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Strongly*-2 Divisor Cordial Labeling of Cycle Related Graphs

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Abstract

A strongly*-2 divisor cordial labeling of a graph G with the vertex set $V(G)$ is a bijection $a : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ such that each edge cd assigned the label 1 if the lower integal value of sum of $a(c)$, $a(d)$ and $a(c)a(d)$ divided by 2 is odd and 0 if lower integal value of sum of $a(c)$, $a(d)$ and $a(c)a(d)$ divided by 2 is even, then the number of edges labeled with 0 and the number of edges labeled with 1 differs by atmost 1 or $|e_a(0) - e_a(1)| \leq 1$ where $e_a(0)$ denotes the number of edges labeled with 0 and $e_a(1)$ denotes the number of edges labeled with 1. A graph which admits a strongly*-2 divisor cordial labeling is called a srongly*-2 divisor cordial graph. In this paper, we prove that the wheel graph W_g and sunflower graph SF_g are strongly*-2 divisor cordial graphs.

Keywords: Bijection, Cordial labeling, Strongly*-graph, Strongly*-2 divisor cordial graph.

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1 Introduction

Graph theory, a mathematical discipline exploring relationships between objects, utilizes structures called graphs. A graph is represented as $G = (V, E)$, where V signifies the vertex set (a finite assortment of elements), and E stands for the edge set (consisting of unordered pairs of distinct elements from V).
In the 1960s, the concept of graph labeling emerged, involving the assignment of integers to vertices, edges, or both, subject to specific conditions. We pursue Harary [1988] for looking over the fundamental terms and notations and Gallian [2018] for a comprehensive understanding of graph labeling.

In 1987, Cahit [1987] hailed cordial labeling. A graph is deemed cordial if it can be labeled with 0s and 1s on its vertices in a way that, when the edges are labeled with the difference of the labels at their endpoints, the difference in the counts of vertices (or edges) labeled with ones and zeros is, at most, one. R. Varatharajan [2011] pioneered the concept of divisor cordial labeling.

A graph is pertained to as a strongly*-graph of order n if it can be labeled with values from 1 to n assigned to its vertices in such a manner that, for any edge labeled with the value $m + z + mz$, where m and z are the labels of its connecting vertices, every edge label is unique. C. Adiga [1999] demonstrated a significant result by proving that all trees, cycles, and grids inherently possess the characteristics of strongly*-graphs.

S.T.A. Mary [2018] establised the strongly*-graph labeling of some more graphs. J. Baskar [2009] have demonstrated that wheels, paths, fans, crowns, and umbrellas can be characterized as vertex strongly*-graphs. M.A. Seoud [2012] offered some technical necessary conditions for a graph to be strongly*-graph.
Stimulated by these, C. Jayasekaran [2020] introduced strongly*-2 divisor cordial labeling. In this paper, we prove that the wheel graph W_g and sunflower graph SF_g are strongly*-2 divisor cordial graphs. Also we delivered the vital definitions that are obligatory for our eternal work.

Definition 1.1. *A wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle.*

Definition 1.2. *The sunflower graph is obtained from a wheel graph with the central vertex b, the cycle $C_g : f_1 f_2 ... f_g f_1$ and additional vertices $j_1, j_2, ..., j_g$ where j_s is joined by edges to f_s, f_{s+1} where f_{s+1} is taken modulo g.*

2 Main Results

Theorem 2.1. *The wheel graph W_g is a strongly*-2 divisor cordial graph.*

Proof. Let $V(W_g) = \{b, f_h : 1 \leq h \leq g\}$ and $E(W_g) = \{bf_h, f_s f_{s+1}, f_g f_1 : 1 \leq h \leq g, 1 \leq s \leq g-1\}$. Then W_g has $g+1$ vertices and $2g$ edges.

Case 1. $g \equiv 0(mod\, 4)$

Define $a : V(W_g) \to \{1, 2, ..., g+1\}$ by $a(b) = 1$ and $a(f_h) = h+1$, $1 \leq h \leq g$. Let a^* be the induced edge label of a and let $g = 4d$, $d \in N$.

Consider the edges $bf_h, 1 \leq h \leq g$. Then $a^*(bf_h)$ = $\lfloor \frac{a(b)+a(f_h)+a(b)a(f_h)}{2} \rfloor$ = $\lfloor \frac{1+h+1+h+1}{2} \rfloor$ = $\lfloor \frac{2h+3}{2} \rfloor$ = $h+1$ which is even if h is odd and odd if h is even. Therefore, the edges $bf_h, 1 \leq h \leq g$ get label 0 if h is odd and 1 if h is even. (1)

For the edges $f_h f_{h+1}, 1 \leq h \leq g-1$, $a^*(f_h f_{h+1})$ = $\lfloor \frac{h+1+h+2+(h+1)(h+2)}{2} \rfloor$ = $\lfloor \frac{h^2+5h+5}{2} \rfloor$ which is even if $h \equiv 0, 3(mod\, 4)$ and odd if $h \equiv 1, 2(mod\, 4)$. Therefore, the edges $f_h f_{h+1}$ get label 0 if $h \equiv 0, 3(mod\, 4)$ and 1 if $h \equiv 1, 2(mod\, 4), 1 \leq h \leq g-1$. (2)

For the edge $f_g f_1$, $a^*(f_g f_1)$ = $\lfloor \frac{g+1+1+g+1}{2} \rfloor$ = $\lfloor \frac{2g+3}{2} \rfloor$ = $g+1$ which is even since g is even. Therefore, the edge $f_g f_1$ gets label 0.

Let us find the number of edges with label 0 and 1. Now $g = 4d$ implies that there are $2g = 8d$ edges. Among these, there are $g = 4d$ edges of the form $bf_h, 1 \leq h \leq g$ and so there are $2d$ even numbers and odd numbers. By (1), there are $2d$ edges get label 0 and $2d$ edges get label 1. Also there are $g-1 = 4d-1$ edges of the form $f_h f_{h+1}, 1 \leq h \leq g-1$ and so there are d numbers which are congruent to h modulo 4, $(h = 1, 2, 3)$ and $d-1$ numbers congruent to 0 modulo 4. By (2), there are $2d-1$ edges get label 0 and $2d$ edges get label 1 and the edge $f_g f_1$ gets label 0. Hence there are $4d$ edges with label 0 and $4d$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

Case 2. $g \equiv 1(mod\, 4)$

Define $a : V(W_g) \to \{1, 2, ..., g+1\}$ by $a(b) = g+1$ and $a(f_h) = h$, $1 \leq h \leq g$. Let a^* be the induced edge label of a and let $g = 4d+1$, $d \in N$.

For the edges $bf_h, 1 \leq h \leq g,\ a^*(bf_h) = \lfloor \frac{g+1+h+(g+1)h}{2} \rfloor = \lfloor \frac{g+1+gh+2h}{2} \rfloor$ = $\lfloor \frac{4d+1+1+(4d+1)h+2h}{2} \rfloor$ = $2d(1+h)$+$\lfloor \frac{3h+2}{2} \rfloor$ which is even if $h \equiv 1, 2(mod\, 4)$ and odd if $h \equiv 0, 3(mod\, 4)$. Therefore, the edges $bf_h, 1 \leq h \leq g$ get label 0 if $h \equiv 1, 2(mod\, 4)$ and 1 if $h \equiv 0, 3(mod\, 4)$. (1)

Consider the edges $f_h f_{h+1}, 1 \leq h \leq g-1$. Then $a^*(f_h f_{h+1})$ = $\lfloor \frac{h+h+1+h(h+1)}{2} \rfloor$ = $\lfloor \frac{h^2+3h+1}{2} \rfloor$ which is even if $h \equiv 0, 1(mod\, 4)$ and odd if $h \equiv 2, 3(mod\, 4)$. Therefore, the edges $f_h f_{h+1}, 1 \leq h \leq g-1$ get label 0 if $h \equiv 0, 1(mod\, 4)$ and 1 if $h \equiv 2, 3(mod\, 4)$. (2)

For the edge $f_g f_1$, $a^*(f_g f_1)$ = $\lfloor \frac{1+g+g}{2} \rfloor$ = $\lfloor \frac{2g+1}{2} \rfloor$ = g which is odd since g is odd and so the edge $f_g f_1$ gets label 1.

Let us find the number of edges with label 0 and 1. Now $g = 4d+1$ implies that there are $2g = 8d+2$ edges. Among these, there are $4d+1$ edges of the form $bf_h, 1 \leq h \leq g$ such that there are d numbers which are congruent to h

modulo 4, $(h = 0, 2, 3)$ and $d+1$ numbers which are congruent to 1 modulo 4. By (1), there are $2d+1$ edges get label 0 and $2d$ edges get label 1. Also there are $g-1 = 4d$ edges of the form $f_hf_{h+1}, 1 \leq h \leq g-1$ and so there are d numbers which are congruent to h modulo 4, $(h = 0, 1, 2, 3)$. By (2), there are $2d$ edges get label 0 and $2d$ edges get label 1 and the edge f_gf_1 gets label 1. Hence, there are $4d+1$ edges with label 0 and $4d+1$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

Case 3. $g \equiv o(mod\ 4)$, $o \in \{2, 3\}$

Define $a : V(W_g) \to \{1, 2, ..., g+1\}$ by $a(b) = 2$, $a(f_1) = 1$, $a(f_{g-1}) = 3$, $a(f_g) = g+1$ and $a(f_h) = h+2$, $2 \leq h \leq g-2$. Let a^* be the induced edge label of a.

Consider the edges $bf_h, 1 \leq h \leq g$. For the edge bf_1, $f^*(bf_1)$ = $\lfloor \frac{2+1+2}{2} \rfloor$ = $\lfloor \frac{5}{2} \rfloor$ = 2 which is even and thereby the edge bf_1 gets label 0.

For the edges $bf_h, 2 \leq h \leq g-2$. $a^*(bf_h)$ = $\lfloor \frac{2+h+2+2(h+2)}{2} \rfloor$ = $4 + \lfloor \frac{3h}{2} \rfloor$ which is even if $h \equiv 0, 3(mod\ 4)$ and odd if $h \equiv 1, 2(mod\ 4)$. Therefore, the edges $bf_h, 2 \leq h \leq g-2$ get label 0 if $h \equiv 0, 3(mod\ 4)$ and 1 if $h \equiv 1, 2(mod\ 4)$. (1)

For the edge bf_{g-1}, $a^*(bf_{g-1})$ = $\lfloor \frac{2+3+6}{2} \rfloor$ = $\lfloor \frac{11}{2} \rfloor$ = 5 which is odd and thereby the edge bf_{g-1} gets label 1.

Consider the edges $f_hf_{h+1}, 1 \leq h \leq g-1$. For the edge f_1f_2, $a^*(f_1f_2)$ = $\lfloor \frac{1+4+4}{2} \rfloor$ = $\lfloor \frac{9}{2} \rfloor$ = 4 which is even and thereby the edge f_1f_2 gets label 0.

For the edges $f_hf_{h+1}, 2 \leq h \leq g-2$, $a^*(f_hf_{h+1})$ = $\lfloor \frac{h+2+h+3+(h+2)(h+3)}{2} \rfloor$ = $\lfloor \frac{h^2+7h+11}{2} \rfloor$ which is even if $h \equiv 2, 3(mod\ 4)$ and odd if $h \equiv 0, 1(mod\ 4)$. Therefore, the edges f_hf_{h+1} get label 0 if $h \equiv 2, 3(mod\ 4)$ and 1 if $h \equiv 0, 1(mod\ 4), 2 \leq h \leq g-2$. (2)

For the edge $f_{g-1}f_g$, $a^*(f_{g-1}f_g)$ = $\lfloor \frac{3+g+1+3(g+1)}{2} \rfloor$ = $\lfloor \frac{4g+7}{2} \rfloor$ = $2g+3$ which is odd and thereby the edge $f_{g-1}f_g$ gets label 1.

Subcase 3.1. $o = 2$

Here $g = 4d+2$. For the edge bf_g, $a^*(bf_g)$ = $\lfloor \frac{2+g+1+2(g+1)}{2} \rfloor$ = $\lfloor \frac{3g+5}{2} \rfloor$ = $\lfloor \frac{12d+11}{2} \rfloor$ = $6d+5$ which is odd and thereby the edge bf_g gets label 1.

For the edge f_gf_1, $a^*(f_gf_1)$ = $\lfloor \frac{g+1+1+g+1}{2} \rfloor$ = $\lfloor \frac{2g+3}{2} \rfloor$ = $g+1$ which is odd since g is even and thereby the edge f_nf_1 gets label 1.

Let us find the number of edges with label 0 and 1. Now $g = 4d+2$ implies that there are $2g = 8d+4$ edges. Among these, there are $g-3 = 4d-1$ edges of the form $bf_h, 2 \leq h \leq g-2$ and so there are d numbers which are congruent to h modulo 4, $(h = 0, 2, 3)$ and $d-1$ numbers congruent to 1 modulo 4 . By (1), there are $2d$ edges get label 0 and $2d-1$ edges get label 1. Also there are $g-3 = 4d-1$ edges of the form $f_hf_{h+1}, 2 \leq h \leq g-2$ and so there are d numbers which are congruent to h modulo 4, $(h = 0, 2, 3)$ and $d-1$ numbers congruent to 1 modulo 4. By (2), there are $2d$ edges get label 0 and $2d-1$ edges get label 1. Also the edges bf_{g-1}, bf_g, $f_{g-1}f_g$ and f_gf_1 get label 1 and the edges bf_1 and f_1f_2 get label

0. Hence, there are $4d+2$ edges with label 0 and $4d+2$ edges with label 1 and so $|e_a(0)-e_a(1)|=0$.

Subcase 3.2. $o=3$

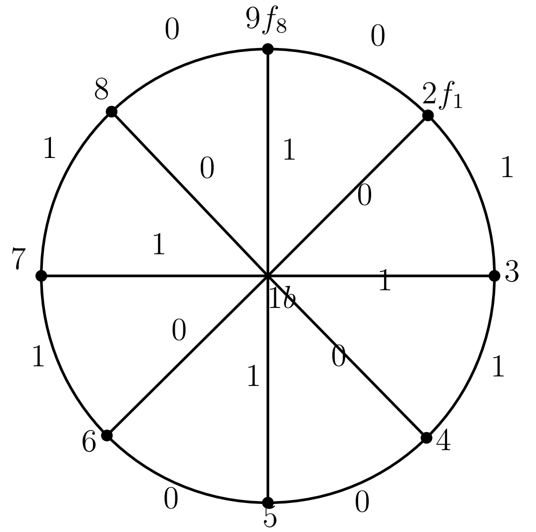
Here $g=4d+3$. For the edge bf_g, as in subcase 2.1., $a^*(bf_g) = \lfloor\frac{3g+5}{2}\rfloor = \lfloor\frac{12d+14}{2}\rfloor = 6d+7$ which is odd and thereby the edge bf_g gets label 1.

For the edge f_gf_1, as in subcase 2.1., $a^*(f_gf_1) = \lfloor\frac{2g+3}{2}\rfloor = g+1$ which is even since g is odd and thereby the edge f_gf_1 gets label 0.

Let us find the number of edges with label 0 and 1. Now $g=4d+3$ implies that there are $2g=8d+6$ edges. Among these, there are $g-3=4d$ edges of the form $bf_h, 2\leq h\leq g-2$ and so there are d numbers which are congruent to h modulo 4, $(0\leq h\leq 3)$. By (1), there are $2d$ edges get label 0 and $2d$ edges get label 1. Also there are $g-3=4d$ edges of the form $f_hf_{h+1}, 2\leq h\leq g-2$ and so there are d numbers which are congruent to h modulo 4, $(0\leq h\leq 3)$. By (2), there are $2d$ edges get label 0 and $2d$ edges get label 1. Also the edges bf_{g-1}, bf_g and $f_{g-1}f_g$ get label 1 and the edges bf_1, f_gf_1 and f_1f_2 get label 0. Hence, there are $4d+3$ edges with label 0 and $4d+3$ edges with label 1 and so $|e_a(0)-e_a(1)|=0$.

Thus, $|e_a(0)-e_a(1)|\leq 1$ and thereby the wheel graph W_g is a strongly*-2 divisor cordial graph. □

Example 2.1. *A strongly*-2 divisor cordial labeling of the graph* W_8 *is shown in Figure 1. Here* $e_a(0)=8$, $e_a(1)=8$ *and hence* $|e_a(0)-e_a(1)|=0$.



$Figure$ 1 : Strongly*-2 divisor cordial labeling of W_8

Theorem 2.2. *The sunflower graph* SF_g *is a strongly*-2 divisor cordial graph.*

Proof. Let $V(SF_g) = \{b, f_h, j_h : 1 \le h \le g\}$ and $E(SF_g) = \{bf_h, f_hj_h, f_sf_{s+1}, f_{s+1}j_s, f_gf_1, j_gj_1 : 1 \le h \le g, 1 \le s \le g-1\}$. Then SF_g has $2g+1$ vertices and $4g$ edges.

Case 1. $g \equiv o(mod\, 4)$, $o \in \{0, 1\}$

Define $a : V(SF_g) \to \{0, 1\}$ by $a(b) = 1$, $a(f_h) = h+1$, $1 \le h \le g$ and $a(j_h) = g+1+h$, $1 \le h \le g$. Let a^* be the induced edge label of a.

For the edges $bf_h, 1 \le h \le g$, $f^*(bf_h) = \lfloor\frac{1+h+1+1(h+1)}{2}\rfloor = \lfloor\frac{2h+3}{2}\rfloor = h+1$ is even if h is odd and odd if h is even. Therefore, the edges $bf_h, 1 \le h \le g$ get label 0 if h is odd and 1 if h is even. (1)

For the edges $f_hf_{h+1}, 1 \le h \le g-1$, $f^*(f_hf_{h+1}) = \lfloor\frac{h+1+h+2+(h+1)(h+2)}{2}\rfloor = \lfloor\frac{h^2+5h+5}{2}\rfloor$ which is even if $h \equiv 0, 3(mod\, 4)$ and odd if $h \equiv 1, 2(mod\, 4)$. Therefore, the edges f_hf_{h+1} get label 0 if $h \equiv 0, 3(mod\, 4)$ and 1 if $h \equiv 1, 2(mod\, 4)$. (2)

Subcase 1.1. $o = 0$

Here $g = 4d$, $d \in N$. For the edge f_gf_1, $a^*(f_gf_1) = \lfloor\frac{g+1+2+2(g+1)}{2}\rfloor = \lfloor\frac{3g+5}{2}\rfloor = \lfloor\frac{12d+5}{2}\rfloor = 6d+2$ which is even and so the edge f_gf_1 gets label 0.

For the edges $f_hj_h, 1 \le h \le g$, $a^*(f_hj_h) = \lfloor\frac{h+1+g+1+h+(h+1)(g+1+h)}{2}\rfloor = \lfloor\frac{2g+gh+h^2+4h+3}{2}\rfloor = \lfloor\frac{2(4d)+(4d)h+h^2+4h+3}{2}\rfloor = 2(4d+dh+h) + \lfloor\frac{h^2+3}{2}\rfloor$ which is even if h is odd and odd if h is even. Therefore, the edges $f_hj_h, 1 \le h \le g$, get label 0 if h is odd and 1 if h is even. (3)

For the edges $f_{h+1}j_h, 1 \le h \le g-1$, $a^*(f_{h+1}j_h) = \lfloor\frac{h+2+g+1+h+(h+2)(g+1+h)}{2}\rfloor = \lfloor\frac{3g+gh+h^2+5h+5}{2}\rfloor = \lfloor\frac{3(4d)+(4d)h+h^2+5h+5}{2}\rfloor = 2d(3+h) + \lfloor\frac{h^2+5h+5}{2}\rfloor$ which is even if $h \equiv 0, 3(mod\, 4)$ and odd if $h \equiv 1, 2(mod\, 4)$. Therefore, the edges $f_{h+1}j_h, 1 \le h \le g-1$ get label 0 if $h \equiv 0, 3(mod\, 4)$ and 1 if $h \equiv 1, 2(mod\, 4)$. (4)

For the edge j_gf_1, $a^*(j_gf_1) = \lfloor\frac{2g+1+2+2(2g+1)2}{2}\rfloor = \lfloor\frac{6g+5}{2}\rfloor = \lfloor\frac{24d+5}{2}\rfloor = 12d+2$ which is even and so the edge j_gf_1 gets label 0.

Let us find the number of edges with label 0 and 1. Now $g = 4d$ implies that there are $4g = 16d$ edges. Among these, there are $g = 4d$ edges of the form bf_h and $f_hj_h, 1 \le h \le g$ and so there are $2d$ odd and $2d$ even numbers. By (1) and (3), there are $4d$ edges get label 0 and $4d$ edges get label 1. Also there are $g-1 = 4d-1$ edges of the form f_hf_{h+1} and $f_{h+1}j_h$, $1 \le h \le g-1$ and so there are d numbers which are congruent to h modulo 4, $(h = 1, 2, 3)$ and $d-1$ numbers which are congruent to 0 modulo 4. By (2) and (4), there are $4d-2$ edges get label 0 and $4d$ edges get label 1. And the edgse f_gf_1 and j_gj_1 get label 0. Hence, there are $8d$ edges with label 0 and $8d$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

Subcase 1.2. $o = 1$

Here $g = 4d+1$, $d \in N$. For the edge f_gf_1, $a^*(f_gf_1) = \lfloor\frac{g+1+2+2(g+1)}{2}\rfloor = \lfloor\frac{3g+5}{2}\rfloor = \lfloor\frac{3(4d+1)+5}{2}\rfloor = 6d+4$ which is even and so the edge f_gf_1 gets label 0.

For the edges $f_hj_h, 1 \leq h \leq g$, $a^*(f_hj_h) = \lfloor\frac{h+1+g+1+h+(h+1)(g+1+h)}{2}\rfloor$ $= \lfloor\frac{2g+gh+h^2+4h+3}{2}\rfloor = \lfloor\frac{2(4d+1)+(4d+1)h+h^2+4h+3}{2}\rfloor = 2d(2+h) + \lfloor\frac{h^2+5h+5}{2}\rfloor$ which is even if $h \equiv 0, 3(mod\ 4)$ and odd if $h \equiv 1, 2(mod\ 4)$. Therefore, the edges $f_hj_h, 1 \leq h \leq g$, get label 0 if $h \equiv 0, 3(mod\,4)$ and 1 if $h \equiv 1, 2(mod\,4)$. (5)

For the edges $f_{h+1}j_h, 1 \leq h \leq g-1$, $a^*(f_{h+1}j_h) = \lfloor\frac{h+2+g+1+h+(h+2)(g+1+h)}{2}\rfloor$ $= \lfloor\frac{3g+gh+h^2+5h+5}{2}\rfloor = \lfloor\frac{3(4d+1)+(4d+1)h+h^2+5h+5}{2}\rfloor = 2d(3+h) + 4 + \lfloor\frac{h^2+6h}{2}\rfloor$ which is even if $h \equiv 0, 2(mod\ 4)$ and odd if $h \equiv 1, 3(mod\ 4)$. Hence, the edges $f_{h+1}j_h, 1 \leq h \leq g-1$, get label 0 if $h \equiv 0, 2(mod\,4)$ and 1 if $h \equiv 1, 3(mod\,4)$.(6)

For the edge j_gf_1, $a^*(j_gf_1) = \lfloor\frac{2g+1+2+(2g+1)2}{2}\rfloor = \lfloor\frac{6g+5}{2}\rfloor = \lfloor\frac{6(4d+1)+5}{2}\rfloor =$ $12d+5$ which is odd and so the edge j_gf_1 gets label 1.

Let us find the number of edges with label 0 and 1. Now $g = 4d+1$ implies that there are $4g = 16d+4$ edges. Among these, there are $g = 4d+1$ edges of the form $bf_h, 1 \leq h \leq g$ and so there are $2d+1$ odd and $2d$ even numbers. By (1), there are $2d+1$ edges get label 0 and $2d$ edges get label 1. Also there are $g-1 = 4d$ edges of the form f_hf_{h+1} and $f_{h+1}j_h, 1 \leq h \leq g-1$ and so there are d numbers which are congruent to h modulo 4, $(0 \leq h \leq g)$. By (2) and (6), there are $4d$ edges get label 0 and $4d$ edges get label 1. There are $g = 4d+1$ edges of the form $f_hj_h, 1 \leq h \leq g$ and so there are d numbers which are congruent to h modulo 4, $(h = 0, 2, 3)$ and there are $d+1$ numbers which are congruent to 1 modulo 4. Hence by (5), $2d+1$ edges get label 1 and $2d$ edges get label 0. And the edge f_gf_1 gets label 0 and the edge j_gf_1 gets label 1. Hence, there are $8d+2$ edges with label 0 and $8d+2$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

Case 2. $g \equiv l(mod\,4), l \in \{2, 3\}$

Define $a : V(SF_g) \to \{1, 2, ..., 2g+1\}$ by $a(b) = 1$, $a(f_h) = 2h$, $1 \leq h \leq g$ and $a(j_h) = 2h+1$, $1 \leq h \leq g$. Let a^* be the induced edge label of a.

For the edges $bf_h, 1 \leq h \leq g$, $a^*(bf_h) = \lfloor\frac{1+2h+2h}{2}\rfloor = \lfloor\frac{4h+1}{2}\rfloor = 2h$ which is even and thereby the edges bf_h get label 0 for $1 \leq h \leq g$. (1)

For the edges $f_hf_{h+1}, 1 \leq h \leq g-1$, $a^*(f_hf_{h+1}) = \lfloor\frac{2h+2h+2+2h(2h+2)}{2}\rfloor$ $= \lfloor\frac{4h^2+8h+2}{2}\rfloor = 2h^2+4h+1$ which is odd. Therefore, the edges f_hf_{h+1} get label 1 for $1 \leq h \leq g-1$. (2)

For the edges $f_hj_h, 1 \leq h \leq g$, $a^*(f_hj_h) = \lfloor\frac{2h+2h+1+2h(2h+1)}{2}\rfloor = \lfloor\frac{4h^2+6h+1}{2}\rfloor$ $= 2h^2+3h$ which is even if h is even and odd if h is odd. Therefore, the edges $f_hj_h, 1 \leq h \leq g$, get label 0 if h is even and 1 if h is odd. (3)

For the edges $f_{h+1}j_h, 1 \leq h \leq g-1$, $a^*(f_{h+1}j_h) = \lfloor\frac{2h+2+2h+1+(2h+2)(2h+1)}{2}\rfloor$ $= \lfloor\frac{4h^2+10h+5}{2}\rfloor = 2h^2+5h+2$ which is even if h is even and odd if h is odd. Therefore, the edges $f_{h+1}j_h, 1 \leq h \leq g-1$, get label 0 if h is even and 1 if h is odd. (4)

Subcase 2.1. $l = 2$

Here $g = 4d + 2$, $d \in N$. For the edge $f_g f_1$, $a^*(f_g f_1)$ = $\lfloor\frac{2g+2+2g(2)}{2}\rfloor$ = $\lfloor\frac{6g+2}{2}\rfloor$ = $\lfloor\frac{6(4d+2)+2}{2}\rfloor$ = $12d + 7$ which is odd and so the edge $f_g f_1$ gets label 1.

For the edge $j_g f_1$, $a^*(j_g f_1)$ = $\lfloor\frac{2g+1+2+(2g+1)2}{2}\rfloor$ = $\lfloor\frac{6g+5}{2}\rfloor$ = $\lfloor\frac{6(4d+2)+5}{2}\rfloor$ = $12d + 8$ which is even and so the edge $j_g f_1$ gets label 0.

Let us find the number of edges with label 0 and 1. Now $g = 4d+2$ implies that there are $4g = 16d+8$ edges. Among these, there are $g = 4d+2$ edges are of the form $bf_h, 1 \leq h \leq g$. By (1), $4d+2$ edges get label 0. There are $g-1 = 4d+1$ edges of the form $f_h f_{h+1}, 1 \leq h \leq g-1$. By (2), $4d+1$ edges get label 1. Also there are $g = 4d+2$ edges of the form $f_h j_h, 1 \leq h \leq g$ and so there are equal odd and even nubers. By (3), there are $2d+1$ edges get label 0 and $2d+1$ edges get label 1. There are $g-1 = 4d+1$ edges of the form $f_{h+1} j_h, 1 \leq h \leq g-1$ and so there are $2d$ even and $2d+1$ odd numbers. By (4), there are $2d$ edges get label 0 and $2d+1$ edges get label 1. And the edge $f_g f_1$ gets label 1 and the edge $j_g f_1$ gets label 0. Hence, there are $8d+4$ edges with label 0 and $8d+4$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$.

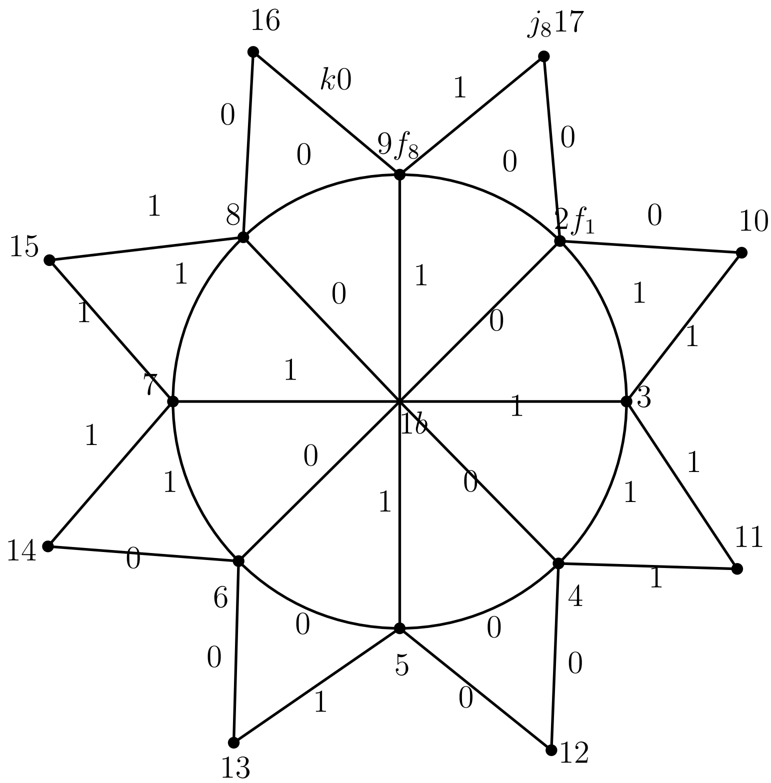
Subcase 2.2. $l = 3$

Here $g = 4d+3$, $d \in N$. For the edge $f_g f_1$, as in subcase 2.1., $a^*(f_g f_1)$ = $\lfloor\frac{6g+2}{2}\rfloor$ = $\lfloor\frac{6(4d+3)+2}{2}\rfloor$= $12d + 10$ which is even and so the edge $f_g f_1$ gets label 0.

For the edge $j_g f_1$, as in subcase 2.1., $a^*(j_g f_1)$ = $\lfloor\frac{6g+5}{2}\rfloor$ = $\lfloor\frac{6(4d+3)+5}{2}\rfloor$ = $12d + 11$ which is odd and so the edge $j_g f_1$ gets label 1.

Let us find the number of edges with label 0 and 1. Now $g = 4d+3$ implies that there are $4g = 16d+12$ edges. Among these, there are $g = 4d+3$ edges of the form $bf_h, 1 \leq h \leq g$. By (1), $4d+3$ edges get label 0. There are $g-1 = 4d+2$ edges of the form $f_h f_{h+1}, 1 \leq h \leq g-1$. By (2), $4d+2$ edges get label 1. Also there are $g = 4d+3$ edges of the form $f_h j_h, 1 \leq h \leq g$ and so there are $2d+2$ odd and $2d+1$ even nubers. By (3), there are $2d+1$ edges get label 0 and $2d+2$ edges get label 1. There are $g-1 = 4d+2$ edges are of the form $f_{h+1} j_h, 1 \leq h \leq g-1$ and so there are equal even and odd numbers. By (4), there are $2d+1$ edges get label 0 and $2d+1$ edges get label 1. And the edge $f_g f_1$ gets label 0 and the edge $j_g f_1$ gets label 1. Hence, there are $8d+6$ edges with label 0 and $8d+6$ edges with label 1 and so $|e_a(0) - e_a(1)| = 0$. □

Example 2.2. *A strongly*-2 divisor cordial labeling of the graph SF_8 is shown in Figure 2. Here $e_a(0) = 16$, $e_a(1) = 16$ and hence $|e_a(0) - e_a(1)| = 0$.*



$Figure\ 2$: Strongly*-2 divisor cordial labeling of SF_8

3 Conclusion

Motivated by the concepts of cordial labeling, divisor cordial labeling and strongly*- graphs in graph theory, we introduce a novel notion termed "strongly*-2 divisor cordial labeling." Our research includes a proof demonstrating that both the wheel graph W_g and the sunflower graph SF_g exhibit the property of being strongly*-2 divisor cordial graphs. Notably, our investigation extends beyond these specific graphs, as we assert that strongly*-2 divisor cordial labeling is applicable to a broader range of graphs. Looking ahead, we envision the development of an algorithm capable of determining whether a given graph admits a strongly*-2 divisor cordial labeling.

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