Some properties and extended Binet's formula for the class of bifurcating Fibonacci sequence

Daksha M Diwan*
Devbhadra V Shah†
Vandana R Patel‡

Abstract

One of the generalizations of Fibonacci sequence is a k-Fibonacci sequence, which is further generalized in several other ways, some by conserving the initial conditions and others by conserving the related recurrence relation. In this paper, we generalize the sequence of k-Fibonacci numbers into the sequence of bifurcating Fibonacci numbers. The Binet-like formula for the terms of these numbers is obtained and further, we obtain several interesting properties related to the sequence.

Keywords: Fibonacci sequence, bifurcating Fibonacci sequence, generalization of Fibonacci sequence, Binet formula, identities related to the Fibonacci sequence.

2010 AMS subject classification: 11B39, 11B83, 11B99.§

---

*Department of Mathematics, Government Engineering College, Gandhinagar, India; dm_diwan@yahoo.com.
† Department of Mathematics, Veer Narmad South Gujarat University, Surat, India; drdvshah@yahoo.com.
‡Department of Mathematics ,Sir P.T.Science College, Modasa, India; vandanapatel483@gmail.com.
§ Received on July 13, 2023. Accepted on January 19, 2024. Published on January 31, 2024. DOI:10.23755/rm.v51i0.1293. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

The well-known sequence $\{F_n\}_{n\geq 0}$ of Fibonacci numbers is a sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, … , where each term is the sum of two preceding terms. The corresponding recurrence relation for the terms of the sequence is $F_n = F_{n-1} + F_{n-2}$; $n \geq 2$ with the initial terms $F_0 = 0, F_1 = 1$. Many papers regarding various generalizations of this sequence have appeared in recent years [2, 3, 12]. There are fundamentally two ways in which the Fibonacci sequence may be generalized; namely, either by maintaining the recurrence relation but altering the first two terms of the sequence from 0, 1 to any arbitrary integers a, b or by preserving the first two terms of the sequence but altering the recurrence relation. The two techniques can even be combined. It is observed that the change in the recurrence relation leads to greater complexity in the properties of the resulting sequence. Edson, Yayenie [8] generalized this sequence into a new class of generalized sequence, which depends on two real parameters used in a recurrence relation. We define further generalizations of this sequence and call it the generalized Fibonacci sequence.

Definition: For any two positive numbers a and b, the class of bifurcating sequences $\left\{f_n^{L(a,\,b)}\right\}(=\{f_n^L\})$ is defined by the recurrence relation $f_n^L = a^{\chi(n)} b^{1-\chi(n)} f_{n-1}^L - f_{n-2}^L$, where $f_0^L = 0, f_1^L = 1$; a, b are any two positive integers and $\chi(n) = \begin{cases} 1; \text{if } n \text{ is odd} \\ 0; \text{if } n \text{ is even} \end{cases}$.

This can be equivalently expressed as

$$f_n^{L(a,\,b)} = \begin{cases} a f_{n-1}^{L(a,\,b)} - f_{n-2}^{L(a,\,b)} \text{ ; when } n \text{ is odd} \\ b f_{n-1}^{L(a,\,b)} - f_{n-2}^{L(a,\,b)} \text{ ; when } n \text{ is even} \end{cases} (n \geq 2) \text{ with } f_0^L = 0, f_1^L = 1 \qquad \textbf{(1.1)}$$

The first few terms of this sequence are shown below:

$\boldsymbol{n}$	$\boldsymbol{f_n^{L(a,\,b)}}$
0	0
1	1
2	b
3	$ab - 1$
4	$ab^2 - 2b$
5	$a^2b^2 - 3ab + 1$
6	$a^2b^3 - 4ab^2 + 3b$
7	$a^3b^3 - 5a^2b^2 + 6ab - 1$

2. Generating function for $\left\{ f_n^{L(a,b)} \right\}$

Generating function is a powerful tool for solving linear homogeneous recurrence relations. In this section, we obtain the generating function for the bifurcating Fibonacci sequence and use it to obtain Binet-like formula for these numbers. We first prove a result which will be required to obtain the generating function.

We first prove an interesting recursive relation which connects three f_n^L's into a linear relation.

Lemma 2.1: $f_{2n+5}^L - (ab-2)f_{2n+3}^L + f_{2n+1}^L = 0$.

Proof: Using definition, we get

$f_{2n+5}^L = af_{2n+4}^L - f_{2n+3}^L = a(bf_{2n+3}^L - f_{2n+2}^L) - f_{2n+3}^L = (ab-1)f_{2n+3}^L - af_{2n+2}^L$.

Then, $f_{2n+5}^L - (ab-2)f_{2n+3}^L + f_{2n+1}^L$

$= (ab-1)f_{2n+3}^L - af_{2n+2}^L - (ab-2)f_{2n+3}^L + f_{2n+1}^L$

$= f_{2n+3}^L - (af_{2n+2}^L - f_{2n+1}^L) = 0.$

We next prove a relation which will be used to derive the generating of f_n^L.

Lemma 2.2: $\sum_{i=1}^{\infty} f_{2i-1}^L x^{2i-1} = \frac{x+x^3}{x^4-(ab-2)x^2+1}$.

Proof: Let $p(x) = \sum_{i=1}^{\infty} f_{2i-1}^L x^{2i-1} = f_1^L x + f_3^L x^3 + f_5^L x^5 + \cdots$.

Then $x^4 p(x) = f_1^L x^5 + f_3^L x^7 + f_5^L x^9 + \cdots$ and

$(ab-2)x^2 p(x) = (ab-2)f_1^L x^3 + (ab-2)f_3^L x^5 + (ab-2)f_5^L x^7 + \cdots$.

Using lemma 2.1, we get $(x^4 - (ab-2)x^2 + 1)p(x) = x + x^3$.

This gives $p(x) = \sum_{i=1}^{\infty} f_{2i-1}^L x^{2i-1} = \frac{x+x^3}{x^4-(ab-2)x^2+1}$.

The next theorem provides the generating function for $\{f_n^L\}$.

Theorem 2.3: The generating function for $\{f_n^L\}$ is given by $f(x) = \frac{x+bx^2+x^3}{x^4-(ab-2)x^2+1}$.

Proof: Define $f(x) = \sum_{i=0}^{\infty} f_i^L x^i = f_0^L + f_1^L x + f_2^L x^2 + f_3^L x^3 + \cdots$.

Then $-axf(x) = \sum_{i=0}^{\infty} f_i^L x^i = -axf_0^L - af_1^L x^2 - af_2^L x^3 - af_3^L x^4 + \cdots$ and

$x^2 f(x) = \sum_{i=0}^{\infty} f_i^L x^i = f_0^L x^2 + f_1^L x^3 + f_2^L x^4 + f_3^L x^5 + \cdots$.

After the rearrangement of terms, we get

$(1 - ax + x^2)f(x)$

$= f_0^L + (f_1^L - af_0^L)x + (f_2^L - af_1^L + f_0^L)x^2 + (f_3^L - af_2^L + f_1^L)x^3 + \cdots$
$= x + (b-a)x^2 + (\,af_2^L - f_1^L -\, af_2^L + f_1^L)x^3 + (bf_3^L - f_2^L - af_3^L + f_2^L)x^4 + \cdots$

Then $(1 - ax + x^2)f(x) = x + (b-a)x\left(\sum_{i=1}^{\infty} f_{2i-1}^L x^{2i-1}\right)$. This gives

$$f(x) = \frac{x}{(1-ax+x^2)} + \frac{(b-a)x}{(1-ax+x^2)}\left\{\frac{x+x^3}{x^4-(ab-2)x^2+1}\right\}$$
$$= \frac{x^5-(ab-2)x^3+x+bx^2+bx^4-ax^2-ax^4}{(1-ax+x^2)(x^4-(ab-2)x^2+1)} = \frac{(x^5-ax^4+x^3)+(bx^2-abx^3+bx^4)+(x-ax^2+x^3)}{(1-ax+x^2)(x^4-(ab-2)x^2+1)}$$
$$= \frac{(x+bx^2+x^3)(1-ax+x^2)}{(1-ax+x^2)(x^4-(ab-2)x^2+1)} = \frac{x(1+bx+x^2)}{(x^4-(ab-2)x^2+1)}.$$

This finally gives the required generating function as $f(x) = \frac{x+bx^2+x^3}{x^4-(ab-2)x^2+1}$.

3.Binet and additional formulae for $\{f_n^L\}$

For roots α and β of the equation $x^2 - abx + ab = 0$, we observe that $\alpha = \left(\frac{ab+\sqrt{a^2b^2-4ab}}{2}\right)$, $\beta = \left(\frac{ab-\sqrt{a^2b^2-4ab}}{2}\right)$. We note the following properties related with α, β:

1) $\alpha + \beta = ab$. (3.1)

2) $\alpha\beta = ab$. (3.2)

3) $\alpha - \beta = \sqrt{a^2b^2 - 4ab}$. (3.3)

4) $\frac{\alpha^2}{ab} = \alpha - 1$. (3.4)

This is since $\alpha^2 - ab\alpha + ab = 0$, we get $\alpha^2 = ab(\alpha - 1)$,

which gives the required result.

5) $\frac{\beta^2}{ab} = \beta - 1$. (3.5)

6) $(\alpha - 1)(\beta - 1) = 1$. (3.6)

This is since $(\alpha - 1)(\beta - 1) = \left(\frac{\alpha^2}{ab}\right)\left(\frac{\beta^2}{ab}\right) = \frac{\alpha^2b^2}{a^2b^2} = 1$.

7) $\beta(\alpha - 1) = \alpha$. (3.7)

Here we note that $\beta(\alpha - 1) = \beta\left(\frac{\alpha^2}{ab}\right) = \frac{(\alpha\beta)\alpha}{ab} = \frac{(ab)\alpha}{ab} = \alpha$.

8) $\alpha(\beta - 1) = \beta$. (3.8)

We now derive an extended Binet's formula for the $\{f_n^L\}$.

Theorem 3.1: $F_n^L = \frac{b^{1-\chi(n)}}{(ab)^{\lfloor n/2 \rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$, where $\chi(n) = \begin{cases} 1; \text{if } n \text{ is odd} \\ 0; \text{if } n \text{ is even} \end{cases}$.

Proof: By theorem 2.3, we have $f(x) = \frac{x+bx^2+x^3}{x^4-(ab-2)x^2+1}$.

This can be written as $f(x) = \frac{x+bx^2+x^3}{x^4-(ab-2)x^2+1} = \frac{(Ax+B)}{x^2-(\alpha-1)} + \frac{(Cx+D)}{x^2-(\beta-1)}$.

Then, $x + bx^2 + x^3 = (Ax+B)\left(x^2-(\beta-1)\right) + (Cx+D)\left(x^2-(\alpha-1)\right)$.

Now comparing various powers of *x* on both sides, we get

$1 = A + C$; $b = B + D$; $-1 = \left(A(\beta-1) + C(\alpha-1)\right)$ and

$0 = -B(\beta-1) - D(\alpha-1)$.

Solving them we get $A = \frac{\alpha}{\alpha-\beta}, B = \frac{b(\alpha-1)}{(\alpha-\beta)}, C = \frac{-\beta}{\alpha-\beta}, D = \frac{-b(\beta-1)}{\alpha-\beta}$.

Substituting these values in $f(x) = \frac{(Ax+B)}{x^2-(\alpha-1)} + \frac{(Cx+D)}{x^2-(\beta-1)}$, we get

$$f(x) = \frac{1}{(\alpha-\beta)}\left[\left\{\frac{(\alpha-1)b+\alpha x}{x^2-(\alpha-1)}\right\} - \left\{\frac{(\beta-1)b+\beta x}{x^2-(\beta-1)}\right\}\right].$$

Now by the Maclaurin's expansion of $\frac{P-Qx}{x^2-R}$, it is observed that

$$\frac{P-Qx}{x^2-R} = \sum_{n=0}^{\infty} QR^{-n-1}x^{2n+1} - \sum_{n=0}^{\infty} PR^{-n-1}x^{2n}.$$

Then $\frac{(\alpha-1)b-(-\alpha x)}{x^2-(\alpha-1)} = \left[\sum_{n=0}^{\infty} \frac{-\alpha}{(\alpha-1)^{n+1}} x^{2n+1} - \sum_{n=0}^{\infty} \frac{(\alpha-1)b}{(\alpha-1)^{n+1}} x^{2n}\right]$ and

$$\frac{-(\beta-1)b-\beta x}{x^2-(\beta-1)} = \left[\sum_{n=0}^{\infty} \frac{\beta}{(\beta-1)^{n+1}} x^{2n+1} - \sum_{n=0}^{\infty} \frac{-b(\beta-1)}{(\beta-1)^{n+1}} x^{2n}\right].$$

$$\begin{aligned}
\text{Then } f(x) &= \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{-\alpha}{(\alpha-1)^{n+1}} + \frac{\beta}{(\beta-1)^{n+1}}\right)x^{2n+1}\right\} \\
&\quad + \frac{b}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{(\beta-1)}{(\beta-1)^{n+1}} - \frac{(\alpha-1)}{(\alpha-1)^{n+1}}\right)x^{2n}\right\} \\
&= \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{-\alpha(\beta-1)^{n+1}+\beta(\alpha-1)^{n+1}}{(\alpha-1)^{n+1}(\beta-1)^{n+1}}\right)x^{2n+1}\right\} \\
&\quad + \frac{b}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{(\beta-1)(\alpha-1)^{n+1}-(\alpha-1)(\beta-1)^{n+1}}{(\beta-1)^{n+1}(\alpha-1)^{n+1}}\right)x^{2n}\right\}
\end{aligned}$$

Now using (3.4) and (3.5) we get

$$\begin{aligned}
f(x) &= \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{\beta\left(\frac{\alpha^2}{ab}\right)^{n+1} - \alpha\left(\frac{\beta^2}{ab}\right)^{n+1}}{\left(\frac{\alpha^2}{ab}\right)^{n+1}\left(\frac{\beta^2}{ab}\right)^{n+1}}\right)x^{2n+1}\right\} \\
&\quad + \frac{b}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{\left(\frac{\beta^2}{ab}\right)\left(\frac{\alpha^2}{ab}\right)^{n+1} - \left(\frac{\alpha^2}{ab}\right)\left(\frac{\beta^2}{ab}\right)^{n+1}}{\left(\frac{\beta^2}{ab}\right)^{n+1}\left(\frac{\alpha^2}{ab}\right)^{n+1}}\right)x^{2n}\right\}
\end{aligned}$$

$$= \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{(ab)^{n+1}(\beta\alpha^{2n+2}-\alpha\beta^{2n+2})}{\alpha^{2n+2}\beta^{2n+2}}\right)x^{2n+1}\right\}$$
$$+\frac{b}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{(ab)^{n}(\alpha^{2n+2}\beta^{2}-\alpha^{2}\beta^{2n+2})}{\alpha^{2n+2}\beta^{2n+2}}\right)x^{2n}\right\}$$
$$= \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{(ab)^{n+1}\alpha\beta(\alpha^{2n+1}-\beta^{2n+1})}{(\alpha\beta)\alpha^{2n+1}\beta^{2n+1}}\right)x^{2n+1}\right\}$$
$$+\frac{b}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{(ab)^{n}(\alpha\beta)^{2}(\alpha^{2n}-\beta^{2n})}{(\alpha\beta)^{2}\alpha^{2n}\beta^{2n}}\right)x^{2n}\right\}$$
$$= \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{(ab)^{n+1}(\alpha\beta)(\alpha^{2n+1}-\beta^{2n+1})}{(\alpha\beta)\alpha^{2n+1}\beta^{2n+1}}\right)x^{2n+1}\right\}$$
$$+\frac{b}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{(ab)^{n}(\alpha\beta)^{2}(\alpha^{2n}-\beta^{2n})}{(\alpha\beta)^{2}\alpha^{2n}\beta^{2n}}\right)x^{2n}\right\}$$

Using (3.2), we get

$$f(x) = \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{(ab)^{n+1}(\alpha^{2n+1}-\beta^{2n+1})}{(ab)^{2n+1}}\right)x^{2n+1}\right\}$$
$$+\frac{b}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{(ab)^{n}(\alpha^{2n}-\beta^{2n})}{(ab)^{2n}}\right)x^{2n}\right\}$$
$$= \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{1}{ab}\right)^{n}(\alpha^{2n+1}-\beta^{2n+1})x^{2n+1}\right\}$$
$$+\frac{b}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left(\frac{(\alpha^{2n}-\beta^{2n})}{(ab)^{n}}\right)x^{2n}\right\}$$

Considering $\chi(n) = \begin{cases} 1; \text{if } n \text{ is odd} \\ 0; \text{if } n \text{ is even} \end{cases}$ we get $f(x) = \sum_{n=0}^{\infty}\frac{b^{1-\chi(n)}}{(ab)^{[n/2]}}\left(\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\right)x^{n}$.

This finally gives the required extended Binet Formula as $f_n^L = \frac{b^{1-\chi(n)}}{(ab)^{[n/2]}}\left(\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\right)$.

We finally prove two the identities for f_{2k}^L and f_{2k+1}^L which expresses them in the series form.

Lemma 3.2:

a) $f_{2k}^L = \frac{b(ab)^{k-1}}{2^{2k-1}}\sum_{r=1}^{\infty}\binom{2k}{2r-1}\left(1-\frac{4}{ab}\right)^{r-1}$

b) $f_{2k+1}^L = \frac{(ab)^{k}}{2^{2k}}\sum_{r=1}^{\infty}\binom{2k+1}{2r-1}\left(1-\frac{4}{ab}\right)^{r-1}$.

Proof*:* By theorem 3.1, we have $f_n^L = \frac{b^{1-\chi(n)}}{(ab)^{[n/2]}}\left(\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\right)$. If we consider $\mu = \frac{b^{1-\chi(n)}}{(ab)^{[n/2]}(\alpha-\beta)}$, then this can be written as $f_n^L = \mu(\alpha^n-\beta^n)$.

Then $f_n^L = \mu\left(\left(\frac{ab+\sqrt{a^2b^2-4ab}}{2}\right)^{n}-\left(\frac{ab-\sqrt{a^2b^2-4ab}}{2}\right)^{n}\right)$

$$= \frac{\mu}{2^n}\left\{(ab+\sqrt{a^2b^2-4ab})^{n}-(ab-\sqrt{a^2b^2-4ab})^{n}\right\}$$
$$= \frac{\mu(ab)^n}{2^n}\left\{\left(1+\sqrt{1-\frac{4}{ab}}\right)^{n}-\left(1-\sqrt{1-\frac{4}{ab}}\right)^{n}\right\}$$

$$= \frac{\mu(ab)^n \times 2}{2^n}\left\{\binom{n}{1}\sqrt{1-\frac{4}{ab}} + \binom{n}{3}\left(\sqrt{1-\frac{4}{ab}}\right)^3 + \binom{n}{5}\left(\sqrt{1-\frac{4}{ab}}\right)^5 + \cdots\right\}$$

$$= \frac{b^{1-\chi(n)}(ab)^n}{(ab)^{[n/2]}\sqrt{a^2b^2-4ab}\; 2^{n-1}}\sqrt{1-\frac{4}{ab}}\left\{\begin{array}{l}\binom{n}{1} + \binom{n}{3}\left(\sqrt{1-\frac{4}{ab}}\right)^2 \\ + \binom{n}{5}\left(\sqrt{1-\frac{4}{ab}}\right)^4 + \cdots\end{array}\right\}$$

$$= \frac{b^{1-\chi(n)}(ab)^{n-1}}{(ab)^{[n/2]}2^{m-1}}\left\{\binom{n}{1} + \binom{n}{3}\left(\sqrt{1-\frac{4}{ab}}\right)^2 + \binom{n}{5}\left(\sqrt{1-\frac{4}{ab}}\right)^4 + \cdots\right\} \qquad \textbf{(3.9)}$$

Now if n is even, say $n = 2k$, then $\chi(n) = 0$. In this case

$$f_{2k}^L = \frac{b(ab)^{2k-1}}{(ab)^k 2^{2k-1}}\left\{\binom{2k}{1} + \binom{2k}{3}\left(\sqrt{1-\frac{4}{ab}}\right)^2 + \binom{2k}{5}\left(\sqrt{1-\frac{4}{ab}}\right)^4 + \cdots\right\}$$

$$= \frac{b(ab)^{k-1}}{2^{2k-1}}\left\{\sum_{r=1}^{\infty}\binom{2k}{2r-1}\left(\sqrt{1-\frac{4}{ab}}\right)^{2(r-1)}\right\}$$

$$= \frac{b(ab)^{k-1}}{2^{2k-1}}\left\{\sum_{r=1}^{\infty}\binom{2k}{2r-1}\left(1-\frac{4}{ab}\right)^{(r-1)}\right\}.$$

Again, if n is odd, say $n = 2r+1$, we have $\chi(n) = 1$.
Then by (3.9), we have

$$f_{2k+1}^L = \frac{(ab)^{(2k+1)-1}}{(ab)^k 2^{2k}}\left\{\begin{array}{l}\binom{2k+1}{1} + \binom{2k+1}{3}\left(\sqrt{1-\frac{4}{ab}}\right)^2 \\ + \binom{2k+1}{5}\left(\sqrt{1-\frac{4}{ab}}\right)^4 + \cdots\end{array}\right\}$$

$$= \frac{(ab)^k}{2^{2k}}\left\{\sum_{r=1}^{\infty}\binom{2k+1}{2r-1}\left(\sqrt{1-\frac{4}{ab}}\right)^{2(r-1)}\right\}$$

$$= \frac{(ab)^k}{2^{2k}}\left\{\sum_{r=1}^{\infty}\binom{2k+1}{2r-1}\left(1-\frac{4}{ab}\right)^{(r-1)}\right\}.$$

4. Conclusions

In this paper, we considered the sequence of 'bifurcating Fibonacci numbers' and obtained its Binet formula for $f_n^{L(a,\,b)}$ and also obtain some of its interesting properties.

References

[1] De Villiers, M., *A Fibonacci generalization and its dual*, International Journal of Mathematics Education, Science and Technology, 31, (2000), 464 – 473.

[2] Diwan D. M., Shah D. V. Explicit and recursive formulae for the class of generalized Fibonacci sequence. International Journal of Advanced Research in Engineering, Science and Management. 1 (10), 1 – 6, July 2015.

[3] Diwan D. M., Shah D. V. Extended Binet's formula for the class of generalized Fibonacci sequences. VNSGU Journal of Science and Technology. 4 (1), 205 – 210, 2015.

[4] Diwan Daksha M., Shah Devbhadra V.: *Extended Binet's formula for the class of generalized Fibonacci sequences*, Proceeding of 19^{th} Annual Cum 4^{th} International conference of Gwalior Academy of Mathematical Sciences, (2014), 109 – 113.

[5] Falcon S., A simple proof of an interesting Fibonacci generalization, International Journal of Mathematics Education, Science and Technology, 35 (2), (2004), 259 – 261.

[6] Falcon S., Plaza, A: *On the Fibonacci K-numbers*, Chaos, Solitons & Fractals, 32 (5), (2007), 1615 – 1624.

[7] Falcon S., Plaza, A.: *On the 3-dimensional k-Fibonacci spirals*, Chaos, Solitons & Fractals, 38 (4), (2008), 993 – 1003.

[8] Falcon S., Plaza, A: The k-Fibonacci hyperbolic functions, Chaos Solitons & Fractals, 38 (4), (2008), 409 – 420.

[9] Falcon S., Plaza, A: The k-Fibonacci sequence and Pascal 2-triangle, Chaos, Solitons & Fractals, (33) (1), (2007), 38 – 49.

[10] Koshy Thomas: *Fibonacci and Lucas Numbers with applications,* John Wiley and Sons, Inc., N. York, 2001.

[11] Marcia Edson, Omer Yayenie: *A new generalization of Fibonacci sequence and extended Binet's formula*, Integers, 9 (6), (2009), 639 – 654.

[12] Verma, Ankur Bala. On Properties of generalized Bi-variate Bi-Periodic Fibonacci polynomials. International journal of Advanced science and Technology. 29 (3), 8065 – 8072, 2020.