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Best proximity point for weak cyclic Kannan type $F-$contraction map and cyclic Chatterjea type $F-$contraction map

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Abstract

In this paper, we introduce an innovative category of contractions termed cyclic Chatterjea type $F-$contraction and weak cyclic Kannan type $F-$contraction. Subsequently, we establish a theorem for determining the best proximity point in a uniformly convex Banach space, specifically focusing on weak cyclic Kannan type $F-$contractions. Furthermore, we extend our investigation to include cyclic Chatterjea type $F-$contractions in uniformly convex Banach spaces. To support our findings, we present illustrative examples.

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2020 AMS subject: weak cyclic Kannan type $F-$contraction, cyclic Chatterjea type $F-$contraction, cyclic Chatterjea type contraction, Kannan type cyclic contraction, $F-$contraction, best proximity point.[1]

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1 Introduction

In 1922, Polish mathematician Banach proved an interesting result known as Banach contraction principle [4] states that a self-mapping T on a complete metric space X has a unique fixed point in X provided that there exists $\alpha \in (0, 1)$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ is satisfied for all $x, y \in X$. many author extended and generalized Banach's result in many directions see [1, 5, 6, 8, 10, 11], Some other extensions of Banach contraction principle have been presented by considering the concept of best proximity point [20]. Let (X, d) be a metric space and A, B be two subsets of X, then $T : A \to B$ be a mapping. In this context the mapping T does not have a fixed point in case of $A \cap B = \phi$. It is reasonable to research a point x, which is closest to Tx, that is $d(x, Tx)$ is minimum. Since $d(x, Tx) \geq d(A, B)$ for all $x \in A$,then $d(x, Tx)$ is at least $d(A, B)$. Best proximity point theorems are related to find a point satisfying $d(x, Tx) = d(A, B)$ which is called a Best proximity point. There are many results about this subject in literature [2, 9, 12, 14, 15, 16, 19].

2 Preliminaries

In 2003, Kirk et al. [13] introduced the notion of cyclical contractive mappings, and generalized Banach fixed point result [4] to the class of cyclic mappings

Theorem 2.1. *[13] Let A and B be nonempty closed subsets of a complete metric space (X, d) and let $T : A \cup B \to A \cup B$ be such that $T(A) \subset B$ and $T(B) \subset A$. Assume that, for all $x \in A$ and $y \in B$*

$$d(Tx, Ty) \leq \alpha d(x, y),$$

where $\alpha \in (0, 1)$. Then, T has a unique fixed point $u \in A \cap B$.

The above mapping is called cyclic. In [7], Eldred and Veeramani are concerned with the case when $A \cap B = \phi$, and in this case they didn't seek for the existence of a fixed point of T but for the existence of a best proximity point. For instance , they [7] presented the following existence best proximity point result for cyclic contractions.

Theorem 2.2. *[7] Let A and B be non-empty closed and convex subsets of a complete metric space (X, d) and let $T : A \cup B \to A \cup B$ be cyclic. Assume that, for all $x \in A$ and $y \in B$*

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)d(A, B).$$

Where $\alpha \in (0,1)$ and $d(A,B) = \inf\{d(x,y), x \in A, y \in B\}$.For $x_0 \in A$,define

$$x_{n+1} = Tx_n$$

for each $n \geq 0$.Then there exists a unique $x \in A$ such that $x_{2n} \to x$ and $d(x,Tx) = d(A,B)$. Here, x is called a best proximity point of T.

In 2011 [17] M.A.petric, introduced a new class of mappings known as weak cyclic Kannan contractions and proved some existence and convergence results for best proximity points in uniformly convex Banach spaces is proved as follows.

Theorem 2.3. *[17] Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Let $T : A \cup B \to A \cup B$ be a weak cyclic Kannan contraction map. Then*
(i) T has a unique best proximity point z in A.
(ii)The sequence $\{T^{2n}x\}$ converges to z for any starting point $x \in A$.
(iii)z is the unique fixed point of T^2.
(iv)Tz is a best proximity point of T in B.

In [21], Dariusz Wardowaski has proved fixed point theorem on $F-$contraction, which is perceived to be one of the most general non-linear contraction in complete metric spaces.

We first define weak cyclic Kannan type $F-$contraction and cyclic Chatterjea type $F-$contraction in this study. We also establish best proximity point result in the setting of uniformly convex Banach space using cyclic Chatterjea type $F-$contractions and prove the existence of best proximity point result in the setting of uniformly convex Banach space using weak cyclic Kannan type $F-$contraction.

Definition 2.1. *[17]Let A and B be nonempty subsets of a metric space (X,d). If a map $T : A \cup B \to A \cup B$ satisfies*
(i) $T(A) \subseteq B$ and $T(B) \subseteq A$;
(ii)there exists a $\alpha \in [0,1/2)$ such that

$$d(Tx,Ty) \leq \alpha[d(x,Tx) + d(y,Ty)] + (1-2\alpha)dist(A,B)$$

for any $x \in A$ and $y \in B$, then T is a called a weak cyclic Kannan contraction on $A \cup B$.

Definition 2.2. *[21]Let $F : R^+ \to R$ be a mapping satisfying*
$(F1)$: F is strictly increasing i.e. for all $\alpha, \beta \in R^+$ such that $F(\alpha) < F(\beta)$ whenever $\alpha < \beta$,
$(F2)$: For each sequence α_n , $n \in N$ of positive real numbers $lim_{n\to\infty}\alpha_n = 0$ if

and only if $lim_{n\to\infty}F(\alpha_n) = -\infty$
$(F3)$ *: There exists $k \in (0,1)$ such that $lim_{\alpha\to 0+}\alpha^k F(\alpha) = 0$.*
A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$, such that

$$\forall x, y \in X, \ \ d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \le F(d(x, y)). \qquad (1)$$

Remark 2.1. *[18] From $(F1)$ and the equation (1) it is clear that every F-contraction T is a contractive mapping i.e.*

$$d(Tx, Ty) < d(x, y), \forall x, y \in X, Tx \neq Ty.$$

Thus every F-contraction is a continuous mapping.

Theorem 2.4. *[21] Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in N}$ is convergent to x^*.*

Definition 2.3. *[3] Let (X, d) be a metric-like space. Let A and B be nonempty subsets of X. Take the cyclic mapping $T : A \cup B \to A \cup B$. T is said to be a cyclic Chatterjea type contraction if*

$$d(Tx, Ty) \le k(d(Tx, y) + d(Ty, x)) + (1 - 4k)d(A, B)$$

for all $x \in A$ and $y \in B$, where $k \in (0, 1/4)$.

Definition 2.4. *Uniformly convex space is a normed vector space such that , for every $0 < \epsilon \le 2$ there is some $\delta > 0$ such that for any two vectors with*

$$(||x - y||) \ge \epsilon \implies ||\frac{x + y}{2}|| \le 1 - \delta.$$

Intuitively, the center of a line segment inside the unit ball must lie deep inside the unit ball unless the segment is short.

Remark 2.2. *Every closed subspace of a uniformly convex Banach space is uniformly convex.*

Lemma 2.1. *[7] Let A be a non-empty mapping closed and convex subset and B be a non-empty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequence in A and $\{y_n\}$ be a sequence in B satisfying:*
(i) $||z_n - y_n|| \to d(A, B)$
(ii) For every $\epsilon > 0$ there exists N_0 such that for all $m > n > N_0$,

$$||x_m - y_n|| \le d(A, B) + \epsilon.$$

Then for every $\epsilon > 0$ there exists N_1 such that for all $m > n > N_1$, $||x_m - z_n|| \le \epsilon$.

Lemma 2.2. *[7] Let A be a non-empty closed and convex subset and B be a non-empty closed subsets of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequence in A and $\{y_n\}$ be a sequence in B satisfying:*
(i) $||x_n - y_n|| \to d(A, B)$,
(ii) $||z_n - y_n|| \to d(A, B)$.
Then $||x_n - z_n||$ converges to 0.

3 Main Results

We first introduce a new concept of weak cyclic Kannan type $F-$contraction. We also prove the existence result for best proximity point of weak cyclic Kannan type $F-$contraction in the setting of uniformly convex Banach space.

Definition 3.1. *Let A and B be two non-empty subsets of a complete metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called weak cyclic Kannan type $F-$contraction if the following conditions hold:*
(a) $T(A) \subseteq B$ and $T(B) \subseteq A$,
(b) $\forall x \in A, y \in B$ and if there exists $\tau > 0$ such that

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq \alpha F(d(x, Tx) + d(y, Ty)) + (1 - 2\alpha)F(d(A, B)),$$

for some $\alpha \in (0, \frac{1}{2})$, provided $d(A, B) > 0$.

Note that (b) implies that T satisfies $\tau + F(d(Tx, Ty)) \leq F(d(x, y))$, for all $x \in A, y \in B$.

Example: Consider the complete metric space $X = R$ with the usual metric d. If we consider the map $F(x) = -\frac{1}{\sqrt{x}}$, $x > 0$ and for any $k \in (\frac{1}{2}, 1)$. Then clearly F satisfies all the conditions from $(F1) - (F3)$.
In this case, any $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$ satisfying the condition

$$d(Tx, Ty) \leq \frac{[d(x, Tx) + d(y, Ty)]}{[2\alpha + (1 - 2\alpha)\sqrt{\frac{d(x,Tx)+d(y,Ty)}{d(A,B)}} + \tau\sqrt{d(x, Tx) + d(y, Ty)}]^2},$$

$\forall x \in A, \forall y \in B, d(Tx, Ty) > 0, \alpha \in (0, \frac{1}{2})$ and $\tau > 0$ provided $d(A, B) > 0$, is a weak cyclic Kannan type $F-$contraction.

Proposition 3.1. *Let A and B be non-empty closed subsets of a complete metric space (X, d). Suppose that mapping $T : A \cup B \to A \cup B$ is weak cyclic Kannan type $F-$contraction. Let $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$ for all $n \geq 0$. Then we have*

$$\lim_{n\to\infty} d(x_n, Tx_n) = d(A, B).$$

Proof. Suppose $x_0 \in A \cup B$. Let a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for all $n \geq 0$. Now,

$$\begin{aligned}
F(d(x_2,x_1)) &= F(d(Tx_1,Tx_0)) \\
&\leq \alpha F[d(x_1,Tx_1)+d(x_0,Tx_0)] + (1-2\alpha)F(d(A,B)) - \tau \\
&< \alpha F[d(x_1,Tx_1)+d(x_0,Tx_0)] + (1-2\alpha)F(d(A,B)) \\
&= \alpha F[d(x_1,x_2)+d(x_0,x_1)] + (1-2\alpha)F(d(A,B)) \\
&< \alpha F[d(x_1,x_2)+d(x_0,x_1)] + (1-2\alpha)F(d(x_1,x_2)) \\
&< \alpha F d(x_1,x_2) + \alpha F(d(x_0,x_1)) + (1-2\alpha)F(d(x_1,x_2)) \\
&< \alpha F(d(x_0,x_1)) + (1-\alpha)F(d(x_1,x_2)) \\
&< \alpha F(d(x_0,x_1)) + (1-\alpha)F(d(A,B)) \\
F(d(x_2,x_1)) - F(d(A,B)) &< \alpha\{F(d(x_0,x_1)) - F(d(A,B))\}.
\end{aligned}$$

Again,

$$\begin{aligned}
F(d(x_3,x_2)) &= F(d(Tx_2,Tx_1)) \\
&\leq \alpha F[d(x_2,Tx_2)+d(x_1,Tx_1)] + (1-2\alpha)F(d(A,B)) - \tau \\
&< \alpha F[d(x_2,Tx_2)+d(x_1,Tx_1)] + (1-2\alpha)F(d(A,B)) \\
&= \alpha F[d(x_2,x_3)+d(x_1,x_2)] + (1-2\alpha)F(d(A,B)) \\
&< \alpha F[d(x_2,x_3)+d(x_1,x_2)] + (1-2\alpha)F(d(x_2,x_3)) \\
&< \alpha F d(x_2,x_3) + \alpha F(d(x_1,x_2)) + (1-2\alpha)F(d(x_2,x_3)) \\
&< \alpha F(d(x_1,x_2)) + (1-\alpha)F(d(x_2,x_3)) \\
&< \alpha F(d(x_1,x_2)) + (1-\alpha)F(d(A,B)) \\
F(d(x_3,x_2)) - F(d(A,B)) &< \alpha\{F(d(x_1,x_2)) - F(d(A,B))\} \\
F(d(x_3,x_2)) - F(d(A,B)) &< \alpha^2\{F(d(x_0,x_1)) - F(d(A,B))\} \\
&\ldots \\
&\ldots
\end{aligned}$$

We say that $\alpha \in (0,1)$,

$$F(d(x_{n+1},x_n)) - F(d(A,B)) < \alpha^n\{F(d(x_0,x_1)) - F(d(A,B))\}.$$

For $n \to \infty$

$$F(d(x_{n+1},x_n)) = F(d(A,B))$$

Since F is increasing.

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = d(A,B)$$
$$\lim_{n\to\infty} d(x_n,Tx_n) = d(A,B)$$

□

Theorem 3.1. *Let A and B be two non-empty closed and convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is weak cyclic Kannan type $F-$contraction map and F is continuous map. Then there exists a best proximity point $x \in A$. Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

Proof. Suppose $d(A, B) \neq 0$. Then by proposition 3.1,

$$\begin{aligned} ||x_{2n} - x_{2n+1}|| &= ||x_{2n} - Tx_{2n}|| \to d(A, B) \\ ||x_{2n+2} - x_{2n+1}|| &= ||Tx_{2n+1} - Tx_{2n}|| \to d(A, B) \\ ||x_{2n+2} - x_{2n+1}|| &= ||T(Tx_{2n}) - Tx_{2n}|| \to d(A, B) \\ ||x_{2n+2} - x_{2n+1}|| &= ||Tx_{2n} - T(Tx_{2n})|| \to d(A, B) \end{aligned}$$

Then, by lemma 2.2

$$||x_{2n} - x_{2n+2}|| \to 0$$

We now show that for every $\epsilon > 0$ there exists N_0 such that for all $m > n > N_0$,

$$||x_{2m} - Tx_{2n}|| \leq d(A, B) + \epsilon$$

Let $\epsilon > 0$, if possible, suppose for all $k \in \mathbf{N}$, there exists $m_k > n_k \geq k$, such that,

$$||x_{2m_k} - Tx_{2n_k}|| \geq d(A, B) + \epsilon$$

and

$$||x_{2(m_k-1)} - Tx_{2n_k}|| \leq d(A, B) + \epsilon.$$

Now,

$$\begin{aligned} d(A, B) + \epsilon &\leq ||x_{2m_k} - Tx_{2n_k}|| \\ &\leq ||x_{2m_k} - x_{2(m_k-1)}|| + ||x_{2(m_k-1)} - Tx_{2n_k}|| \\ &\leq ||x_{2m_k} - x_{2(m_k-1)}|| + d(A, B) + \epsilon. \end{aligned}$$

Thus,

$$\lim_{k\to\infty} ||x_{2m_k} - Tx_{2n_k}|| = d(A, B) + \epsilon.$$

as F is continuous.

$$
\begin{aligned}
\lim_{k\to\infty} F(||x_{2m_k} - Tx_{2n_k}||) &\leq \lim_{k\to\infty} F(||Tx_{2m_k+1} - Tx_{2n_k+2}||)\\
&\leq \lim_{k\to\infty} \{\alpha F(||x_{2m_k+1} - Tx_{2m_k+1}||\\
&+ ||x_{2n_k+2} - Tx_{2n_k+2}||)\}\\
&+ (1-2\alpha)F(d(A,B)) - \tau\\
&\leq \lim_{k\to\infty} \{\alpha F(||x_{2m_k+1} - Tx_{2m_k+1}||)\\
&+ \alpha F(||x_{2n_k+2} - Tx_{2n_k+2}||)\}\\
&+ (1-2\alpha)F(d(A,B)) - \tau\\
&< \lim_{k\to\infty} \{\alpha F(||x_{2m_k+1} - Tx_{2m_k+1}||)\\
&+ \alpha F(||x_{2n_k+2} - Tx_{2n_k+2}||)\}\\
&+ (1-2\alpha)F(d(A,B))\\
&= \lim_{k\to\infty} \{\alpha F(||Tx_{2m_k} - Tx_{2m_k+1}||)\\
&+ \alpha F(||Tx_{2n_k+1} - Tx_{2n_k+2}||)\}\\
&+ (1-2\alpha)F(d(A,B))\\
&< \lim_{k\to\infty} \{\alpha\{\alpha F(||x_{2m_k} - Tx_{2m_k}||\\
&+ ||x_{2m_k+1} - Tx_{2m_k+1}||)+\\
&(1-2\alpha)F(d(A,B))\} + \alpha\{\alpha F(||x_{2n_k+1} - Tx_{2n_k+1}||\\
&+ ||x_{2n_k+2} - Tx_{2n_k+2}||)\\
&+ (1-2\alpha)F(d(A,B))\}\} + (1-2\alpha)F(d(A,B))\\
&< \{\alpha^2 F(d(A,B)+\epsilon) + \alpha^2 F(d(A,B)+\epsilon)\\
&+ \alpha(1-2\alpha)F(d(A,B))\}\\
&+ \{\alpha^2 F(d(A,B)+\epsilon) + \alpha^2 F(d(A,B)+\epsilon)\\
&+ \alpha(1-2\alpha)F(d(A,B))\} + (1-2\alpha)F(d(A,B))\\
&< 4\alpha^2 F(d(A,B)+\epsilon) + (\alpha - 2\alpha^2 + \alpha - 2\alpha^2\\
&+ 1 - 2\alpha)F(d(A,B))\\
&< 4\alpha^2 F(d(A,B)+\epsilon) + (1-4\alpha^2)F(d(A,B))\\
&< 4\alpha^2 F(d(A,B)+\epsilon) + (1-4\alpha^2)F(d(A,B))\\
F(d(A,B)+\epsilon) &< 4\alpha^2 F(d(A,B)+\epsilon) + (1-4\alpha^2)F(d(A,B))\\
F(d(A,B)+\epsilon)(1-4\alpha^2) &< (1-4\alpha^2)F(d(A,B))\\
F(d(A,B)+\epsilon) &< F(d(A,B)).
\end{aligned}
$$

Hence T is best proximity point. □

Now we introduce a new concept of cyclic Chatterjea type $F-$contraction. We also prove the existence result for best proximity point of cyclic Chatterjea type $F-$contraction in the setting of uniformly convex Banach space.

Definition 3.2. *Let A and B be two non-empty subsets of a complete metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called cyclic Chatterjea type $F-$contraction if the following conditions hold:*
(a)$T(A) \subseteq B$ and $T(B) \subseteq A$,
(b)$\forall x \in A, y \in B$ and if there exists $\tau > 0$ such that

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq \alpha F(d(Tx, y) + d(Ty, x)) + (1-4\alpha)F(d(A, B))$$

,
for some $\alpha \in (0, \frac{1}{4})$, provided $d(A, B) > 0$.

Note that (b) implies that T satisfies $\tau + F(d(Tx, Ty)) \leq F(d(Tx, y) + d(Ty, x))$, for all $x \in A, y \in B$.

Example: Consider the complete metric space $X = R$ with the usual metric d. If we consider the map $F(x) = -\frac{1}{\sqrt{x}}$, $x > 0$ and for any $k \in (\frac{1}{2}, 1)$. Then clearly F satisfies all the conditions from $(F1) - (F3)$.
Let us define $X = \{0, 1\}$ and $d(0, 0) = d(1, 1) = 2$ and $d(0, 1) = d(1, 0) = 1$ and take $\alpha \in (0, \frac{1}{4})$,

Let $A = \{0\}$ and $B = \{1\}$. Note that $d(A, B) = 1$ and A, B are closed in (X, d)

Consider $T : A \cap B \to A \cap B$ defined by $T0 = 1$ and $T1 = 0$

Clearly, T is cyclic. Let $x \in A$ and $y \in B$, that is $x = 0$ and $y = 1$ satisfying the condition

$$d(Tx, Ty) \leq \frac{1}{[4\alpha + (1-4\alpha)\sqrt{\frac{d(Tx,y)+d(Ty,x)}{d(A,B)}} + \tau\sqrt{d(Tx, y) + d(Ty, x)}]^2}[d(Tx, y) + d(Ty, x)],$$

$\forall x \in A, \forall y \in B, d(Tx, Ty) > 0, \alpha \in (0, \frac{1}{4})$ and $\tau > 0$ provided $d(A, B) > 0$, is a cyclic Chatterjea type $F-$contraction.

Proposition 3.2. *Let A and B be non-empty closed subsets of a complete metric space (X, d). Suppose that mapping $T : A \cup B \to A \cup B$ is a cyclic Chatterjea type $F-$contraction. Let $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$ for all $n \geq 0$. Then we have*

$$\lim_{n\to\infty} d(x_n, Tx_n) = d(A, B).$$

Proof. Suppose $x_0 \in A \cup B$. Let a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for all $n \geq 0$. Now,

$$\begin{aligned}
F(d(x_2, x_1)) &= F(d(Tx_1, Tx_0)) \\
&\leq \alpha F[d(Tx_1, x_0) + d(Tx_0, x_1)] + (1 - 4\alpha)F(d(A, B)) - \tau \\
&< \alpha F[d(Tx_1, x_0) + d(Tx_0, x_1)] + (1 - 4\alpha)F(d(A, B)) \\
&< \alpha F[d(x_2, x_0) + d(x_1, x_1)] + (1 - 4\alpha)F(d(A, B)) \\
&< \alpha F[\{d(x_2, x_1) + d(x_1, x_0)\} + \{d(x_1, x_2) + d(x_2, x_1)\}] \\
&\quad + (1 - 4\alpha)F(d(A, B)) \\
&< \alpha F[d(x_1, x_2) + d(x_1, x_0) + d(x_1, x_2) + d(x_1, x_2)] \\
&\quad + (1 - 4\alpha)F(d(x_1, x_2)) \\
&< \alpha F(d(x_1, x_0)) + 3\alpha F(d(x_1, x_2)) + (1 - 4\alpha)F(d(x_1, x_2)) \\
&< \alpha F[d(x_1, x_0)] + (1 - \alpha)F(d(x_1, x_2)) \\
&< \alpha F[d(x_0, x_1)] + (1 - \alpha)F(d(A, B)) \\
F(d(x_2, x_1)) - F(d(A, B)) &< \alpha\{F(d(x_0, x_1)) - F(d(A, B))\}.
\end{aligned}$$

Again,

$$\begin{aligned}
F(d(x_3, x_2)) &= F(d(Tx_2, Tx_1)) \\
&\leq \alpha F[d(Tx_2, x_1) + d(Tx_1, x_2)] + (1 - 4\alpha)F(d(A, B)) - \tau \\
&< \alpha F[d(x_3, x_1) + d(x_2, x_2)] + (1 - 4\alpha)F(d(A, B)) \\
&< \alpha F[\{d(x_3, x_2) + d(x_2, x_1)\} + \{d(x_2, x_3) + d(x_3, x_2)\}] \\
&\quad + (1 - 4\alpha)F(d(A, B)) \\
&< \alpha F[3d(x_3, x_2) + d(x_2, x_1)] + (1 - 4\alpha)F(d(x_3, x_2)) \\
&< \alpha F(d(x_2, x_1)) + (1 - \alpha)F(d(x_3, x_2)) \\
&< \alpha F(d(x_2, x_1)) + (1 - \alpha)F(d(A, B)) \\
F(d(x_3, x_2)) - F(d(A, B)) &< \alpha\{F(d(x_2, x_1)) - F(d(A, B))\} \\
F(d(x_3, x_2)) - F(d(A, B)) &< \alpha^2\{F(d(x_1, x_0)) - F(d(A, B))\} \\
&< \alpha^2\{F(d(x_0, x_1)) - F(d(A, B))\} \\
&\cdots \\
&\cdots
\end{aligned}$$

We say that $\alpha \in (0, 1)$,

$$F(d(x_{n+1}, x_n)) - F(d(A, B)) < \alpha^n\{F(d(x_0, x_1)) - F(d(A, B))\}.$$

For $n \to \infty$

$$F(d(x_{n+1}, x_n)) = F(d(A, B))$$

Since F is increasing.

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B)$$
$$\lim_{n\to\infty} d(x_n, Tx_n) = d(A, B).$$

□

Theorem 3.2. *Let A and B be two non-empty closed and convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is cyclic Chatterjea type $F-$contraction map and F is continuous map. Then there exists a best proximity point $x \in A$. Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

Proof. Suppose $d(A, B) \neq 0$. Then by proposition 3.2

$$||x_{2n} - x_{2n+1}|| = ||x_{2n} - Tx_{2n}|| \to d(A, B)$$
$$||x_{2n+2} - x_{2n+1}|| = ||Tx_{2n+1} - Tx_{2n}|| \to d(A, B)$$

Then, by lemma 2.2

$$||x_{2n} - x_{2n+2}|| \to 0$$

We now show that for every $\epsilon > 0$ there exists N_0 such that for all $m > n > N_0$,

$$||x_{2m} - Tx_{2n}|| \leq d(A, B) + \epsilon$$

Let $\epsilon > 0$, if possible, suppose for all $k \in \mathbf{N}$, there exists $m_k > n_k \geq k$, such that

$$||x_{2m_k} - Tx_{2m_k}|| \geq d(A, B) + \epsilon$$

and

$$||x_{2(m_k-1)} - Tx_{2n_k}|| \leq d(A, B) + \epsilon$$

Now,

$$\begin{aligned} d(A, B) + \epsilon &\leq ||x_{2m_k} - Tx_{2n_k}|| \\ &\leq ||x_{2m_k} - x_{2(m_k-1)}|| + ||x_{2(m_k-1)} - x_{2n_k}|| \\ &\leq ||x_{2m_k} - x_{2(m_k-1)}|| + d(A, B) + \epsilon \end{aligned}$$

Thus,

$$\lim_{k\to\infty} ||x_{2m_k} - Tx_{2n_k}|| = d(A, B) + \epsilon$$

as F is continuous.

$$
\begin{aligned}
\lim_{k\to\infty} F(||x_{2m_k} - Tx_{2n_k}||) &\leq \lim_{k\to\infty} F(||Tx_{2m_k+1} - Tx_{2n_k+2}||) \\
&\leq \lim_{k\to\infty} \{\alpha F(||Tx_{2m_k+1} - x_{2n_k+2}||+||Tx_{2n_k+2} - x_{2m_k+1}||)\} \\
&+ (1-4\alpha)F(d(A,B)) - \tau \\
&< \lim_{k\to\infty} \{\alpha F(||Tx_{2m_k+1} - Tx_{2n_k+1}||+||Tx_{2n_k+2} - Tx_{2m_k}||)\} \\
&+ (1-4\alpha)F(d(A,B)) \\
&< \lim_{k\to\infty} [\alpha\{\alpha F(||Tx_{2m_k+1} - x_{2n_k+1}||+||Tx_{2n_k+1} - x_{2m_k+1}||) \\
&+ (1-4\alpha)F(d(A,B))\} + \alpha\{\alpha F(||Tx_{2n_k+2} - x_{2m_k}|| \\
&+ ||Tx_{2m_k} - x_{2n_k+2}||) + (1-4\alpha)F(d(A,B))\}] \\
&+ (1-4\alpha)F(d(A,B)) \\
&\leq \{\alpha^2 F(d(A,B)+\epsilon) + \alpha^2 F(d(A,B)+\epsilon) + \alpha(1-4\alpha) \\
&F(d(A,B))\} + \lim_{k\to\infty} \alpha^2 F(\{||Tx_{2n_k+2} - x_{2m_k+2}||+||x_{2m_k+2} \\
&- x_{2m_k+1}|| \\
&+ ||x_{2m_k+1} - x_{2m_k}||\} + \{||Tx_{2m_k} - x_{2n_k}||+||x_{2n_k} \\
&- x_{2n_k+1}||+||x_{2n_k+1} - x_{2n_k+2}||\}) + \alpha(1-4\alpha)F(d(A,B)) \\
&+ (1-4\alpha)F(d(A,B)) \\
&\leq \{\alpha^2 F(d(A,B)+\epsilon) + \alpha^2 F(d(A,B)+\epsilon) + \alpha(1-4\alpha) \\
&F(d(A,B)) + \lim_{k\to\infty} \alpha^2 F(\{||Tx_{2n_k+2} - x_{2m_k+2}||+||Tx_{2m_k+1} \\
&- x_{2m_k+1}||+||Tx_{2m_k} - x_{2m_k}||\} + \{||Tx_{2m_k} - x_{2n_k}||+||x_{2n_k} \\
&- Tx_{2n_k}||+||x_{2n_k+1} - Tx_{2n_k+1}||\}) + \alpha(1-4\alpha) \\
&F(d(A,B)) + (1-4\alpha)F(d(A,B)) \\
&\leq \{\alpha^2 F(d(A,B)+\epsilon) + \alpha^2 F(d(A,B)+\epsilon) + \alpha(1-4\alpha) \\
&F(d(A,B)) + \alpha^2 F(\{(d(A,B)+\epsilon) + (d(A,B)+\epsilon) \\
&+ (d(A,B)+\epsilon)\} + \{(d(A,B)+\epsilon) + (d(A,B)+\epsilon) \\
&+ (d(A,B)+\epsilon)\}) + \alpha(1-4\alpha)F(d(A,B)) \\
&+ (1-4\alpha)F(d(A,B)) \\
&\leq 8\alpha^2 F(d(A,B)+\epsilon) + \alpha(1-4\alpha+1-4\alpha)F(d(A,B)) \\
&+ (1-4\alpha)F(d(A,B)) \\
&\leq 8\alpha^2 F(d(A,B)+\epsilon) + (2\alpha - 8\alpha^2)F(d(A,B)) + \\
&(1-4\alpha)F(d(A,B)) \\
&\leq 8\alpha^2 F(d(A,B)+\epsilon) + (2\alpha - 8\alpha^2 + 1 - 4\alpha)F(d(A,B)) \\
&\leq 8\alpha^2 F(d(A,B)+\epsilon) + (1 - 8\alpha^2 - 2\alpha)F(d(A,B)) \\
&< 8\alpha^2 F(d(A,B)+\epsilon) + (1 - 8\alpha^2)F(d(A,B)) \\
F(d(A,B)+\epsilon)(1-8\alpha^2) &< (1-8\alpha^2)F(d(A,B)) \\
F(d(A,B)+\epsilon) &< F(d(A,B)).
\end{aligned}
$$

Hence T is best proximity point. $\square$

4 Conclusion

In this study, we have introduced the concepts of weak cyclic Kannan type F-contraction and cyclic Chatterjea type F-contraction. By employing these novel contraction mappings, we have successfully determined the best proximity point theorem within a uniformly convex Banach space. To further illustrate the effectiveness of our approach, we have presented a series of illustrative examples.

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References

[1] J. Ahmad, A. Al-Rawashdeh, and A. Azam. New fixed point theorems for generalized f-contractions in complete metric spaces. *Fixed Point Theory and Applications*, 2015(1):1–18, 2015.

[2] M. A. Alghamdi, N. Shahzad, F. Vetro, et al. Best proximity points for some classes of proximal contractions. In *Abstract and Applied Analysis*, volume 2013. Hindawi, 2013.

[3] H. Aydi and A. Felhi. Best proximity points for cyclic kannan-chatterjea-ciric type contractions on metric-like spaces. *J. Nonlinear Sci. Appl*, 9(5): 2458–2466, 2016.

[4] S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta mathematicae*, 3(1):133–181, 1922.

[5] D. W. Boyd and J. S. Wong. On nonlinear contractions. *Proceedings of the American Mathematical Society*, 20(2):458–464, 1969.

[6] P. Das and L. K. Dey. Fixed point of contractive mappings in generalized metric spaces. *Mathematica Slovaca*, 59:499–504, 2009.

[7] A. A. Eldred and P. Veeramani. Existence and convergence of best proximity points. *Journal of Mathematical Analysis and Applications*, 323(2):1001–1006, 2006.

[8] K. Fan. Extensions of two fixed point theorems of fe browder. In *Set-Valued Mappings, Selections and Topological Properties of 2x: Proceedings of the conference held at the State University of New York at Buffalo May 8–10, 1969*, pages 12–16. Springer, 2006.

[9] M. Gabeleh and C. Vetro. A note on best proximity point theory using proximal contractions. *Journal of Fixed Point Theory and Applications*, 20:1–11, 2018.

[10] N. Hussain, M. Kutbi, and P. Salimi. Fixed point theory in α-complete metric spaces with applications. In *Abstract and Applied Analysis*, volume 2014, pages 1–11. Hindawi Limited, 2014.

[11] M. Jleli and B. Samet. A new generalization of the banach contraction principle. *Journal of inequalities and applications*, 2014:1–8, 2014.

[12] W. K. Kim and K. H. Lee. Existence of best proximity pairs and equilibrium pairs (vol 316, pg 433, 2006). *JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS*, 329(2):1482–1483, 2007.

[13] W. A. Kirk, S. Reich, and P. Veeramani. Proximinal retracts and best proximity pair theorems. 2003.

[14] C. Mongkolkeha and P. Kumam. Some common best proximity points for proximity commuting mappings. *Optimization Letters*, 7:1825–1836, 2013.

[15] C. Mongkolkeha, Y. J. Cho, and P. Kumam. Best proximity points for geraghty's proximal contraction mappings. *Fixed Point Theory and Applications*, 2013(1):1–17, 2013.

[16] H. K. Nashine, P. Kumam, and C. Vetro. Best proximity point theorems for rational proximal contractions. *Fixed Point Theory and Applications*, 2013 (1):1–11, 2013.

[17] M. A. Petric. Best proximity point theorems for weak cyclic kannan contractions. *Filomat*, 25(1):145–154, 2011.

[18] H. Piri and P. Kumam. Some fixed point theorems concerning f-contraction in complete metric spaces. *Fixed Point Theory and Applications*, 2014:1–11, 2014.

[19] V. S. Raj. A best proximity point theorem for weakly contractive non-self-mappings. *Nonlinear Analysis: Theory, Methods & Applications*, 74(14): 4804–4808, 2011.

[20] S. Sadiq Basha. Extensions of banach's contraction principle. *Numerical Functional Analysis and Optimization*, 31(5):569–576, 2010.

[21] D. Wardowski. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed point theory and applications*, 2012(1):1–6, 2012.