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G-Supra and G-Infra space

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Abstract

The main idea of this paper is to generate supra and infra topologies from simple undirected graphs. For this, we have introduced two new operators namely supra and infra operators which are defined on the power set of the vertex set of a graph. Moreover, we have also proved that the supra operator satisfying Kuratowski's closure axiom will yield a topology. Further it was extended to develop the concept of connectedness and separation axioms on G-supra and G-infra spaces.

Keywords: supra operator, infra operator, G-supra topology, G-infra topology.
2020 AMS subject classifications: 05C40, 54C50, 54D10 [1]

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1 Introduction

Topology is a relatively new branch of mathematics; most of the research in topology has been done since 1900. Graph theory is another application oriented area of mathematics originated from konigsberg bridge problem. These two theories were connectively studied by many researchers. In 1983, Diesto and Gervacio [S.Diesto and S.Gervacio, 1983] proved that, a simple undirected graph generates a topology on its vertex set. In 1986, Guerrero and Gervacio [R.Guerrero and S.Gervacio, 1986] investigated further the graphs which generate the discrete and indiscrete topology.

In 1983, Mashhour et al.[A. S.Mashhour and F.H.Khedr, 1983] emerged a new idea of supra topological spaces by relaxing the finite intersection axiom of topological space.In 2015, Adel. M. Al-Odhari.[M.Adel and Al-Ddhari, 2015] introduced infra topological spaces by dropping arbitrary union axiom of topological space. Many researchers have induced topology from undirected graphs by generating base or subbase form the vertex set of graphs.

This made an intention to generate supra and infra topologies from graphs. In order to induce these types of topology, we have defined the supra and infra operators on the power set of the vertex set of a simple undirected graph. The properties of these operators were studied. Certain types of graphs namely null graph, complete graph and disconnected graph with complete components satisfies $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$ and $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$. So these graphs also generates topological space. Moreover, we have also proved that the disconnected graphs generates the disconnected G-supra and G-infra topologies. Using this operator we have made an attempt to define and charactereize separation axioms both in G-supra and G-infra spaces.

2 Preliminaries

In this section, we recalled some basic definitions related to graph theory and topological spaces.

Definition 2.1. *[Deo, 2016] A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \cdots\}$ called vertices, and another set $E = \{e_1, e_2, \cdots\}$, whose elements are called edges, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k.*

Definition 2.2. *[Deo, 2016] A graph with a finite number of vertices as well as a finite number of edges is called a finite graph; otherwise, it is an infinite graph.*

Definition 2.3. *[Deo, 2016] In a graph, an edge having the same vertex as both its end vertices is called a self-loop. Also more than one edge associated with a given pair of vertices are referred to as parallel edges. A graph that has neither self-loops nor parallel edges is called a simple graph.*

Definition 2.4. *[Deo, 2016] A vertex having no incident edge is called an isolated vertex. In a graph $G = (V, E)$, it is possible for the edge set E to be empty. Such a graph, without any edges, is called a null graph or totally disconnected graph.*

Definition 2.5. *[Deo, 2016] A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a circuit. A circuit is also called a cycle.*

Definition 2.6. *Munkres [2022] A topology on a set X is a collection τ of subsets of X having the following properties.(i) ϕ and X are in τ. (ii) The union of the elements of any subcollection of τ is in τ. (iii) The intersection of the elements of any finite subcollection of τ is in τ. A set X for which a topology τ has been specified is called a topological space. A subset U of X is an open set of X if U belongs to the collection τ.*

Definition 2.7. *[Munkres, 2022] The collection of all subsets of X is a topology on X, it is called the discrete topology. The collection consisting of X and ϕ only is also a topology on X, it is called the indiscrete topology.*

Definition 2.8. *[Munkres, 2022] A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space is said to be connected if there does not exist a separation of X. A space is totally disconnected if its only connected subspaces are one-point sets. If X has the discrete topology, then X is totally disconnected.*

Definition 2.9. *[A. S.Mashhour and F.H.Khedr, 1983] A subclass $\tau^* \subset \mathcal{P}(X)$ is called a supra topology on X if $X \in \tau^*$ and τ^* is closed under arbitrary union. (X, τ^*) is called a supra topological space. The members of τ^* are called the supra open sets.*

Definition 2.10. *[M.Adel and Al-Ddhari, 2015] An Infra-topological space on a non empty set X is a collection τ_{iX} of subsets of X such that (i) $\phi, X \in \tau_{iX}$. (ii) The intersection of the elements of any sub collection of τ_{iX} is in X.*

Throughout this paper a graph G means a finite simple undirected graph.

3 Supra operator on graphs

In this section, we have defined supra operator and G-supra topology on graphs.

Definition 3.1. *Let $G = (V, E)$ be a graph and let $\mathcal{P}(V)$ is the power set of V. The function $\mathcal{S} : \mathcal{P}(V) \to \mathcal{P}(V)$ defined by $\mathcal{S}(A) = A \cup \{v \in A^c/uv \in E, \forall u \in A\}$ is called the supra operator on $\mathcal{P}(V)$.*

Definition 3.2. *The G-supra topology μ_G on the vertex set V of a graph G is defined as the collection $\{\mathcal{S}(A) : A \subseteq V\}$. The elements of μ_G are called μ_G-open sets and its complements are called μ_G-closed sets.*

Theorem 3.1. ***(Properties of supra operator)*** *Let $G = (V, E)$ be a graph. Let A and B are subsets of V. Then,*

(i) $\mathcal{S}(\phi) = \phi$

(ii) $\mathcal{S}(V) = V$

(iii) $\mathcal{S}(A) \subseteq \mathcal{S}(B)$ *; when* $A \subseteq B$

(iv) $A \subseteq \mathcal{S}(A), \forall A \subseteq V$

(v) $\mathcal{S}(A \cup B) = \mathcal{S}(A) \cup \mathcal{S}(B)$

(v) $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A) \cap \mathcal{S}(B)$

Proof. (i) and (ii) it is obvious.
(iii) Let $A, B \subseteq V$ such that $A \subseteq B$.
Let $x \in \mathcal{S}(A)$. Then $x \in A \cup \{v \in A^c/uv \in E, \forall u \in A\}$
$\implies x \in A$ or $x \in \{v \in A^c/uv \in E, \forall u \in A\}$
$\implies x \in A$ or there exist $u \in A$ such that $uv \in E$
$\implies x \in B$ or there exist $u \in B$ such that $uv \in E$, since $A \subseteq B$
$\implies x \in B$ or $x \in \{v \in B^c/uv \in E, \forall u \in B\}$
$\implies x \in B \cup \{v \in B^c/uv \in E, \forall u \in B\} \implies x \in \mathcal{S}(B)$
Hence $\mathcal{S}(A) \subseteq \mathcal{S}(B)$
(iv) It is clear from the definition.
(v) Let $x \in \mathcal{S}(A \cup B)$
$\iff x \in (A \cup B) \cup \{v \in (A \cup B)^c/uv \in E, \forall u \in (A \cup B)\}$
$\iff x \in (A \cup B)$ or $x \in \{v \in (A \cup B)^c/uv \in E, \forall u \in (A \cup B)\}$
$\iff x \in A$ or $x \in B$ or there exist $u \in (A \cup B)$ such that $ux \in E$
$\iff x \in A$ or $x \in B$ or there exist $u \in A$ such that $ux \in E$ or
there exist $u \in B$ such that $ux \in E$
$\iff x \in A$ or $x \in B$ or $x \in \{v \in A^c/uv \in E, \forall u \in A\}$ or

$x \in \{v \in B^c/uv \in E, \forall u \in B\}$
$\iff x \in A$ or $x \in \{v \in A^c/uv \in E, \forall u \in A\}$
or $x \in B$ or $x \in \{v \in B^c/uv \in E, \forall u \in B\}$
$\iff x \in A \cup \{v \in A^c/uv \in E, \forall u \in A\}$ or $x \in B \cup \{v \in B^c/uv \in E, \forall u \in B\}$
$\iff x \in \mathcal{S}(A)$ or $x \in \mathcal{S}(B)$
$\iff x \in \mathcal{S}(A) \cup \mathcal{S}(B)$
Hence $\mathcal{S}(A \cup B) = \mathcal{S}(A) \cup \mathcal{S}(B)$.
(vi) $A \cap B \subseteq A$ and $A \cap B \subseteq B$
$\implies \mathcal{S}(A \cap B) \subseteq \mathcal{S}(A)$ and $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(B)$
$\implies \mathcal{S}(A \cap B) \subseteq \mathcal{S}(A) \cap \mathcal{S}(B)$ □

Theorem 3.2. *The collection $\mu_G = \{\mathcal{S}(A) : A \subseteq V\}$ forms a G-supra topology on the vertex set V of G.*

Proof. (i) Since $\mathcal{S}(\phi) = \phi$ and $\mathcal{S}(V) = V$. Hence ϕ, $V \in \mu_G$.
(ii) Let $\{G_i\}$ be any collection of open sets in μ_G. By definition, each $G_i = \mathcal{S}(A_i)$, where $A_i \subseteq V$.
$\therefore \bigcup_{i\in\Lambda} G_i = \bigcup_{i\in\Lambda} [\mathcal{S}(A_i)] = \mathcal{S}(\bigcup_{i\in\Lambda} A_i) = \mathcal{S}(A_k)$ [Since $\bigcup_{i\in\Lambda} A_i = A_k$; $A_k \subseteq V$]
$= G_k \in \mu_G$. Hence arbitrary union of μ_G-open sets is μ_G-open. □

Example 3.1. *Consider the graph $G = (V, E)$ in figure 1 with vertex set $V = \{a, b, c, d\}$. $\mu_G = \{\phi, \{a, b\}, \{a, c, d\}, V\}$ is a G-supra topology on the vertex set V of G.*

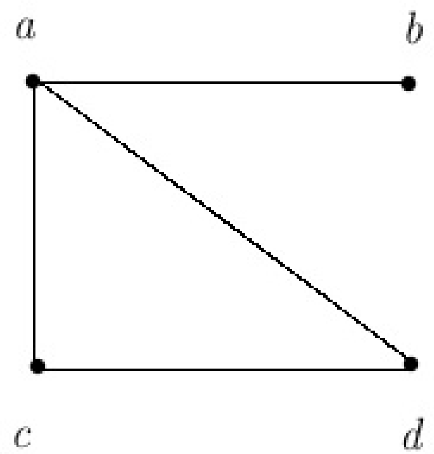

Figure 1: Graph for G-supra space

Proposition 3.1. *For a complete graph $G = (V, E)$, $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$, $\forall A \subseteq V$.*

Proof. Let $G = (V, E)$ be a complete graph. Since G is complete, each vertex is adjacent to every other vertices. Let $v \in V$. Then $\mathcal{S}(\{v\}) = V$, $\forall v \in V$. Then $\mathcal{S}(A) = V$, for every non empty subset A of V, which implies $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(V) = V = \mathcal{S}(A)$. Also $\mathcal{S}(\phi) = \phi \implies \mathcal{S}(\mathcal{S}(\phi)) = \mathcal{S}(\phi)$. Hence $\mathcal{S}(\mathcal{S}(A))$ = $\mathcal{S}(A), \forall A \subseteq V$. □

Proposition 3.2. *For a null graph $G = (V, E)$, $\mathcal{S}(A) = A$, $\forall A \subseteq V$.*

Proof. Let $G = (V, E)$ be a null graph with 'n' vertices. Since G is a null graph, each vertex is not adjacent to any other vertices. Let $v \in V$. Then $\mathcal{S}(\{v\}) = \{v\}$, $\forall v \in V$. Let $u, v \in V$. Then $\mathcal{S}(\{u, v\}) = \mathcal{S}(\{u\}) \cup \mathcal{S}(\{v\}) = \{u, v\}$. Similarly $\mathcal{S}(\{v_1, v_2, \cdots, v_n\}) = \mathcal{S}(\{v_1\}) \cup \mathcal{S}(\{v_2\}) \cup \cdots \cup \mathcal{S}(\{v_n\}) = \{v_1, v_2, \cdots, v_n\}$. Also $\mathcal{S}(\phi) = \phi$. Hence $\mathcal{S}(A) = A$, $\forall A \subseteq V$. □

Remark 3.1. *From the above proposition, for a null graph $G = (V, E)$, $\mathcal{S}(\mathcal{S}(A))$ = $\mathcal{S}(A)$, $\forall A \subseteq V$.*

Remark 3.2. *The following example shows that the property $\mathcal{S}(\mathcal{S}(A))$ = $\mathcal{S}(A)$, $\forall A \subseteq V$ does not hold true for all graphs G with vertex set V.*

Example 3.2. *Consider the graph G in figure 1, with vertex set $V = \{a, b, c, d\}$. Here $\mathcal{S}(\{b\}) = \{a, b\}$ and $\mathcal{S}(\{a, b\}) = V$. Hence $\mathcal{S}(\mathcal{S}(A)) \neq \mathcal{S}(A), \forall A \subseteq V$.*

Theorem 3.3. *A graph $G = (V, E)$ is complete if and only if $\mu_G = \{\phi, V\}$.*

Proof. Let $G = (V, E)$ be a complete graph. Then each vertex is adjacent to every other vertices. Also $\mathcal{S}(\phi) = \phi$ and $\mathcal{S}(A) = V$, for every non empty subset A of V. Hence ϕ and V are the only μ_G-open sets. Therefore a complete graph G generates an indiscrete topology. Conversely, let $\mu_G = \{\phi, V\}$. Then for each $A \subseteq V, \mathcal{S}(A) = \phi$ or $\mathcal{S}(A) = V$. When $A = \phi$, $\mathcal{S}(\phi) = \phi$. When $A = \{u\}, \mathcal{S}(\{u\}) = \{u\} \cup \{v \in A^c / uv \in E\} \neq \phi$. Therefore the only possibility is $\mathcal{S}(\{u\}) = V$, which means that u is adjacent to every other vertices in G. Since $u \in V$ is arbitrary, $\mathcal{S}(\{x\}) = V, \forall x \in V$. Hence each vertex is adjacent to every other vertices in G. Hence G is complete. □

Theorem 3.4. *$G = (V, E)$ is a null graph if and only if μ_G is discrete topology.*

Proof. Let $G = (V, E)$ be a null graph. By proposition 3.2, $\mathcal{S}(A) = A$, for every subset A of V. Hence every subset of V is a μ_G-open set. Therefore a null graph G generates discrete topology. Conversely, let μ_G be discrete topology. Then for each $x \in V, \{x\} \in \mu_G$. From the definition of $\mathcal{S}(A)$, x is not adjacent to any other vertex in G. Hence G is a null graph. □

Theorem 3.5. *Let $G = (V, E)$ be a disconnected graph with all its components are complete, then $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$, where A is a subset of V.*

Proof. Let $G_1, G_2, \cdots, G_n$ be the 'n' components of a disconnected graph G. Let V_i be the vertex set of G_i. Since each G_i is complete, $\mathcal{S}(\mathcal{S}(A_i)) = \mathcal{S}(A_i)$, for every subset A_i of V_i. Let A be a subset of V such that its elements belong to the components $G_1, G_2, \cdots, G_m$ where $m \leq n$. Now $\mathcal{S}(A) = A \cup \{v \in A^c/uv \in E, \forall u \in A\} = \mathcal{S}(V_1) \cup \mathcal{S}(V_2) \cup \cdots \cup \mathcal{S}(V_m)$(Since $u \in A \implies u \in V_i$, for anyone of i = 1, 2, $\cdots$,n) = $\mathcal{S}(\mathcal{S}(V_1)) \cup \mathcal{S}(\mathcal{S}(V_2)) \cup \cdots \cup \mathcal{S}(\mathcal{S}(V_m))$ = $\mathcal{S}(\mathcal{S}(V_1) \cup \mathcal{S}(V_2) \cup \cdots \cup \mathcal{S}(V_m))$ = $\mathcal{S}(\mathcal{S}(A))$. Hence for every subset A of V, $\mathcal{S}(\mathcal{S}(A)) = \mathcal{S}(A)$. □

Theorem 3.6. *Let $G = (V, E)$ be a disconnected graph with all its components are complete then the collection $\mu_G = \{\mathcal{S}(A) : A \subseteq V\}$ forms a topology τ_G on the vertex set V of G.(i.e) $\mu_G = \tau_G$.*

Proof. Let $G = (V, E)$ be a disconnected graph 'n' complete components. Let V_i be the vertex set of the component G_i, $i = 1, 2, \cdots, n$. By theorem 3.2, $\phi, V \in \tau_G$ and arbitrary union of μ_G-open sets is μ_G-open. Let $x \in V$, then $x \in V_i$, for some $i = 1, 2, \cdots, n$. Now $\mathcal{S}(\{x\}) = V_i, i = 1, 2, \cdots, n$. For any subsets A and B of V, we have $\mathcal{S}(A) \cup \mathcal{S}(B) = \mathcal{S}(A \cup B)$. Therefore for any non empty subset A of V, $\mathcal{S}(A) = V_i$ or $\mathcal{S}(A)$ = $\bigcup_{i\in\lambda} V_i$, $\lambda = \{1, 2, \cdots, n\}$. Suppose $\mathcal{S}(A)$ and $\mathcal{S}(B)$ are disjoint, then $\mathcal{S}(A) \cap \mathcal{S}(B) = \phi \in \mu_G$. Also when $\mathcal{S}(A) \subset \mathcal{S}(B)$, then $\mathcal{S}(A) \cap \mathcal{S}(B) = \mathcal{S}(A) \in \mu_G$ and when $\mathcal{S}(A) \cap \mathcal{S}(B) \neq \phi$ then $\mathcal{S}(A) \cap \mathcal{S}(B) = (\bigcup_{i\in\lambda} V_i) \cap (\bigcup_{j\in\lambda} V_j) = \bigcup_{k\in\lambda} V_j, \in \mu_G$. Therefore the finite intersection of elements of μ_G is in μ_G. Hence μ_G is a topology on the vertex set V of G. Hence $\mu_G = \tau_G$. □

Remark 3.3. *For null graph, complete graph and disconnected graph with complete components, the collection $\{\mathcal{S}(A) : A \subseteq V\}$ forms a topology. Also, for these type of graphs the supra operator obeys Kuratowski's closure axioms.*

4 Connectedness on G-supra topology

In this section, the concept of connectedness on graphs and G-supra space were compared.

Theorem 4.1. *If the graph G is disconnected then the G-supra topology μ_G generated by G is also disconnected.*

Proof. Let $G = (V, E)$ be a disconnected graph with 'n' components namely $G_1, G_2, \cdots, G_n$. Let $V_1, V_2, \cdots, V_n$ be the vertex set of $G_1, G_2, \cdots, G_n$ respectively. Clearly $V_1, V_2, \cdots, V_n$ are subsets of V and $V_1 \cup V_2 \cup \cdots \cup V_n = V$ and $V_1 \cap V_2 \cap \cdots \cap V_n = \phi$. Since each components G_i are connected, $\mathcal{S}(V_i) = V_i$, for $i = 1, 2, \cdots, n$. Also each V_i is a μ_G-open set, there exist separation of V. Hence (μ_G, V) is disconnected. □

Remark 4.1. *If the graph G is connected then the G-supra topology μ_G generated by G need not be connected, which can be illustrated by the following example.*

Example 4.1. *Consider the connected graph $G = (V, E)$ in figure.2 with vertex set $V = \{a, b, c, d\}$. The G-supra topology generated by the above graph G is*

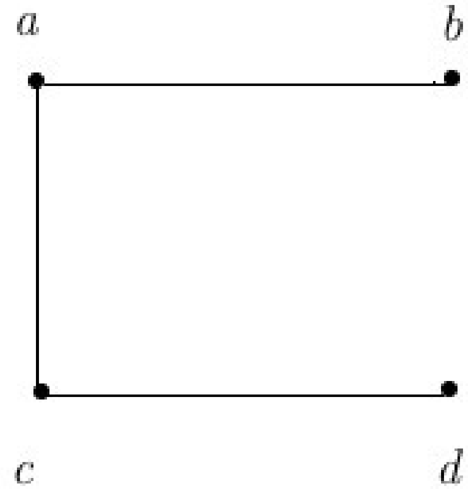

Figure 2: Graph for disconnected G-supra space

$\mu_G = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, V\}$*, which is disconnected.*

Proposition 4.1. *The G-supra topology μ_G generated by a totally disconnected graph is also totally disconnected.*

Proof. Let $G = (V, E)$ be a totally disconnected graph. Then $\mathcal{S}(A) = A, \forall A \subseteq V$. Hence the supra topology μ_G generated by G is discrete. Since every discrete topology is totally disconnected, μ_G is totally disconnected.□

5 Separation axioms in G-supra space

In this section, G-supra T_0-space, T_1-space and T_2-space were defined and illustrated with examples.

Definition 5.1. *Let μ_G be a G-supra topology generated by the vertex set V of a graph G. (V, μ_G) is called G-supra T_0-space if for every pair of distinct vertices there exist a μ_G-open set containing one but not other.*

Definition 5.2. *Let μ_G be a G-supra topology generated by the vertex set V of a graph G. (V, μ_G) is called G-supra T_1-space if for every pair of distinct vertices $u, v \in V$ there exist two μ_G-open sets X and Y such that $u \in X$ but $v \notin X$ and $v \in Y$ but $u \notin Y$.*

Example 5.1. *Let G be a graph in figure 3 with vertex set $V = \{a, b, c\}$. The G-*

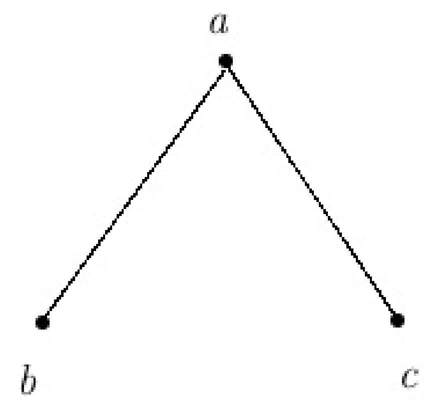

Figure 3: Graph for G-supra T_0-space

supra topology μ_G on the vertex set V of G is $\mu_G = \{\phi, \{a, b\}, \{a, c\}, V\}$. Since for each pair of distinct vertices, there exist a μ_G-open set containing one but not other. Hence (V, μ_G) is a G-supra T_0-space.

Example 5.2. *Let G be a graph in figure 4 with vertex set $V = \{a, b, c, d\}$. The*

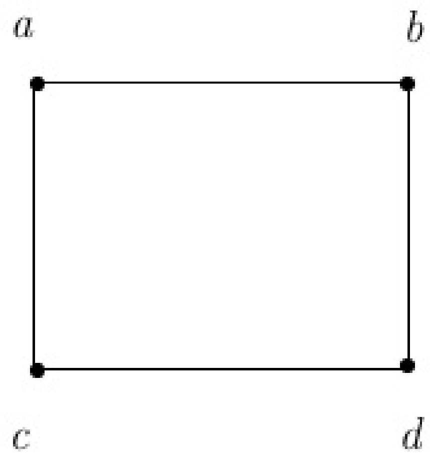

Figure 4: Graph for G-supra T_1-space

G-supra topology μ_G on the vertex set V of G is $\mu_G = \{\phi, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, V\}$. Since for each pair of distinct vertices $u, v \in V$, there exist two μ_G-open sets X and Y such that $u \in X$ but $v \notin X$ and $v \in Y$ but $u \notin Y$. Hence (V, μ_G) is a G-supra T_1-space.

Definition 5.3. *Let μ_G be a G-supra topology generated by the vertex set V of a graph G. (V, μ_G) is called G-supra T_2-space if for every pair of distinct vertices $u, v \in V$ there exist disjoint μ_G-open sets X and Y such that $u \in X$ and $v \in Y$.*

Example 5.3. *Let G be a graph in figure 5, with vertex set $V = \{a, b, c, d, e, f\}$.*

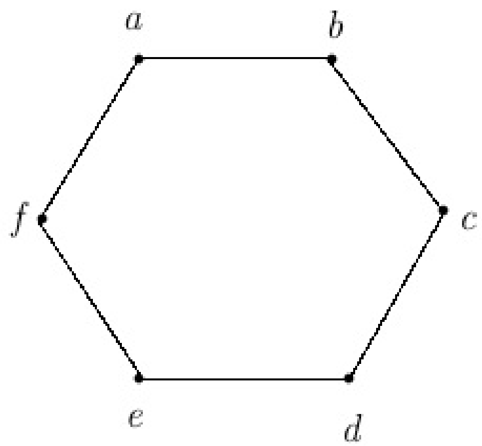

Figure 5: Graph for G-supra T_2-space

The G-supra topology μ_G on the vertex set V of G is $\mu_G = \{\phi, \{a, b, c\}, \{a, b, f\}, \{a, e, f\}, \{b, c, d\}, \{c, d, e\}, \{d, e, f\}, \{a, b, c, d\}, \{a, b, c, f\}, \{a, b, e, f\}, \{a, d, e, f\}, \{b, c, d, e\}, \{c, d, e, f\}, \{a, b, c, d, e\}, \{a, b, c, d, f\}, \{a, b, c, e, f\}, \{a, b, d, e, f\}, \{a, c, d, e, f\}, \{b, c, d, e, f\}, V\}$. Since for each pair of distinct vertices $u, v \in V$ there exist disjoint μ_G-open sets X and Y such that $u \in X$ and $v \in Y$. Hence (V, μ_G) is a G-supra T_2-space.

Theorem 5.1. *Let $G = C_n$, $n \geq 6$ be a cycle graph, then (V, μ_G) is a G-supra T_2-space.*

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$, $n \geq 6$ be the vertex set of G. Then $\mathcal{S}(\{v_i\}) = \{v_{i-1}, v_i, v_{i+1}\}$, for i = 1, 2, $\cdots$, n. Since G is a cycle, for $i = 1, v_{i-1} = v_n$ and for $i = n, v_{i+1} = v_1, v_{i+2} = v_2$ and so on. Now $\mathcal{S}(\{v_i\}) = \{v_{i-1}, v_i, v_{i+1}\}$ and $\mathcal{S}(\{v_{i+3}\}) = \{v_{i+2}, v_{i+3}, v_{i+4}\}$ are disjoint μ_G-open sets seperates 9 pairs of dictinct vertices of G. Hence there exist disjoint μ_G-open sets X and Y for every pair of distinct vertices v_i and v_j such that $v_i \in X$ and $v_j \in Y$. Hence (V, μ_G) is a G-supra T_2-space. □

Remark 5.1. *For C_3, C_4 and C_5, there does not exist disjoint μ_G-open sets for any pair of distinct vertices.*

6 Infra operator on graphs

In this section, G-infra topology was induced by the infra operator and its properties were studied.

Definition 6.1. *Let $G = (V, E)$ be a graph and let $\mathcal{P}(V)$ is the power set of V. The function $\mathcal{I} : \mathcal{P}(V) \to \mathcal{P}(V)$ defined by $\mathcal{I}(A) = [\mathcal{S}(A^c)]^c$is called the infra operator on $\mathcal{P}(V)$.*

Definition 6.2. *The G-infra topology δ_G on the vertex set V of a graph G is defined as the collection $\{\mathcal{I}(A) : A \subseteq V\}$. The elements of δ_G are called δ_G-open sets and its complemets are called δ_G-closed sets.*

Theorem 6.1. ***(Properties of infra operator)*** *Let $G = (V, E)$ be a graph. Let A and B are subsets of V. Then,*

(i) $\mathcal{I}(\phi) = \phi$

(ii) $\mathcal{I}(V) = V$

(iii) $\mathcal{I}(A) \subseteq \mathcal{I}(B)$ *; when* $A \subseteq B$

(iv) $\mathcal{I}(A) \subseteq A, \forall A \subseteq V$

(v) $\mathcal{I}(A \cap B) = \mathcal{I}(A) \cap \mathcal{I}(B)$

(vi) $\mathcal{I}(A) \cup \mathcal{I}(B) \subseteq \mathcal{I}(A \cup B)$

Proof. (i) $\mathcal{I}(\phi) = [\mathcal{S}(\phi^c)]^c = [\mathcal{S}(V)]^c = V^c = \phi$.
(ii) $\mathcal{I}(V) = [\mathcal{S}(V^c)]^c = [\mathcal{S}(\phi)]^c = \phi^c = V$.
(iii) Let $A, B \subseteq V$ such that $A \subseteq B$.
Let $x \in \mathcal{I}(A)$. Then $x \in [\mathcal{S}(A^c)]^c$. $\implies x \notin \mathcal{S}(A^c)$.
Since $A \subseteq B, B^c \subseteq A^c \implies \mathcal{S}(B^c) \subseteq \mathcal{S}(A^c)$.
$\therefore x \notin \mathcal{S}(B^c) \implies x \in [\mathcal{S}(B^c)]^c \implies x \in \mathcal{I}(B)$. Hence $\mathcal{S}(A) \subseteq \mathcal{I}(B)$.
(iv) Let $x \in \mathcal{I}(A)$. Then $x \in [\mathcal{S}(A^c)]^c \implies x \notin \mathcal{S}(A^c) \implies x \notin A^c$
$\implies x \in A$. Hence $\mathcal{I}(A) \subseteq A$.
(v) $\mathcal{I}(A \cap B) = [\mathcal{S}(A \cap B)^c]^c = [\mathcal{S}(A^c \cup B^c)]^c = [\mathcal{S}(A^c) \cup \mathcal{S}(B^c)]^c = [\mathcal{S}(A^c)]^c \cap [\mathcal{S}(B^c)]^c = \mathcal{I}(A) \cap \mathcal{I}(B)$.
(vi) $A \subseteq A \cup B$ and $B \subseteq A \cup B$
$\implies \mathcal{I}(A) \subseteq \mathcal{I}(A \cup B)$ and $\mathcal{I}(B) \subseteq \mathcal{I}(A \cup B)$
$\implies \mathcal{I}(A) \cup \mathcal{I}(B) \subseteq \mathcal{I}(A \cup B)$. □

Theorem 6.2. *The collection $\delta_G = \{\mathcal{I}(A) : A \subseteq V\}$ forms a G-infra topology on the vertex set V of G.*

Proof. (i) Since $\mathcal{I}(\phi) = \phi$ and $\mathcal{I}(V) = V$. Hence $\phi, V \in \delta_G$.
(ii) Let $\{G_i\}$ be a collection of open sets in δ_G. By definition each $G_i = \mathcal{I}(A_i)$, where $A_i \subseteq V$.
$\therefore \bigcap_{i\in\Lambda} G_i = \bigcap_{i\in\Lambda} [\mathcal{I}(A_i)] = \mathcal{I}(\bigcap_{i\in\Lambda} A_i) = \mathcal{I}(A_k)$ [Since $\bigcap_{i\in\Lambda} A_i = A_k$; $A_k \subseteq V$]
$= G_k \in \mu_G$. Hence intersection of δ_G-open sets is δ_G-open. □

Example 6.1. *Consider the graph $G = (V, E)$ in figure 1.*
$\delta_G = \{\phi, \{b\}, \{c, d\}, V\}$ is a G-infra topology on the vertex set V of G.

Proposition 6.1. *For a complete graph $G = (V, E)$, $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$, $\forall A \subseteq V$.*

Proof. $\mathcal{I}(\phi) = \phi \implies \mathcal{I}(\mathcal{I}(\phi)) = \mathcal{I}(\phi)$. Since G is a complete graph, $\mathcal{S}(A) = V$, $\forall A \subset V$. Let A be a proper subset of V. Then $\mathcal{I}(A) = [\mathcal{S}(A^c)]^c = V^c = \phi \implies \mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(\phi) = \phi = \mathcal{I}(A)$. Also $\mathcal{I}(V) = V \implies \mathcal{I}(\mathcal{I}(V)) = \mathcal{I}(V)$. Hence $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$, $\forall A \subseteq V$. □

Proposition 6.2. *For a null graph $G = (V, E)$, $\mathcal{I}(A) = A$, $\forall A \subseteq V$.*

Proof. Let $G = (V, E)$ be a null graph with 'n' vertices. Since $\mathcal{S}(A) = A, \forall A \subseteq V$. Now $\mathcal{I}(A) = [\mathcal{S}(A^c)]^c = [A^c]^c$ = A, $\forall A \subseteq V$. □

Remark 6.1. *From the above proposition, for a null graph $G = (V, E)$, $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$, $\forall A \subseteq V$.*

Remark 6.2. *The following example shows that the property $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$, $\forall A \subseteq V$ does not hold for all graphs G with vertex set V.*

Example 6.2. *Consider the graph G in figure 1, with vertex set $V = \{a, b, c, d\}$. Here $\mathcal{I}(\{a, c, d\}) = \{c, d\}$ and $\mathcal{I}(\{c, d\}) = \phi$. Hence $\mathcal{I}(\mathcal{I}(A) \neq \mathcal{I}(A)$, $\forall A \subseteq V$.*

Theorem 6.3. *Let $G = (V, E)$ be a disconnected graph with all its components are complete, then $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$, where A is a subset of V.*

Proof. Let $G_1, G_2, \cdots, G_n$ be the 'n' components of a disconnected graph G. Let V_i be the vertex set of G_i. Since each G_i is complete, $\mathcal{I}(\mathcal{I}(A_i)) = \mathcal{I}(A_i)$, for every subset A_i of V_i. Let A be a sub set of V.
Case(i). Suppose $A = V_1 \cup V_2 \cup \cdots \cup V_m, m \leq n$. Then $\mathcal{I}(A) = [\mathcal{S}(V_1 \cup V_2 \cup \cdots \cup V_m)^c]^c = [\mathcal{S}(V_{m+1} \cup V_{m+2} \cup \cdots \cup V_n)]^c = [\mathcal{S}(V_{m+1}) \cup \mathcal{S}(V_{m+1}) \cup \cdots \cup \mathcal{S}(V_n)]^c = [V_{m+1} \cup V_{m+2} \cup \cdots \cup V_n]^c = V_1 \cup V_2 \cup \cdots \cup V_m = A$. $\therefore \mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$.
Case(ii). Suppose $A = U_1 \cup U_2 \cup \cdots \cup U_m$, where $U_j \subseteq V_j$, $j = 1, 2, \cdots, m$ and

$m \leq n$. Then $\mathcal{I}(A) = \mathcal{I}(U_1 \cup U_2 \cup \cdots \cup U_m) = [\mathcal{S}(U_1 \cup U_2 \cup \cdots \cup U_m)^c]^c$
= $[\mathcal{S}\{(V_1 - U_1) \cup (V_2 - U_2) \cup \cdots (V_m - U_m) \cup V_{m+1} \cup V_{m+2} \cup \cdots \cup V_n\}]^c$
=$[\mathcal{S}(V_1 - U_1) \cup \mathcal{S}(V_2 - U_2) \cup \cdots \mathcal{S}(V_m - U_m) \cup \mathcal{S}(V_{m+1}) \cup \mathcal{S}(V_{m+2}) \cup \cdots \cup \mathcal{S}(V_n)]^c$
=$[\mathcal{S}(W_1) \cup \mathcal{S}(W_2) \cup \cdots \mathcal{S}(W_m) \cup \mathcal{S}(V_{m+1}) \cup \mathcal{S}(V_{m+2}) \cup \cdots \cup \mathcal{S}(V_n)]^c$
$[whereW_j = V_j - U_j; j = 1, 2, \cdots, m.$ Also $W_j \subset V_j]$
= $[V_1 \cup V_2 \cup \cdots \cup V_m \cup V_{m+1} \cup \cdots \cup V_n]^c = V^c = \phi$.
Hence $\mathcal{I}(A) = \phi \implies \mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(\phi) = \phi = \mathcal{I}(A)$. $\therefore \mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$.
Case(iii). Suppose $A = V_1 \cup V_2 \cup \cdots \cup U_{k+1} \cup U_{k+2} \cup \cdots \cup U_m$,
where $k < m \leq n$ and $U_j \subset V_j, j = k+1, k+2, \cdots, m$. Then
$\mathcal{I}(A) = \mathcal{I}(V_1 \cup V_2 \cup \cdots \cup U_{k+1} \cup U_{k+2} \cup \cdots \cup U_m) = [\mathcal{S}(V_1 \cup V_2 \cup \cdots \cup U_{k+1} \cup U_{k+2} \cup \cdots \cup U_m)^c]^c = [\mathcal{S}\{(V_{k+1} - U_{k+1}) \cup \mathcal{S}\{(V_{k+2} - U_{k+2}) \cdots \cup (V_m - U_m) \cup V_{m+1} \cup V_{m+2} \cup \cdots \cup V_n\}]^c = [\mathcal{S}\{W_{k+1} \cup W_{k+2} \cup \cdots \cup W_m \cup V_{m+1} \cup V_{m+2} \cup \cdots \cup V_n\}]^c$, $[where$ $W_j = V_j - U_j, j = k+1, k+2, \cdots, m.$ Since $W_j \subset V_j \implies \mathcal{S}(W_j) = V_j]$.Now
$\mathcal{I}(A) = [V_{k+1} \cup V_{k+2} \cup \cdots \cup V_m \cup V_{m+1} \cup \cdots \cup V_n]^c = V_1 \cup V_2 \cup \cdots \cup V_k$.
Hence $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(V_1 \cup V_2 \cup \cdots \cup V_k) = [\mathcal{S}(V_1 \cup V_2 \cup \cdots \cup V_k)^c]^c = [\mathcal{S}(V_{k+1} \cup V_{k+2} \cup \cdots \cup V_m \cup V_{m+1} \cup \cdots \cup V_n)]^c = [\mathcal{S}(V_{k+1}) \cup \mathcal{S}(V_{k+1}) \cup \cdots \cup \mathcal{S}(V_m) \cup \mathcal{S}(V_{m+1}) \cup \cdots \mathcal{S}(V_n)]^c = [V_{k+1} \cup V_{k+2} \cup \cdots \cup V_m \cup V_{m+1} \cup \cdots V_n]^c = V_1 \cup V_2 \cup \cdots \cup V_k = \mathcal{I}(A)$.
$\therefore \mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$. □

Theorem 6.4. *Let $G = (V, E)$ be a a graph and let $A \subseteq V$, then the collections $\mu_G = \{\mathcal{S}(A)\}$ and $\delta_G = \{\mathcal{I}(A)\}$ are complement to each other.*

Proof. Let A be the subset of the vertex set V of G. Put $A^c = B$. Since $A, B \subseteq V$, $\mathcal{S}(A)$, $\mathcal{S}(B) \in \mu_G$. Now, for $\mathcal{S}(B) \in \mu_G$, we have $[\mathcal{S}(B)]^c = [\mathcal{S}(A^c)]^c = \mathcal{I}(A) \in \delta_G$. Also for $\mathcal{I}(B) \in \delta_G$, we have $[\mathcal{I}(B)]^c$ = $[[\mathcal{S}(B^c)]^c]^c = [[\mathcal{S}(A)]^c]^c = \mathcal{S}(A) \in \mu_G$. Hence every μ_G-open set is the complement of δ_G-open set and vice-versa. □

Theorem 6.5. *Let $G = (V, E)$ be a disconnected graph with all its components are complete then the collection $\delta_G = \{\mathcal{I}(A) : A \subseteq V\}$ forms a topology τ_G on the vertex set V of G. (i.e) $\delta_G = \tau_G$.*

Proof. By theorem 6.2, $\phi, V \in \delta_G$ and intersection of finite number of δ_G-open sets is δ_G-open. Now by theorem 6.4, every δ_G-open set is μ_G-closed set. Also by theorem 3.6, finite intersection of μ_G-open sets is μ_G-open. This gives arbitrary union of δ_G-open sets is δ_G-open. □

Theorem 6.6. *A graph $G = (V, E)$ is complete if and only if $\delta_G = \{\phi, V\}$.*

Proof. Clearly, by theorem 3.3 and theorem 6.4. □

Theorem 6.7. *$G = (V, E)$ is a null graph if and only if δ_G is discrete topology.*

Proof. Clearly, by theorem 3.4 and theorem 6.4. □

Remark 6.3. *For null graph, complete graph and disconnected graph with complete components, the collection $\{\mathcal{I}(A) : A \subseteq V\}$ forms a topology. Also, for these type of graphs the infra operator obeys the axioms of interior operator.*

7 Connectedness on G-infra space

In this section, we have compared the concept of connectedness on graphs and G-supra space.

Theorem 7.1. *If the graph G is disconnected then the G-infra topology δ_G generated by G is also disconnected.*

Proof. Let $G = (V, E)$ be a disconnected graph with 'n' components namely $G_1, G_2, \cdots, G_n$. Let $V_1, V_2, \cdots, V_n$ be the vertex set of $G_1, G_2, \cdots, G_n$ respectively. Clearly $V_1, V_2, \cdots, V_n$ are subsets of V and $V_1 \cup V_2 \cup \cdots \cup V_n = V$ and $V_1 \cap V_2 \cap \cdots \cap V_n = \phi$.–(*). Now $\mathcal{I}(V_i) = [\mathcal{S}(V_i)^c]^c = [\mathcal{S}(V_1 \cup V_2 \cup \cdots \cup V_{i-1} \cup V_{i+1} \cup \cdots \cup V_n)]^c$ = $[\mathcal{S}(V_1) \cup \mathcal{S}(V_2) \cup \cdots \cup \mathcal{S}(V_{i-1}) \cup \mathcal{S}(V_{i+1}) \cdots \cup \mathcal{S}(V_n)]^c = [V_1 \cup V_2 \cup \cdots \cup V_{i-1} \cup V_{i+1} \cup \cdots \cup V_n]^c = V_i$. Since each $\mathcal{I}(V_i) = V_i$, for $i = 1, 2, \cdots, n$ is a $\delta_G - openset$, (δ_G, V) is disconnected (by*). □

Remark 7.1. *If the graph G is connected then the G-infra topology δ_G generated by G need not be connected, which can be illustrated by the following example.*

Example 7.1. *Consider the connected graph $G = (V, E)$ in figure.2 with vertex set $V = \{a, b, c, d\}$. The G-infra topology generated G is $\delta_G = \{\phi, \{b\}, \{d\}, \{a, b\}, \{c, d\}, V\}$, which is disconnected.*

Proposition 7.1. *The G-infra topology δ_G generated by a totally didconnected graph is also totally disconnected.*

Proof. Let $G = (V, E)$ be a totally disconnected graph. Then $\mathcal{I}(A) = A, \forall A \subseteq V$. Hence the infra topology δ_G generated by G is a discrete. Since every discrete topology is totally disconnected, δ_G is totally disconnected. □

8 Separation axioms in G-infra space

In this section, we have defined and discussed the separation axioms in G-infra space.

Definition 8.1. *Let δ_G be a G-infra topology generated by the vertex set V of a graph G. (V, δ_G) is called G-infra T_0-space if for every pair of distinct vertices there exist a δ_G-open set containing one but not other.*

Example 8.1. *Consider the graph G in figure 2. The G-infra topology δ_G on the vertex set V of G is $\delta_G = \{\phi, \{b\}, \{d\}, \{a, b\}, \{c, d\}, V\}$. Since for each pair of distinct vertices, there exist a δ_G-open set containing one but not other. Hence (V, δ_G) is a G-infra T_0-space.*

Definition 8.2. *Let δ_G be a G-infra topology generated by the vertex set V of a graph G. (V, δ_G) is called G-infra T_1-space if for every pair of distinct vertices $u, v \in V$ there exist two δ_G-open sets X and Y such that $u \in X$ but $v \notin X$ and $v \in Y$ but $u \notin Y$.*

Example 8.2. *Consider the graph G in figure 6, with vertex set $V = \{a, b, c, d\}$. The G-infra topology δ_G on the vertex set V of G is $\delta_G\{\phi, \{b\}, \{c\}, \{d\}, \{b, d\},$*

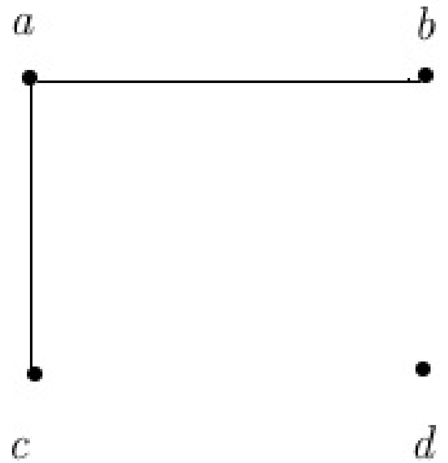

Figure 6: Graph generates G-infra T_1-space

$\{c, d\}, \{a, b, c\}, V\}$. Since for each pair of distinct vertices $u, v \in V$, there exist two δ_G-open sets X and Y such that $u \in X$ but $v \notin X$ and $v \in Y$ but $u \notin Y$. Hence (V, δ_G) is a G-infra T_1-space.

Definition 8.3. *Let δ_G be a G-infra topology generated by the vertex set V of a graph G. (V, δ_G) is called G-infra T_2-space if for every pair of distinct vertices $u, v \in V$ there exist disjoint δ_G-open sets X and Y such that $u \in X$ and $v \in Y$.*

Example 8.3. *Consider the graph G in figure 4. The G-infra topology δ_G on the vertex set V of G is $\delta_G = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, V\}$. Since for each pair of distinct vertices $u, v \in V$ there exist disjoint δ_G-open sets X and Y such that $u \in X$ and $v \in Y$. Hence (V, δ_G) is a G-infra T_2-space.*

Theorem 8.1. *Let $G = C_n$, $n \geq 4$ be a cycle graph, then (V, μ_G) is a G-infra T_2-space.*

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$, $n \geq 4$ be the vertex set of G. Let $A = \{v_n, v_1, v_2\}$. Then $\mathcal{I}(A) = [\mathcal{S}(A^c)]^c = [\mathcal{S}(\{v_3, v_4, \cdots, v_{n-1}\})]^c = [\{v_2, v_3, \cdots, v_n\}]^c = \{v_1\}$. For $i = 2, 3, \cdots, (n-1)$, $\mathcal{I}(\{v_{i-1}, v_i, v_{1+1}\}) = [\mathcal{S}(\{v_{i-1}, v_i, v_{1+1}\}^c)]^c = [\mathcal{S}(\{v_1, v_2, \cdots, v_{i-2}, v_{i+2}, \cdots, v_n\})]^c = [\{v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n\}]^c = \{v_i\}$. When $A = \{v_{n-1}, v_n, v_{n+1}\}$, $\mathcal{I}(A) = [\mathcal{S}(A^c)]^c = [\mathcal{S}(\{v_2, v_3, \cdots, v_{n-2}\})]^c = [\{v_1, v_2, \cdots, v_{n-1}\}]^c = \{v_n\}$. Hence each $\{v_i\}$, for $i = 1, 2, \cdots, n$ is a δ_G-open set. Therefore for every pair of distinct vertice $v_i, v_j \in V$ there exist disjoint δ_G-open sets $\{v_i\}$ and $\{v_j\}$ containing them respectively. Hence (V, μ_G) is a G-infra T_2-space. □

9 Conclusion

By employing supra and infra operator we have generated G-supra and G-infra topologies. Some properties of this operators were studied. Also we have proved that supra and infra operators obeys the criterion Kuratowski's closure axioms and interior operator respectively in some special types of graph namely null graph, complete graph and disconnected graph with all its components are complete. Also we have proved that the G-supra and G-infra topologies are complement to each other. In addition we have discussed the connectedness and separation axioms in G-supra and G-infra spaces. In future, this study can be extended on some other weaker forms of topology and also on directed graphs.

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