Ratio Mathematica Volume 48, 2023

Relatively Prime Detour Domination Number of Switching Graphs

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Abstract

In this paper, we introduce the concept of relatively prime detour domination number for switching graph. If a set $S \subseteq V$ is a detour set, a dominating set with at least two elements, and has $(deg(u), deg(v)) = 1$ for each pair of vertices u and v, then it is said to be a relatively prime detour dominating set of a graph G. The relatively prime detour domination number of a graph G is known as $\gamma_{rpdn}(G)$ and represents the lowest cardinality of a relatively prime detour dominating set. First, we explain the concept of a switching graph before producing some conclusions based on the switching graphs of helm, fan, complete, spider, and sunlet graphs that have relatively prime detour domination numbers.

Keywords: Domination, Detour domination, Relatively prime domination, Relatively prime detour domination, Switching graphs.

2020 AMS subject classifications: 05C12, 05C69.[1]

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[1] Received on . Accepted on . Published on December 31, 2023 . DOI: 10.23755/rm.v48i0.1289. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Consider a finite undirected graph with no loops or multiple edges, adhering to graph theoretic terminology as outlined in the book by Chartrand and Lesniak[3]. The detour distance, denoted as $D(u, w)$, between vertices u and w is defined as the length of the longest $u - w$ path within a connected graph. A path with a length of $u - w$ is referred to as a detour of length $D(u, w)$, and an $u - w$ path with a length of $D(u, w)$ is termed an $u - w$ detour [1].

The closed interval $I_D[u, w]$ encompasses all vertices situated on some $u - w$ detour of the graph G. For a set $S \subseteq V$, $I_D[S]$ is represented as $I_D[u, w]$ for every $u, w \in S$. A set S of vertices is deemed a detour set if $I_D[S]$ covers the entire vertex set V, and the detour number $dn(G)$ is defined as the minimum cardinality of such a detour set. A detour set with cardinality $dn(G)$ is specifically referred to as a minimum detour set [2].

A set S, a subset of the vertex set $V(G)$, is recognized as a dominating set of the graph G if every vertex in $V(G)$ not in S is adjacent to at least one vertex in S. The domination number of G is denoted by $\gamma(G)$ and signifies the smallest size of a dominating set in G; such a set is referred to as a γ-set. The exploration of the domination number in graph G was conducted in [4]. The concept of detour domination number in a graph was initially introduced in [8]. In a graph $G = (V, E)$, the degree of a vertex u is defined as the count of edges incident with u, denoted by $deg(u)$.

A set S, a subset of V, is considered a relatively prime dominating set if a set S is dominant set and it contains at least two elements, and for every pair of vertices u and w in S, the greatest common divisor of their degrees is 1 (i.e., $(deg(u), deg(w)) = 1$). The smallest cardinality of such a relatively prime dominating set in a graph G is termed the relatively prime domination number and is denoted by $\gamma_{rpd}(G)$ [7]. The concept of switching in graphs, initially introduced by Lint and Seidel [9], is also discussed.

This paper establishes that a dn-set is a minimal detour dominating set, and it introduces the notion of the relatively prime detour domination number for a switching graph, determining the corresponding value denoted as $\gamma_{rpdn}(G^v)$.

Definition 1.1. *[6] A set $S \subseteq V$ is said be relatively prime detour dominating set of a graph G if it is a detour set and a dominating set with at least two elements and for every pair of vertices u and v such that $(deg(u), deg(v)) = 1$. The relatively prime detour domination number is represented by the symbol $\gamma_{rpdn}(G)$ and defines the lowest cardinality of a relatively prime detour dominating set of order $\gamma_{rpdn}(G)$ is said to as a γ_{rpdn}- set of G. If the relatively prime detour dominating set does not exist, then the relatively prime detour domination number is zero.*

Definition 1.2. *[5]For a finite undirected graph $G(V, E)$ and a vertex $v \in V$, the* *switching of G by v is defined as the graph G^v by removing all edges incident to v and adding edges which are not adjacent to v.*

Example 1.1. *Consider the graph G and G^{v_1} are given in Figure 1. Here, $S = \{v_1, v_4, v_6\}$ is a γ_{rpdn}- set of G^{v_1}. Thus $\gamma_{rpdn}(G^{v_1}) = 3$.*

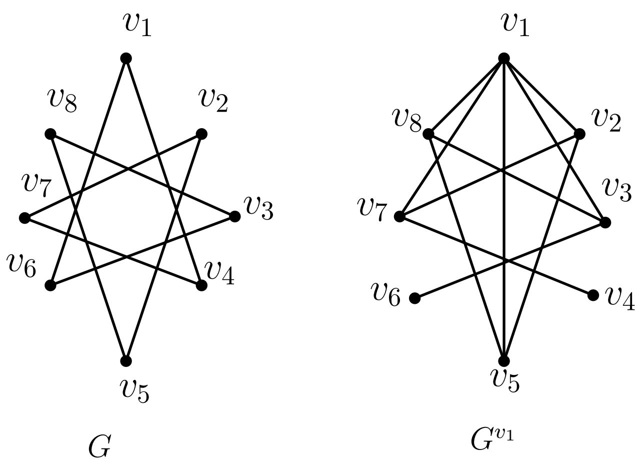


Figure 1: G and Switching of G by v_1 is G^{v_1}

2 Relatively Prime Detour Domination Number of Some Switching Graphs

Theorem 2.1. *Let v be a vertex of helm graph H_n. Then for $n \geq 4$,*

$$\gamma_{rpdn}\left(H_n^v\right) = \begin{cases} 2 & if \;\; deg_{H_n}(v) = 1 \; and \; 2n-3 \not\equiv 0 (mod\, 15) \\ 0 & otherwise \end{cases}$$

.

Proof. Let $v_1 v_2 ... v_{n-1} v_1$ be the cycle C_{n-1}. Add a vertex u which is adjacent to $v_r, 1 \leq r \leq n$. The resultant graph is the wheel W_n. Add u_r, which is adjacent to v_r, for $1 \leq r \leq n-1$. The resultant graph is the helm graph H_n. Then $deg_{H_n}(u) = n-1$, $deg_{H_n}(v_r) = 4$ and $deg_{H_n}(u_r) = 1$ for each $r \in \{1, 2, ..., n\}$.
Case 1. $deg_{H_n}(v) = n-1$

Clearly, v is u and the graph H_n and H_n^u are given in Figure 2. Clearly, $E(H_n^u) = \{uu_r, u_r v_r, v_1 v_{n-1}, v_s v_{s+1} : 1 \leq r \leq n-1, 1 \leq s \leq n-2\}$. In H_n^u, $deg(u) = n-1$, $deg(u_r) = 2$ and $deg(v_r) = 3$, $1 \leq r \leq n-1$

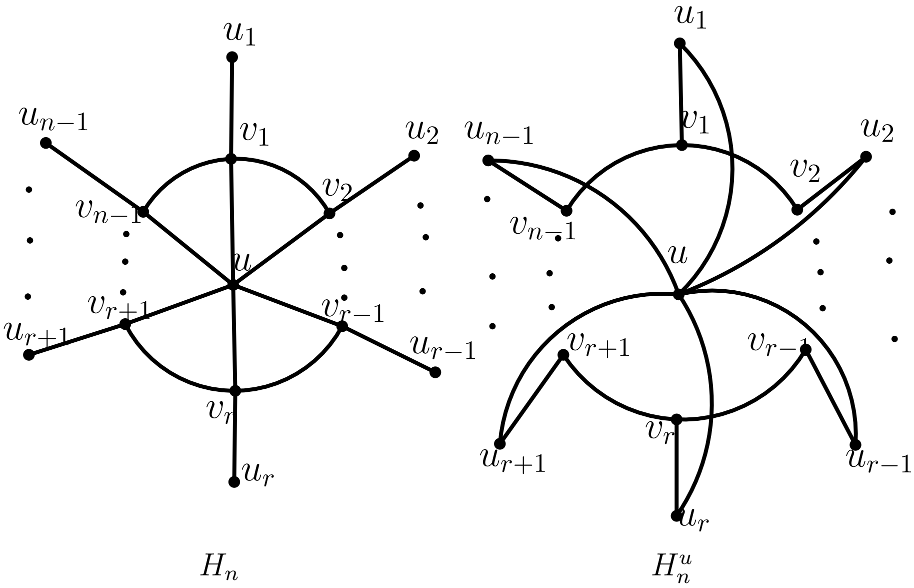


Figure 2: Helm graph H_n and switching of the helm graph by u

and also u is adjacent to all u_r and non-adjacent to all v_r, $1 \leq r \leq n-1$. Since $H_n^u[\{v_1, v_2, ..., v_{n-1}\}] = C_{n-1}$, $S = \{u, v_1, v_3, ..., v_{n-2}\}$ is a γ- set and also $I_D[S] = V(H_n^u)$. Therefore, it follows that S is a γ_{dn}- set of H_n^u. Since S cannot be a γ_{rpdn}- set because the degree sequence of its elements is $(n-1, 3, 3, ..., 3)$, $\gamma_{rpdn}\,(H_n^u) = 0$.

Case 2. $deg_{H_n}(v) = 4$

Here v is v_r, for some r, $1 \leq r \leq n$ the graph $H_n^{v_r}$ is given in Figure 3. Clearly, $H_n^{v_r}$ is a disconnected graph with the isolated vertex u_r and so $\gamma_{rpdn}\,(H_n^{v_r}) = 0$.

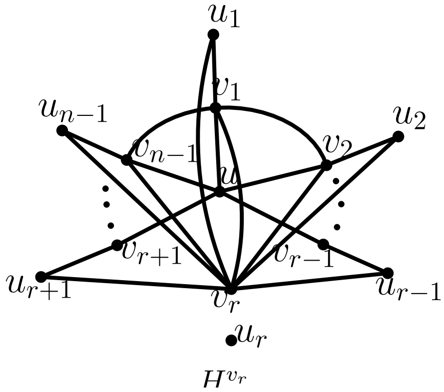


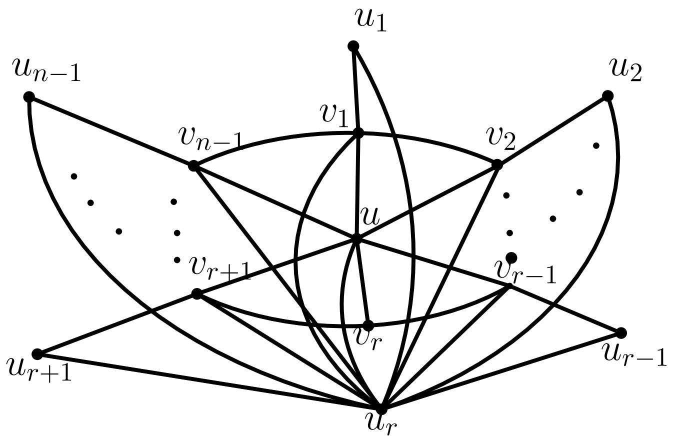
Figure 3: Switching of the helm graph H_n by v_r

Case 3. $deg_{H_n}(v) = 1$ and $2n - 3 \not\equiv 0(mod\, 15)$

In this case v is u_r for some r, for $1 \leq r \leq n-1$ the graph $H_n^{u_r}$ is given in Figure 4. Clearly, $E(H_n^{u_r}) = \{uv_l, v_tv_{t+1}, v_{n-1}v_1, v_ku_k,\ u_ru, u_rv_k, u_ru_k : 1 \leq l \leq n-1, 1 \leq k \leq n-1,\ 1 \leq t \leq n-2$ and $k \neq r\}$ and in $H_n^{u_r}$, $deg(u_r) = 2n-3, deg(v_r) = 3, deg(u) = n$, $deg(u_k) = 2$ and $deg(v_k) = 5$. In $H_n^{u_i}$, u_i is non-adjacent to v_i and v_{i-1} is adjacent to v_i and thereby $S = \{u_i, v_{i-1}\}$ is a γ- set of $H_n^{u_i}$ and also $I_D[S] = V({H_n}^v)$. Hence, S is a γ_{dn}- set of $H_n^{u_i}$. Now $(deg(u_i), deg(v_{i-1})) = (2n-3, 5) = 1$ if and only if $2n-3 \not\equiv 0(mod\, 5)$. Therefore, S is a γ_{rpdn}- set if and only if $2n-3 \not\equiv 0(mod\, 5)$ and hence $\gamma_{rpdn}\,(H_n^{u_i}) = 2$ if and only if $2n-3 \not\equiv 0(mod\, 5)$. . . . (1)

Also in $H_n^{u_r}$, u_r is non-adjacent to v_r and thereby $S^* = \{u_r, v_r\}$ is a γ-set of $H_n^{u_r}$ and also $I_D[S^*] = V({H_n}^v)$. Hence, S^* is a γ_{dn}- set of $H_n^{u_i}$. Now $(deg(u_r), deg(v_r)) = (2n-3, 3) = 1$ if and only if $2n-3 \not\equiv 0(mod\, 3)$. Therefore, S^* is a γ_{rpdn}- set if and only if $2n-3 \not\equiv 0(mod\, 3)$ and hence $\gamma_{rpdn}\,(H_n^{u_r}) = 2$ if and only if $2n-3 \not\equiv 0(mod\, 3)$. . . . (2)

Since $2n-3$ is odd, from (1) and (2), we have $\gamma_{rpdn}\,(H_n^{u_r}) = 2$ if and only if $2n-3 \not\equiv 0(mod\, 15)$. □



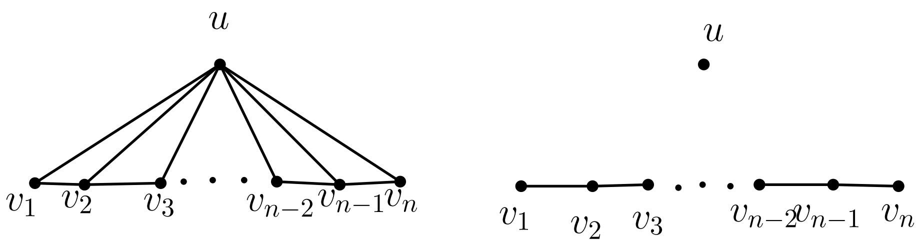
$H_n^{u_r}$

Figure 4: Switching of the helm graph H_n by u_r

Theorem 2.2. *Let v be any vertex of the fan graph $F_{1,n}$. Then*

$$\gamma_{rpdn}\left(F_{1,n}^{v}\right)=\begin{cases}2 & if\ \ n>2\ and\ either\ deg_{F_{1,n}}(v)=2\ or\ deg_{F_{1,n}}(v)=3,\ v\\ & \quad is\ non\ adjacent\ to\ a\ vertex\ of\ degree\ 2\ in\ F_{1,n}\ and\\ & \quad n\ is\ even\\ 3 & if\ \ v\ is\ adjacent\ to\ a\ vertex\ of\ degree\ 2\ in\ F_{1,n}\\ & \quad and\ n>2\ is\ even\\ 0 & otherwise\end{cases}$$

Proof. Let u be a vertex of K_1 and $v_1v_2...v_n$ be a path P_n. The join $P_n+K_1=F_{1,n}$ is a fan graph with $deg_{F_{1,n}}(u)=n, deg_{F_{1,n}}(v_1)=deg_{F_{1,n}}(v_n)=2$ and $deg_{F_{1,n}}(v_r)=3, 2\leq r\leq n-1$. Let v be any vertex of $F_{1,n}$ and let $F_{1,n}^{v}$ be the vertex switching of $F_{1,n}$ with respect to v.



$F_{1,n}$ $\qquad$ $F_{1,n}^{v}$

Figure 5: Fan graph $F_{1,n}$ and switching of the fan graph by v

Case 1. $n=2$

Then $F_{1,2}=C_3$ and for any vertex v in $F_{1,2}$, $F_{1,2}^{v}=K_2\cup K_1$, is a disconnected graph and so $\gamma_{rpdn}\left(F_{1,n}^{v}\right)=0$.

Case 2. $n>2$

Subcase 2.1. $deg_{F_{1,n}}(v)=n$

Here v is u. The graph $F_{1,n}$ and $F_{1,n}^{v}=P_n\cup K_1$ are given in Figure 5 which is a disconnected graph and so $\gamma_{rpdn}\left(F_{1,n}^{v}\right)=0$.

Subcase 2.2. $deg_{F_{1,n}}(v)=2$

Clearly, v is either v_1 or v_n and $F_{1,n}^{v_1}\cong F_{1,n}^{v_n}$. Let v be v_1. Now $E(F_{1,n}^{v})=\{uv_r, vv_k, v_sv_{s+1}: 2\leq r\leq n, 2\leq s\leq n-1, 3\leq k\leq n\}$ and in $F_{1,n}^{v}$, $deg(v)=n-2, deg(u)=n-1, deg(v_2)=2, deg(v_n)=3$ and $deg(v_l)=4, 3\leq l\leq n-1$. The graph $F_{1,n}^{v_1}$ is given in Figure 6. In $F_{1,n}^{v_1}$, v is non-adjacent to only v_2 and u and the vertices u and v_2 are adjacent and so $S = \{u, v\}$ is a γ- set. Also $I_D[u, v] = V[F_{1,n}^{v_1}]$. Hence, S is a γ_{rpdn}- set since $(deg(u), deg(v)) = (n-1, n-2) = 1$ and thereby $\gamma_{rpdn}\left(F_{1,n}^{v_1}\right) = 2$.

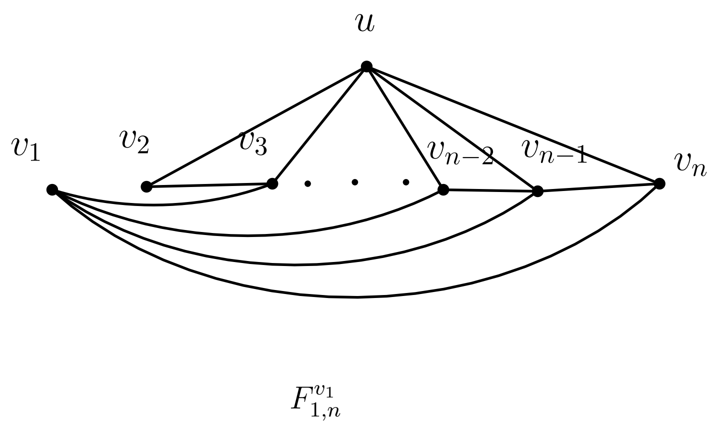


Figure 6: Switching of the fan graph $F_{1,n}$ by v_1

Subcase 2.3. $deg_{F_{1,n}}(v) = 3$
Subcase 2.3.1. v is adjacent to a vertex of degree 2 in $F_{1,n}$

Clearly, v is either v_2 or v_{n-1} and $F_{1,n}^{v_2} \cong F_{1,n}^{v_{n-1}}$. Let v be v_2. The graph $F_{1,n}^{v_2}$ is given in Figure 7. Now $E(F_{1,n}^{v_2}) = \{uv_r, vv_s, v_kv_{k+1} : 1 \leq r \leq n, r \neq 2, 4 \leq s \leq n, 3 \leq k \leq n-1\}$ and in G^v, $deg(v) = n-3, deg(u) = n-1, deg(v_1) = 1, deg(v_3) = 2$ $deg(v_n) = 3$ and $deg(u_k) = 4, 4 \leq k \leq n-1$.

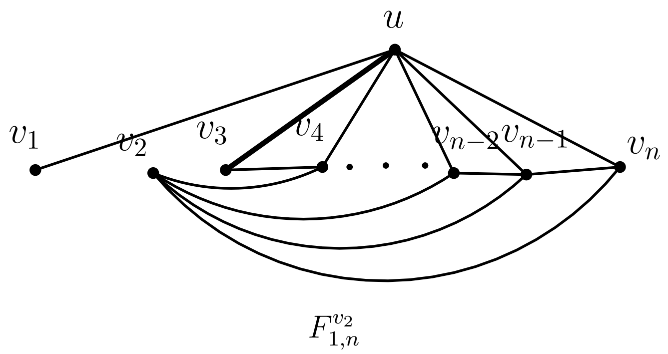


Figure 7: Switching of the fan graph $F_{1,n}$ by v_2

In $F_{1,n}^{v_2}$, v is non-adjacent to v_1, v_3 and u and u is non-adjacent to only vertex v_2 and so $S = \{u, v\}$ is a γ- set for $F_{1,n}^{v_2}$. But $I_D[u, v] \neq V[F_{1,n}^{v_2}]$. For the set $S_1 =$

$S \cup \{v_1\}$, $I_D[S_1] = V[F_{1,n}^{v_2}]$ and so S_1 is a γ_{dn}- set of G^v. Now $(deg(u), deg(v)) = (n-1, n-3) = 1$ if and only if n is even. Thus, S is a γ_{rpdn}- set if and only if n is even and hence $\gamma_{rpdn}\left(F_{1,n}^{v_2}\right) = 3$ if and only if n is even.

Subcase 2.3.2. v is non-adjacent to a vertex of degree 2 in $F_{1,n}$

Here v is v_r, for some r, $3 \leq r \leq n-2$. The graph $F_{1,n}^{v_r}$ is given in Figure 8. Now $E(F_{1,n}^{v_r}) = \{v_r v_s, v_k v_{k+1}, uv_l : 1 \leq r, s \leq n, s \neq r \pm 1, 1 \leq l, k \leq n-1, l \neq r, k \neq r-1, r\}$ and $deg(v_r) = n-3, deg(u) = n-1$ and $deg(v_{r-1}) = 2 = deg(v_{r+1})$. In $F_{1,n}^{v_r}$, v_r is non-adjacent to v_{r-1}, v_{r+1} and u and u is non-adjacent to v_r and so $S = \{u, v_r\}$ is aγ- set for $F_{1,n}^{v_r}$. Also $I_D[u, v_r] = V[F_{1,n}^{v_r}]$. Hence, S is a γ_{dn}- set and also $(deg(u), deg(v)) = (n-1, n-3) = 1$ if and only if n is even. Thus, S is a γ_{rpdn}- set if and only if n is even and hence $\gamma_{rpdn}\left(F_{1,n}^{v_r}\right) = 2$ if and only if n is even. □

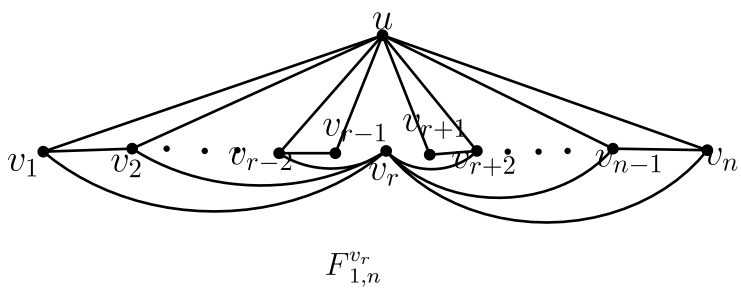


Figure 8: Switching of the fan graph $F_{1,n}$ by v_r

Theorem 2.3. *If v is a vertex of the complete graph K_n, then $\gamma_{rpdn}(K_n^v) = 0$.*

Proof. Since each vertex in K_n is a full vertex, K_n^v is disconnected graph and so $\gamma_{rpdn}(K_n^v) = 0$. □

Theorem 2.4. *Let G be a spider graph $K_{1,n,n}$ with a centre u and let v be any vertex of G. Then*

$$\gamma_{rpdn}(G^v) = \begin{cases} n+1 & if \;\; v \; is \; the \; centre \; vertex \\ 0 & if \;\; v \; is \; a \; to \; support \; vertex \\ 2 & if \;\; v \; is \; an \; end \; vertex \end{cases}$$

.

Proof. Let u be the central vertex and $v_1, v_2, ..., v_n$ be the end vertices of $K_{1,n}$, and let u_r, $1 \leq r \leq n$ be the respective vertices connected with v_r, $1 \leq r \leq n$. The spider graph $K_{1,n,n}$ is the resulting graph, and its vertex set $V(G) = \{u, u_r, v_r : 1 \leq r \leq n\}$ and $E(G) = \{uu_r, u_rv_r : 1 \leq r \leq n\}$. Let v be any vertex of G. Then v is either the centre vertex or an end vertex or a support vertex.

Case 1. v is the centre vertex

Here v is u. The graph $K_{1,n,n}$ and $K^v_{1,n,n}$ are given in Figure 9. Clearly, G^v is a spider graph with end vertices u_r, $1 \leq r \leq n$ and support vertices v_r, $1 \leq r \leq n$. Now $S = \{v, u_1, u_2, ..., u_n\}$ is a γ- set and $I_D[S] = V[G^v]$. In G^v, $deg(v) = n$ and $deg(u_r) = 1, 1 \leq r \leq n$. This gives that S is a γ_{rpdn}- set of G^v and hence $\gamma_{rpdn}(G^v) = n + 1$.

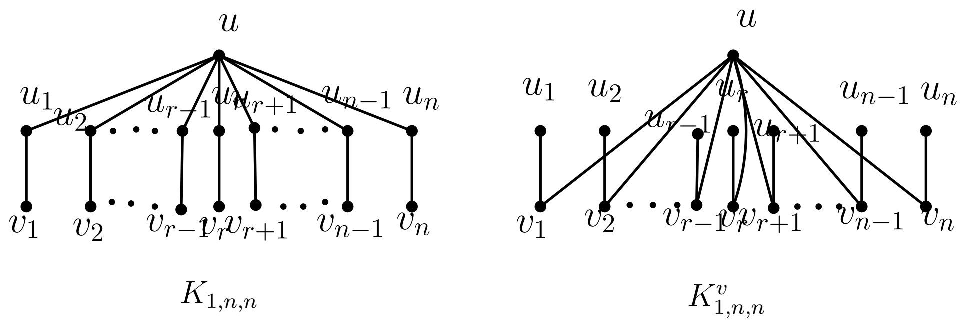


Figure 9: Spider graph $K_{1,n,n}$, Switching of the spider graph $K_{1,n,n}$ by v

Case 2. v is an end vertex

Here v is v_r, for some $r, 1 \leq r \leq n$. Clearly, $G^{v_1} \cong G^{v_2} \cong ... \cong G^{v_n}$. Let v be v_1. The graph G^{v_1} is given in Figure 10.

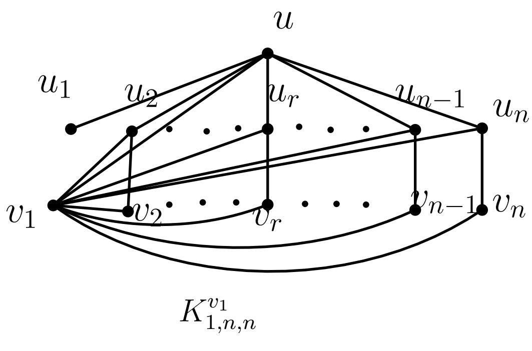


Figure 10: Switching of the spider graph $K_{1,n,n}$ by v_1

Now $E(G^v) = \{vu, vu_r, vv_k, uv_r, u_lv_l : 1 \leq r \leq n, 2 \leq k, l \leq n\}$ and in G^v, $deg(v) = 2n-1$, $deg(u_1) = 1$ $deg(v_r) = 2, deg(u_r) = 3$ and $deg(u) = n+1$, $2 \leq r \leq n$. In G^v, v is adjacent all the vertices other than u_1 and so $S = \{v, u_1\}$ is a γ- set and also $I_D[v, u_1] = V[G^v]$. Hence, S is a γ_{rpdn}- set of G^v since $(deg(v), deg(u_1)) = 1$ and thereby $\gamma_{rpdn}(G^v) = 2$.
Case 3. v is a support vertex

Then v is u_r, for some $r, 1 \leq r \leq n$. Clearly, $G^{u_1} \cong G^{u_2} \cong ... \cong G^{u_n}$ and G^v is a disconnected graph and so $\gamma_{rpdn}(G^v) = 0$. □

Theorem 2.5. *Let v be any vertex of sunlet graph S_n. Then $\gamma_{rpdn}(S_n^v) = 0$.*

Proof. The sunlet graph S_n is obtained from a cycle $C_n : u_1u_2...u_nu_1$ by attaching a pendent vertex v_r at each vertex u_r of the cycle, $1 \leq r \leq n$. Clearly, $V(G) = \{u_r, v_r : 1 \leq r \leq n\}$ and $E(G) = \{u_rv_r, u_ru_{r\pm1} : 1 \leq r \leq n\}$.
Case 1. v is an end vertex

Then v is v_r, for some $r, 1 \leq r \leq n$. The graphs S_n and S_n^v are given in Figure 11. Clearly, $G^{v_1} \cong G^{v_2} \cong ... \cong G^{v_n}$. So let v be v_r. In G^{v_r}, the distinct degrees of the vertices are $2(n-1)$, 4 and 2 and hence $\gamma_{rpdn}(G^v) = 0$.

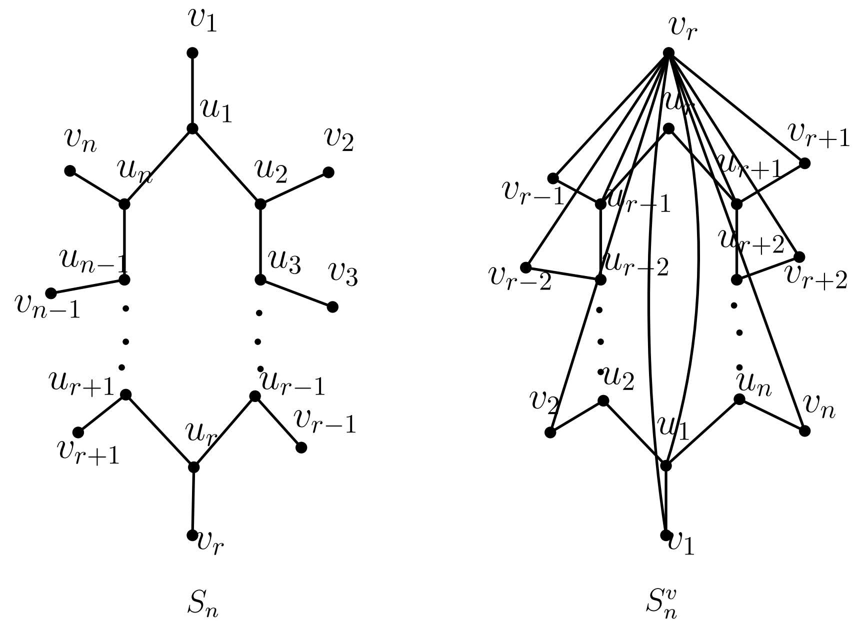


Figure 11: Sunlet graph S_n and switching of the sunlet graph S_n by v

Case 2. v is not an end vertex

Here $v = u_r$, for some r, $1 \leq r \leq n$. Clearly, G^v is disconnected graph with the isolated vertex v_r. Hence, $\gamma_{rpdn}(G^v) = 0$. □

3 Conclusion

Motivated by the concepts of detour sets and relatively prime dominating sets, we present the relatively prime detour dominating set specifically tailored for switching graphs. Our investigation extends to determining the relatively prime detour domination numbers for switching graphs derived from helm graphs, fan graphs, complete graphs, spider graphs, and sunlet graphs. The validity of our findings is bolstered by illustrative examples. Notably, our investigation extends beyond these specific graphs, as we aasert that relatively prime detour domination number is applicable to a broader range of graphs. Looking ahead, we envisage the development of an algorithm capable of determining whether a given graphs have been relatively prime detour domination number.

Acknowledgements

The authors express their gratitude to the management- Ratio Mathematica for their constant support towards the successful completion of this work. We wish to thank the anonymous reviewers for the valuable suggestions and comments.

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