# Numerical solution of pollutant transport model with source term by RDTM

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**Abstract**

The focus of this paper is the study of mathematical model for the analysis of diffusion and transport processes of chemicals in river systems. In the present study, we have introduced a one-dimensional model that is described by time-dependent convection-diffusion differential equations. The focus of our study is the potential pollution of the river, both with and without a specific pollution source. The current investigation aims to analyze the impact of two different input source functions, namely constant and linear forms. In order to validate the model, we conducted a study on the diffusion and transport of chemicals, specifically focusing on $NO_3$ and $PO_4$ of the Tapi river. Reduced differential transform method (RDTM) is used to obtain solutions. For the accuracy of the solution, the convergence of solution function is examined in each case obtained from RDTM. The pollutant concentration at various distances and time levels has been shown numerically and graphically in each case. The pollution levels with and without sources are compared in 2D and 3D graphs. The methodology proposed in this study for river pollution prediction using a 1D pollution model can be applied to other rivers.

**Keywords**: Source, Pollutant transport equation; Reduced differential transform method; Convergence

**2020 AMS subject classifications**: 35A22,35C10, 35G05,35G10 [1]

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# 1 Introduction

Water, an essential element of the environment, plays a vital part in various ecological processes. However, the quality of both surface water and groundwater has been steadily declining over time. This degradation can be attributed to natural factors and human activities. Various natural factors influence water quality, including hydrological, atmospheric, climatic, topographical, and lithological factors. These factors play a significant role in shaping the overall quality of water in various aquatic systems. Hydrological factors, such as the flow rate and volume of water, can impact water quality by affecting the transport and dilution of pollutants. Atmospheric factors, such as temperature and precipitation, can influence water quality through the deposition of airborne pollutants and the alteration of water chemistry. Climatic factors, including seasonal variations and long-term climate patterns, can also impact water quality by influencing factors such as [Magesh et al., 2013, Kut et al., 2019].

In the modern era, there has been a conspicuous increase in the difficulties encountered by developing countries in maintaining water quality while attempting to improve water supply and sanitation. Many wealthy countries have faced substantial obstacles recently in their attempts to maintain or improve the quality of their water supplies. These difficulties are generally caused by problems like eutrophication and nutrient enrichment, which have negative impacts on the quality of the water. Additionally, these countries have had significant difficulties in meeting the needs of their expanding populations for water and wastewater services [Ortega et al., 2016, Abbasi and Abbasi, 2012].

The application of partial differential equations (PDEs) has been prevalent in formulating a diverse array of applied problems due to the inherent relationship between real-world phenomena and these equations. In numerous practical scenarios, such as the advection-diffusion equation, a significant partial differential equation (PDE) commonly observed in groundwater flows, the availability of ideal information is seldom encountered. The accuracy of our information regarding the soil's permeability, the source term's magnitude, and inflow or outflow conditions is questionable [Mirzaee et al., 2021].

Multiple methods approach the heat equation and advection–diffusion equation. The octic B-spline collocation, Exponential B-spline, Finite element and operator splitting methods improve the approximate solution of the heat and advection–diffusion equation by taking into consideration the inadequacies of previous numerical studies with high error accuracy. MATLAB simplifies tiresome computations [Jena and Gebremedhin, 2021, Mohammadi, 2013, Mojtabi and Deville, 2015, Bahar and Gürarslan, 2017].

The primary aim of the present study is to comprehensively integrate various components, including mathematical modelling, transformation methods, approximations, and computer implementation, to investigate the time-dependent transport of substances in a water medium during the eutrophication process. The implementation of the proposed mathematical models and methods can be easily accomplished using the collected initial data. The model that has been developed offers an effective model for expeditious assessment of water quality in river systems. Furthermore, this concept can be extended to other disciplines within science and engineering. A model has been developed to analyze the distribution of polluting substances in rivers, considering multiple sources and types of pollutants. One-dimensional advection-diffusion models have been created. The presence of pollution sources in the studied river section is hypothesized. These sources may include point or volume sources such as pipe outlets used by industries to release industrial wastes into the river, underground sources, or other rivers that flow into the section.

For the present study, the water samples are collected from the Tapi river, Surat, India to develop the mathematical model for predicting pollutant level at different distance and time only.

In section 3, we describe the mathematical formulation of the problem. Section 4 presents a comprehensive overview of the fundamental concepts that underpin the RDTM. Section 5 contains the detailed process of converging the analytic series solution proposed by RDTM. Section 6 discusses the numerical outcomes and the method's convergence to evaluate its effectiveness. The utilization of 2D and 3D plots provides a visual depiction of the solutions that have been acquired. In section 7, a concise overview of the conclusion is presented.

## 2 Water Sample Collection

This investigation examines the Tapi river from the Pal-Umra bridge to the ONGC bridge in Surat. On November 12, 2022, seven 1 Litre capacity samples were taken from the beginning point of the study area in identically sized bottles. Using these water samples, $NO_3$ and $PO_4$ water pollutants concentration were analysed. This gathered information was transmitted to Pollucon Laboratories Pvt. Ltd., Nr. Navjivan Circle, Surat, Gujarat, for $NO_3$(Nitrate) and $PO_4$(Phosphate) pollutants concentration. Figure 1 show some pictures of research area.

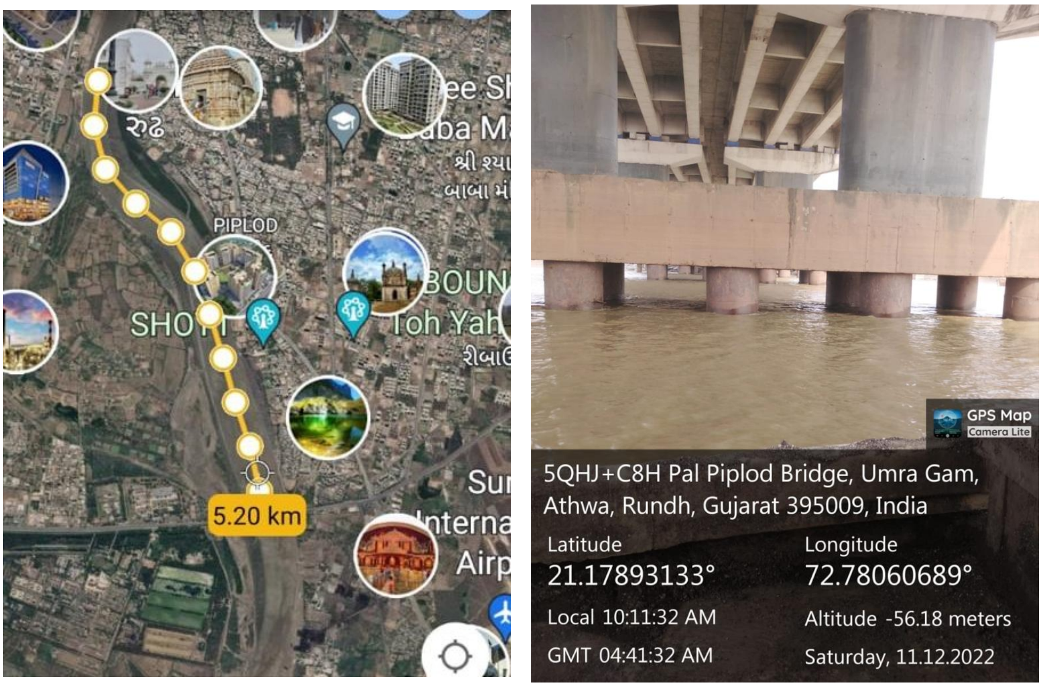


Figure 1: Research Area

| $x^*(km)$ | $NO_3$ concentration $(mg/L)$ | $PO_4$ concentration $(mg/L)$ |
|---|---|---|
| 0 | 0.59 | 2.81 |
| 0.5 | 1.01 | 2.73 |
| 1 | 0.97 | 3.88 |
| 1.5 | 1.25 | 3.92 |
| 2.5 | 0.21 | 2.78 |
| 3.5 | 0.57 | 2.9 |
| 5 | 2.48 | 2.91 |

Table 1: Initial data of pollutant's concentration

Table 1 give numerical values of initial concentration of $NO_3$ and $PO_4$ pollutants in research area.

# 3 Mathematical Formulation of the problem

The entry of pollutants into the subsurface is primarily facilitated by two mechanisms. The first mechanism is known as advection, which occurs due to the movement of groundwater. The second mechanism is dispersion, which is caused by a combination of mechanical mixing and molecular diffusion. The impact of molecular diffusions is often disregarded in the context of small seepage velocities [Yadav and Kumar, 2019]. The advection-diffusion equation in one dimension can be represented mathematically as a second order partial differential equation of parabolic nature. It can be expressed as:

$$A^* \frac{\partial c}{\partial t^*} = \frac{\partial}{\partial x^*} \left( A^* D_x^* \frac{\partial c}{\partial x^*} \right) - A^* v^* \frac{\partial c}{\partial x^*} - A^* kc + A^* f^* \tag{1}$$

where $c$ $[ML^{-3}]$is the pollutant's solute concentration at any point $x^*$ $[L]$ and time $t^*$ $[T]$while it is being transported through the medium along the flow field. $D_x^*$ $[L^2T^{-1}]$ is Longitudinal Diffusion coefficient. $v^*$ $[LT^{-1}]$ is the unsteady uniform seepage flow velocity along longitudinal direction. $k$ $[T^-1]$ is coefficient of non-conservation. $A^*$ $[L^2]$ is Flow area of cross section of river. $f^*$ $[ML^{-3}T^{-1}]$ is Source term. $x^*$ is longitudinal distance from starting point of research area. $t^*$ is Time dimension.The area of the river's cross-section is treated as a constant throughout the study and $k = 0$ [Kachiashvili et al., 2007]. Then equation (1) convert in the given form:

$$\frac{\partial c}{\partial t^*} = D_x^* \frac{\partial^2 c}{\partial x^{*2}} - v^* \frac{\partial c}{\partial x^*} + f^* \tag{2}$$

The presentation of the non-dimensional variables [Rubbab et al., 2016]:

$$x = \frac{x^*}{L}, t = \frac{D_x^* t^*}{L^2}, v = \frac{v^* L}{D_x^*}, S = \frac{L^2 f}{D_x^*} \tag{3}$$

Here $L$ is the length of research portion of river. We derive dimensional less advection-diffusion equation as

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - v \frac{\partial c}{\partial x} + S(x,t) \tag{4}$$

To analyze this problem, we have used the concentration of pollutant substances $NO_3$ and $PO_4$ of river Tapi. In this paper, We have taken $5\,km$ Research area of tapi river from ONGC bridge to Pal Umra bridge. So here, river's length($L$) is $5\,km$. Using collected data of pollutants $NO_3$ and $PO_4$ concentration, at time $t = 0$ (any fixed time), the concentration is assumed as a polynomial form with good statistical indices.

In this article, the concentration level of contamination is discussed for both without and with various kind of input sources. Various cases have been examined to determine the concentration function's value and visual depiction. We're looking at two distinct sorts of known sources to see the pollution level in river. Here, $S$ denotes the rate of the pollutant that enters the beginning border of the portion of the river that is being examined per unit of time and per unit of volume. According to the value of $S(x,t)$, these two types of input sources include constant and linear source.

Initial function (5) for $NO_3$ pollutant concentration function

$$c(x,0) = -1.4880e{-}05\,x^5 + 8.1728e{-}04\,x^4 - 0.0139\,x^3 + 0.0728\,x^2 + 0.0045\,x + 0.6140 \tag{5}$$

Initial function (6) for $PO_4$ pollutant concentration function

$$c(x,0) = -3.9488e{-}05\,x^5 + 0.0023\,x^4 - 0.0428\,x^3 + 0.3006\,x^2 - 0.5051\,x + 2.791 \tag{6}$$

# 4 Reduced Differential Transform Method

Let $p(\Psi,t)$ be a two-variable function. Suppose $p(\Psi,t)$ is written as $p(\Psi,\tau) = u(\Psi)v(t).p(\Psi,\tau)$ can be represented as the following using the features of the differential transform:

$$p(\Psi,t) = \sum_{i=0}^{\infty} U_{(i)}\Psi^i \sum_{j=0}^{\infty} V_{(j)}t^j = \sum_{k=0}^{\infty} P_k(\Psi)t^k \tag{7}$$

where $P_k(\Psi)$ is referred to as the t-dimensional spectrum function of $p(\Psi,t)$.

$$P_n(\Psi) = \frac{1}{n!}\left[\frac{\partial^n}{\partial t^n}p(\Psi,t)\right]_{t=0} \tag{8}$$

The original function is denoted by the lowercase $[p(\Psi,t)]$, whereas the altered function is denoted by the capital $[P(\Psi,t)]$. The way to define the differential inverse transform of $P_k(\Psi)$ is as follows:

$$p(\Psi,t) = \sum_{n=0}^{\infty} P_n(\Psi)t^n \tag{9}$$

From Equations (8) and (9), we get

$$p(\Psi,t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\partial^n}{\partial t^n}p(\Psi,t)\right]_{t=0} t^n \tag{10}$$

Let us consider the following nonlinear PDE, to understand the basic concept of

RDTM.

$$Tp(\Psi,t)+Kp(\Psi,t)+Op(\Psi,t)=f(\Psi,t) \tag{11}$$

with initial condition $p(\Psi,0)=\Omega(\Psi)$
, where $T=\frac{\partial}{\partial t}$, $Kp(\Psi,t)$ is a linear term that has partial derivatives, while $Op(\Psi,t)$ is a non-linear term, and $f(\Psi,t)$ is a source term [Keskin and Oturanc, 2009].
By applying the transform on equation (11), we get

$$(k+1)P_{k+1}(\Psi)=F_k(\Psi)-KP_k(\Psi)-OP_k(\Psi) \tag{12}$$

where $P_k(\Psi)$,$F_k(\Psi)$,$KP_k(\Psi)$ and $OP_k(\Psi)$ are transform of $p(\Psi,t)$,$f(\Psi,t)$,$Kp(\Psi,t)$ and $Op(\Psi,t)$ respectively. We are able to write this down based on the initial condition.

$$P_0(\omega)=\Omega(\Psi) \tag{13}$$

From equations (12) and (13), we get the values of $P_k(\Psi)$. After that, an approximation solution is produced by carrying out an inverse transformation on the set of values $\{P_k(\Psi)\}_{k=0}^{n}$. This transformation yields an approximation solution as

$$\tilde{p}_n(\Psi,t)=\sum_{k=0}^{n}P_k(\Psi)t^k \tag{14}$$

where n is the order of approximation answer. Consequently, the exact solution is given by [Al-Amr, 2014],

$$p(\Psi,t)=\lim_{n\to\infty}\tilde{p}_n(\Psi,t) \tag{15}$$

| Original function | Transformation of function |
|---|---|
| $p(\Psi, t)$ | $P_k(\Psi) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} p(\Psi, t)\right]_{t=0}$ |
| $\gamma m(\Psi, t) \pm \delta n(\Psi, t)$ | $\gamma M_k(\Psi) + \delta N_k(\Psi)$ |
| $\Psi^m t^n$ | $\Psi^m \delta(k-n)$ |
| $\Psi^m t^n p(\Psi, t)$ | $\Psi^m P_{k-n}(\Psi)$ |
| $a(\Psi, t) = b(\Psi, t)c(\Psi, t)$ | $A_k(\Psi) = \sum_{r=0}^{k} B_r(\Psi) C_{m-r}(\Psi)$ |
| $\frac{\partial^r}{\partial t^r} p(\Psi, t)$ | $\frac{(k+r)!}{k!} P_{k+r}(\Psi)$ |
| $\frac{\partial}{\partial \Psi} p(\Psi, t)$ | $\frac{\partial}{\partial \Psi} P_k(\Psi)$ |

Table 2: Transform Table[Maisuria and Tandel, 2023, Maisuria et al., 2023, Aljahdaly and Alharbi, 2022, Ganji et al., 2016]

Table 2 represents the fundamental operations of the reduced differential transform method.

# 5 Convergence of RDTM

**Theorem 5.1.** *If $\sum_{m=0}^{\infty} P_m t^m$ is given series [Maisuria and Tandel, 2023, Moosavi Noori and Taghizadeh, 2021, Maisuria et al., 2023, Saeed and Mustafa, 2017],*

[1] $\exists\, 0 < \delta < 1 \ni \frac{\|P_{m+1}\|}{\|P_m\|} \leq \delta \Rightarrow$ *given series is convergent.*

[2] $\exists\, \delta > 1 \ni \frac{\|P_{m+1}\|}{\|P_m\|} \geq \delta \Rightarrow$ *given series is divergent.*

**Proof.** Let $(C[E], \|.\|)$ represent the Banach space that contains all continuous functions on $E$ that satisfy the norm $\|.\|$. Also, let's suppose that $\|P_0(\Psi)\| \leq K$, where $K$ is an integer in the positive range. Let $\{\kappa_m\}_{m=0}^{\infty}$ be a partial sum

$$\kappa_m = P_0 + P_1 + P_2 + ... + P_m$$

If we can prove that $\{\kappa_m\}_{m=0}^{\infty}$ is a Cauchy sequence in Banach Space, then we can conclude that the sequence of partial sum is convergent in Banach Space. As a result, at this point, we shall demonstrate that the series of partial sums follows the Cauchy sequence. We take

$$\|\kappa_{m+1} - \kappa_m\| = \|P_{m+1}\| \leq \delta\|P_m\| \leq ... \leq \delta^{m+1}\|P_0\| \leq \delta^{m+1}K$$

Therefore,$\forall m, n \in \mathbb{N}, m \geq n$ ,We have

$$\begin{aligned} \|\kappa_m - \kappa_n\| &= \|(\kappa_m - \kappa_{m-1}) + (\kappa_{m-1} - \kappa_{m-2}) + ... + (\kappa_{n+1} - \kappa_n)\| \\ &\leq \|\kappa_m - \kappa_{m-1}\| + \|\kappa_{m-1} - \kappa_{m-2}\| + ... + \|\kappa_{n+1} - \kappa_n\| \\ &\leq \tfrac{1-\delta^{m-n}}{1-\delta}\delta^{n+1}\|P_0\| \end{aligned}$$

Now here $0 < \delta < 1$, We obtain

$$\lim_{m,n\to\infty} \|\kappa_m - \kappa_n\| = 0$$

Hence $\{\kappa_m\}_{m=0}^{\infty}$ is Cauchy sequence in Banach Space. Therefore, given series is convergent.

# 6 Result and discussion

## 6.1 $NO_3$ pollutant

$(a)$ **Without pollutant source**

In this case, $S(x,t) = 0$ in equation (4). Solving equation (4) by RDTM with initial function (5), we get

$$c(x,t) = M_0 + M_1 t + M_2 t^2 + M_3 t^3 + ...$$

where

$$\begin{aligned} M_0 &= -1.4880e-05\,x^5 + 8.1728e-04\,x^4 - 0.0139\,x^3 + 0.0728\,x^2 \\ &\quad +0.0045\,x + 0.6140 \\ M_1 &= 3.7200e-05\,x^4 - 0.0019\,x^3 + 0.0307\,x^2 - 0.1564\,x + 0.1434 \\ M_2 &= -3.7200e-05\,x^3 + 0.0017\,x^2 - 0.0211\,x + 0.0698 \\ M_3 &= 1.8600e-05\,x^2 - 6.3184e-04\,x + 0.0046 \end{aligned}$$

Here,

$$\frac{\|M_1\|}{\|M_0\|} = 0.2335 < 1, \frac{\|M_2\|}{\|M_1\|} = 0.4869 < 1, \frac{\|M_3\|}{\|M_2\|} = 0.0665 < 1$$

Therefore, given solution function $c(x,t)$ is converge to exact solution.

| $x\backslash t$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.628653 | 0.64352 | 0.659796 | 0.677508 | 0.696684 | 0.717352 | 0.739538 | 0.763271 | 0.788577 | 0.815485 |
| 0.2 | 0.629702 | 0.643033 | 0.657731 | 0.673823 | 0.691336 | 0.710298 | 0.730734 | 0.752674 | 0.776142 | 0.801167 |
| 0.3 | 0.632103 | 0.643958 | 0.657139 | 0.671673 | 0.687585 | 0.704903 | 0.723654 | 0.743864 | 0.765559 | 0.788767 |
| 0.4 | 0.635777 | 0.646216 | 0.657939 | 0.670974 | 0.685347 | 0.701083 | 0.71821 | 0.736753 | 0.756739 | 0.778195 |
| 0.5 | 0.640646 | 0.649726 | 0.660051 | 0.671646 | 0.684539 | 0.698754 | 0.714318 | 0.731256 | 0.749595 | 0.769361 |
| 0.6 | 0.646633 | 0.654411 | 0.663395 | 0.673609 | 0.68508 | 0.697833 | 0.711894 | 0.727289 | 0.744042 | 0.76218 |
| 0.7 | 0.653666 | 0.660197 | 0.667895 | 0.676785 | 0.686892 | 0.698242 | 0.710859 | 0.724769 | 0.739996 | 0.756567 |
| 0.8 | 0.66167 | 0.66701 | 0.673478 | 0.681099 | 0.689899 | 0.699901 | 0.711132 | 0.723616 | 0.737377 | 0.752441 |
| 0.9 | 0.670577 | 0.674778 | 0.680069 | 0.686477 | 0.694024 | 0.702736 | 0.712637 | 0.723751 | 0.736104 | 0.749719 |
| 1 | 0.680316 | 0.683431 | 0.6876 | 0.692846 | 0.699196 | 0.706672 | 0.715298 | 0.7251 | 0.736101 | 0.748325 |

Table 3: $NO_3$ pollutant concentration at different $x$ and $t$ for $v = 0.5$

The values of the nitrate pollutant concentration at different values of $x$ and $t$ for this case are shown in table 3.

$(b)$ **Constant Input pollutant source**

In this case, $S(x,t) = \frac{\alpha}{(x+L)^3}$, $(\alpha > 0)$ in equation (4) then solving equation (4) by RDTM with initial function (5), we get

$$c(x,t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + ...$$

where

$$\begin{aligned}
A_0 &= -1.4880e-05\,x^5 + 8.1728e-04\,x^4 - 0.0139\,x^3 + 0.0728\,x^2 \\
&\quad +0.0045\,x + 0.6140 \\
A_1 &= \tfrac{2}{(x+5)^3} - 0.1564\,x + 0.0307\,x^2 - 0.0019\,x^3 + 3.7200e-05\,x^4 + 0.1434 \\
A_2 &= \tfrac{3\,x+39}{2(x+5)^5} - 0.0211\,x + 0.0017\,x^2 - 3.7200e-05\,x^3 + 0.0698 \\
A_3 &= \tfrac{x^2+30\,x+245}{(x+5)^7} - 6.3184e-04\,x + 1.8600e-05\,x^2 + 0.0046
\end{aligned}$$

Here,

$$\frac{\|A_1\|}{\|A_0\|} = 0.2596 < 1, \frac{\|A_2\|}{\|A_1\|} = 0.4772 < 1, \frac{\|A_3\|}{\|A_2\|} = 0.1023 < 1$$

Hence, it can be observed that the solution function $c(x,t)$ converges towards the exact solution.

| $x\backslash t$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.630221 | 0.646786 | 0.664906 | 0.684627 | 0.705992 | 0.729045 | 0.753831 | 0.780393 | 0.808775 | 0.839021 |
| 0.2 | 0.631179 | 0.646105 | 0.662533 | 0.680502 | 0.700055 | 0.721234 | 0.74408 | 0.768636 | 0.794942 | 0.823041 |
| 0.3 | 0.633496 | 0.646853 | 0.661657 | 0.677948 | 0.695765 | 0.715148 | 0.736137 | 0.758771 | 0.783091 | 0.809135 |
| 0.4 | 0.637093 | 0.648946 | 0.662196 | 0.676878 | 0.693032 | 0.710695 | 0.729905 | 0.750699 | 0.773116 | 0.797194 |
| 0.5 | 0.64189 | 0.652305 | 0.664065 | 0.677208 | 0.691769 | 0.707784 | 0.72529 | 0.744323 | 0.764919 | 0.787115 |
| 0.6 | 0.647811 | 0.656849 | 0.667186 | 0.678855 | 0.691891 | 0.706329 | 0.722204 | 0.739551 | 0.758404 | 0.778798 |
| 0.7 | 0.654781 | 0.662505 | 0.671479 | 0.681739 | 0.693316 | 0.706245 | 0.72056 | 0.736292 | 0.753476 | 0.772146 |
| 0.8 | 0.662728 | 0.669196 | 0.67687 | 0.685782 | 0.695965 | 0.707451 | 0.720272 | 0.73446 | 0.750047 | 0.767066 |
| 0.9 | 0.671581 | 0.676851 | 0.683283 | 0.690909 | 0.699759 | 0.709866 | 0.721259 | 0.73397 | 0.74803 | 0.76347 |
| 1 | 0.68127 | 0.685399 | 0.690647 | 0.697045 | 0.704624 | 0.713412 | 0.723441 | 0.734741 | 0.74734 | 0.76127 |

Table 4: $NO_3$ pollutant concentration at different $x$ and $t$ for $\alpha = 2$ and $v = 0.5$

The values of nitrate pollutant concentration at various values of $x$ and $t$ with constant input pollutant source are presented in table 4.

$(c)$ **Linear Input pollutant source**

In this case, $S(x,t) = \frac{\alpha t}{(x+L)^3}$, $(\alpha > 0)$ in equation (4) then solving equation (4) by RDTM with initial function (5), we get

$$c(x,t) = P_0 + P_1 t + P_2 t^2 + P_3 t^3 + ...$$

where

$$\begin{aligned}
P_0 &= -1.4880e-05\,x^5 + 8.1728e-04\,x^4 - 0.0139\,x^3 + 0.0728\,x^2 \\
&\quad +0.0045\,x + 0.6140 \\
P_1 &= 3.7200e-05\,x^4 - 0.0019\,x^3 + 0.0307\,x^2 - 0.1564\,x + 0.1434 \\
P_2 &= \tfrac{1}{(x+5)^3} - 0.0211\,x + 0.0017\,x^2 - 3.7200e-05\,x^3 + 0.0698 \\
P_3 &= \tfrac{x+13}{2(x+5)^5} - 6.3184e-04\,x + 1.8600e-05\,x^2 + 0.0046
\end{aligned}$$

Here,

$$\frac{\|P_1\|}{\|P_0\|} = 0.2335 < 1, \frac{\|P_2\|}{\|P_1\|} = 0.5427 < 1, \frac{\|P_3\|}{\|P_2\|} = 0.0864 < 1$$

Therefore, the solution function $c(x,t)$ apparently exhibits convergence towards the exact solution.

| x\t | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.628731 | 0.643837 | 0.660526 | 0.678836 | 0.698806 | 0.720476 | 0.743883 | 0.769068 | 0.796068 | 0.824922 |
| 0.2 | 0.629775 | 0.643331 | 0.658418 | 0.675072 | 0.693331 | 0.713233 | 0.734815 | 0.758114 | 0.783168 | 0.810015 |
| 0.3 | 0.632172 | 0.64424 | 0.657787 | 0.672849 | 0.689463 | 0.707665 | 0.727491 | 0.748977 | 0.772159 | 0.797074 |
| 0.4 | 0.635842 | 0.646481 | 0.65855 | 0.672084 | 0.687117 | 0.703685 | 0.721822 | 0.741565 | 0.762947 | 0.786004 |
| 0.5 | 0.640707 | 0.649977 | 0.660628 | 0.672694 | 0.686209 | 0.701207 | 0.717723 | 0.73579 | 0.755442 | 0.776713 |
| 0.6 | 0.646691 | 0.654649 | 0.663941 | 0.674599 | 0.686658 | 0.70015 | 0.715108 | 0.731565 | 0.749554 | 0.769109 |
| 0.7 | 0.653721 | 0.660422 | 0.668412 | 0.677722 | 0.688385 | 0.700432 | 0.713895 | 0.728807 | 0.7452 | 0.763106 |
| 0.8 | 0.661723 | 0.667223 | 0.673967 | 0.681986 | 0.691311 | 0.701973 | 0.714004 | 0.727434 | 0.742295 | 0.758617 |
| 0.9 | 0.670626 | 0.67498 | 0.680534 | 0.687318 | 0.695363 | 0.704699 | 0.715356 | 0.727365 | 0.740757 | 0.75556 |
| 1 | 0.680364 | 0.683623 | 0.688041 | 0.693645 | 0.700466 | 0.708533 | 0.717876 | 0.728524 | 0.740507 | 0.753854 |

Table 5: $NO_3$ pollutant concentration at different $x$ and $t$ for $\alpha = 2$ and $v = 0.5$

The values of nitrate pollutant concentration at different values of $x$ and $t$ with linear input pollutant source are shown in table 5.

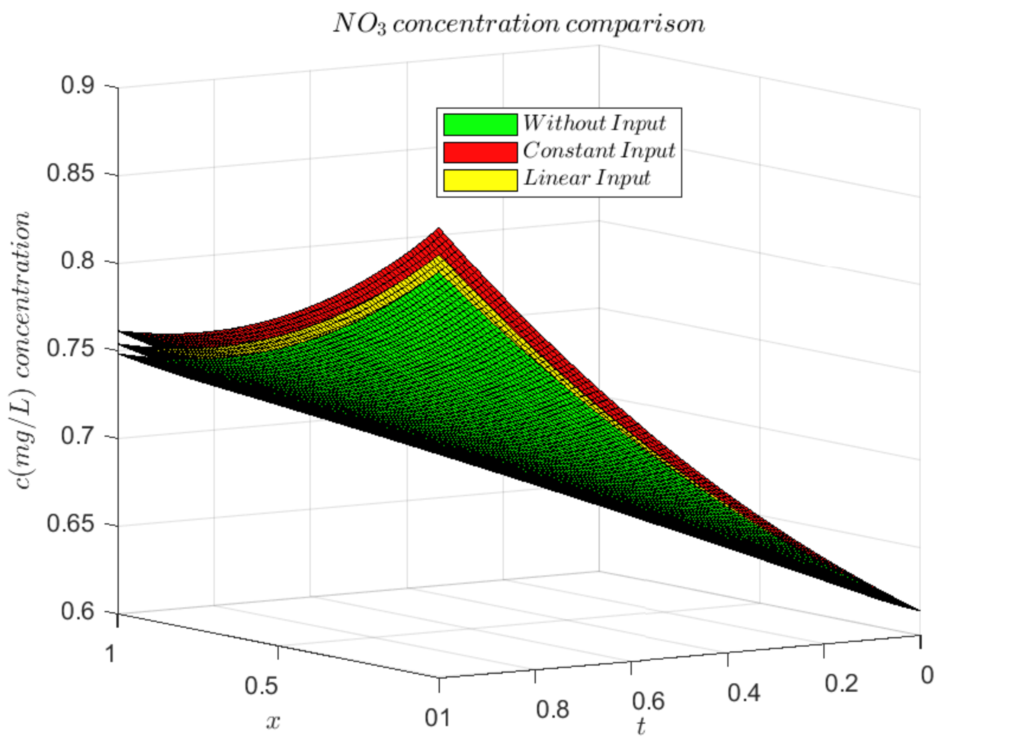


Figure 2: 3D Comparison of $NO_3$ pollutant concentration for $\alpha = 2$,$v = 0.5$

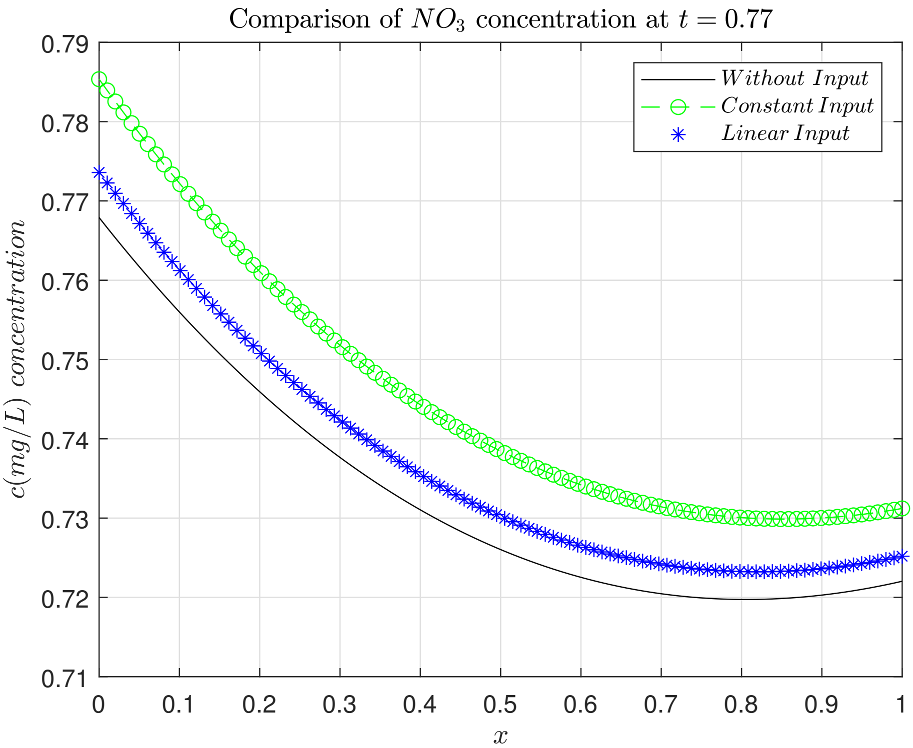


Figure 3: $NO_3$ pollutant concentration comparison for $t = 0.77$,$\alpha = 2$,$v = 0.5$

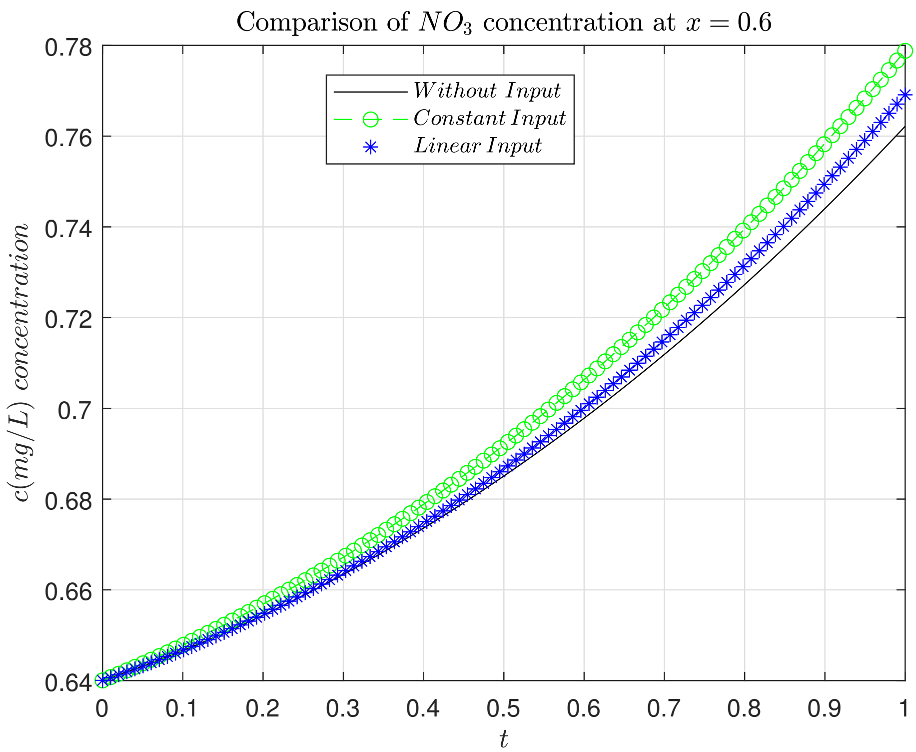


Figure 4: $NO_3$ pollutant concentration comparison for $x = 0.6$,$\alpha = 2$,$v = 0.5$

Figure 2 give the 3D graphical comparison of solution function obtained for all cases. Also figures 3 and 4 give the 2D graphical comparison of nitrate pollutant level obtained for all cases for fixed $t = 0.77$ and $x = 0.6$ respectively.

## 6.2 $PO_4$ pollutant

$(a)$ **Without pollutant source**

In this case, $S(x,t) = 0$ in equation (4). Solving equation (4) by RDTM with initial function (6), we get

$$c(x,t) = N_0 + N_1 t + N_2 t^2 + N_3 t^3 + ...$$

where

$$\begin{aligned}
N_0 &= -3.9488e-05\,x^5 + 0.0023\,x^4 - 0.0428\,x^3 + 0.3006\,x^2 - 0.5051\,x + 2.791 \\
N_1 &= 9.8720e-05\,x^4 - 0.0053\,x^3 + 0.0912\,x^2 - 0.5575\,x + 0.8538 \\
N_2 &= -9.8720e-05\,x^3 + 0.0046\,x^2 - 0.0615\,x + 0.2306 \\
N_3 &= 4.9360e-05\,x^2 - 0.0017\,x + 0.0133
\end{aligned}$$

Here,

$$\frac{\|N_1\|}{\|N_0\|} = 0.0063 < 1, \frac{\|N_2\|}{\|N_1\|} = 0.0025 < 1, \frac{\|N_3\|}{\|N_2\|} = 0.0013 < 1$$

Hence, the solution function $c(x,t)$ shows convergence toward the exact solution.

| x\t | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 2.82561 | 2.912336 | 3.00371 | 3.099809 | 3.200714 | 3.306501 | 3.417251 | 3.533042 | 3.653952 | 3.78006 |
| 0.2 | 2.778455 | 2.859695 | 2.94546 | 3.035828 | 3.130876 | 3.230681 | 3.335323 | 3.444878 | 3.559424 | 3.679039 |
| 0.3 | 2.736988 | 2.812921 | 2.893257 | 2.978075 | 3.067451 | 3.161461 | 3.260183 | 3.363692 | 3.472066 | 3.58538 |
| 0.4 | 2.700962 | 2.771762 | 2.846848 | 2.926296 | 3.010181 | 3.098578 | 3.191564 | 3.289214 | 3.391604 | 3.498809 |
| 0.5 | 2.670135 | 2.735976 | 2.805986 | 2.880239 | 2.958811 | 3.041776 | 3.129208 | 3.221182 | 3.317773 | 3.419056 |
| 0.6 | 2.644271 | 2.705323 | 2.770428 | 2.83966 | 2.913094 | 2.990802 | 3.072859 | 3.159338 | 3.250312 | 3.345856 |
| 0.7 | 2.62314 | 2.679568 | 2.739936 | 2.804318 | 2.872785 | 2.94541 | 3.022267 | 3.103428 | 3.188965 | 3.278951 |
| 0.8 | 2.606515 | 2.658483 | 2.71428 | 2.773977 | 2.837647 | 2.90536 | 2.977189 | 3.053206 | 3.133481 | 3.218087 |
| 0.9 | 2.594176 | 2.641845 | 2.693232 | 2.748409 | 2.807447 | 2.870416 | 2.937386 | 3.008429 | 3.083615 | 3.163015 |
| 1 | 2.585908 | 2.629434 | 2.676571 | 2.727389 | 2.781957 | 2.840346 | 2.902624 | 2.968861 | 3.039128 | 3.113493 |

Table 6: $PO_4$ pollutant concentration at different $x$ and $t$ for $v = 0.5$

The values of phosphate pollutant concentration level at different values of $x$ and $t$ for this case are shown in table 6.

$(b)$ **Constant Input pollutant source**

In this case, $S(x,t) = \frac{\alpha}{(x+L)^3}$, $(\alpha > 0)$ in equation (4) then solving equation (4) by RDTM with initial function (6), we get

$$c(x,t) = U_0 + U_1 t + U_2 t^2 + U_3 t^3 + ...$$

where

$$\begin{aligned} U_0 &= -3.9488e-05\,x^5+0.0023\,x^4-0.0428\,x^3+0.3006\,x^2-0.5051\,x+2.791 \\ U_1 &= \tfrac{2}{(x+5)^3}-0.5575\,x+0.0912\,x^2-0.0053\,x^3+9.8720e-05\,x^4+0.8538 \\ U_2 &= \tfrac{3\,x+39}{2(x+5)^5}-0.0615\,x+0.0046\,x^2-9.8720e-05\,x^3+0.2306 \\ U_3 &= \tfrac{x^2+30\,x+245}{(x+5)^7}-0.0017\,x+4.9360e-05\,x^2+0.0133 \end{aligned}$$

Here,

$$\frac{\|U_1\|}{\|U_0\|}=0.0063<1, \frac{\|U_2\|}{\|U_1\|}=0.0025<1, \frac{\|U_3\|}{\|U_2\|}=0.0013<1$$

Hence, the solution function $c(x,t)$ converges to exact solution.

| x\t | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 2.827177 | 2.915601 | 3.00882 | 3.106928 | 3.210022 | 3.318195 | 3.431544 | 3.550164 | 3.674149 | 3.803596 |
| 0.2 | 2.779932 | 2.862768 | 2.950262 | 3.042507 | 3.139595 | 3.241618 | 3.348669 | 3.46084 | 3.578224 | 3.700913 |
| 0.3 | 2.738382 | 2.815815 | 2.897775 | 2.98435 | 3.075631 | 3.171706 | 3.272666 | 3.3786 | 3.489597 | 3.605748 |
| 0.4 | 2.702278 | 2.774493 | 2.851105 | 2.9322 | 3.017866 | 3.10819 | 3.203259 | 3.30316 | 3.407981 | 3.517809 |
| 0.5 | 2.671379 | 2.738555 | 2.81 | 2.885801 | 2.966041 | 3.050806 | 3.140181 | 3.234249 | 3.333098 | 3.43681 |
| 0.6 | 2.645449 | 2.707761 | 2.774219 | 2.844906 | 2.919904 | 2.999298 | 3.083169 | 3.1716 | 3.264674 | 3.362474 |
| 0.7 | 2.624255 | 2.681875 | 2.74352 | 2.809271 | 2.879209 | 2.953414 | 3.031968 | 3.114951 | 3.202445 | 3.294529 |
| 0.8 | 2.607573 | 2.660669 | 2.717672 | 2.77866 | 2.843713 | 2.91291 | 2.986329 | 3.06405 | 3.146151 | 3.232712 |
| 0.9 | 2.59518 | 2.643918 | 2.696446 | 2.752841 | 2.813182 | 2.877545 | 2.946008 | 3.018647 | 3.095541 | 3.176766 |
| 1 | 2.586862 | 2.631402 | 2.679619 | 2.731588 | 2.787385 | 2.847086 | 2.910767 | 2.978502 | 3.050367 | 3.126439 |

Table 7: $PO_4$ pollutant concentration at different $x$ and $t$ for $\alpha=2$ and $v=0.5$

Table 7 gives the values of phosphate pollutant concentration level at different values of $x$ and $t$ with constant input source.

$(c)$ **Linear Input pollutant source**

In this case, $S(x,t)=\frac{\alpha t}{(x+L)^3}$, $(\alpha>0)$ in equation (4) then solving equation (4) by RDTM with initial function (6), we get

$$c(x,t)=Q_0+Q_1t+Q_2t^2+Q_3t^3+...$$

where

$$\begin{aligned}
Q_0 &= -3.9488e-05\,x^5 + 0.0023\,x^4 - 0.0428\,x^3 + 0.3006\,x^2 - 0.5051\,x + 2.791 \\
Q_1 &= 9.8720e-05\,x^4 - 0.0053\,x^3 + 0.0912\,x^2 - 0.5575\,x + 0.8538 \\
Q_2 &= \tfrac{1}{(x+5)^3} - 0.0615\,x + 0.0046\,x^2 - 9.8720e-05\,x^3 + 0.2306 \\
Q_3 &= \tfrac{x+13}{2(x+5)^5} - 0.0017\,x + 4.9360e-05\,x^2 + 0.0133
\end{aligned}$$

Here,

$$\frac{\|Q_1\|}{\|Q_0\|} = 0.0063 < 1, \frac{\|Q_2\|}{\|Q_1\|} = 0.0025 < 1, \frac{\|Q_3\|}{\|Q_2\|} = 0.0013 < 1$$

Hence, the solution function $c(x,t)$ is convergent.

| x\t | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 2.825687 | 2.912653 | 3.004439 | 3.101137 | 3.202836 | 3.309625 | 3.421596 | 3.538839 | 3.661442 | 3.789497 |
| 0.2 | 2.778528 | 2.859994 | 2.946147 | 3.037077 | 3.13287 | 3.233617 | 3.339403 | 3.450319 | 3.56645 | 3.687887 |
| 0.3 | 2.737057 | 2.813202 | 2.893905 | 2.979252 | 3.069329 | 3.164223 | 3.264019 | 3.368805 | 3.478666 | 3.593688 |
| 0.4 | 2.701027 | 2.772028 | 2.847459 | 2.927405 | 3.011951 | 3.10118 | 3.195177 | 3.294026 | 3.397812 | 3.506619 |
| 0.5 | 2.670196 | 2.736227 | 2.806563 | 2.881287 | 2.960481 | 3.044229 | 3.132613 | 3.225716 | 3.32362 | 3.426408 |
| 0.6 | 2.64433 | 2.70556 | 2.770973 | 2.84065 | 2.914671 | 2.993119 | 3.076072 | 3.163614 | 3.255825 | 3.352785 |
| 0.7 | 2.623195 | 2.679793 | 2.740453 | 2.805254 | 2.874277 | 2.9476 | 3.025304 | 3.107467 | 3.194169 | 3.285489 |
| 0.8 | 2.606568 | 2.658697 | 2.71477 | 2.774865 | 2.83906 | 2.907433 | 2.980061 | 3.057024 | 3.138399 | 3.224263 |
| 0.9 | 2.594226 | 2.642047 | 2.693697 | 2.749251 | 2.808786 | 2.872378 | 2.940105 | 3.012043 | 3.088268 | 3.168856 |
| 1 | 2.585955 | 2.629626 | 2.677012 | 2.728188 | 2.783227 | 2.842207 | 2.905201 | 2.972285 | 3.043534 | 3.119023 |

Table 8: $PO_4$ pollutant concentration at different $x$ and $t$ for $\alpha = 2$ and $v = 0.5$

Table 8 gives the values of phosphate pollutant concentration level at different values of $x$ and $t$ with linear input pollutant source.

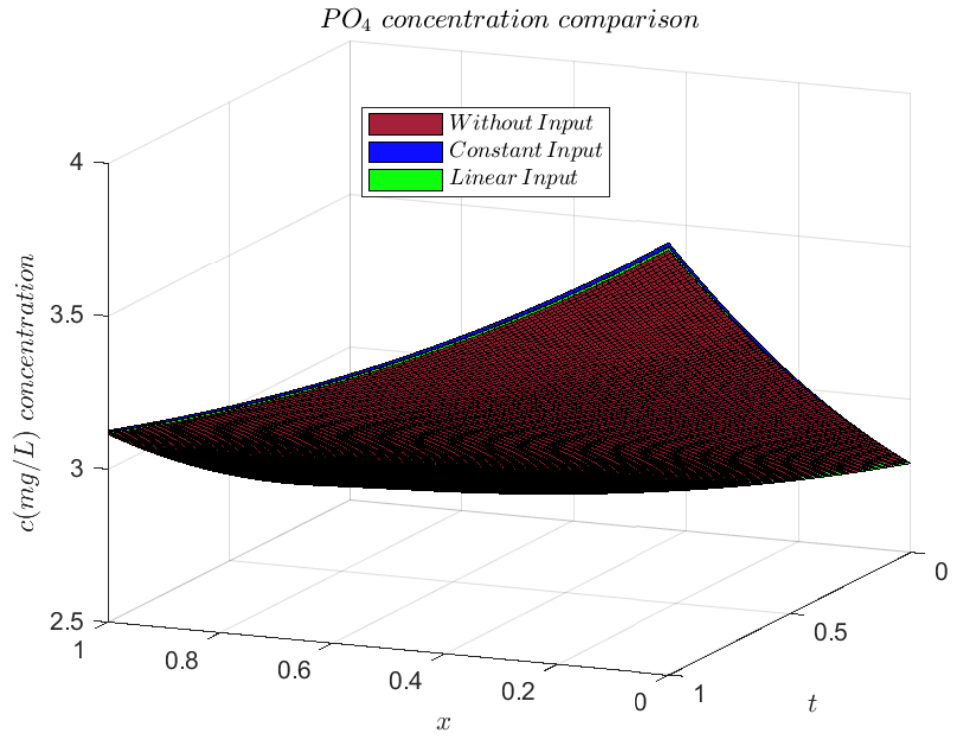


Figure 5: 3D Comparison of $PO_4$ pollutant concentration for $\alpha = 2$,$v = 0.5$

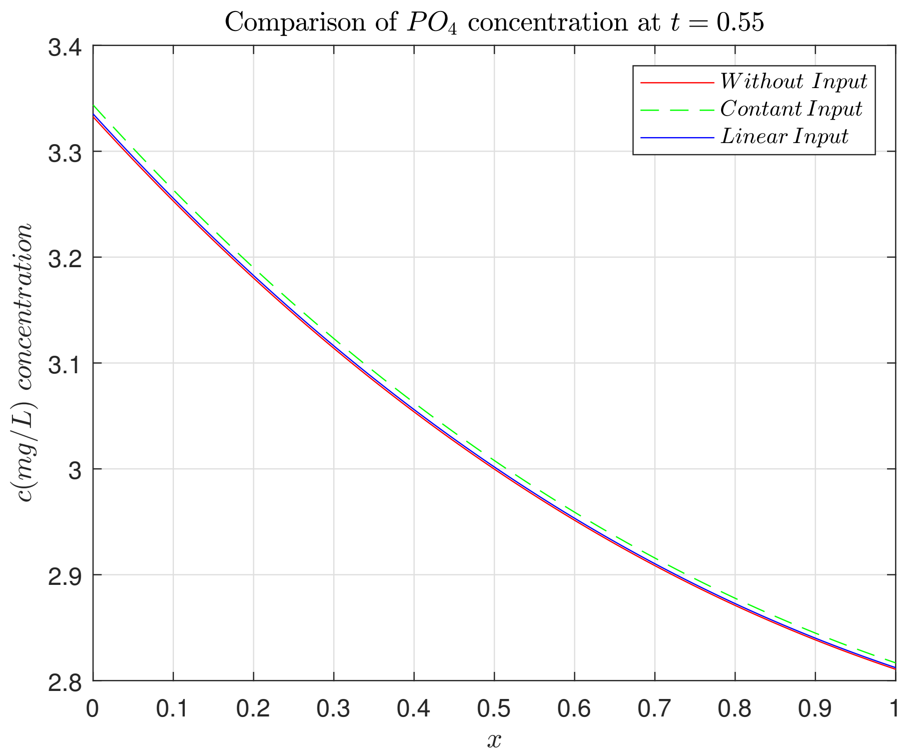


Figure 6: $\mathrm{PO}_4$ pollutant concentration comparison for $t = 0.55$,$\alpha = 2$,$v = 0.5$

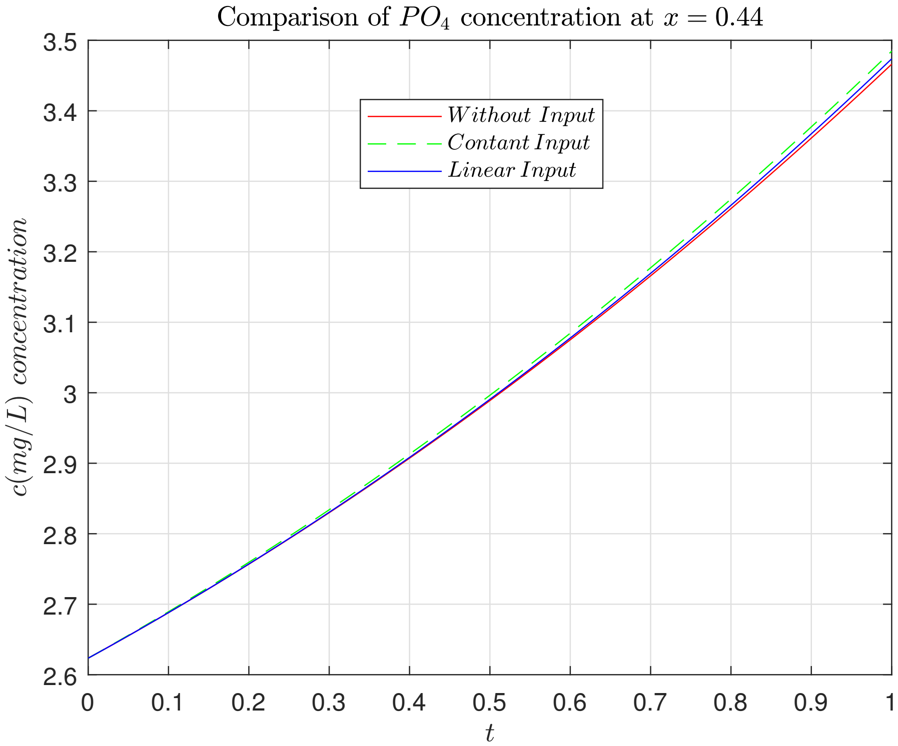


Figure 7: $PO_4$ pollutant concentration comparison for $x = 0.44$,$\alpha = 2$,$v = 0.5$

Figure 5 give the 3D graphical comparison of phosphate pollutant level obtained for all cases. Also figures 6 and 7 give the 2D graphical comparison of phosphate pollutant level obtained for all cases for fixed $t = 0.55$ and $x = 0.44$ respectively.

# 7 Conclusion

The analysis of the current problem has been conducted for two distinct cases: one in the absence of an input source and another with the presence of an input source. This research focuses on the identification and mitigation of two distinct types of input pollution sources: linear and constant. The objective is to propose effective solutions for addressing these specific forms of pollution in order to enhance the overall quality and reliability of the input data. By examining the characteristics and impacts of linear and constant input pollution, this study aims to contribute to the development of methods and approaches that can minimize the negative effects associated with these sources. Through our observations, it has been noted that the concentration value obtained from a constant input source exceeds the concentration value acquired from a linear input source across various combinations of distance and time. In the context of concentration, it has been observed that the acquisition of a higher value is possible when the input source maintains linearity compared to situations where the input source is eliminated across various distances and time intervals. Based on our analysis, it can be inferred that the presence of an input source in the model leads to a higher level of pollutants compared to the model without an input source. In this study, the Reduced Differential Transform Method (RDTM) is employed to smoothly handle the problem of solving a one-dimensional advection-diffusion equation with a source term. The initial condition is derived from the collected data by fitting a polynomial function. The potential expansion of this approach encompasses the inclusion of advection-diffusion equations within the realms of both two and three-dimensional systems.

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