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# Supra $\beta$-Paracompact Spaces

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**Abstract**

In this paper, authors present the concept of supra $\beta$-compact spaces and studied some basic properties of them. Further, supra $\beta$-locally finite collection and supra $\beta$-compact spaces are studied. Also, Some results regarding supra $\beta$-paracompact spaces are studied.
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# 1 Introduction

A space M is said to be paracompact if every open cover of M has a locally finite open refinement. Many researchers introduced and investigated the properties of $\alpha$-paracompact [1], $P_1$-paracompact [11] and $S_1$-paracompact [13] spaces replacing the open cover by $\alpha$-open, pre-open and semi-open cover in original definition.

Mashhour et al. [12] introduced the supra topological space and studied the properties of $S$-continuous maps, $S^*$-continuous maps in supra topology. Motivation for supra topology come from the need to obtain various examples that

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satisfy some properties on the finite sets, for instance, the only topology on a finite set that satisfies a $T_1$-space is the discrete topology, where as there are many different types of supra topologies that satisfy a condition of $T_1$-space.

Several authors, investigated the some topological notions on supra topological spaces such as compact spaces, [[2], [3]] and separation axioms [[4], [8]]. Some of the published paper showed that, any topological results became true on supra topology, but some fails to prove the results. Thus, supra topology provides a framework which is general enough to solve practical problems and to model phenomena.

Jassim T. H. [10] introduced the concept of supra compactness in supra topology and came out with many intersecting results. Paracompactness on the supra topology was introduced and studied by T.M.Al-shami [5] in 2020.

In this paper, authors are introduced and studied the concept of $\beta$-compact spaces, $\beta$-locally finite collection and $\beta$-paracompact spaces in supra topology. We discuss the main properties of these spaces.

## 2 Preliminaries

Throughout this paper (P, $\mu$) and (Q, $\sigma$) (or simply P and Q) always mean supra topological spaces on which no separation axioms are assumed unless explicitly stated.

**Definition 2.1.** *[12] A sub collection $\mu$ of $2^p$ is called a supra topology on $P \neq \phi$ provided that it contains P and is closed under arbitrary union.*
*A pair $(P, \mu)$ is called a supra topological spaces.*
*Every member of $\mu$ is called a supra open set and its complement is called a supra closed sets.*

**Definition 2.2.** *[12] A map $s : (P, \mu) \to (Q, \sigma)$ is said to be an $S^*$-continuous if the inverse image of every supra open set in Q is supra open set in P.*

**Definition 2.3.** *[2] Let $A_1$ be a subset of a supra topological space $(P, \mu)$. The family $\mu_{A_1} = \{A_1 \cap G_1 : G_1 \in \mu\}$ is called a supra relative topology on $A_1$.*
*A pair $(A_1, \mu_{A_1})$ is called a supra subspace of $(P, \mu)$.*

**Definition 2.4.** *[2] A space $(P, \mu)$ is called a supra compact if every supra open cover of P has a finite subcover.*

**Definition 2.5.** *[7] Let $U_1 = \{U_i : i \in I\}$ and $V_1 = \{V_j : j \in J\}$ be two covers of P. The collection $U_1$ is said to be a refinement of $V_1$ provided that for each $U_i \in U_1$, there are some $V_j \in V_1$ such that $U_i \subseteq V_j$.*

**Definition 2.6.** *[7] The collection $U_1 = \{U_i : i \in I\}$ of subsets of a space P is said to be a locally finite provided that each point $p \in P$ has a neighborhood $W$ such that $W \cap U_i \neq \phi$ holds for finitely many values of i.*

**Definition 2.7.** *[7] If the collection $\{U_i : i \in I\}$ is locally finite, then the collection $\{ cl(U_i) : i \in I\}$ is locally finite.*

**Definition 2.8.** *[9] A function $s : (P, \mu) \to (Q, \sigma)$ is said to be*
*(i) supra $\beta$-continuous if the inverse image of each open set in Q is supra $\beta$-open set in P.*
*(ii) supra $\beta$-irresolute if the inverse image of every supra $\beta$-open set in Q is supra $\beta$-open in P.*
*(iii) pre supra $\beta$-closed if the image of supra $\beta$-open set in P is supra $\beta$-open in Q.*

# 3 supra $\beta$-compact spaces

**Definition 3.1.** *A collection $\{A_i : i \in \Lambda\}$ of a supra $\beta$-open subsets of the space $(P, \mu)$ is called a supra $\beta$-open cover of the subsets B of P provided $B \subseteq \cup_{i\in\Lambda} A_i$.*

**Definition 3.2.** *A supra topological space $(P, \mu)$ is called supra $\beta$-compact provided that every supra $\beta$-open cover of P has a finite subcover.*

**Remark 3.1.** *Every finite supra topological space is supra $\beta$-compact.*

**Remark 3.2.** *A finite union of supra $\beta$-compact subsets of the space $(P, \mu)$ is supra $\beta$-compact.*

**Definition 3.3.** *A subset B of a supra topological space $(P, \mu)$ is said to be supra $\beta$-compact relative to $(P, \mu)$ if for every collection $\{A_i : i \in I\}$ of supra $\beta$-open subsets of P with $B \subseteq \cup\{A_i : i \in I\}$, there exists a finite subset $I_0$ of I such that $B \subseteq \cup\{A_i : i \in I_0\}$.*

**Theorem 3.1.** *Every supra $\beta$-compact space is supra compact.*
**Proof:** *Let $\{A_i : i \in \lambda\}$ be supra open cover of P. As every supra open set is supra $\beta$-open, so $\{A_i : i \in \lambda\}$ is a supra $\beta$-open cover of P. As P is supra $\beta$-compact, $\{A_i : i \in \lambda\}$ has a finite subcover. Thus, P is supra compact.*

**Definition 3.4.** *A subset $A_1$ of a supra space $(P, \mu)$ is said to be supra $\beta$-compact relative to P if every supra $\beta$-open cover of $A_1$ is reducible to a finite subcover.*

**Theorem 3.2.** *Every supra $\beta$-closed subset of a supra $\beta$-compact space is supra $\beta$-compact with respect to the space $\mu$.*
**Proof:** *Let $S = \{G_i : i \in \lambda\}$ be a supra $\beta$-open cover of the supra $\beta$-closed subset $F$ of the space P. Then $F^c = P \setminus F$ is supra $\beta$-open with $F \subseteq \cup\{G_i : i \in \lambda\}$. Let $S^* = \{G_i : i \in \lambda\} \cup \{F^c\}$ be a supra $\beta$-open cover of P, that is $P = \cup S^* = \cup\{G_i : i \in \lambda\} \cup \{F^c\}$. Then $S^*$ is reducible to a finite subcover $G_1 \cup G_2 \cup .... \cup G_n \cup F^c$. Hence $G \subseteq G_1 \cup G_2 \cup ..... \cup G_n$. Thus, every supra $\beta$-open cover of S contains a finite subcover. Thus, G is supra $\beta$-compact relative to $\mu$.*

**Theorem 3.3.** *A space P is supra $\beta$-compact iff every collection of supra $\beta$-closed subsets has a finite intersection property has a non empty intersection.*

**Theorem 3.4.** *Supra $\beta$-irresolute image of a supra $\beta$-compact set is supra $\beta$-compact.*
**Proof:** *Let $S : (P, \mu) \to (Q, \sigma)$ be a supra $\beta$-irresolute map and $A_1$ be a supra $\beta$-compact subset of P. Let $\{G_i : i \in \lambda\}$ be supra $\beta$-open cover of Q. Then $A_1 \subseteq \cup_i S^{-1}(G_i)$. As S is supra $\beta$-irresolute, $S^{-1}(G_i)$ is supra $\beta$-open cover. Since $A_i$ is supra $\beta$-compact, $A_1 \subseteq \cup_{i=1}^{n} S^{-1}(G_i)$. Thus, $S(A_1) \subseteq \cup_{i=1}^{n} G_i$ and so $S(A_1)$ is supra $\beta$-compact.*

# 4 Supra $\beta$-locally finite collection

**Definition 4.1.** *A space $(P, \mu)$ is said to be Supra paracompact[2] if every supra open cover has a supra open locally finite refinement.*

**Definition 4.2.** *A collection $\Xi_1 = \{F_\alpha : \alpha \in \lambda\}$ of subsets of a space P is said to be supra $\beta$-locally finite provided that for each $p \in P$, there exists a supra $\beta$-open sets $U_1 \in P$ with $U_1$ intersects $F_\alpha$ at most finitely many values of $\alpha$.*

**Example 4.1.** *Let us consider $P = \{p_1, p_2, r_1, r_2\}$ and $\mu = \{P, \phi, \{p_1, p_2, r_1\}, \{p_1, p_2, r_2\}, \{p_1, r_1, r_2\}, \{p_2, r_1, r_2\}, \{p_1, p_2\}, \{r_1, r_2\}, \{p_1, r_1\}, \{p_2, r_2\}, \{r_1, p_2\}\}$ be a supra topology on P. We can verify that, the subset $A = \{p_1, p_2\}$ is a supra $\beta$-locally finite in P.*

**Lemma 4.1.** *Let $\Xi = \{F_\alpha : \alpha \in \lambda\}$ be a collection of subsets of a space $(P, \mu)$. Then the following properties holds:*
*(i) if $\Xi = \{F_\alpha : \alpha \in \lambda\}$ is supra $\beta$-locally finite collection of subsets of P with $G_\alpha \subseteq F_\alpha$ holds for each $\alpha \in \lambda$, then $G_1 = \{G_\alpha : \alpha \in \lambda\}$ is also supra $\beta$-locally finite.*
*(ii) $\Xi$ is supra $\beta$-locally finite if and only if $\{\beta\text{-}cl(F_\alpha) : \alpha \in \lambda\}$ is also supra $\beta$-locally finite.*
*(iii) if $\Xi$ is supra $\beta$-locally finite then $\cup\beta\text{-}cl(F_\alpha) = \beta\text{-}cl(\cup F_\alpha)$.*

**Proof:** *(i) Follows from the definition of supra $\beta$-locally finite collection.*
*(ii) Assume that $\Xi$ is supra $\beta$-locally finite. Then, for each $p \in P$, $\exists$ a $\beta$-open set $U_1$ containing the point p which meets finitely many of the supra sets of $F_\alpha$, that is $F_{1_\alpha}, F_{2_\alpha}, F_{3_\alpha}....F_{n_\alpha}$. But $F_{\alpha_k} \subseteq \beta$-cl$(F_{\alpha_k})$ holds for each $k = 1, 2, ...n$. Then $U_1$ meets many supra $\beta$-open sets, say $\beta$-cl$(F_{\alpha_1})$, $\beta$-cl$(F_{\alpha_2}), ....\beta$-cl$(F_{\alpha_k})$. Thus $U_1$ meets finitely many of the supra sets $\beta$-cl$F_\alpha$. Thus, $\beta$-cl$(F_\alpha)$ is supra $\beta$-locally finite.*
*On the other hand, let $p \in P$. Then, there exists supra $\beta$-open set $U_1$ containing the point p which meets finitely many of the supra sets $\beta$-cl$(F_\alpha)$, that is $\beta$-cl$(F_{\alpha_1}), \beta$-cl$(F_{\alpha_2})....\beta$-cl$(F_{\alpha_n})$. Thus, $U_1 \cap \beta$-cl$(F_{\alpha_k}) \neq \phi$ holds for each $k = 1, 2, ....n$. Let $q \in U_1$ and $q \in \beta$-cl$(F_{\alpha_k})$. Then for every supra $\beta$-open set $V_1$ containing the point q such that $V_1 \cap F_{\alpha_k} \neq \phi$. Since $U_1$ is supra $\beta$-open set in P containing the point p, then $U_1 \cap F_{\alpha_k} \neq \phi$. Thus, $\Xi$ is supra $\beta$-locally finite.*
*(iii) It is obvious.*

# 5 Supra $\beta$-paracompact spaces

**Definition 5.1.** *A space $(P, \mu)$ is said to be supra $\beta$-paracompact if every supra open cover has a supra $\beta$-locally finite supra $\beta$-open refinement.*

**Remark 5.1.** *Every supra compact space is supra $\beta$-paracompact.*

**Remark 5.2.** *The existence of supra topology does not forms a supra $\beta$-paracompact spaces follows from the following example.*

**Example 5.1.** *Let us consider a supra topology $(P, \mu)$, where $\mu = \{\phi, G \subseteq Z : 2 \in G \text{ and } 3 \in G\}$ be a supra topology on set of all integers. Let $U_1$ be an infinite supra open cover of the space Z. Then any supra open refinement of $U_1$ contains an infinite supra pen sets containing the points 2 and 3. Thus, we can not find a locally finite open refinement for a supra open cover $U_1$. Thus, $(Z, \mu)$ is not supra $\beta$-paracompact.*

**Theorem 5.1.** *Union of any two supra $\beta$-paracompact space is supra $\beta$-paracompact.*
**Proof:** *Let $W_1 = \{w_i : i \in \lambda\}$ be a supra open cover of the set $A_1 \cup B_1$ where $A_1$ and $B_1$ are supra paracompact spaces. Then $W_1$ is a supra open cover of the sets $A_1$ and $B_1$. Since $W_1$ has a supra $\beta$-locally finite open refinement say $U_1$ and $V_1$ of the sets $A_1$ and $B_1$ respectively. Again, from lemma 4.1, we have $U_1 \cup V_1$ is supra $\beta$-loaclly finite refinement of $A_1 \cup B_1$. Thus, $A_1 \cup B_1$ is supra $\beta$-paracompact.*

**Remark 5.3.** *[9] Union of supra open and supra $\beta$-open set is again supra $\beta$-open set.*

**Theorem 5.2.** *Every supra closed subset of a supra $\beta$-paracompact space is supra $\beta$-paracompact.*

**Proof:** *Let $U_1 = \{U_i : i \in \lambda\}$ be a supra open cover of the supra closed set $F_1$. Then $U_1 \cup F_1^c$ is supra open cover of the space P. Since P is supra $\beta$-paracompact, then $U_1 \cup F_1^c$ has a supra open supra $\beta$-locally finite open refinement. Thus $U_1$ has a supra open, supra $\beta$-locally finite refinement of $F_1$. Thus, $F_1$ is supra $\beta$-paracompact.*

**Theorem 5.3.** *Let $s : (P, \mu) \to (Q, \sigma)$ be surjective function. The S is supra pre $\beta$-closed if and only if for every $q \in Q$ and every supra $\beta$-open set $U_q \in (P, \mu)$ contains $s^{-1}(q)$, there exists a supra $\beta$-open set $V_1 \in Q$ with $s^{-1}(V_1) \subseteq U_1$.*

**Proof:** *Let us prove the necessity part first. Let $q \in Q$ and $U_1$ be supra $\beta$-open set in P with $s^{-1}(q) \subseteq U_1$. Put $V_1 = Q \setminus s(P \setminus U_1)$. Then from hypothesis, $V_1$ is supra $\beta$-open set in Q containing q with $q \in V_1$ and $s^{-1}(V_1) \subseteq U_1$.*

*Let us consider the sufficiency part. Let M be supra $\beta$-closed set in P with $q \in Q \setminus s(M)$. Then, $s^{-1}(q) \subset P \setminus M$. From hypothesis, there exists a supra $\beta$-open set $V_{1_q}$ such that $s^{-1}(V_{1_q}) \subseteq Q \setminus s(M)$. Thus, $Q \setminus s(M) = \cup\{V_{1_q} : q \in Q \setminus s(M)\}$ and so $Q \setminus s(M)$ is supra $\beta$-open set in Y. Hence s(M) is supra $\beta$-closed in Q.*

**Theorem 5.4.** *Let $s : (P, \mu) \to (Q, \sigma)$ be supra continuous, supra open and supra pre $\beta$-closed surjective function with $s^{-1}(q)$ is supra $\beta$-compact for each $q \in Q$. Then Q is supra $\beta$-paracompact if P is supra $\beta$-paracompact.*

**Proof:** *Let $\Psi = \{U_\alpha : \alpha \in \lambda\}$ be an supra open cover of Q. Then $s^{-1}(\Psi) = \{s^{-1}(U_\alpha) : \alpha \in \lambda\}$ be an supra open cover of the supra $\beta$-paracompact space P and so it has a locally finite supra $\beta$-open refinement, that is $\nu = \{V_\gamma : \gamma \in B\}$. As s is supra pre $\beta$-open, the collection $s(\nu) = \{s(V_\gamma) : \gamma \in B\}$ is a supra $\beta$-open refinement of $\Psi$.*

*Further, we have to prove that the collection is supra $\beta$-locally finite in Q.*

*Let $q \in Q$. Then for each $p \in s^{-1}(q)$, there exists a supra $\beta$-open set $U_p$ such that $U_p$ intersects at most finitely many members of $\nu$. Then the collection $\{U_p : p \in s^{-1}(q)\}$ is a supra $\beta$-open cover of $s^{-1}(q)$. Thus, there exists a finite supra subset M of $s^{-1}(q)$ with $s^{-1}(q) \subset \cup U_p$. By the definition of supra pre $\beta$-closed function, there exists a supra $\beta$-open set $G_q$ containing q such that $s^{-1}(G_q) \subseteq \cup U_p$. Thus, $f^{-1}(G_q)$ intersects at most finitely many members of $\Psi$. Thus, $G_q$ intersects at most finitely many members of $s(\Psi)$. Hence $s(\Psi)$ is supra $\beta$-locally finite in Q.*

**Theorem 5.5.** *Let $s : (P, \mu) \to (Q, \sigma)$ be supra $\beta$-irresolute, supra closed surjective function with $s^{-1}(q)$ is compact for each $q \in Q$. Then P is supra $\beta$-paracompact if Q is supra $\beta$-paracompact.*

**Proof:** *Let $\Psi = \{U_\alpha : \alpha \in \lambda\}$ be an supra open cover of P. As $s^{-1}(q)$ is supra compact, there exists a supra finite subset $M(q)$ such that $s^{-1}(q) \subseteq \cup_{\alpha \in M(q)} U_\alpha$. Since s is supra closed, there exists a supra open set $V_q$ containing q such that*

*$s^{-1}(q) \subseteq \cup_{\alpha \in M(q)} U_\alpha$. Thus, $\{V_q : q \in Q\}$ is a supra open cover of the supra $\beta$-paracompact space Q and so it has a supra locally finite supra $\beta$-open refinement say $W = \{w_\gamma : \gamma \in B\}$. Since s is supra $\beta$-irresolute, $\{s^{-1}(w_\gamma) : \gamma \in B\}$ is a supra $\beta$-locally finite supra $\beta$-open cover of P. Then for each $\gamma \in B$, there exists a member $q(\gamma) \in Q$ such that $W_\gamma \subseteq V_{q_\gamma}$. Thus, $s^{-1}(W_\gamma) \subseteq s^{-1}(V_{q_\gamma}) \subseteq \cup_{\alpha \in M(q_\gamma))} U_\alpha = F_{q_\gamma}$. Put $\Omega = \{s^{-1}(W_\gamma) \wedge F_{q_\gamma} : \gamma \in B, q(\gamma) \in Q\}$, where $s^{-1}(W_\gamma) \wedge F_{q_\gamma} = \{s^{-1}(W_\gamma) \cap U_\alpha : \gamma \in B, \alpha \in M(q_\gamma))\}$. By remark 5.1, each set of $\Omega$ is supra $\beta$-open subset of P.Thus, the family $\Omega$ is supra $\beta$-locally finite supra $\beta$-open refinement of $\Psi$. Thus, P is supra $\beta$-paracompact.*

# 6 Discussion and Conclusion

Topology developed as a field of study of geometry and set theory, through analysis of such concepts as space, dimensions and transformation. The topological structures are modelled suitably in the field of Computer graphics, Pattern recognition, Artificial intelligence, Data mining, Rough set theory, Information systems, Quantum physics etc.

It is noted that the supra topological frame can be more convenient to solve some practical problems and to model some phenomena. Also, the possibility of applying semi-open sets to deal with some problems on digital topology.

In this way, this paper introduce some main concepts in supra topological spaces. We firstly introduce a supra $\beta$-compact spaces and some related properties and point out the relationship among them. Further, we introduce supra $\beta$-locally finite collection and and several results of these spaces in supra topological spaces. In the end, we presented the concept of supra $\beta$-paracompact spaces and several properties concerning to these spaces in supra topological spaces.

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