Ratio Mathematica Volume 49, 2023

Some Fixed Point Results in Neutrosophic b-Metric Spaces

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Abstract

Several fixed point theorems in the Neutrosophic b-Metric Spaces are presented in this study. We provide a necessary condition for a sequence to be Cacuhy in the Neutrosophic b-Metric Space, which is a significant result. Additionally, we present a large number of fixed point theorems in the Neutrosophic b-Metric Spaces under the well-known contraction conditions.
Keywords: Fixed point; Neutrosophic b-Metric Space.
2020 AMS subject classifications: 54H25; 05C40. [1]

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[1] Received on October 20, 2023. Accepted on November 30, 2023. Published on December 30, 2023. DOI: 10.23755/rm.v49i0.1434. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Due to the wide range of applications it has in science, economics and game theory, fixed point theory has grown in importance in mathematics. Since the publication of Banach's fundamental finding in 1922, metric fixed point theory has been the subject of active scientific research. It is commonly known that the fixed point theory's main conclusion-the Banach contraction principle-has been applied and developed in a wide range of ways. Bakhtin presented the idea of b-metric space in 1989 as a generalization of metric space and demonstrated a b-metric space equivalent to the Banach contraction principle. Czerwik[5] expanded the results of the b-metric space in 1993. Since then, other articles, including [4], have addressed the generalization of the well-known Banach fixed point theorem in the b-metric space. A countable of the t-norm and contraction condition were used in 2021 by Jeyaraman and Poovaragavan[8] to show that the sequence in generalized fuzzy b-metric spaces is a Cauchy sequence. Since the paper's release, Zadeh's [13] Fuzzy Sets(FSs) theory has a significant impact on all scientific disciplines. Although this idea is crucial for real-life circumstances, there weren't enough timely solutions for some issues. Such issues have recently become the subject of new quests. For these situations, Atanassov[1] pioneered Intuituionistic Fuzzy Sets(IFS). The term "Neutrosophic Set" (NS) refers to a novel interpretation of Smarandache's definition of the term "Classical Set". Interesting and helpful for problem-solving is Atanassov's [1] IFS theory. The fuzzy set described by Zadeh[13] is really extended by the IFS. The IFS theory has been used in a variety of fields recently, including logic programming [2], decision-making issues, medical diagnosis[3] etc. The idea of Contractive mappings in Neutrosophic b-metric spaces in fixed point theorems was used in 2022 by Shakila and Jeyaraman[9].

The main idea of this paper is to describe a contraction for Neutrosophic b-Metric Space and show a very helpful lemma that guarantees that a sequence $\{x_n\}$ is a Cauchy sequence using the idea of a countable extension of the t-norm. The proofs of numerous well-known fixed-point theorems are made simpler using this lemma. Some of them are discussed in the paper's main section.

2 Preliminaries

Proposition 2.1. *Let $(\varsigma_n)_{n\in\mathbb{N}}$ be a sequence of numbers from $[0,1]$ such that $\lim\limits_{n\to\infty} \varsigma_n = 1$, and let $*$ be a t-norm of Ω-type.*
Then $\lim\limits_{n\to\infty} *_{i=n}^{\infty}\varsigma_i = \lim\limits_{n\to\infty} *_{i=1}^{\infty}\varsigma_{n+i} = 1$.

Definition 2.1. *[9] A 7-tuple $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ is said to be a Neutrosophic b-metric space (NbMS), if Λ is an arbitrary set, $b \geq 1$ is a given real number, $*$ is*

a continuous triangular norm, $\diamond$ is a continuous triangular conorm, Ξ, Ω and Υ are nonempty sets on $\Lambda^2 \times [0, \infty)$ meeting the aforementioned prerequisites: For all $\varsigma, \vartheta, \delta \in \Lambda$ and $\varpi, \mu > 0$:
(a) $\Xi(\varsigma, \vartheta, \varpi) + \Omega(\varsigma, \vartheta, \varpi) + \Upsilon(\varsigma, \vartheta, \varpi) \leq 3$;
(b) $0 \leq \Xi(\varsigma, \vartheta, \varpi) \leq 1; 0 \leq \Omega(\varsigma, \vartheta, \varpi) \leq 1; 0 \leq \Upsilon(\varsigma, \vartheta, \varpi) \leq 1$;
(c) $\Xi(\varsigma, \vartheta, 0) > 0$;
(d) $\Xi(\varsigma, \vartheta, \varpi) = 1$ *iff* $\varsigma = \vartheta$;
(e) $\Xi(\varsigma, \vartheta, \varpi) = \Xi(\vartheta, \varsigma, \varpi)$;
(f) $\Xi(\varsigma, \delta, b(\varpi + \mu)) \geq * \left(\Xi\left(\varsigma, \vartheta, \frac{\varpi}{b}\right), \Xi\left(\vartheta, \delta, \frac{\mu}{b}\right)\right)$;
(g) $\Xi(\varsigma, \vartheta, .) : (0, \infty) \to [0, 1]$ *is continuous;*
(h) $\Omega(\varsigma, \vartheta, 0) < 1$;
(i) $\Omega(\varsigma, \vartheta, \varpi) = 0$ *iff* $\varsigma = \vartheta$;
(j) $\Omega(\varsigma, \vartheta, \varpi) = \Omega(\vartheta, \varsigma, \varpi)$;
(k) $\Omega(\varsigma, \delta, b(\varpi + \mu)) \leq \diamond \left(\Omega\left(\varsigma, \vartheta, \frac{\varpi}{b}\right), \Omega\left(\vartheta, \delta, \frac{\mu}{b}\right)\right)$;
(l) $\Omega(\varsigma, \vartheta, .) : (0, \infty) \to [0, 1]$ *is continuous;*
(m) $\Upsilon(\varsigma, \vartheta, 0) < 1$;
(n) $\Upsilon(\varsigma, \vartheta, \varpi) = 0$ *iff* $\varsigma = \vartheta$;
(o) $\Upsilon(\varsigma, \vartheta, \varpi) = \Upsilon(\vartheta, \varsigma, \varpi)$;
(p) $\Upsilon(\varsigma, \delta, b(\varpi + \mu)) \leq \diamond \left(\Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{b}\right), \Upsilon\left(\vartheta, \delta, \frac{\mu}{b}\right)\right)$;
(q) $\Upsilon(\varsigma, \vartheta, .) : (0, \infty) \to [0, 1]$ *is continuous;*

Here, $\Xi(\varsigma, \vartheta, \varpi), \Omega(\varsigma, \vartheta, \varpi)$ and $\Upsilon(\varsigma, \vartheta, \varpi)$ denote the degree of nearness, the degree of non-nearness and the degree of neutralness between ς and ϑ with respect to ϖ respectively.

Since a neutrosophic b-metric is a neutrosophic metric when $b = 1$, the class of NbMS is essentially bigger than that of neutrosophic metric spaces.

The following illustration demonstrates that a neutrosophic b-metric space on Λ does not necessarily have to be a neutrosophic metric space on Λ.

Example 2.1. *[6] Let $\Xi(\varsigma, \vartheta, \varpi) = e^{\frac{-|\varsigma-\vartheta|^\theta}{\varpi}}, \Omega(\varsigma, \vartheta, \varpi) = 1 - e^{\frac{-|\varsigma-\vartheta|^\theta}{\varpi}}$, $\Upsilon(\varsigma, \vartheta, \varpi) = 1 - e^{\frac{|\varsigma-\vartheta|^\theta}{\varpi}}$ where $\theta > 1$. Then Ξ, Ω, Υ is a neutrosophic b-metric with $b = 2^{\theta-1}$. For $\theta = 2$, it is clear from the prior example that $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ is not a neutrosophic metric space.*

Example 2.2. *[6] Let $\Xi(\varsigma, \vartheta, \varpi) = e^{\frac{-d(\varsigma,\vartheta)}{\varpi}}$ or $\Xi(\varsigma, \vartheta, \varpi) = \frac{\varpi}{\varpi+d(\varsigma,\vartheta)}$, $\Omega(\varsigma, \vartheta, \varpi) = 1-e^{\frac{-d(\varsigma,\vartheta)}{\varpi}}$ or $\Omega(\varsigma, \vartheta, \varpi) = 1-\frac{\varpi}{\varpi+d(\varsigma,\vartheta)}$ and $\Upsilon(\varsigma, \vartheta, \varpi) = 1-e^{\frac{d(\varsigma,\vartheta)}{\varpi}}$ or $\Upsilon(\varsigma, \vartheta, \varpi) = 1 - \frac{\varpi+d(\varsigma,\vartheta)}{\varpi}$ where d is a b-metric on Λ. Then it is easy to show that Ξ, Ω and Υ is a neutrosophic b-metric.*

Definition 2.2. *[6] A function $g : \mathbb{R} \to \mathbb{R}$ is called b-nondecreasing if $\varsigma > b\vartheta$ implies $g(\varsigma) \geq g(\vartheta)$ for all $\varsigma, \vartheta \in \mathbb{R}$.*

Lemma 2.1. *[9] Let $\Xi(\varsigma,\vartheta,.),\Omega(\varsigma,\vartheta,.)$ and $\Upsilon(\varsigma,\vartheta,.)$ be a NbMS. Then $\Xi(\varsigma,\vartheta,\varpi)$ is b-nondecreasing with respect to ϖ, $\Omega(\varsigma,\vartheta,\varpi)$ and $\Upsilon(\varsigma,\vartheta,\varpi)$ are b-non-increasing with respect to ϖ, for all $\varsigma,\vartheta\in\Lambda$.*

Definition 2.3. *[9] Let $(\Lambda,\Xi,\Omega,\Upsilon,*,\diamond,b)$ be a neutrosophic b-metric space. For $\varpi>0$, the open ball $\Psi(\varsigma,r,\varpi)$ with centre $\varsigma\in\Lambda$ and radius $0<r<1$ is defined as*
$\Psi(\varsigma,r,\varpi)=\{\vartheta\in\Lambda:\Xi(\varsigma,\vartheta,\varpi)>1-r,\Omega(\varsigma,\vartheta,\varpi)<r$ and $\Upsilon(\varsigma,\vartheta,\varpi)<r\}$.
A sequence $\{\varsigma_n\}$:
(a) Converges to ς if $\Xi(\varsigma_n,\vartheta,\varpi)\to 1,\Omega(\varsigma_n,\vartheta,\varpi)\to 0$ and $\Upsilon(\varsigma_n,\vartheta,\varpi)\to 0$ as $n\to\infty$, for each $\varpi>0$. In this case we write $\lim\limits_{n\to\infty}\varsigma_n=\varsigma$;
(b) is called a Cauchy sequence if for all $0<\epsilon<1$ and $\varpi>0$, there exists $n_0\in\mathbb{N}$ such that $\Xi(\varsigma_n,\varsigma_m,\varpi)>1-\epsilon,\Omega(\varsigma_n,\varsigma_m,\varpi)<\epsilon$ and $\Upsilon(\varsigma_n,\varsigma_m,\varpi)<\epsilon$ for all $n,m\geq n_0$.

Definition 2.4. *[9] If every Cauchy sequence in NbMS is convergent, NbMS is complete.*

Lemma 2.2. *[9] In a NbMS $(\Lambda,\Xi,\Omega,\Upsilon,*,\diamond,b)$ we have:*
(i) If a sequence $\{\varsigma_n\}$ in Λ converges to ς, then ς is unique.
(ii) If a sequence $\{\varsigma_n\}$ in Λ converges to ς, then it is a Cauchy sequence.

Proposition 2.2. *Let $(\Lambda,\Xi,\Omega,\Upsilon,*,\diamond,b)$ be a NbMS and suppose that $\{\varsigma_n\}$ Converges to ς. Then we have*

$$\begin{aligned}
\Xi\left(\varsigma,\vartheta,\frac{\varpi}{b}\right)&\leq\lim_{n\to\infty}\sup\Xi(\varsigma_n,\vartheta,\varpi)\leq\Xi\left(\varsigma,\vartheta,b\varpi\right),\\
\Xi\left(\varsigma,\vartheta,\frac{\varpi}{b}\right)&\leq\lim_{n\to\infty}\inf\Xi(\varsigma_n,\vartheta,\varpi)\leq\Xi(\varsigma,\vartheta,b\varpi),\\
\Omega\left(\varsigma,\vartheta,\frac{\varpi}{b}\right)&\geq\lim_{n\to\infty}\inf\Omega(\varsigma_n,\vartheta,\varpi)\geq\Omega\left(\varsigma,\vartheta,b\varpi\right),\\
\Omega\left(\varsigma,\vartheta,\frac{\varpi}{b}\right)&\geq\lim_{n\to\infty}\sup\Omega(\varsigma_n,\vartheta,\varpi)\geq\Omega(\varsigma,\vartheta,b\varpi)\text{ and}\\
\Upsilon\left(\varsigma,\vartheta,\frac{\varpi}{b}\right)&\geq\lim_{n\to\infty}\inf\Upsilon(\varsigma_n,\vartheta,\varpi)\geq\Upsilon\left(\varsigma,\vartheta,b\varpi\right),\\
\Upsilon\left(\varsigma,\vartheta,\frac{\varpi}{b}\right)&\geq\lim_{n\to\infty}\sup\Upsilon(\varsigma_n,\vartheta,\varpi)\geq\Upsilon(\varsigma,\vartheta,b\varpi)
\end{aligned}$$

Example 2.3. *Let $\Lambda=[0,\infty),\Xi(\varsigma,\vartheta,\varpi)=e^{\frac{-d(\varsigma,\vartheta)}{\varpi}},\Omega(\varsigma,\vartheta,\varpi)=1-e^{\frac{-d(\varsigma,\vartheta)}{\varpi}}$ and $\Upsilon(\varsigma,\vartheta,\varpi)=1-e^{\frac{d(\varsigma,\vartheta)}{\varpi}}*=*_\theta,\diamond=\diamond_\theta$ and $d(\varsigma,\vartheta)=\begin{cases}0, & \varsigma=\vartheta,\\ 2|\varsigma-\vartheta|, & \varsigma,\vartheta\in[0,1)\\ \frac{1}{2}|\varsigma-\vartheta| & \text{otherwise}.\end{cases}$*
Then $(\Lambda,\Xi,\Omega,\Upsilon,*,\diamond,b)$ is a NbMS with $b=4$. The b-metric d in this example is

taken from [12]. Note that the Neutrosophic b-metric Ξ, Ω, Υ is not continuous. Let us observe that

$$\lim_{n\to\infty} \Xi\left(1, 1-\frac{1}{n}, \varpi\right) = \lim_{n\to\infty} e^{-\frac{1}{2n\varpi}} = 1, \Xi(1,1,\varpi) = 1,$$
$$\lim_{n\to\infty} \Omega\left(0, \frac{1}{n}, \varpi\right) = \lim_{n\to\infty}\left(1 - e^{-\frac{2}{n\varpi}}\right) = 0, \Omega(0,0,\varpi) = 0 \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\left(0, \frac{1}{n-1}, \varpi\right) = \lim_{n\to\infty}\left(1 - e^{-\frac{1}{\varpi}\left(\frac{2}{n-1}\right)}\right) = 0, \Upsilon(0,0,\varpi) = 0.$$

But,

$$\lim_{n\to\infty} \Xi\left(0, 1-\frac{1}{n}, \varpi\right) = \lim_{n\to\infty} e^{\frac{-1}{\varpi}(2)\left(\frac{1}{n}-1\right)} = e^{-\frac{2}{\varpi}},$$
$$\Xi(0,1,\varpi) = e^{-\frac{1}{2\varpi}},$$
$$\lim_{n\to\infty} \Omega\left(1, \frac{1}{n}, \varpi\right) = \lim_{n\to\infty}\left(1 - e^{\frac{-1}{\varpi}(2)\left(1-\frac{1}{n}\right)}\right) = 1 - e^{-\frac{2}{\varpi}},$$
$$\Omega(1,0,\varpi) = 1 - e^{-\frac{1}{2\varpi}} \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\left(1, \frac{1}{n-1}, \varpi\right) = \lim_{n\to\infty}\left(1 - e^{\frac{1}{\varpi}(2)\left(\frac{1}{1-\frac{1}{n}}\right)}\right) = 1 - e^{\frac{2}{\varpi}},$$
$$\Upsilon(1,0,\varpi) = 1 - e^{\frac{1}{2\varpi}}.$$

$\Xi(\varsigma, \vartheta, \varpi), \Omega(\varsigma, \vartheta, \varpi)$ *and* $\Upsilon(\varsigma, \vartheta, \varpi)$ *is not continuous.*

3 Main results

With extra conditions, we shall continue to employ a NbMS in the sense of Definition (2.1) $\lim_{\varpi\to\infty} \Xi(\varsigma, \vartheta, \varpi) = 1$, $\lim_{\varpi\to\infty} \Omega(\varsigma, \vartheta, \varpi) = 0$ and $\lim_{\varpi\to\infty} \Upsilon(\varsigma, \vartheta, \varpi) = 0$.

Lemma 3.1. *Let $\{\varsigma_n\}$ be a sequence in a NbMS $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$. Let's say that there is $\mu \in \left(0, \frac{1}{b}\right)$ such that*

$$\Xi(\varsigma_n, \varsigma_{n+1}, \varpi) \geq \Xi\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right),$$
$$\Omega(\varsigma_n, \varsigma_{n+1}, \varpi) \leq \Omega\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right) \text{ and}$$
$$\Upsilon(\varsigma_n, \varsigma_{n+1}, \varpi) \leq \Upsilon\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right), n \in \mathbb{N}, \varpi > 0 \tag{1}$$

and there exists $\varsigma_0, \varsigma_1 \in \Lambda$ *and* $\rho \in (0,1)$ *such that*

$$\begin{aligned}
&\lim_{n\to\infty} *_{i=n}^{\infty}\Xi\left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i}\right) = 1,\\
&\lim_{n\to\infty} \diamond_{i=n}^{\infty}\Omega\left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i}\right) = 0\, and\\
&\lim_{n\to\infty} \diamond_{i=n}^{\infty}\Upsilon\left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i}\right) = 0, \varpi > 0.
\end{aligned} \tag{2}$$

Then $\{\varsigma_n\}$ *is a Cauchy sequence.*

Proof. Let $\tau \in (\mu b, 1)$. Clearly, $\sum_{i=1}^{\infty} \tau^i$ is convergent and a thing exists $n_0 \in \mathbb{N}$ such that $\sum_{i=n}^{\infty} \tau^i < 1$ for all $n > n_0$. Let $n > m > n_0$. Since Ξ is b-nondecreasing, Ω and Υ are non-increasing also by (f), (k) and (p) of (Definition: 2.1) for every $\varpi > 0$, we have

$$\begin{aligned}
\Xi(\varsigma_n, \varsigma_{n+m}, \varpi) &\geq \Xi\left(\varsigma_n, \varsigma_{n+m}, \frac{\varpi\sum_{i=n}^{n+m-1}\tau^i}{b}\right)\\
&\geq *\left(\Xi\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau^n}{b^2}\right), \Xi\left(\varsigma_{n+1}, \varsigma_{n+m}, \frac{\varpi\sum_{i=n}^{n+m-1}\tau^i}{b^2}\right)\right)\\
&\geq *\left(\begin{array}{c}\Xi\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau^n}{b^2}\right), *\left(\Xi\left(\varsigma_{n+1}, \varsigma_{n+2}, \frac{\varpi\tau^{n+1}}{b^3}\right), \cdots\right.\\ \left.\Xi\left(\varsigma_{n+m-1}, \varsigma_{n+m}, \frac{\varpi\tau^{n+m-1}}{b^{m+1}}\right)\cdots\right)\end{array}\right)
\end{aligned}$$

By (1), it is evident that $\Xi(\varsigma_n, \varsigma_{n+1}, \varpi) \geq \Xi\left(\varsigma_0, \varsigma_1, \frac{\varpi}{\mu^n}\right), n \in \mathbb{N}, \varpi > 0$, and since $n > m$ and $b > 1$, we have

$$\begin{aligned}
\Xi(\varsigma_n, \varsigma_{n+m}, \varpi) &\geq *\left(\begin{array}{c}\Xi\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^n}{b^2\mu^n}\right), *\left(\Xi\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^{n+1}}{b^3\mu^{n+1}}\right), \cdots\right.\\ \left.\Xi\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^{n+m-1}}{b^{m+1}\mu^{n+m-1}}\right)\right)\end{array}\right)\\
&\geq *_{i=n}^{n+m-1}\Xi\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^i}{b^{i-n+2}\mu^i}\right)\\
&\geq *_{i=n}^{n+m-1}\Xi\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^i}{b^i\mu^i}\right) \geq *_{i=n}^{\infty}\Xi\left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i}\right), \text{where} \rho = \frac{b\mu}{\tau}.\\
\Omega(\varsigma_n, \varsigma_{n+m}, \varpi) &\leq \Omega\left(\varsigma_n, \varsigma_{n+m}, \frac{\varpi\sum_{i=n}^{n+m-1}\tau^i}{b}\right)\\
&\leq \diamond\left(\Omega\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau^n}{b^2}\right), \Omega\left(\varsigma_{n+1}, \varsigma_{n+m}, \frac{\varpi\sum_{i=n}^{n+m-1}\tau^i}{b^2}\right)\right)
\end{aligned}$$

$$\leq \diamond \left(\begin{array}{c} \Omega\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau^n}{b^2}\right), \diamond\left(\Omega\left(\varsigma_{n+1}, \varsigma_{n+2}, \frac{\varpi\tau^{n+1}}{b^3}\right), \cdots \right. \\ \left. \Omega\left(\varsigma_{n+m-1}, \varsigma_{n+m}, \frac{\varpi\tau^{n+m-1}}{b^{m+1}}\right)\cdots\right) \end{array} \right)$$

By (1), it is evident that $\Omega(\varsigma_n, \varsigma_{n+1}, \varpi) \leq \Omega\left(\varsigma_0, \varsigma_1, \frac{\varpi}{\mu^n}\right), n \in \mathbb{N}, \varpi > 0$, and since $n > m$ and $b > 1$, we have

$$\begin{aligned}
\Omega(\varsigma_n, \varsigma_{n+m}, \varpi) &\leq \diamond \left(\begin{array}{c} \Omega\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^n}{b^2\mu^n}\right), \diamond\left(\Omega\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^{n+1}}{b^3\mu^{n+1}}\right), \cdots \right. \\ \left. \Omega\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^{n+m-1}}{b^{m+1}\mu^{n+m-1}}\right)\cdots\right) \end{array} \right) \\
&\leq \diamond_{i=n}^{n+m-1}\Omega\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^i}{b^{i-n+2}\mu^i}\right) \\
&\leq \diamond_{i=n}^{n+m-1}\Omega\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^i}{b^i\mu^i}\right) \\
&\leq \diamond_{i=n}^{\infty}\Omega\left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i}\right), \text{ where } \rho = \frac{b\mu}{\tau}. \\
\Upsilon(\varsigma_n, \varsigma_{n+m}, \varpi) &\leq \Upsilon\left(\varsigma_n, \varsigma_{n+m}, \frac{\varpi\sum_{i=n}^{n+m-1}\tau^i}{b}\right) \\
&\leq \diamond\left(\Upsilon\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau^n}{b^2}\right), \Upsilon\left(\varsigma_{n+1}, \varsigma_{n+m}, \frac{\varpi\sum_{i=n}^{n+m-1}\tau^i}{b^2}\right)\right) \\
&\leq \diamond \left(\begin{array}{c} \Upsilon\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau^n}{b^2}\right), \diamond\left(\Upsilon\left(\varsigma_{n+1}, \varsigma_{n+2}, \frac{\varpi\tau^{n+1}}{b^3}\right), \cdots \right. \\ \left. \Upsilon\left(\varsigma_{n+m-1}, \varsigma_{n+m}, \frac{\varpi\tau^{n+m-1}}{b^{m+1}}\right)\cdots\right) \end{array} \right)
\end{aligned}$$

By (1) it follows that $\Upsilon(\varsigma_n, \varsigma_{n+1}, \varpi) \leq \Upsilon\left(\varsigma_0, \varsigma_1, \frac{\varpi}{\mu^n}\right), n \in \mathbb{N}, \varpi > 0$, and since $n > m$ and $b > 1$, we have

$$\begin{aligned}
\Upsilon(\varsigma_n, \varsigma_{n+m}, \varpi) &\leq \diamond \left(\begin{array}{c} \Upsilon\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^n}{b^2\mu^n}\right), \diamond\left(\Upsilon\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^{n+1}}{b^3\mu^{n+1}}\right), \cdots \right. \\ \left. \Upsilon\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^{n+m-1}}{b^{m+1}\mu^{n+m-1}}\right)\cdots\right) \end{array} \right) \\
&\leq \diamond_{i=n}^{n+m-1}\Upsilon\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^i}{b^{i-n+2}\mu^i}\right) \\
&\leq \diamond_{i=n}^{n+m-1}\Upsilon\left(\varsigma_0, \varsigma_1, \frac{\varpi\tau^i}{b^i\mu^i}\right) \\
&\leq \diamond_{i=n}^{\infty}\Upsilon\left(\varsigma_0, \varsigma_1, \frac{\varpi}{\rho^i}\right), \text{ where } \rho = \frac{b\mu}{\tau}.
\end{aligned}$$

Since $\rho \in (0, 1)$, by (2) follows, it is inferred that $\{\varsigma_n\}$ is a Cauchy sequence. $\square$

Corolary 3.1. *Let $\{\varsigma_n\}$ be a sequence in a NbMS $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$, and let $*$ be of Ω - type. If it's there $\mu \in \left(0, \frac{1}{b}\right)$ like that*

$$\begin{aligned}
&\Xi(\varsigma_n, \varsigma_{n+1}, \varpi) \geq \Xi\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right), \\
&\Omega(\varsigma_n, \varsigma_{n+1}, \varpi) \leq \Omega\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right) \text{ and} \\
&\Upsilon(\varsigma_n, \varsigma_{n+1}, \varpi) \leq \Upsilon\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right), n \in \mathbb{N}, \varpi > 0,
\end{aligned} \tag{3}$$

then $\{\varsigma_n\}$ is a Cauchy sequence.

Lemma 3.2. *If for some $\mu \in (0, 1)$ and $\varsigma, \vartheta \in \Lambda$,*

$$\begin{aligned}
&\Xi(\varsigma, \vartheta, \varpi) \geq \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \\
&\Omega(\varsigma, \vartheta, \varpi) \leq \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) \text{ and} \\
&\Upsilon(\varsigma, \vartheta, \varpi) \leq \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right),
\end{aligned} \tag{4}$$

where $\varpi > 0$, then $\varsigma = \vartheta$

Proof. Condition (4) implies that $\Xi(\varsigma, \vartheta, \varpi) \geq \lim_{n\to\infty} \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu^n}\right) = 1, \Omega(\varsigma, \vartheta, \varpi) \leq \lim_{n\to\infty} \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu^n}\right) = 0$ and $\Upsilon(\varsigma, \vartheta, \varpi) \leq \lim_{n\to\infty} \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu^n}\right) = 0$, where $n \in \mathbb{N}, \varpi > 0$ and by (d), (i) and (n) of Definition (2.1) it follows that $\varsigma = \vartheta$. □

Theorem 3.1. *Let $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ be a Complete Neutrosophic b-Metric space (CNbMS) and let $g : \Lambda \to \Lambda$. Let's say that there is $\mu \in \left(0, \frac{1}{b}\right)$ such that*

$$\begin{aligned}
&\Xi(g\varsigma, g\vartheta, \varpi) \geq \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \\
&\Omega(g\varsigma, g\vartheta, \varpi) \leq \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) \text{ and} \\
&\Upsilon(g\varsigma, g\vartheta, \varpi) \leq \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right)
\end{aligned} \tag{5}$$

where $\varsigma, \vartheta \in \Lambda, \varpi > 0$ and there exists $\varsigma_0 \in \Lambda$ and $\rho \in (0, 1)$ such that

$$\lim_{n\to\infty} *_{i=n}^{\infty} \Xi\left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i}\right) = 1,$$

$$\lim_{n\to\infty} \diamond_{i=n}^{\infty} \Omega\left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i}\right) = 0 \, and$$
$$\lim_{n\to\infty} \diamond_{i=n}^{\infty} \Upsilon\left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i}\right) = 0 \, where \, \varpi > 0. \tag{6}$$

Then $\mathfrak{g}$ *has a one and only fixed point in* Λ.

Proof. Let $\varsigma_0 \in \Lambda$ and $\varsigma_{n+1} = g\varsigma_n, n \in \mathbb{N}$. If we take $\varsigma = \varsigma_n$ and $\vartheta = \varsigma_{n-1}$ in (5), then we have

$$\Xi(\varsigma_n, \varsigma_{n+1}, \varpi) \geq \Xi\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Omega(\varsigma_n, \varsigma_{n+1}, \varpi) \leq \Omega\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right) \text{ and}$$
$$\Upsilon(\varsigma_n, \varsigma_{n+1}, \varpi) \leq \Upsilon\left(\varsigma_{n-1}, \varsigma_n, \frac{\varpi}{\mu}\right)$$

where $n \in \mathbb{N}, \varpi > 0$, and Lemma (3.1) implies that $\{\varsigma_n\}$ is a Cauchy sequence. Since $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ is complete, a thing exists $\varsigma \in \Lambda$ like that $\lim_{n\to\infty} \varsigma_n = \varsigma$ and

$$\lim_{n\to\infty} \Xi(\varsigma, \varsigma_n, \varpi) = 1, \lim_{n\to\infty} \Omega(\varsigma, \varsigma_n, \varpi) = 0 \, \text{and}$$
$$\lim_{n\to\infty} \Upsilon(\varsigma, \varsigma_n, \varpi) = 0, \text{ where} \varpi > 0. \tag{7}$$

Conditions (5) and by (f), (k) and (p) of Definition (2.1) are used to show that ς is a fixed point for g.

$$\Xi(g\varsigma, \varsigma, \varpi) \geq *\left(\Xi\left(g\varsigma, \varsigma_n, \frac{\varpi}{2b}\right), \Xi\left(\varsigma_n, \varsigma, \frac{\varpi}{2b}\right)\right)$$
$$\geq *\left(\Xi\left(\varsigma, \varsigma_{n-1}, \frac{\varpi}{2b\mu}\right), \Xi\left(\varsigma_n, \varsigma, \frac{\varpi}{2b}\right)\right),$$
$$\Omega(g\varsigma, \varsigma, \varpi) \leq \diamond\left(\Omega\left(g\varsigma, \varsigma_n, \frac{\varpi}{2b}\right), \Omega\left(\varsigma_n, \varsigma, \frac{\varpi}{2b}\right)\right)$$
$$\leq \diamond\left(\Omega\left(\varsigma, \varsigma_{n-1}, \frac{\varpi}{2b\mu}\right), \Omega\left(\varsigma_n, \varsigma, \frac{\varpi}{2b}\right)\right) \text{ and}$$
$$\Upsilon(g\varsigma, \varsigma, \varpi) \leq \diamond\left(\Upsilon\left(g\varsigma, \varsigma_n, \frac{\varpi}{2b}\right), \Upsilon\left(\varsigma_n, \varsigma, \frac{\varpi}{2b}\right)\right)$$
$$\leq \diamond\left(\Upsilon\left(\varsigma, \varsigma_{n-1}, \frac{\varpi}{2b\mu}\right), \Upsilon\left(\varsigma_n, \varsigma, \frac{\varpi}{2b}\right)\right) \text{ for all } \varpi > 0.$$

By (7), as $n \to \infty$, we get
$\Xi(g\varsigma, \varsigma, \varpi) \geq *(1, 1) = 1, \Omega(g\varsigma, \varsigma, \varpi) \leq \diamond(0, 0) = 0$ and
$\Upsilon(g\varsigma, \varsigma, \varpi) \leq \diamond(0, 0) = 0$.
Suppose that ς and ϑ are fixed points for g. By (5) we have

$\Xi(\varsigma,\vartheta,\varpi) = \Xi(g\varsigma,g\vartheta,\varpi) \geq \Xi\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right)$,
$\Omega(\varsigma,\vartheta,\varpi) = \Omega(g\varsigma,g\vartheta,\varpi) \leq \Omega\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right)$ and
$\Upsilon(\varsigma,\vartheta,\varpi) = \Upsilon(g\varsigma,g\vartheta,\varpi) \leq \Upsilon\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right), \varpi > 0$, and through Lemma (3.2) suggests that $\varsigma = \vartheta$. □

Example 3.1. *Let* $\Lambda = [0,1]$. *By Example (2.1), for* $\theta = 2$. *Thus, it follows* $(\Lambda,\Xi,\Omega,\Upsilon,*,\diamond,b)$ *is a NbMS with* $b = 2$ *and Neutrosophic b-Metric* $\Xi(\varsigma,\vartheta,\varpi) = e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}}, \Omega(\varsigma,\vartheta,\varpi) = 1 - e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}}, \Upsilon(\varsigma,\vartheta,\varpi) = 1 - e^{\frac{(\varsigma-\vartheta)^2}{\varpi}}, \varsigma,\vartheta \in \Lambda, \varpi > 0.$ *Let* $g(\varsigma) = \omega\varsigma, \omega < \frac{\sqrt{2}}{2}, \varsigma \in \Lambda$. *Then*

$$\Xi(g\varsigma,g\vartheta,\varpi) = e^{\frac{-\omega^2(\varsigma-\vartheta)^2}{\varpi}} \geq e^{\frac{-\mu(\varsigma-\vartheta)^2}{\varpi}} = \Xi\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right),$$
$$\Omega(g\varsigma,g\vartheta,\varpi) = 1 - e^{\frac{-\omega^2(\varsigma-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{-\mu(\varsigma-\vartheta)^2}{\varpi}} = \Omega\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right) \text{ and}$$
$$\Upsilon(g\varsigma,g\vartheta,\varpi) = 1 - e^{\frac{\omega^2(\varsigma-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{\mu(\varsigma-\vartheta)^2}{\varpi}} = \Upsilon\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right) \varsigma,\vartheta \in \Lambda, \varpi > 0,$$

for $\frac{1}{b} > \mu > \omega^2$. *As a result, Theorem (3.1)'s condition (5) is met and g has a singular fixed point in* Λ.

Theorem 3.2. *Let* $(\Lambda,\Xi,\Omega,\Upsilon,*,\diamond,b)$ *is a CNbMS and let* $g : \Lambda \to \Lambda$. *Let's say that there is* $\mu \in \left(0,\frac{1}{b}\right)$ *like that*

$$\Xi(g\varsigma,g\vartheta,\varpi) \geq \min\left\{\Xi\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right),\Xi\left(g\varsigma,\varsigma,\frac{\varpi}{\mu}\right),\Xi\left(g\vartheta,\vartheta,\frac{\varpi}{\mu}\right)\right\},$$
$$\Omega(g\varsigma,g\vartheta,\varpi) \leq \max\left\{\Omega\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right),\Omega\left(g\varsigma,\varsigma,\frac{\varpi}{\mu}\right),\Omega\left(g\vartheta,\vartheta,\frac{\varpi}{\mu}\right)\right\} \text{ and}$$
$$\Upsilon(g\varsigma,g\vartheta,\varpi) \leq \max\left\{\Upsilon\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right),\Upsilon\left(g\varsigma,\varsigma,\frac{\varpi}{\mu}\right),\Upsilon\left(g\vartheta,\vartheta,\frac{\varpi}{\mu}\right)\right\} \tag{8}$$

for all $\varsigma,\vartheta \in \Lambda, \varpi > 0$ *and there exists* $\varsigma_0 \in \Lambda$, *and* $\rho \in (0,1)$ *such that*

$$\lim_{n\to\infty} *_{i=n}^{\infty}\Xi\left(\varsigma_0,g\varsigma_0,\frac{\varpi}{\rho^i}\right) = 1,$$
$$\lim_{n\to\infty} \diamond_{i=n}^{\infty}\Omega\left(\varsigma_0,g\varsigma_0,\frac{\varpi}{\rho^i}\right) = 0 \text{ and}$$
$$\lim_{n\to\infty} \diamond_{i=n}^{\infty}\Omega\left(\varsigma_0,g\varsigma_0,\frac{\varpi}{\rho^i}\right) = 0 \text{for all} \, \varpi > 0. \tag{9}$$

Then g has a one and only fixed point in Λ.

Proof. Let $\varsigma_0 \in \Lambda$ and $\varsigma_{n+1} = g\varsigma_n, n \in \mathbb{N}$. By (8) with $\varsigma = \varsigma_n$ and $\vartheta = \varsigma_{n-1}$, for every $n \in \mathbb{N}$ and every $\varpi > 0$, we have

$$\begin{aligned}\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) &\geq \min\left\{\Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right)\right\}\\ &\geq \min\left\{\Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right)\right\}\end{aligned}$$

If $\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) \geq \Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), n \in \mathbb{N}, \varpi > 0$, therefore Lemma (3.2) requires that $\varsigma_n = \varsigma_{n+1}, n \in \mathbb{N}$.
So, $\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) \geq \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), n \in \mathbb{N}, \varpi > 0$, and according to Lemma (3.1), $\{\varsigma_n\}$ is a Cauchy sequence.

$$\begin{aligned}\Omega(\varsigma_{n+1}, \varsigma_n, \varpi) &\leq \max\left\{\Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right)\right\}\\ &\leq \max\left\{\Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right)\right\}\end{aligned}$$

If $\Omega(\varsigma_{n+1}, \varsigma_n, \varpi) \leq \Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), n \in \mathbb{N}, \varpi > 0$, therefore Lemma (3.2) requires that $\varsigma_n = \varsigma_{n+1}, n \in \mathbb{N}$.
So, $\Omega(\varsigma_{n+1}, \varsigma_n, \varpi) \leq \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), n \in \mathbb{N}, \varpi > 0$, and according to Lemma (3.1), $\{\varsigma_n\}$ is a Cauchy sequence.

$$\begin{aligned}\Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) &\leq \max\left\{\Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right)\right\}\\ &\leq \max\left\{\Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right)\right\}\end{aligned}$$

If $\Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) \leq \Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), n \in \mathbb{N}, \varpi > 0$, therefore Lemma (3.2) requires that $\varsigma_n = \varsigma_{n+1}, n \in \mathbb{N}$.
So, $\Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) \leq \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), n \in \mathbb{N}, \varpi > 0$, and according to Lemma (3.1), $\{\varsigma_n\}$ is a Cauchy sequence.
Consequently, there is $\varsigma \in \Lambda$ like that

$$\begin{aligned}&\lim_{n\to\infty} \varsigma_n = \varsigma \text{ and } \lim_{n\to\infty} \Xi(\varsigma, \varsigma_n, \varpi) = 1,\\ &\lim_{n\to\infty} \Omega(\varsigma, \varsigma_n, \varpi) = 0 \text{ and}\\ &\lim_{n\to\infty} \Upsilon(\varsigma, \varsigma_n, \varpi) = 0, \text{ where } \varpi > 0. \qquad (10)\end{aligned}$$

Let's demonstrate that ς is g's fixed point. Let $\tau_1 \in (\mu b, 1)$ and $\tau_2 = 1 - \tau_1$. By (8) we have

$$\begin{aligned}
\Xi(g\varsigma, \varsigma, \varpi) &\geq * \left(\Xi\left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b}\right), \Xi\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right)\right) \\
&\geq * \left(\min\left\{\begin{array}{c} \Xi\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Xi\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), \\ \Xi\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right) \end{array}\right\}, \Xi\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right)\right)
\end{aligned}$$

Taking $n \to \infty$ and using (10), we get

$$\begin{aligned}
\Xi(g\varsigma, \varsigma, \varpi) &\geq * \left(\min\left\{1, \Xi\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), 1\right\}, 1\right) \\
&= * \left(\Xi\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), 1\right) = \Xi\left(\varsigma, g\varsigma, \frac{\varpi}{\rho}\right), \varpi > 0
\end{aligned}$$

where $\rho = \frac{b\mu}{\tau_1} \in (0, 1)$. So, $\Xi(g\varsigma, \varsigma, \varpi) \geq \Xi\left(g\varsigma, \varsigma, \frac{\varpi}{\rho}\right), \varpi > 0$, and by Lemma (3.2) it follows that $g\varsigma = \varsigma$. Imagine that ς and ϑ are fixed points for g, that is $g\varsigma = \varsigma$ and $g\vartheta = \vartheta$. We obtain condition (8)

$$\begin{aligned}
&\Xi(g\varsigma, g\vartheta, \varpi) \\
&\quad \geq \min\left\{\Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Xi\left(\varsigma, g\varsigma, \frac{\varpi}{\mu}\right), \Xi\left(\vartheta, g\vartheta, \frac{\varpi}{\mu}\right)\right\} \\
&\quad = \min\left\{\Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), 1, 1\right\} = \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) = \Xi\left(g\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \\
&\Omega(g\varsigma, \varsigma, \varpi) \\
&\quad \leq \diamond \left(\Omega\left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b}\right), \Omega\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right)\right) \\
&\quad \leq \diamond \left(\max\left\{\begin{array}{c} \Omega\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Omega\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), \\ \Omega\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right) \end{array}\right\}, \Omega\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right)\right)
\end{aligned}$$

Taking $n \to \infty$ and using (10), we get

$$\begin{aligned}
\Omega(g\varsigma, \varsigma, \varpi) &\leq \diamond \left(\max\left\{0, \Omega\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), 0\right\}, 0\right) \\
&= \diamond \left(\Omega\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), 0\right) \\
&= \Omega\left(\varsigma, g\varsigma, \frac{\varpi}{\rho}\right), \varpi > 0,
\end{aligned}$$

where $\rho = \frac{b\mu}{\tau_1} \in (0,1)$. So, $\Omega(g\varsigma, \varsigma, \varpi) \leq \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\rho}\right), \varpi > 0$, and by Lemma (3.2) it follows that $g\varsigma = \varsigma$.
Imagine that ς and ϑ are fixed points for g, that is $g\varsigma = \varsigma$ and $g\vartheta = \vartheta$. We obtain condition (8)

$$\begin{aligned}
&\Omega(g\varsigma, g\vartheta, \varpi)\\
&\quad\leq \max\left\{\Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Omega\left(\varsigma, g\varsigma, \frac{\varpi}{\mu}\right), \Omega\left(\vartheta, g\vartheta, \frac{\varpi}{\mu}\right)\right\}\\
&\quad= \max\left\{\Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), 0, 0\right\} = \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) = \Omega\left(g\varsigma, g\vartheta, \frac{\varpi}{\mu}\right)\\
&\Upsilon(g\varsigma, \varsigma, \varpi)\\
&\quad\leq \diamond\left(\Upsilon\left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b}\right), \Upsilon\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right)\right)\\
&\quad\leq \diamond\left(\max\left\{\begin{array}{c}\Upsilon\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Upsilon\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right),\\ \Upsilon\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right)\end{array}\right\}, \Upsilon\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right)\right)
\end{aligned}$$

Taking $n \to \infty$ and using (10), we get

$$\begin{aligned}
\Upsilon(g\varsigma, \varsigma, \varpi) &\leq \diamond\left(\max\left\{0, \Upsilon\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), 0\right\}, 0\right)\\
&= \diamond\left(\Upsilon\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), 0\right)\\
&= \Upsilon\left(\varsigma, g\varsigma, \frac{\varpi}{\rho}\right), \varpi > 0
\end{aligned}$$

where $\rho = \frac{b\mu}{\tau_1} \in (0,1)$. So, $\Upsilon(g\varsigma, \varsigma, \varpi) \leq \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\rho}\right), \varpi > 0$, and by Lemma (3.2) it follows that $g\varsigma = \varsigma$.
Imagine that ς and ϑ are fixed points for g, that is $g\varsigma = \varsigma$ and $g\vartheta = \vartheta$. We obtain condition (8)

$$\begin{aligned}
\Upsilon(g\varsigma, g\vartheta, \varpi) &\leq \max\left\{\Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Upsilon\left(\varsigma, g\varsigma, \frac{\varpi}{\mu}\right), \Upsilon\left(\vartheta, g\vartheta, \frac{\varpi}{\mu}\right)\right\}\\
&= \max\left\{\Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), 0, 0\right\} = \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right) = \Upsilon\left(g\varsigma, g\vartheta, \frac{\varpi}{\mu}\right)
\end{aligned}$$

for $\varpi > 0$, and it is implied from Lemma (3.2) that $g\varsigma = g\vartheta$, that is $\varsigma = \vartheta$. □

Example 3.2. *Let* $\Lambda = (0,2), \Xi(\varsigma, \vartheta, \varpi) = e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}}, \Omega(\varsigma, \vartheta, \varpi) = 1 - e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}}$, $\Upsilon(\varsigma, \vartheta, \varpi) = 1 - e^{\frac{(\varsigma-\vartheta)^2}{\varpi}}, * = *_\theta$ *and* $\diamond = \diamond_\theta$. *Then* $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond, b)$ *is a*

CNbMS with $b = 2$. Let $g(\varsigma) = \begin{cases} 2-\varsigma, & \varsigma \in (0,1) \\ 1, & \varsigma \in [1,2) \end{cases}$

Case 1: *If $\varsigma, \vartheta \in [1,2)$ then $\Xi(g\varsigma, g\vartheta, \varpi) = 1, \Omega(g\varsigma, g\vartheta, \varpi) = 0$ and $\Upsilon(g\varsigma, g\vartheta, \varpi) = 0$, $\varpi > 0$, and conditions (8) are met.*

Case 2: *If $\varsigma \in [1,2)$ and $\vartheta \in (0,1)$, then, for $\mu \in \left(\frac{1}{4}, \frac{1}{2}\right)$, we have*

$$\begin{aligned}
\Xi(g\varsigma, g\vartheta, \varpi) &= e^{\frac{-(1-\vartheta)^2}{\varpi}} \geq e^{\frac{-4\mu(1-\vartheta)^2}{\varpi}} = \Xi\left(g\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \\
\Omega(g\varsigma, g\vartheta, \varpi) &= 1 - e^{\frac{-(1-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{-4\mu(1-\vartheta)^2}{\varpi}} = \Omega\left(g\varsigma, \vartheta, \frac{\varpi}{\mu}\right) \\
\Upsilon(g\varsigma, g\vartheta, \varpi) &= 1 - e^{\frac{(1-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{4\mu(1-\vartheta)^2}{\varpi}} = \Upsilon\left(g\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \varpi > 0.
\end{aligned}$$

Case 3: *As in the earlier instance, for $\mu \in \left(\frac{1}{4}, \frac{1}{2}\right)$, we have*

$$\begin{aligned}
\Xi(g\varsigma, g\vartheta, \varpi) &\geq \Xi\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \\
\Omega(g\varsigma, g\vartheta, \varpi) &\leq \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right) \text{ and} \\
\Upsilon(g\varsigma, g\vartheta, \varpi) &\leq \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \varsigma \in (0,1), \vartheta \in [1,2), \varpi > 0.
\end{aligned}$$

Case 4: *If $\varsigma, \vartheta \in (0,1)$ then for $\mu \in \left(\frac{1}{4}, \frac{1}{2}\right)$,*

$$\begin{aligned}
\Xi(g\varsigma, g\vartheta, \varpi) &\geq e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}} \geq e^{\frac{-(1-\vartheta)^2}{\varpi}} \geq e^{\frac{-4\mu(1-\vartheta)^2}{\varpi}} \\
&= \Xi\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \varsigma > \vartheta, \varpi > 0, \\
\Xi(g\varsigma, g\vartheta, \varpi) &\geq \Xi\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \varsigma < \vartheta, \varpi > 0, \\
\Omega(g\varsigma, g\vartheta, \varpi) &\leq 1 - e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{-(1-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{-4\mu(1-\vartheta)^2}{\varpi}} \\
&= \Omega\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \varsigma > \vartheta, \varpi > 0, \\
\Omega(g\varsigma, g\vartheta, \varpi) &\leq \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \varsigma < \vartheta, \varpi > 0 \text{ and} \\
\Upsilon(g\varsigma, g\vartheta, \varpi) &\leq 1 - e^{\frac{(\varsigma-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{(1-\vartheta)^2}{\varpi}} \leq 1 - e^{\frac{4\mu(1-\vartheta)^2}{\varpi}} \\
&= \Upsilon\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \varsigma > \vartheta, \varpi > 0, \\
\Upsilon(g\varsigma, g\vartheta, \varpi) &\leq \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \varsigma < \vartheta, \varpi > 0.
\end{aligned}$$

So conditions (8) are satisfied for all $\varsigma, \vartheta \in \Lambda, \varpi > 0$*, and it follows from Theorem (3.2) that* $\varsigma = 1$ *is a singular fixed point for* g*.*

Theorem 3.3. *Let* $(\Lambda, \Xi, \Omega, \Upsilon, *_{\min}, \Diamond_{\max})$ *be a CNbMS, and let* $g : \Lambda \to \Lambda$*. If for some* $\mu \in \left(0, \frac{1}{b^2}\right)$*,*

$$\Xi(g\varsigma, g\vartheta, \varpi) \geq \min\left\{\begin{array}{c} \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Xi\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Xi\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \Xi\left(g\varsigma, \vartheta, \frac{2\varpi}{\mu}\right), \Xi\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \end{array}\right\},$$

$$\Omega(g\varsigma, g\vartheta, \varpi) \leq \max\left\{\begin{array}{c} \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Omega\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \Omega\left(g\varsigma, \vartheta, \frac{2\varpi}{\mu}\right), \Omega\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \end{array}\right\} \textit{and}$$

$$\Upsilon(g\varsigma, g\vartheta, \varpi) \leq \max\left\{\begin{array}{c} \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Upsilon\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \Upsilon\left(g\varsigma, \vartheta, \frac{2\varpi}{\mu}\right), \Upsilon\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \end{array}\right\}, \quad (11)$$

where $\varsigma, \vartheta \in \Lambda, \varpi > 0$ *then* g *has a unique fixed point in* Λ*.*

Proof. Let $\varsigma_0 \in \Lambda$ and $\varsigma_{n+1} = g\varsigma_n, n \in \mathbb{N}$. By (11) with $\varsigma = \varsigma_n$ and $\vartheta = \varsigma_{n-1}$, using (f, k and p) and the assumption that $* = *_{\min}$ and $\Diamond = \Diamond_{\max}$ we have

$$\begin{aligned}
&\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) \\
&\quad \geq \min\left\{\begin{array}{c} \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \\ \min\left\{\Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right), \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right)\right\}, \Xi\left(\varsigma_n, \varsigma_n, \frac{\varpi}{\mu}\right) \end{array}\right\} \\
&\quad \geq \min\left\{\Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right), \Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right)\right\}, n \in \mathbb{N}, \varpi > 0. \\
&\Omega(\varsigma_{n+1}, \varsigma_n, \varpi) \\
&\quad \leq \max\left\{\begin{array}{c} \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \\ \max\left\{\Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right), \Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right)\right\}, \Omega\left(\varsigma_n, \varsigma_n, \frac{\varpi}{\mu}\right) \end{array}\right\} \\
&\quad \leq \max\left\{\Omega\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right), \Omega\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right)\right\}, n \in \mathbb{N}, \varpi > 0 \text{ and} \\
&\Upsilon(\varsigma_{n+1}, \varsigma_n, \varpi) \\
&\quad \leq \max\left\{\begin{array}{c} \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \\ \max\left\{\Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right), \Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right)\right\}, \Upsilon\left(\varsigma_n, \varsigma_n, \frac{\varpi}{\mu}\right) \end{array}\right\} \\
&\quad \leq \max\left\{\Upsilon\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right), \Upsilon\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right)\right\}, \text{ where } n \in \mathbb{N}, \varpi > 0.
\end{aligned}$$

As in the proof of Theorem (3.2), it is inferred from Lemma (3.2) and Corollary (3.1) that
$\Xi(\varsigma_{n+1},\varsigma_n,\varpi)\geq\Xi\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{b\mu}\right),\Omega(\varsigma_{n+1},\varsigma_n,\varpi)\leq\Omega\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{b\mu}\right)$ and
$\Upsilon(\varsigma_{n+1},\varsigma_n,\varpi)\leq\Upsilon\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{b\mu}\right) n\in\mathbb{N},\varpi>0$, and $\{\varsigma_n\}$ is a Cauchy sequence. Thus, there is $\varsigma\in\Lambda$ like that $\lim_{n\to\infty}\varsigma_n=\varsigma$ and

$$\begin{aligned}&\lim_{n\to\infty}\Xi(\varsigma,\varsigma_n,\varpi)=1,\ \lim_{n\to\infty}\Omega(\varsigma,\varsigma_n,\varpi)=0\,\text{and}\\&\lim_{n\to\infty}\Upsilon(\varsigma,\varsigma_n,\varpi)=0,\varpi>0.\end{aligned}\tag{12}$$

Let $\tau_1\in(b^2\mu,1)$ and $\tau_2=1-\tau_1$. By (11) and (f, k and p) for $*=*_{\min},\diamond=\diamond_{\max}$ we have

$$\begin{aligned}&\Xi(g\varsigma,\varsigma,\varpi)\\&\geq\min\left\{\Xi\left(g\varsigma,g\varsigma_n,\frac{\varpi\tau_1}{b}\right),\Xi\left(g\varsigma_n,\varsigma,\frac{\varpi\tau_2}{b}\right)\right\}\\&\geq\min\left\{\min\left\{\begin{array}{c}\Xi\left(\varsigma,\varsigma_n,\frac{\varpi\tau_1}{b\mu}\right),\Xi\left(\varsigma,g\varsigma,\frac{\varpi\tau_1}{b\mu}\right),\\\Xi\left(\varsigma_n,\varsigma_{n+1},\frac{\varpi\tau_1}{b\mu}\right),\\\min\left\{\Xi\left(g\varsigma,\varsigma,\frac{\varpi\tau_1}{b^2\mu}\right),\Xi\left(\varsigma,\varsigma_n,\frac{\varpi\tau_1}{b^2\mu}\right)\right\},\\\Xi\left(\varsigma_n,\varsigma_{n+1},\frac{\varpi\tau_1}{b\mu}\right)\end{array}\right\},\Xi\left(\varsigma_{n+1},\varsigma,\frac{\varpi\tau_2}{b}\right)\right\}\end{aligned}$$

$$\begin{aligned}&\Omega(g\varsigma,\varsigma,\varpi)\\&\leq\max\left\{\Omega\left(g\varsigma,g\varsigma_n,\frac{\varpi\tau_1}{b}\right),\Omega\left(g\varsigma_n,\varsigma,\frac{\varpi\tau_2}{b}\right)\right\}\\&\leq\max\left\{\max\left\{\begin{array}{c}\Omega\left(\varsigma,\varsigma_n,\frac{\varpi\tau_1}{b\mu}\right),\Omega\left(\varsigma,g\varsigma,\frac{\varpi\tau_1}{b\mu}\right),\\\Omega\left(\varsigma_n,\varsigma_{n+1},\frac{\varpi\tau_1}{b\mu}\right),\\\max\left\{\Omega\left(g\varsigma,\varsigma,\frac{\varpi\tau_1}{b^2\mu}\right),\Omega\left(\varsigma,\varsigma_n,\frac{\varpi\tau_1}{b^2\mu}\right)\right\},\\\Omega\left(\varsigma_n,\varsigma_{n+1},\frac{\varpi\tau_1}{b\mu}\right)\end{array}\right\},\Omega\left(\varsigma_{n+1},\varsigma,\frac{\varpi\tau_2}{b}\right)\right\}\end{aligned}$$

$$\begin{aligned}&\Upsilon(g\varsigma,\varsigma,\varpi)\\&\leq\max\left\{\Upsilon\left(g\varsigma,g\varsigma_n,\frac{\varpi\tau_1}{b}\right),\Upsilon\left(g\varsigma_n,\varsigma,\frac{\varpi\tau_2}{b}\right)\right\}\\&\leq\max\left\{\max\left\{\begin{array}{c}\Upsilon\left(\varsigma,\varsigma_n,\frac{\varpi\tau_1}{b\mu}\right),\Upsilon\left(\varsigma,g\varsigma,\frac{\varpi\tau_1}{b\mu}\right),\\\Upsilon\left(\varsigma_n,\varsigma_{n+1},\frac{\varpi\tau_1}{b\mu}\right),\\\max\left\{\Upsilon\left(g\varsigma,\varsigma,\frac{\varpi\tau_1}{b^2\mu}\right),\Upsilon\left(\varsigma,\varsigma_n,\frac{\varpi\tau_1}{b^2\mu}\right)\right\},\\\Upsilon\left(\varsigma_n,\varsigma_{n+1},\frac{\varpi\tau_1}{b\mu}\right)\end{array}\right\},\Upsilon\left(\varsigma_{n+1},\varsigma,\frac{\varpi\tau_2}{b}\right)\right\}\end{aligned}$$

for all $n \in \mathbb{N}, \varpi > 0$. Taking $n \to \infty$ and using (12), we get

$$\begin{aligned}
&\Xi(g\varsigma, \varsigma, \varpi) \\
&\geq \min\left\{\min\left\{1, \Xi\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), 1, \min\left\{\Xi\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right), 1\right\}, 1\right\}, 1\right\} \\
&= \Xi\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right), \varpi > 0, \\
&\Omega(g\varsigma, \varsigma, \varpi) \\
&\leq \max\left\{\max\left\{0, \Omega\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), 0, \max\left\{\Omega\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right), 0\right\}, 0\right\}, 0\right\} \\
&= \Omega\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right), \varpi > 0, \\
&\Upsilon(g\varsigma, \varsigma, \varpi) \\
&\leq \max\left\{\max\left\{0, \Upsilon\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), 0, \max\left\{\Upsilon\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right), 0\right\}, 0\right\}, 0\right\} \\
&= \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right), \varpi > 0,
\end{aligned}$$

and by lemma (3.2) with $\rho = \frac{b^2\mu}{\tau_1} \in (0, 1)$. Thus, it follows $g\varsigma = \varsigma$.
With reference to condition (11), two fixed points $\varsigma = g\varsigma$ and $\vartheta = g\vartheta$, we have

$$\begin{aligned}
&\Xi(g\varsigma, g\vartheta, \varpi) \\
&\geq \min\left\{\begin{array}{c} \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Xi\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Xi\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \min\left\{\Xi\left(g\varsigma, \varsigma, \frac{\varpi}{b\mu}\right), \Xi\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right)\right\}, \Xi\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \end{array}\right\} \\
&= \min\left\{\Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), 1, 1, \min\left\{1, \Xi\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right)\right\}, \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right)\right\} \\
&= \Xi\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right) = \Xi\left(g\varsigma, g\vartheta, \frac{\varpi}{b\mu}\right), \varpi > 0, \\
&\Omega(g\varsigma, g\vartheta, \varpi) \\
&\leq \max\left\{\begin{array}{c} \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Omega\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \max\left\{\Omega\left(g\varsigma, \varsigma, \frac{\varpi}{b\mu}\right), \Omega\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right)\right\}, \Omega\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \end{array}\right\} \\
&= \max\left\{\Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), 0, 0, \max\left\{0, \Omega\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right)\right\}, \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right)\right\} \\
&= \Omega\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right) = \Omega\left(g\varsigma, g\vartheta, \frac{\varpi}{b\mu}\right), \varpi > 0, \\
&\Upsilon(g\varsigma, g\vartheta, \varpi)
\end{aligned}$$

$$\leq \max \left\{ \begin{array}{c} \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Upsilon\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \max\left\{\Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{b\mu}\right), \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right)\right\}, \Upsilon\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right) \end{array} \right\}$$
$$= \max\left\{\Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), 0, 0, \max\left\{0, \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right)\right\}, \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right)\right\}$$
$$= \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{b\mu}\right) = \Upsilon\left(g\varsigma, g\vartheta, \frac{\varpi}{b\mu}\right), \varpi > 0,$$

and it is implied from Lemma (3.2) that $\varsigma = \vartheta$. □

Theorem 3.4. *Let* $(\Lambda, \Xi, \Omega, \Upsilon, *, \diamond), * \geq *_\theta, \diamond \leq \diamond_\theta$*, be a CNbMS, and let* $g : \Lambda \to \Lambda$. *Suppose that for some* $\mu \in \left(0, \frac{1}{b^2}\right)$*,*

$$\Xi(g\varsigma, g\vartheta, \varpi) \geq \min \left\{ \begin{array}{c} \Xi\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Xi\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Xi\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \sqrt{\Xi\left(g\varsigma, \vartheta, \frac{2\varpi}{\mu}\right), \Xi\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right)} \end{array} \right\}$$
$$\Omega(g\varsigma, g\vartheta, \varpi) \leq \max \left\{ \begin{array}{c} \Omega\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Omega\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Omega\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \sqrt{\Omega\left(g\varsigma, \vartheta, \frac{2\varpi}{\mu}\right), \Omega\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right)} \end{array} \right\}$$
$$\Upsilon(g\varsigma, g\vartheta, \varpi) \leq \max \left\{ \begin{array}{c} \Upsilon\left(\varsigma, \vartheta, \frac{\varpi}{\mu}\right), \Upsilon\left(g\varsigma, \varsigma, \frac{\varpi}{\mu}\right), \Upsilon\left(g\vartheta, \vartheta, \frac{\varpi}{\mu}\right), \\ \sqrt{\Upsilon\left(g\varsigma, \vartheta, \frac{2\varpi}{\mu}\right), \Upsilon\left(\varsigma, g\vartheta, \frac{\varpi}{\mu}\right)} \end{array} \right\}, \quad (13)$$

$\varsigma, \vartheta \in \Lambda, \varpi > 0$ *and there exists* $\varsigma_0 \in \Lambda$ *and* $\rho \in (0,1)$ *such that*

$$\lim_{n\to\infty} *_{i=n}^{\infty} \Xi\left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i}\right) = 1,$$
$$\lim_{n\to\infty} \diamond_{i=n}^{\infty} \Omega\left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i}\right) = 0 \textit{ and}$$
$$\lim_{n\to\infty} \diamond_{i=n}^{\infty} \Omega\left(\varsigma_0, g\varsigma_0, \frac{\varpi}{\rho^i}\right) = 0 \textit{ for all } \varpi > 0. \quad (14)$$

Then g *has a unique fixed point in* Λ.

Proof. Let $\varsigma_0 \in \Lambda$ and $\varsigma_{n+1} = g\varsigma_n, n \in \mathbb{N}$. Taking $\varsigma = \varsigma_n$ and $\vartheta = \varsigma_{n-1}$ in condition (13), by using (f, k and p) and $* \geq *_\theta, \diamond \leq \diamond_\theta$ we have

$$\Xi(\varsigma_{n+1}, \varsigma_n, \varpi) \geq \min \left\{ \begin{array}{c} \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{\mu}\right), \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{\mu}\right), \\ \sqrt{\Xi\left(\varsigma_{n+1}, \varsigma_n, \frac{\varpi}{b\mu}\right) . \Xi\left(\varsigma_n, \varsigma_{n-1}, \frac{\varpi}{b\mu}\right), \Xi\left(\varsigma_n, \varsigma_n, \frac{\varpi}{\mu}\right)} \end{array} \right\},$$

$$\Omega(\varsigma_{n+1},\varsigma_n,\varpi) \leq \max\left\{\begin{array}{c} \Omega\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{\mu}\right),\Omega\left(\varsigma_{n+1},\varsigma_n,\frac{\varpi}{\mu}\right),\Omega\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{\mu}\right), \\ \sqrt{\Omega\left(\varsigma_{n+1},\varsigma_n,\frac{\varpi}{b\mu}\right).\Omega\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{b\mu}\right)},\Omega\left(\varsigma_n,\varsigma_n,\frac{\varpi}{\mu}\right) \end{array}\right\},$$

$$\Upsilon(\varsigma_{n+1},\varsigma_n,\varpi) \leq \max\left\{\begin{array}{c} \Upsilon\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{\mu}\right),\Upsilon\left(\varsigma_{n+1},\varsigma_n,\frac{\varpi}{\mu}\right),\Upsilon\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{\mu}\right), \\ \sqrt{\Upsilon\left(\varsigma_{n+1},\varsigma_n,\frac{\varpi}{b\mu}\right).\Upsilon\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{b\mu}\right)},\Upsilon\left(\varsigma_n,\varsigma_n,\frac{\varpi}{\mu}\right) \end{array}\right\},$$

$n \in \mathbb{N}, \varpi > 0$. Since $\Xi(\varsigma,\vartheta,\varpi)$ is b-nondecreasing in ϖ and $\sqrt{(a.b)} \geq \min\{a,b\}$, $\Omega(\varsigma,\vartheta,\varpi)$ and $\Upsilon(\varsigma,\vartheta,\varpi)$ is b-nonincreasing in ϖ and $\sqrt{(a.b)} \leq \max\{a,b\}$ we obtain that

$$\begin{aligned} \Xi(\varsigma_{n+1},\varsigma_n,\varpi) &\geq \min\left\{\Xi\left(\varsigma_{n+1},\varsigma_n,\frac{\varpi}{b\mu}\right),\Xi\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{b\mu}\right)\right\}, \\ \Omega(\varsigma_{n+1},\varsigma_n,\varpi) &\leq \max\left\{\Omega\left(\varsigma_{n+1},\varsigma_n,\frac{\varpi}{b\mu}\right),\Omega\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{b\mu}\right)\right\} \text{ and} \\ \Upsilon(\varsigma_{n+1},\varsigma_n,\varpi) &\leq \max\left\{\Upsilon\left(\varsigma_{n+1},\varsigma_n,\frac{\varpi}{b\mu}\right),\Upsilon\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{b\mu}\right)\right\}. \end{aligned}$$

By Lemma (3.2) and (3.1) it follows that

$$\begin{aligned} \Xi\left(\varsigma_{n+1},\varsigma_n,\varpi\right) &\geq \Xi\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{b\mu}\right), \\ \Omega\left(\varsigma_{n+1},\varsigma_n,\varpi\right) &\leq \Omega\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{b\mu}\right) \text{ and} \\ \Upsilon\left(\varsigma_{n+1},\varsigma_n,\varpi\right) &\leq \Upsilon\left(\varsigma_n,\varsigma_{n-1},\frac{\varpi}{b\mu}\right). \end{aligned}$$

where $n \in \mathbb{N}, \varpi > 0$, and $\{\varsigma_n\}$ is a Cauchy sequence. Since $(\Lambda,\Xi,\Omega,\Upsilon,*,\diamond)$ is a complete space, there exists $\varsigma \in \Lambda$ such that $\lim_{n\to\infty} \varsigma_n = \varsigma$ and

$$\begin{aligned} &\lim_{n\to\infty} \Xi(\varsigma,\varsigma_n,\varpi) = 1, \\ &\lim_{n\to\infty} \Omega(\varsigma,\varsigma_n,\varpi) = 0 \text{ and} \\ &\lim_{n\to\infty} \Upsilon(\varsigma,\varsigma_n,\varpi) = 0,\ \varpi > 0. \end{aligned} \tag{15}$$

Let $\tau_1 \in (b^2\mu, 1)$ and $\tau_2 = 1 - \tau_1$. By (13) and (f, k and p) for $* = *_\theta$ and $\diamond \leq \diamond_\theta$ we have

$$\begin{aligned} &\Xi(g\varsigma,\varsigma,\varpi) \\ &\geq *\left(\Xi\left(g\varsigma,g\varsigma_n,\frac{\varpi\tau_1}{b}\right),\Xi\left(g\varsigma_n,\varsigma,\frac{\varpi\tau_2}{b}\right)\right) \end{aligned}$$

$$\geq * \left(\min \left\{ \begin{array}{c} \Xi\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Xi\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), \\ \Xi\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right), \\ \sqrt{\Xi\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right) . \Xi\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu}\right)}, \\ \Xi\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right) \end{array} \right\}, \Xi\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right) \right).$$

$$\geq * \left(\min \left\{ \begin{array}{c} \Xi\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Xi\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), \\ \Xi\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right), \\ \min\left\{\Xi\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right), \Xi\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu}\right)\right\}, \\ \Xi\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right), \Xi\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right) \end{array} \right\} \right),$$

$$\Omega(g\varsigma, \varsigma, \varpi)$$
$$\leq \diamond \left(\Omega\left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b}\right), \Omega\left(g\varsigma_n, \varsigma, \frac{\varpi\tau_2}{b}\right)\right)$$

$$\leq \diamond \left(\max \left\{ \begin{array}{c} \Omega\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Omega\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), \\ \Omega\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right), \\ \sqrt{\Omega\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right) . \Omega\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu}\right)}, \\ \Omega\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right) \end{array} \right\}, \Omega\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right) \right),$$

$$\leq \diamond \left(\max \left\{ \begin{array}{c} \Omega\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Omega\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), \\ \Omega\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right), \\ \max\left\{\Omega\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right), \Omega\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu}\right)\right\}, \\ \Omega\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right), \Omega\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right) \end{array} \right\} \right) \text{ and}$$

$$\Upsilon(g\varsigma, \varsigma, \varpi)$$
$$\leq \diamond \left(\Upsilon\left(g\varsigma, g\varsigma_n, \frac{\varpi\tau_1}{b}\right), \Upsilon\left(g\varsigma_n, \varsigma, \frac{\varpi\tau_2}{b}\right)\right)$$

$$\leq \diamond \left(\max \left\{ \begin{array}{c} \Upsilon\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b\mu}\right), \Upsilon\left(\varsigma, g\varsigma, \frac{\varpi\tau_1}{b\mu}\right), \\ \Upsilon\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right), \\ \sqrt{\Upsilon\left(g\varsigma, \varsigma, \frac{\varpi\tau_1}{b^2\mu}\right) . \Upsilon\left(\varsigma, \varsigma_n, \frac{\varpi\tau_1}{b^2\mu}\right)}, \\ \Upsilon\left(\varsigma_n, \varsigma_{n+1}, \frac{\varpi\tau_1}{b\mu}\right) \end{array} \right\}, \Upsilon\left(\varsigma_{n+1}, \varsigma, \frac{\varpi\tau_2}{b}\right) \right),$$

$$\leq \diamond\left(\max\left\{\begin{array}{c}\Upsilon\left(\varsigma,\varsigma_n,\frac{\varpi\tau_1}{b\mu}\right),\Upsilon\left(\varsigma,g\varsigma,\frac{\varpi\tau_1}{b\mu}\right),\\ \Upsilon\left(\varsigma_n,\varsigma_{n+1},\frac{\varpi\tau_1}{b\mu}\right),\\ \max\left\{\Upsilon\left(g\varsigma,\varsigma,\frac{\varpi\tau_1}{b^2\mu}\right),\Upsilon\left(\varsigma,\varsigma_n,\frac{\varpi\tau_1}{b^2\mu}\right)\right\},\\ \Upsilon\left(\varsigma_n,\varsigma_{n+1},\frac{\varpi\tau_1}{b\mu}\right),\Upsilon\left(\varsigma_{n+1},\varsigma,\frac{\varpi\tau_2}{b}\right)\end{array}\right\}\right),$$

for all $\vartheta\in\mathbb{N}$ and $\varpi>0$. Taking $n\to\infty$ and using (15), we get

$$\begin{aligned}&\Xi\left(g\varsigma,\varsigma,\varpi\right)\\ &\geq *\left(\min\left\{1,\Xi\left(\varsigma,g\varsigma,\frac{\varpi\tau_1}{b\mu}\right),1,\min\left\{\Xi\left(g\varsigma,\varsigma,\frac{\varpi\tau_1}{b^2\mu}\right),1\right\},1\right\},1\right)\\ &=\Xi\left(g\varsigma,\varsigma,\frac{\varpi\tau_1}{b^2\mu}\right),\\ &\Omega\left(g\varsigma,\varsigma,\varpi\right)\\ &\leq \diamond\left(\max\left\{0,\Omega\left(\varsigma,g\varsigma,\frac{\varpi\tau_1}{b\mu}\right),0,\max\left\{\Omega\left(g\varsigma,\varsigma,\frac{\varpi\tau_1}{b^2\mu}\right),0\right\},0\right\},0\right)\\ &=\Omega\left(g\varsigma,\varsigma,\frac{\varpi\tau_1}{b^2\mu}\right)\text{ and}\\ &\Upsilon\left(g\varsigma,\varsigma,\varpi\right)\\ &\leq \diamond\left(\max\left\{0,\Upsilon\left(\varsigma,g\varsigma,\frac{\varpi\tau_1}{b\mu}\right),0,\max\left\{\Upsilon\left(g\varsigma,\varsigma,\frac{\varpi\tau_1}{b^2\mu}\right),0\right\},0\right\},0\right)\\ &=\Upsilon\left(g\varsigma,\varsigma,\frac{\varpi\tau_1}{b^2\mu}\right),\varpi>0\end{aligned}$$

and by Lemma (3.2) with $\rho=\frac{b^2\mu}{\tau_1}\in(0,1)$ as a result,$g\varsigma=\varsigma$.
Imagine that ς and ϑ are fixed points for g. By condition (13) we have

$$\begin{aligned}&\Xi\left(g\varsigma,g\vartheta,\varpi\right)\\ &\geq *\left(\begin{array}{c}\Xi\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right),\Xi\left(g\varsigma,\varsigma,\frac{\varpi}{\mu}\right),\Xi\left(g\vartheta,\vartheta,\frac{\varpi}{\mu}\right),\\ \sqrt{\Xi\left(g\varsigma,\varsigma,\frac{\varpi}{b\mu}\right),\Xi\left(\varsigma,\vartheta,\frac{\varpi}{b\mu}\right)},\Xi\left(\varsigma,g\vartheta,\frac{\varpi}{\mu}\right)\end{array}\right)\\ &\geq *\left(\Xi\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right),1,1,\min\left\{1,\Xi\left(\varsigma,\vartheta,\frac{\varpi}{b\mu}\right)\right\},\Xi\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right)\right)\\ &=\Xi\left(\varsigma,\vartheta,\frac{\varpi}{b\mu}\right)=\Xi\left(g\varsigma,g\vartheta,\frac{\varpi}{\mu}\right),\\ &\Omega\left(g\varsigma,g\vartheta,\varpi\right)\end{aligned}$$

$$\leq \diamond\left(\begin{array}{c}\Omega\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right),\Omega\left(g\varsigma,\varsigma,\frac{\varpi}{\mu}\right),\Omega\left(g\vartheta,\vartheta,\frac{\varpi}{\mu}\right),\\ \sqrt{\Omega\left(g\varsigma,\varsigma,\frac{\varpi}{b\mu}\right),\Omega\left(\varsigma,\vartheta,\frac{\varpi}{b\mu}\right)},\Omega\left(\varsigma,g\vartheta,\frac{\varpi}{\mu}\right)\end{array}\right)$$
$$\leq \diamond\left(\Omega\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right),0,0,\max\left\{0,\Omega\left(\varsigma,\vartheta,\frac{\varpi}{b\mu}\right)\right\},\Omega\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right)\right)$$
$$=\Omega\left(\varsigma,\vartheta,\frac{\varpi}{b\mu}\right)=\Omega\left(g\varsigma,g\vartheta,\frac{\varpi}{\mu}\right),\text{ and}$$
$$\Upsilon\left(g\varsigma,g\vartheta,\varpi\right)$$
$$\leq \diamond\left(\begin{array}{c}\Upsilon\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right),\Upsilon\left(g\varsigma,\varsigma,\frac{\varpi}{\mu}\right),\Upsilon\left(g\vartheta,\vartheta,\frac{\varpi}{\mu}\right),\\ \sqrt{\Upsilon\left(g\varsigma,\varsigma,\frac{\varpi}{b\mu}\right),\Upsilon\left(\varsigma,\vartheta,\frac{\varpi}{b\mu}\right)},\Upsilon\left(\varsigma,g\vartheta,\frac{\varpi}{\mu}\right)\end{array}\right)$$
$$\leq \diamond\left(\Upsilon\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right),0,0,\max\left\{0,\Upsilon\left(\varsigma,\vartheta,\frac{\varpi}{b\mu}\right)\right\},\Upsilon\left(\varsigma,\vartheta,\frac{\varpi}{\mu}\right)\right)$$
$$=\Upsilon\left(\varsigma,\vartheta,\frac{\varpi}{b\mu}\right)=\Upsilon\left(g\varsigma,g\vartheta,\frac{\varpi}{\mu}\right),\varpi>0.$$

Therefore, it follows from Lemma (3.2) that $\varsigma=\vartheta$. □

Example 3.3. *Let* $\Lambda=\{0,1,3\}$, $\Xi(\varsigma,\vartheta,\varpi)=e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}}$, $\Omega(\varsigma,\vartheta,\varpi)=1-e^{\frac{-(\varsigma-\vartheta)^2}{\varpi}}$ *and* $\Upsilon(\varsigma,\vartheta,\varpi)=1-e^{\frac{(\varsigma-\vartheta)^2}{\varpi}}$, $*=*_\theta,\diamond=\diamond_\theta$. *Then* $(\Lambda,\Xi,\Omega,\Upsilon,*,\diamond)$ *is a CNbMS with* $b=2$. *Define the function* $g:\Lambda\to\Lambda$ *as follows:* $g_0=g_1=1,g_3=0$. *Observe that if* $\varsigma=\vartheta$ *or* $\varsigma,\vartheta\in[0,1]$, *then* $\Xi(g\varsigma,g\vartheta,\varpi)=1,\Omega(g\varsigma,g\vartheta,\varpi)=0$ *and* $\Upsilon(g\varsigma,g\vartheta,\varpi)=0,\varpi>0$ *and condition (13) is fulfilled.*
Case: 1 *Let* $\varsigma=0$ *and* $\vartheta=1$. *Then since* $\mu\in\left(\frac{1}{9},\frac{1}{4}\right)$, *we have that*

$$\Xi(g\varsigma,g\vartheta,\varpi)=e^{\frac{-1}{\varpi}}\geq\min\left\{e^{\frac{-\mu}{\varpi}},e^{\frac{-\mu}{\varpi}},1,1,e^{\frac{-\mu}{\varpi}}\right\},$$
$$\Omega(g\varsigma,g\vartheta,\varpi)=1-e^{\frac{-1}{\varpi}}\leq\max\left\{1-e^{\frac{-\mu}{\varpi}},1-e^{\frac{-\mu}{\varpi}},0,0,1-e^{\frac{-\mu}{\varpi}}\right\},\textit{ and}$$
$$\Upsilon(g\varsigma,g\vartheta,\varpi)=1-e^{\frac{1}{\varpi}}\leq\max\left\{1-e^{\frac{-\mu}{\varpi}},1-e^{\frac{-\mu}{\varpi}},0,0,1-e^{\frac{-\mu}{\varpi}}\right\}.$$

Case: 2 *Let* $\varsigma=0$ *and* $\vartheta=3$. *Then since* $\mu\in\left(\frac{1}{9},\frac{1}{4}\right)$, *we have that*

$$\Xi(g\varsigma,g\vartheta,\varpi)=e^{\frac{-1}{\varpi}}\geq\min\left\{e^{\frac{-9\mu}{\varpi}},e^{\frac{-\mu}{\varpi}},e^{\frac{-9\mu}{\varpi}},e^{\frac{-\mu}{\varpi}},1\right\},$$
$$\Omega(g\varsigma,g\vartheta,\varpi)=1-e^{\frac{-1}{\varpi}}\leq\max\left\{1-e^{\frac{-9\mu}{\varpi}},1-e^{\frac{-\mu}{\varpi}},1-e^{\frac{-9\mu}{\varpi}},1-e^{\frac{-\mu}{\varpi}},0\right\},\textit{ and}$$
$$\Upsilon(g\varsigma,g\vartheta,\varpi)=1-e^{\frac{1}{\varpi}}\leq\max\left\{1-e^{\frac{-9\mu}{\varpi}},1-e^{\frac{-\mu}{\varpi}},1-e^{\frac{-9\mu}{\varpi}},1-e^{\frac{-\mu}{\varpi}},0\right\}.$$

Case: 3 *Let* $\varsigma = 1$ *and* $\vartheta = 3$. *Then selecting* $\mu \in \left(\frac{1}{9}, \frac{1}{4}\right)$, *we have that*

$$\Xi\left(g\varsigma, g\vartheta, \varpi\right) = e^{\frac{-1}{\varpi}} \geq \min\left\{e^{\frac{-4\mu}{\varpi}}, 1, e^{\frac{-9\mu}{\varpi}}, e^{\frac{-\mu}{\varpi}}, e^{\frac{-\mu}{\varpi}}\right\},$$

$$\Omega\left(g\varsigma, g\vartheta, \varpi\right) = 1 - e^{\frac{-1}{\varpi}} \leq \max\left\{1 - e^{\frac{-4\mu}{\varpi}}, 0, 1 - e^{\frac{-9\mu}{\varpi}}, 1 - e^{\frac{-\mu}{\varpi}}, 1 - e^{\frac{-\mu}{\varpi}}\right\} \text{ and}$$

$$\Upsilon\left(g\varsigma, g\vartheta, \varpi\right) = 1 - e^{\frac{1}{\varpi}} \leq \max\left\{1 - e^{\frac{4\mu}{\varpi}}, 0, 1 - e^{\frac{9\mu}{\varpi}}, 1 - e^{\frac{\mu}{\varpi}}, 1 - e^{\frac{\mu}{\varpi}}\right\}.$$

Therefore we conclude that $\mu \in \left(\frac{1}{9}, \frac{1}{4}\right)$, *condition (13) is fullfilled for* $\varsigma, \vartheta \in \Lambda$ *and* $\varpi > 0$.
Therefore, $\varsigma = 1$ *is a one and only fixed point and all conditions of Theorem (3.4) are met.*

4 Conclusion

An extension of the t-norm that is countable is used in this paper. Using this condition of expansion and contraction, we can show that the sequence in NbMS is a Cauchy sequence. By using our findings, some fixed point theorems are additionally established.

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