow A_{d_j}, i \neq j$ when $d_i \mid d_j \text{ or } d_j \mid d_i$ and $A_{d_i} \not\leftrightarrow A_{d_j}, i \neq j$ when

$d_i \nmid d_j$ *and* $d_j \nmid d_i$. As the clique number is size of largest complete subgraph of graph. note that, as $p_1 < p_i, \forall i$ then $| < p_1 > | > | < p_i > |$ Hence

$$\begin{aligned} cl(\mathbb{D}_0[Z_n]) &= |A_{p_1}| + |A_{p_1p_2}| + \cdots + |A_{p_1p_2\cdots p_{(m-1)}}| \\ &= \phi(\frac{n}{p_1}) + \phi(\frac{n}{p_1p_2}) + \cdots + \phi(\frac{n}{p_1p_2\cdots p_{(m-1)}}) \\ &= (p_2-1)(p_3-1)\cdots(p_m-1) + (p_3-1)\cdots(p_m-1) + \cdots + (p_m-1) \\ &= \sum_{i=1}^{k-1} \prod_{k=i+1}^{m} (p_k - 1) \end{aligned}$$

Case III: Let $n = p_1^{k_1}p_2^{k_2}p_3^{k_3}\cdots p_m^{k_m}, k_1 \geq 2$. Notice that $p_1^{k_1} \nmid p_1p_2$ but $p_1^{k_1} \mid p_1^{k_1}p_2$, Also $p_1^{k_1}p_2^{k_2} \nmid p_1^{k_1}p_2p_3$ but $p_1^{k_1}p_2^{k_2} \mid p_1^{k_1}p_2^{k_2}p_3$ and so on. Since set $T_{p_1} = \cup_{i=1}^{k_1} A_{p_1^i}$, so instead of using set A_{d_i} we must use T_{d_i}. Therefore

$$\begin{aligned} cl(\mathbb{D}_0[Z_n]) &= |T_{p_1}| + |T_{p_1^{k_1}p_2}| + |T_{p_1^{k_1}p_2^{k_2}p_3}| + \cdots + |T_{p_1^{k_1}p_2^{k_2}p_3^{k_3}\cdots p_m}| \\ &= p_1^{k_1-1}p_2^{k_2-1}\cdots p_m^{k_m-1}(p_2-1)(p_3-1)\cdots(p_m-1) + p_2^{k_2-1}\cdots p_m^{k_m-1}(p_3-1) \\ &+ \cdots(p_m-1) + p_3^{k_3-1}\cdots p_m^{k_m-1}(p_4-1)\cdots(p_m-1) + \cdots + p_m^{k_m-1} - 1 \\ &= \sum_{i=1}^{m} \prod_{j=i}^{m} p_j^{k_j-1}(p_{j+1}-1) \end{aligned}$$

Example 6.1. *For* $n = 2 \times 3 \times 5$ *then* $cl(\mathbb{D}_0[Z_n]) = 12$

Let $n = 2 \times 3 \times 5$ then by theorem 6.1, $cl(\mathbb{D}_0[Z_n]) = \sum_{i=1}^{k-1} \prod_{k=i+1}^{m} (p_k - 1)$
$\therefore\ cl(\mathbb{D}_0[Z_n] = (p_2-1)(p_3-1) + (p_3-1) = 2 \times 4 + 4 = 12$

Example 6.2. *For* $n = 2^2 \times 3^2 \times 5^2$ *then* $cl(\mathbb{D}_0[Z_n]) = 304$

As $2^2 \nmid 2 \cdot 3,\ \ 2^2 \cdot 3^2 \nmid 2^2 \cdot 3 \cdot 5^2$ Thus,

$$\begin{aligned} cl(\mathbb{D}_0[Z_n] &= |T_2| + |T_{2^2 \cdot 3}| + |T_{2^2 \cdot 3^2 \cdot 5}| \\ &= 2 \cdot 3 \cdot 5(2 \times 4) + 3 \cdot 5(4) + 5 - 1 \\ &= 304 \end{aligned}$$

Theorem 6.2. *If R is ring of integer modulo n then* $cl(\mathbb{D}_0[Z_n] = \chi(\mathbb{D}_0[Z_n], \forall\ n$.

Proof: Let R be ring of integer modulo n and $cl(\mathbb{D}_0[Z_n] = m$. Let $\{u_1, u_2, u_3, \cdots, u_m\}$ be maximal clique. Assume that there exists a vertex u_i, $i \neq m$ with i-color different from other color. Then the set $\{u_1, u_2, u_3, \cdots u_m\} \cup \{u_i\}$ becomes maximal clique. Which is contradiction. Hence $cl(\mathbb{D}_0[Z_n] = \chi(\mathbb{D}_0[Z_n], \forall\ n$.

Theorem 6.3. *If R is ring of integer modulo n then following results are equivalent.*
(i) $\mathbb{D}_0[Z_n]$ *is perfect.*
(ii) $\mathbb{D}_0[Z_n]$ *does not contain* K_5 *as a subgraph.*
(iii) $n \leq 15, n \neq 14$.

Proof: $(i) \Rightarrow (ii)$ Let $\mathbb{D}_0[Z_n]$ is perfect and assume that $\mathbb{D}_0[Z_n]$ have K_5 as a subgraph with vertices u_1, u_2, u_3, u_4, u_5 Then $u_1 \sim u_2 \sim u_3 \sim u_4 \sim u_5$ is an induced 5-cycle in $\mathbb{D}_0[Z_n]$, is a contradiction to perfectness. Hence $\mathbb{D}_0[Z_n]$ does not contain K_5.
$(ii) \Rightarrow (iii)$ If $\mathbb{D}_0[Z_n]$ does not contain K_5. Then by definition 2.1, It is easy to show that $n \leq 15, n \neq 14$.
$(i) \Rightarrow (iii)$ follows from $\mathbb{D}_0[Z_n]$ has an induced cycle if and only if $n \geq 15, n \neq 14$.

Corolary 6.1. *If $n \neq 15, n \geq 14$ then $cl(\mathbb{D}_0[Z_n]$ is weakly perfect graph.*

Proof: The proof is followed by theorem 6.5.

7 Vertex and Edge Connectivity in $\mathbb{D}_0[Z_n]$:

In this section, vertex and edge connectivity in $\mathbb{D}[Z_n]$ are calculated.

Theorem 7.1. *If R is ring of integer modulo n then a vertex connectivity of zero divisor graph $\kappa(\mathbb{D}[Z_n]) = p^{k-1} - 2$.*

Proof: Since the vertex set of $\mathbb{D}_0[R] = Z^*(R)$, set of non-zero zero divisor. If $n = p^k$ then $p, p^2, p^3, \cdots p^k$ are proper divisors of n. Therefore $V = \cup A_{d_i} = A_p \cup A_{p^2} \cup A_{p^3} \cup \cdots \cup A_{p^{k-1}}$ and $|V| = p^{k-1} - 1$. As the graph corresponding to these vertices is complete of size $p^{k-1} - 1$. deletion of one or two vertices does not give a disconnected graph. If we delete $p^{k-1} - 2$ vertices then only one vertex remains and it results to disconnected graph. Thus minimum number of vertices required to obtain disconnected graph are $p^{k-1} - 2$. Hence $\kappa(\mathbb{D}[Z_n]) = p^{k-1} - 2$.

Theorem 7.2. *If R is ring of integer modulo n and $n = p_1 p_2 \cdots p_m, p_1 < p_2 < \ldots < p_m$ then vertex connectivity $\kappa(\mathbb{D}[Z_n]) = p_1 p_2 \cdots p_{(m-1)} - \phi(p_1 p_2 \cdots p_{(m-1)}) - 1$*

Proof: Let for the divisor $d_1, d_2, d_3, \cdots, d_k$ of n,vertex set V, $V = \cup A_{d_i}$. Here notice that $|A_{p_m}| < |A_{p_i}|, \ \forall i = 1, 2, ..., (m-1)$. As by theorem 3.4, vertices of set A_{p_m} are adjacent to each other vertex of A_{p_m} and all vertices of set A_{d_j} where $gcd(d_j, p_m) = p_m$. So deletion of vertices of A_{d_j} leaves vertices of A_{p_m} isolated. Thus graph is disconnected when we remove minimal set $A_{d_j}, gcd(d_j, p_m) = p_m, d_j \neq p_m$. Therefore vertex connectivity of the graph,

$$\begin{aligned}
\kappa(\mathbb{D}[Z_n]) &= \sum_j |A_{d_j}|, gcd(d_j, p_m) = p_m, d_j \neq p_m \\
&= |A_{p_1 p_m}| + |A_{p_2 p_m}| + \cdots + |A_{p_{m-1} p_m}| + |A_{p_1 p_2 p_m}| + \cdots + \cdots \\
&+ |A_{p_1 p_2 \cdots p_{m-2} p_m}| \\
&= p_1 p_2 \cdots p_{(m-1)} - \phi(p_1 p_2 \cdots p_{(m-1)}) - 1
\end{aligned}$$

Corolary 7.1. *If $n = p_1 p_2$, then vertex connectivity of $\mathbb{D}_0[Z_n]$ is zero.*

Proof: Since for $n = p_1 p_2$, $\mathbb{D}_0[Z_n]$ is disconnected graph. Hence vertex connectivity is 0.

$$\begin{aligned}
\kappa(\mathbb{D}_0[Z_n]) &= \sum_j |A_{d_j}|, gcd(d_j, p_2) = p_2, d_j \neq p_2 \\
&= p_1 - \phi(p_1) - 1 = 0
\end{aligned}$$

Corolary 7.2. *If $n = p_1 p_2 p_3$, then vertex connectivity of $\mathbb{D}_0[Z_n]$ is $p_1 + p_2 - 2$.*

Proof: Let $n = p_1 p_2 p_3$, then from theorem 7.2,

$$\begin{aligned}
\kappa(\mathbb{D}_0[Z_n]) &= \sum_j |A_{d_j}|, gcd(d_j, p_3) = p_3, d_j \neq p_3 \\
&= |A_{p_1 p_3}| + |A_{p_2 p_3}| = (p_2 - 1) + (p_1 - 1) = p_1 + p_2 - 2
\end{aligned}$$

Theorem 7.3. *If* $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}, p_1 < p_2 < \ldots < p_m$ *then vertex connectivity* $\kappa(\mathbb{D}_0[Z_n]) = \sum_j |T_{d_j}|, gcd(d_j, p_m) = p_m, d_j \neq p_m$

Proof: If $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}, p_1 < p_2 < \ldots < p_m$ then vertex set V can be divide such that , $V = \cup T_{d_i}, d_i \mid n$. Here also we found that $p_1^{k_1} \nmid p_j^{k_j}, j = 2, 3, 4, \cdots, m$ but $T_{p_1} \not\leftrightarrow T_{p_j}$, since $p_1^{k_1} \mid p_1^{k_1} p_j^{k_j}, \forall j = 2, 3, 4, \cdots, m$ and $p_m^{k_m} \nmid p_m^{k_m - 1} p_i, i = 1, 2, 3 \cdots (m-1)$. Therefore each vertex of T_{p_m} is adjacent to other vertices of A_{p_m} and some vertices are adjacent with T_{d_j}, where $gcd(d_j, p_m) = p_m$. As the vertex connectivity is minimum number of vertices whose removal disconnected the graph are $\sum_i T_{d_j}, gcd(d_j, p_m) = p_m, d_j \neq p_m$. Hence,

$$\begin{aligned}
\kappa(\mathbb{D}_0[Z_n]) &= \sum_j |T_{d_j}|,\ gcd(d_j, p_m) = p_m, d_j \neq p_m^s, s = 1, 2, 3, \cdots k_m \\
&= |T_{p_1 p_m}| + |T_{p_2 p_m}| + \cdots + |T_{p_{m-1} p_m}| + |T_{p_1 p_2 p_m}| + \cdots + \cdots \\
&+ |T_{p_1 p_2 \cdots p_{m-2} p_m}|
\end{aligned}$$

since $T_{p_1^{k_1} p_m^{k_m}} \subset T_{p_1 p_m}, T_{p_1^{k_1} p_2^{k_2} p_m^{k_m}} \subset T_{p_1 p_2 p_m}$ and so on.

Corolary 7.3. *If* $n = p_1^{k_1} p_2^{k_2}, p_1 < p_2$ *then* $\kappa(\mathbb{D}_0[Z_n]) = p_1^{k_1 - 1} p_2^{k_2 - 1} - 1$.

Proof: Let $n = p_1^{k_1} p_2^{k_2}, p_1 < p_2$, from theorem 7.4,

$$\begin{aligned}
\kappa(\mathbb{D}_0[Z_n]) &= \sum_j |T_{d_j}|, gcd(p_2, d_j) = p_2, d_j \neq p_2^s, s = 1, 2, \cdots k_2 \\
&= |T_{p_1 p_2}| = p_1^{k_1 - 1} p_2^{k_2 - 1} - 1
\end{aligned}$$

Corolary 7.4. *If* $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}, p_1 < p_2 < p_3$*, then* $\kappa(\mathbb{D}_0[Z_n]) = p_1^{k_1 - 1} p_2^{k_2 - 1} p_3^{k_3 - 1}(p_1 + p_2 - 1)$.

Proof: Let $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}, p_1 < p_2 < p_3$, from theorem 7.4,

$$\begin{aligned}
\kappa(\mathbb{D}_0[Z_n]) &= \sum_j |T_{d_j}|, gcd(p_3, d_j) = p_3, d_j \neq p_3^s, s = 1, 2, 3, \cdots k_3 \\
&= |T_{p_1 p_3}| + |T_{p_2 p_3}| = p_1^{k_1 - 1} p_2^{k_2 - 1} p_3^{k_3 - 1}(p_1 + p_2 - 1)
\end{aligned}$$

Theorem 7.4. *If* $s = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}, p_1 < p_2 < \cdots < p_m$ *and* $t = p_1 p_2 \cdots p_m$ *then vertex connectivity of the graph,*

$$\kappa(\mathbb{D}_0[Z_s]) = p_1^{k_1 - 1} p_2^{k_2 - 1} \cdots p_m^{k_m - 1} \kappa(\mathbb{D}_0[Z_t]) - 1, t \neq p_1 p_2$$

Proof: Theorem 6.2 and 6.3 follows the proof.

Theorem 7.5. *If* $n = p^k$ *then edge connectivity* $\kappa_e(\mathbb{D}_0[Z_n]) = p^{k-1} - 2$.

Proof: Let R be ring of integer modulo n and $n = p^k$. As the vertex set $V = Z^*[R]$, and $|V| = p^{k-1} - 1$. Since every vertex incident to $p^{k-1} - 2$ edges. So deletion of this $p^{k-1} - 2$ with respect to fixed vertex u, the graph becomes disconnected and vertex u becomes isolated. Hence $\kappa_e(\mathbb{D}_0[Z_{p^k}]) = p^{k-1} - 2$.

Theorem 7.6. *If* $n = p_1 p_2 \cdots p_m, p_1 < p_2 < \cdots < p_m$ *then edge connectivity* $\kappa_e(\mathbb{D}_0[Z_n]) = p_1 p_2 \cdots p_{(m-1)} - 1$.

Proof: We have the partition of the vertex set, $V = \cup A_{d_i}, d_i \mid n$. Clearly the prime p_m is largest, then the size of set A_{p_m} will be less. So degree of vertices of set A_{p_m} is less than the degree of vertices of other set $A_{p_i}, i = 1, 2, 3, \cdots (m-1)$. Therefore removal of vertex of set A_{p_m} gives disconnected graph. Since by theorem 3.4, each vertex of A_{p_m} is adjacent with each other and adjacent to vertices of set A_{d_j}, where $gcd(d_j, p_m) = p_m$. If we remove all edges incident to the fixed vertex $u \in A_{p_m}$ then u becomes isolated. Thus edge connectivity,

$$\begin{aligned}
\kappa_e(\mathbb{D}_0[Z_n]) &= d(u) \\
&= (\sum_j |A_{d_j}|, gcd(d_j, p_m) = p_m) - 1 \\
&= |A_{p_m}| - 1 + |A_{p_1p_m}| + |A_{p_2p_m}| + \cdots + |A_{p_{m-1}p_m}| + |A_{p_1p_2p_m}| \\
&+ \cdots + \cdots + |A_{p_1p_2\cdots p_{m-2}p_m}| \\
&= p_1p_2\cdots p_{(m-1)} - 1
\end{aligned}$$

Corolary 7.5. *If $n = p_1p_2$, then edge connectivity of $\mathbb{D}_0[Z_n]$ is zero.*

Proof: The proof follows from theorem 7.6.

Corolary 7.6. *If $n = p_1p_2p_3$, then edge connectivity of $\mathbb{D}_0[Z_n]$ is $\kappa_e(\mathbb{D}_0[Z_n]) = p_1p_2 - 1$.*

Proof: Let $n = p_1p_2p_3$, then from theorem 7.6, $\kappa_e(\mathbb{D}_0[Z_n]) = d(p_3) = |A_{p_3}| - 1 + |A_{p_1p_3}| + |A_{p_2p_3}| = p_1p_2 - 1$

Theorem 7.7. *If $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}, p_1 < p_2 < \cdots < p_m$ then edge connectivity $\kappa_e(\mathbb{D}_0[Z_n]) = \phi(p_1^{k_1}p_2^{k_2}\cdots p_m^{k_{(m-1)}}) \times [p_m^{k_m} - 1] + p_1^{k_1}p_2^{k_2}\cdots p_m^{k_{(m-1)}} - 2$.*

Proof: Let $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}, p_1 < p_2 < \cdots < p_m$. we should note that $p_m^{k_m} \nmid p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m-1}$ and $p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$. Clearly the degree of vertex of $T_{p_m^{k_m}}$ is smaller than degree of all vertices of other set. Since $p_m^{k_m} \in V$, So removal of all edges incident to vertex $p_m^{k_m}$ makes graph disconnected. Therefore minimum number of edges whose removal disconnected the graph are ,

$$\begin{aligned}
\kappa_e(\mathbb{D}_0[Z_n]) &= d(p_m^{k_m}) \\
&= \phi(p_1^{k_1}p_2^{k_2}\cdots p_m^{k_{(m-1)}}) \times [p_m^{k_m} - 1] + p_1^{k_1}p_2^{k_2}\cdots p_m^{k_{(m-1)}} - 2
\end{aligned}$$

Corolary 7.7. *If $n = p_1^{k_1}p_2^{k_2}$, then edge connectivity $\kappa_e(\mathbb{D}_0[Z_n]) = \phi(p_1^{k_1}) \times [p_2^{k_2} - 1] + p_1^{k_1} - 2$*

Proof: Let $n = p_1^{k_1}p_2^{k_2}$, then from theorem 7.13,

$$\kappa_e(\mathbb{D}_0[Z_n]) = \phi(p_1^{k_1}) \times [p_2^{k_2} - 1] + p_1^{k_1} - 2$$

Corolary 7.8. *If $n = p_1^{k_1}p_2^{k_2}p_3^{k_3}$, then edge connectivity $\kappa_e(\mathbb{D}_0[Z_n]) = \phi(p_1^{k_1}p_2^{k_2}) \times [p_3^{k_3} - 1] + p_1^{k_1}p_2^{k_2} - 2$*

Proof: Let $n = p_1^{k_1}p_2^{k_2}p_3^{k_3}$, then from theorem 7.13,

$$\begin{aligned}
\kappa_e(\mathbb{D}_0[Z_n]) &= d(p_3^{k_3}) \\
&= \phi(p_1^{k_1}p_2^{k_2}) \times [p_3^{k_3} - 1] + p_1^{k_1}p_2^{k_2} - 2
\end{aligned}$$

Following table shows vertex connectivity, edge connectivity of $\mathbb{D}_0[Z_n]$ for some values n.

n	$\kappa(\mathbb{D}_0[Z_n])$	$\kappa_e(\mathbb{D}_0[Z_n])$
$n = 2^{15} = 32768$	16382	16382
$n = 3^{10} = 59049$	19681	19681
$n = 11 \times 13$	0	0
$n = 5 \times 7 \times 11$	10	33
$n = 2^4 \times 5^3$	199	206

Table 1: vertex and edge connectivity of $(\mathbb{D}_0[Z_n])$

Discussion and Conclusion

In this research article, The structure of modified divisor graph $\mathbb{D}_0[R]$ of ring R is introduced and discussed. Moreover, various interplay between Z_n and $\mathbb{D}_0[Z_n]$ were discussed. In this article, we characterized the value of n, for which the graph $\mathbb{D}_0[Z_n]$ is perfect, Eulerian, Hamiltonian, planar, regular, complete and connected. Apart from this, we determine diameter, girth, clique number,vertex connectivity and edge connectivity of the $\mathbb{D}_0[Z_n]$ for all natural values of n.

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Conflict of interest

There is no conflict of interest.

References

S. Akbari and Mohammadian. On the zero-divisor graph of a commutative ring. *Journal of algebra*, 274(2):847–855, 2004.

I. Beck. Colouring of a commutative ring. *Journal of Algebra*, 116:208–226, 1998.

N. L. B.S. Reddy, R. S. Jain. Eulerian of zero divisor graph $\gamma(z_n)$. *arXiv preprint arXiv*, 2020.

D. F.Anderson and P. S.Livingston. The zero-divisor graph of a commutative ring. *Journal of Algebra*, 217:434–447, 1999.

A. Das. Subspace inclusion graph of a vector space. *Communications in Algebra*, 44:4724–4731, 2016.

Khandare, Jogdand, and Muneshwar. On the divisor graph of finite commutative ring. *J. Math. Comput. Sci.*, 12:37–41, 2022.

M. K. M. S. Rahman. On hamiltonian cycles and hamiltonian paths. *Information Processing Letters*, 49:37–41, 2005.

P. Mathil, B. Baloda, and J. Kumar. On the cozero-divisor graphs associated to rings. *AKCE International Journal of Graphs and Combinatorics*, 19(3):238–248, 2022.

K. B. R. A. Muneshwar. Some properties of open subset intersection graph of a topological space. *Journal of Information and Optimization Sciences*, 22:1007–1018, 2021.

Redmond and S. P. The zero-divisor graph of a non-commutative ring. *International J. Commutative Rings*, 1:203–211, 2002.

S. P. Redmond. An ideal-based zero-divisor graph of a commutative ring. *Communications in Algebra*, 31(9):4425–4443, 2003.

A. M. R. M. S. Akabari, M. Hababi. On the inclusion graph of ring. *Electronic notes in discrete math*, 45:73–78, 2014.

D. West. *Introduction to graph theory*. Prentice hall, 2001.