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Weaker forms of Contra Continuous Multifunctions

Rajeshwari K.*
T.D.Rayanagoudar†
Sarika M. Patil‡
P. S. Mirajakar§

Abstract

The purpose of this paper is to study the properties of αgs-continuous multifunctions.
Some basic characterizations, preservation theorems and several properties concerning upper and lower contra αgs-continuous multifunctions are investigated. Further, the basic properties of upper and lower almost contra αgs-continuous multifunctions are studied.
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*Research Scholar, R and D Department, Bharathiar University, Coimbatore-641046, Tamilnadu, India; rajeshwaribhat10@yahoo.com
†Department of Mathematics, Government First Grade College, Rajnagar, Hubballi-580032, Karnataka, India; rgoudar1980@gmail.com
‡Department of Mathematics, Government First Grade College, Rajnagar, Hubballi-580032, Karnataka, India; sarupatil@rediffmail.com
§Department of Mathematics, P. C. Jabin Science College, Vidyanagar, Hubballi-580031, Karnataka, India; psmirajakar@gmail.com
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1 Introduction

The research in topology over last two decades has reached a high level in many directions. Topological methods are widely used in many other branches of modern mathematics such as differential equation, functional analysis, classical mechanics, general theory of relativity, mathematical economics, quantum theory, biology etc. One of the important and basic topics in the theory of classical point set topology and in several branches of mathematics, which has been investigated by many authors is continuity of functions. This concept has been extended to the setting of multifunctions.

The concept of upper and lower multifunctions were introduced and studied by Berge [2]. After this work several authors have given the several weaker and strong forms of continuity of multifunctions [1], [7], [8], [9], [10], [11], [15]). In this direction, we will continue the study of new type of multifunction called αgs-continuous multifunctions and study some of basic properties.

In this paper, (P, τ) and (Q, σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. A multifunction $f : (P, \tau) \to (Q, \sigma)$ is a point to set correspondence, and we always assume that $f(p) \neq \phi$, for every point $p \in P$. For each $A \subset P$ and each $B \subset Q$, $f(A) = \cup\{f(p) : p \in A\}$, $f^+(B) = \{p \in P : f(p) \subset B\}$ and $f^-(B) = \{p \in P : f(p) \cap B \neq \phi\}$.

A multifunction $f : P \to Q$ is said to be upper semi continuous [13] (resp. lower semi continuous) at a point $p \in P$, and each open set $V \subseteq Q$ with $f(p) \subset V$ ($f(p) \cap V \neq \phi$) there is an open neighborhood U of p such that $f(U) \subseteq V$ ($f(a) \cap V \neq \phi$ for each $a \in U$). Then f is semi continuous if and only if f is upper and lower semi continuous.

In this paper, the first section deals with the concept of Upper and Lower contra αgs-continuous multifunctions in topological spaces. Several characterizations of these functions have been introduced. The second section contains the properties of Upper and Lower Almost contra αgs-continuous functions in topological spaces. Many properties concerning to these functions have been introduced in this paper.

2 Preliminaries

Definition 2.1. *[12] A subset A of a topological space (P, τ) is called αgs-closed if αcl(A)$\subseteq U$ whenever $A \subseteq U$ and U is gs-open in P.*

The family of all αgs-open (resp. αgs-closed) sets of P is denoted by αgs-O(P) (resp. αgs-C(P)).
If $B \in \alpha$gs-O(P) then $P - B \in \alpha$gs-C(P).

Definition 2.2. *[12] Let $A \subset P$. Then*
(i) αgs-interior of A is denoted by αgs-int(A) and is defined as αgs-int(A) = $\cup\{G : G \subset A$, where $G \in \alpha$gs-O(P)$\}$.
(ii) αgs-closure of A is denoted by αgs-cl(A) and is defined as αgs-cl(A) = $\cap\{G : A \subset G$, where $G \in \alpha$gs-C(P)$\}$.

Definition 2.3. *[12] Let $f : P \to Q$ is a function. Then f is said to be*
(i) αgs-continuous, if $f^{-1}(U) \in \alpha$gs-C(P) for every $U \in C(Q)$.
(ii) αgs-irresolute, if $f^{-1}(U) \in \alpha$gs-C(P) for every $U \in \alpha$gs-C(Q).

Lemma 2.1. *[10] For a multifunction $f : (P, \tau) \to (Q, \sigma)$ we have the following:*
(i) $G_f^+(A \times B) = A \cap f^+(B)$
(ii) $G_f^-(A \times B) = A \cap f^-(B)$.
For any subsets A of P and B of Q, we have $G_f : P \to P \times Q$ is defined as $G_f(p) = \{p\} \times f(p)$ for every $p \in P$.

3 Upper and Lower Contra αgs-continuous Multifunctions

Definition 3.1. *A multifunction $f : (P, \tau) \to (Q, \sigma)$ is called a*
(i) Upper αgs-continuous if for each $p \in P$, $V \in (Q, f(p))$ there exists $U \in \alpha$gs-$O(P, p)$ such that $f(U) \subseteq V$.
(ii) Lower αgs-continuous if for each $p \in P$, $V \in O(Q, f(p))$ with $f(p) \cap V \neq \phi$, then there exists $U \in \alpha$gs-$O(P, p)$ such that $U \in f^{-1}(V)$.

Definition 3.2. *A function $f : P \to Q$ is said to be Contra αgs-continuous if for each $p \in P$ and $V \in O(Q, f(p))$ there exists $V \in \alpha$gs-$O(Q, f(p)$ such that $f(U) \subseteq V$.*

Definition 3.3. *A multifunction $f : (P, \tau) \to (Q, \sigma)$ is called a*
(i) Upper contra αgs-continuous if for each $p \in P$, $V \in C(Q, f(p))$ there exists $U \in \alpha$gs-$O(P, p)$ such that $f(U) \subseteq V$.
(ii) Lower contra αgs-continuous if for each $p \in P$, $V \in C(Q)$ with $f(p) \cap V \neq \phi$, then there exists $U \in \alpha$gs-$O(P, p)$ such that $U \in f(V)$.

Theorem 3.1. *The following statements are equivalent for a multifunction in $f : (P, \tau) \to (Q, \sigma)$*
(i) f is upper contra αgs-continuous

(ii) $f^+(V) \in \alpha gp$*-*$O(P)$*, for every* $V \in C(Q)$
(iii) for each $V \in O(Q)$*,* $\Rightarrow f^-(V) \in \alpha gs$*-*$C(Q)$
(iv) for each $p_1 \in P$ *and each* $K \in C(Q, f(p_1))$*, there exists* $U \in \alpha gs$*-*$O(P, p_1)$ *with* $q_1 \in U$*, then* $p_1(q_1) \subseteq K$*.*
Proof: $(i) \Leftrightarrow (ii)$*: Let* $V \in C(Q)$ *and* $p_1 \in f^+(V)$ *as* p *is upper contra* αgs*-continuous, there exists* $U \in \alpha gs$*-*$O(P, p_1)$ *with* $f(U) \subseteq V$*. Thus* $f^+(V) \in O(P)$*. We can prove the other part similarly.*
$(ii) \Rightarrow (iii)$*: We have* $f^+(Q \setminus V) = P \setminus f^-(V)$ *for every* $V \subseteq Q$*.*
$(iii) \Rightarrow (iV)$*: Obvious from the definition.*

Theorem 3.2. *The following statements are equivalent for a multifunction in* $f : (P, \tau) \rightarrow (Q, \sigma)$
(i) f *is lower almost contra* αgs*-continuous.*
(ii) $f^+(V) \in \alpha gs$*-*$O(P)$*, for each* $V \in C(Q)$*.*
(iii) $f^-(K) \in \alpha gs$*-*$C(P)$*, for each* $K \in O(Q)$
(iv) for each $p \in Q$ *and each* $K \in C(P)$ *with* $f(p_1) \cap K \neq \phi$ *there exists* $U \in \alpha gs$*-*$O(P, p_1)$ *such that* $q_1 \in U$ *then* $p_1(q_1) \subseteq K \neq \phi$*.*
Proof: *The proof is similar to the Theorem 3.1.*

Definition 3.4. *Let* $K \in P$*. Then* K *is said to be strongly S-closed(*αgs*-compact) relative to* P *if every cover of* K *by closed (resp.* αgs*-open) sets of* P *has a finite subcover.*
A space P *is said to be strongly S-closed(*αgs*-compact) if* P *is strongly S-closed relative to* P*.*

Theorem 3.3. *Let* $f : (P, \tau) \rightarrow (Q, \sigma)$ *be an upper contra* αgs*-continuous surjective multifunction and* $f(p)$ *is strongly S-closed relative to* Q *holds for each* $p \in P$*. Then* $f(A)$ *is strongly S-closed if* A *is* αgs*-compact relative to* P*.*
Proof: *Let* $\{A_i : i \in \alpha\}$ *be any cover of* $f(A)$ *by closed sets in* P*. Then for each* $p \in A$ *there exists a finite subset* $\alpha(p)$ *of* α *such that* $f(p) \subseteq \cup\{A_i : i \in \alpha(p)\}$*. Let* $V(p) = \cup\{A_i : i \in \alpha(p)\}$*, then* $f(p) \subseteq V(P)$ *and so there exist* $U(P) \in \alpha gs$*-*$O(P, p)$ *with* $p(U(p)) \subseteq V(P)$ *as* $\{U(P) : p \in A\}$ *is cover of* A *by* αgs*-open sets in* P*. Then, there exist finite number of points say* $p_1, p_2, ...p_m$ *such that* $A \subseteq \cup\{U(p_i) : i = 1, 2, ...n\}$*. Hence* $f(A) \subseteq f(\bigcup_{i=1}^{n} U(p_i)) \subseteq \bigcup_{i=1}^{n} f(U(p_n)) \subseteq \bigcup_{i=1}^{n} V(p_i) \subseteq \bigcup_{i=1}^{n} \bigcup_{i=\alpha(p_i)} V_i$*. Thus* $f(A)$ *is strongly S-closed relative to* Q*.*

Lemma 3.1. *[12] Let* $A, B \subseteq P$*. Then*

(i) If $A \in \alpha gs$*-*$O(P)$ *and* $B \in O(P)$ *then* $A \cap B \in \alpha gs$*-*$O(P)$*.*

(ii) If $A \in \alpha gs$*-*$O(B)$ *and* $B \in \alpha gs$*-*$O(P)$ *then* $A \in \alpha gs$ *-*$O(P)$*.*

Theorem 3.4. *Let $f : (P, \tau) \to (Q, \sigma)$ be a multifunction with $V \subseteq O(P)$. If f is upper contra αgs-continuous (lower contra αgs-continuous) then $f_{|v} : U \to Q$ is upper contra αgs-continuous(lower contra αgs-continuous).*
Proof: *Let $V \subseteq C(Q)$ with $p \in U$ with $p \in P_{|v^{-1}}(V)$ by lower contra αgs-continuous of f. Then there exist $G \in \alpha$gs-$O(P, p)$ such that $G \subseteq P^{-1}(V)$ and so $p \in G \cap U \in \alpha gs\ O(A)$ with $G \cap U \subseteq F_{|U^{-1}}(V)$ which satisfies the conditions of lower contra αgs-continuous multifunction.*
Similarly, we can show for upper contra αgs-continuous multifunction.

Theorem 3.5. *Let $\{U_i : i \in \alpha\}$ be an open cover of a space P. A function $p : (P, \tau) \to (Q, \sigma)$ is upper contra αgs-continuous if and only if the restriction $P_{|U_i} : U_i \to Q$ is upper contra αgs-continuous for each $i \in \alpha$.*
Proof: *Let p is upper contra αgs-continuous and $i \in \alpha$. Let $P \in U_i$ and $V \subseteq C(N, P_{|U_i}(p_i)))$ as p is upper contra αgs-continuous and $P(m) = P_{|U_i}(p_1)$. Then there exists $G \in \alpha$gs-**O(P,** p_1) with $P(G) \subseteq V$. Put $U = G \cap U_i$, then $p_1 \in U \in \alpha$gs-$O(U, p_1)$ and $P_{|U_i}(U) = P(U) \subseteq V$. Hence $P_{|U_i}$ is upper contra αgs-continuous.*
Conversely, let $m \in P$, $V \in \alpha O(Q)$. Then there exist $i < \alpha$ with $p_1 \in U_i$ by upper contra αgs continuity of p and $P(p_1) = P_{|U_i}(p_1)$. Then there exists $U \in \alpha$gs-$O(U_i, p_1)$ with $P_{|U_i}(p_1) \subseteq V$. Thus $V \in \alpha$gs-$\boldsymbol{O}(P, p_1)$ and $P(U) \subseteq V$. Hence p is upper contra αgs-continuous.

Theorem 3.6. *The set of points p of P at which a multifunction $f : (P, \tau) \to (Q, \sigma)$ is not upper contra αgs-continuous (lower contra αgs-continuous) is identical with the union of the αgs-frontiers of the upper(lower) inverse images of a closed sets containing (meeting) $f(p)$.*
Proof: *Let $p \in P$ at which f is not upper contra αgs-continuous. Then there exist $V \in C(Q)$ with $U \cap (P \setminus f^+(V)) \neq \phi$ holds for each $U \in \alpha$gs-**O(P,** p) and so $p \in \alpha$gs-cl$(P \setminus f^+(V))$. As $p \in f^+(V)$, so $p \in \alpha$gs-cl$(f^+(Q))$ and so $p \in \alpha$gs-Fr$(f^+(V))$. On the other hand, let $V \in C(Q, f(p))$ with $p \in \alpha$gs-Fr$(f^+(V))$. Assume that f is upper contra αgs-continuous at point p, then there exists $U \in \alpha$gs-**O(P,** p) with $f(U) \subset V$. Thus, $p \in U \subset \alpha$gs-int$(f^+(V))$, which contradicts the fact that $p \in \alpha$gs-Fr$(f^+(V))$. Thus, f is not upper contra αgs-continuous.*
Similarly, we can prove for lower contra αgs-continuous.

4 Upper and Lower Almost Contra αgs-Continuous Multifunctions

Definition 4.1. *A multifunction $f : (P, \tau) \to (Q, \sigma)$ is said to be*
(i) upper almost contra αgs-continuous if for each $p \in P$ and $V \in RC(P)$ with

*$p \in f^+(V)$, there exists $U \in \alpha$gs-**O(P, p)** with $U \subset f^+(V)$.*
*(ii) lower almost contra αgs-continuous if for each $p \in P$ and $V \in RC(P)$ with $p \in f^-(V)$, there exists $U \in \alpha$gs-**O(P, p)** with $U \subset f^-(V)$.*

Theorem 4.1. *Every upper almost contra αgs-continuous (lower almost contra αgs-continuous) multifunction is upper (lower) weakly αgs-continuous.*
Proof: *Let $p \in P$ and $V \in O(Q)$ with $f(p) \subset V$, that is $cl(V)$ is regular closed with $f(p) \subseteq cl(V)$. By upper almost contra αgs-continuity, there exists $U \in \alpha$gs-**O(P, p)** with $U \subset f^+(cl(V))$ and so f is upper weakly αgs-continuous.*

Example 4.1. *Let $P = Q = \{p_1, p_2, p_3\}$. Let $\tau = \{P, \phi, \{p_1\}, \{p_1, p_3\}\}$ and $\sigma = \{Q, \phi, \{p_1\}, \{p_2\}, \{p_1, p_2\}, \{p_1, p_3\}\}$. Let $A = \{p_1, p_2\}$ and $U = \{p_1, p_2\}$. Then $p(U) = \{p_1, p_2\} \subset cl(A) = P$. Then f is weakly αgs-continuous but not upper almost contra αgs-continuous. For $V = \{p_2\}$ and $U = \{p_1, p_2\}$, we have $U \notin f^+(int(cl(V))$.*

Theorem 4.2. *Every upper contra αgs-continuous (lower contra αgs-continuous) multifunction is upper almost contra αgs-continuous (lower almost contraαgs-continuous).*
Proof: *Follows from the definition 4.1.*

Theorem 4.3. *The following are equivalent for a multifunction $f : (P, \tau) \rightarrow (Q, \sigma)$:*
(i) f is upper almost contra αgs-continuous
*(ii) for every $A \subset RC(Q)$, $f^+(A) \subset \alpha$gs-**O(Q)***
*(iii) for every $U \subset RO(Q)$, $f^-(U) \in \alpha$gs-**C(P)***
*(iv) for every $A \subset O(Q)$, $f^-(int(cl(A))) \in \alpha$gs-**C(P)***
*(v) for every $A \subset O(Q)$, $f^+(cl(int(A))) \in \alpha$gs-**O(P)**.*
*(vi) for all $p \in P$ and $V \in SO(Q)$ with $f(p) \subset V$, there exists $U \in \alpha$gs-**O(P, p)** such that $f(U) \subset cl(V)$.*
*(vii) $f^+(V) \subset \alpha$gs-**int**$(f^+(cl(V)))$ holds for all $V \in SO(Q)$.*
Proof: *$(i) \rightarrow (ii)$: Let $A \subset RC(Q)$ with $p \in f^+(A))$. As f is upper almost contra αgs-continuous, there exists $U \in \alpha$gs-**O(P, p)** such that $U \subset f^+(A)$. Thus, $f^+(A) \in \alpha$gs-**O(P)**.*
$(ii) \rightarrow (i)$: Follows from the definition.
$(ii) \Leftrightarrow (iii)$ and $(iv) \Leftrightarrow (v)$: We have $f^+(Q \setminus A) = P \setminus f^-(A)$ holds for all $A \subset Q$.
*$(iii) \Leftrightarrow (iv)$: Let $A \in O(Q)$. As $int(cl(A)) \in RO(Q)$ and so $f^-(int(cl(A)) \in \alpha$gs-**C(P)**.*
Similarly, we can show $(v) \Leftrightarrow (ii)$.
*$(vi) \rightarrow (vii)$: Let $V \in SO(Q)$ and $p \in f^+(V)$. Then, $f(p) \subset V$. By definition, there exists $U \in \alpha$gs-**O(P, p)** such that $f(U) \subset cl(V)$, that is $p \in U \subset f^+(cl(V))$.*

Thus, $p \in \alpha gs$-int$(f^+(cl(V)))$ and $f^+(V) \subset \alpha gs$-int$(f^+(cl(V)))$.
$(vii) \to (ii)$: Let $A \in RC(Q)$. As $A \in SO(Q)$ and from (vii), $f^+(A) \subset \alpha gs$-int$(f^+(cl(A)))$. Thus, $f^+(A) \in \alpha gs$-O(P).
$(ii) \to (vi)$: Let $p \in P$ and $V \in SO(Q)$ with $f(p) \subset V$. As $cl(V) \in RC(Q)$, then there exists $A \in \alpha gs$-O(P, p) with $A \subset f^+(cl(A))$. Thus, $f(A) \subset cl(V)$.

Lemma 4.1. *[14] If A and B are disjoint compact subsets of an Urysohn space P, then there exist $U \in O(P)$ and $V \in O(Q)$ with $A \subset U$, $B \subset V$ with $cl(U) \cap cl(V) = \phi$.*

Theorem 4.4. *Let $f : (P, \tau) \to (Q, \sigma)$ be upper almost contra αgs-continuous injective multifunction into an Urysohn space Q with $f(p)$ is compact for each point $p \in P$. Then P is αgs-T_2 space.*
Proof: *Let $p, q \in P$ with $f(p) \cap f(q) = \phi$ by injectiveness of f, where $f(p)$ and $f(q)$ are disjoint compact sets in P. Then there exist $V_1, V_2 \in O(P)$ with $f(p) \subset V_1$ and $f(q) \subset V_2$ such that $cl(V_1) \cap cl(V_2) = \phi$. Since $cl(V_1)$ and $cl(V_2)$ are regular closed sets, then by upper almost contra αgs-continuity of f, there exist $U_1 \in \alpha gs$-O(P, p) and $U_2 \in \alpha gs$-O(P, q) with $f(U_1) \subset cl(V_1)$, $f(U_2) \subset cl(V_2)$. Thus, $U_1 \cap U_2 = \phi$. Thus P is αgs-T_2 space.*

Discussion and Conclusion

Topology developed as a field of study of geometry and set theory, through analysis of such concepts as space, dimensions and transformation. The topological structures are modelled suitably in the field of Computer graphics, Pattern recognition, Artificial intelligence, Data mining, Rough set theory, Information systems, Quantum physics etc.

Continuous functions stands among the most fundamental point in the whole of the Mathematical Science. Many different forms of stronger and weaker forms of functions have been introduced over the years. As a generalization of closed sets, Levine introduced the concept of generalized closed sets which are weaker than closed sets in topological spaces.

In this way, this paper deals with the properties of weaker forms of continuous functions contra and almost contra continuous multifunctions in topological spaces. We have defined the new weaker froms of continuous multifunctions called contra and almost contra αgs-continuous multifunctions in topological spaces. The properties and preservation theorems on upper and lower contra αgs-continuous multifunctions are introduced. Further, upper and lower almost contra αgs-continuous multifunctions are also introduced and studied basic characterizations of these multifunctions in topological spaces.

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