Ratio Mathematica Volume 51, 2024

On an abstract nonlinear functional second order Volterra integrodifferential equation

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Abstract

In this paper, we prove the existence, uniqueness and boundedness of solutions of a nonlinear functional second order Volterra integrodifferential equation in a general Banach space. The theory of strongly continuous cosine family, modified version of Banach contraction mapping principle and Pachpatte's integral inequality are main techniques employed in our analysis.

Keywords: Volterra integro-differential equation, Cosine family, Modified version of Banach contraction mapping principle, Pachpatte's integral inequality.

2020 AMS subject classifications: 45D05, 34G20, 45N05, 45P05, 37C25. [1]

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[1]Received on June 28, 2023. Accepted on February 10, 2024. Published on March 1, 2024. DOI: 10.23755/rm.v51i0.1252. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

In what follows, the symbol X stands for a Banach space with norm $\|\cdot\|$. Let $C = C([-r, 0], X),\ 0 < r < \infty$, be the Banach space of all continuous functions ψ from $[-r, 0]$ to X with supremum norm

$$\|\psi\|_C = \sup\{\|\psi(\theta)\| : -r \leq \theta \leq 0\}.$$

Let $B = C\,([-r, T],\, X), T > 0$ denotes the Banach space of all continuous functions $x : [-r,\, T] \to X$ with supremum norm $||\, x\, ||_B = sup\ \{||\, x(t)\, || : -r \leq t \leq T\}$. For any $x \in B$ and $t \in [0, T], x_t$ is the element of C given by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. Now, we study the abstract nonlinear functional second order Volterra integrodifferential equation of the form

$$x''(t) = Ax(t) + f\left(t, x_t, \int_0^t k(t,s)g(s, x_s)ds\right), \quad t \in [0, T] \tag{1}$$

$$x_0(t) = \phi(t), \quad -r \leq t \leq 0, \qquad x'(0) = \delta \tag{2}$$

where A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in Banach space $X,\ f : [0, T] \times C \times X \to X,\ k : [0, T] \times [0, T] \to \mathbb{R},\ g : [0, T] \times C \to X$ are continuous functions, ϕ and δ are given elements of $C = C([-r, 0], X)$ and X respectively.

The equations of these types (1)-(2) and their special forms serve as an abstract formulation of many partial differential equations or partial integrodifferential equations arising in variety of physical phenomena, see, for example [2, 3, 6, 7, 11] and some of the references given therein. The problems of existence, uniqueness and other properties of solutions of various special forms of (1)-(2) have been studied by using different techniques by many authors, see, for example [1, 4, 5, 8, 9, 10, 14, 15, 17, 19, 20, 21, 22] and some of the references cited therein. Our endeavour is to generalize some results obtained by A Pazy [15] and C.C. Travis and G. F. Webb [20]. To treat second order abstract differential equations directly is worthwhile rather than to convert into first order systems, see, for example Fitzgibbon [11]. In [11], Fitzgibbon employed the second order abstract equation for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. Our work in this paper is motivated by the interesting results obtained by Fattorini H. O. in [10] and is influenced by the work of Travis C. C. and Webb G. F. [21]. The aim of the present paper is to investigate the existence, uniqueness and boundedness of solution to second order Volterra functional integrodifferential equations (1)-(2) by using the theory of cosine and sine families of operators, modified version of contraction mapping principle and Pachpatte's integral inequality.

The paper is organized as follows: Section 2 presents the preliminaries and statements of our results. In section 3, we investigate Theorems 2.1 and 2.2. Finally, section 4 exhibits an example to illustrate the applications of our results.

2 Preliminaries and statement of results

Before proceeding to the statements of our main results, we setforth some preliminaries from [12, 18, 20] and hypotheses used in our further discussion.

Definition 2.1. *A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space X is called a strongly continuous cosine family if and only if*

(a) $C(0) = I$ *(I is the identity operator);*

(b) $C(t)x$ *is strongly continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in X$*;*

(c) $C(t+s) + C(t-s) = 2C(t)C(s)$ *for all* $t,\ s \in \mathbb{R}$*.*

The associated strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}. \tag{3}$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is the operator $A : X \longrightarrow X$ defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in X : C(.)x \in C^2(\mathbb{R}, X)\}$.

Definition 2.2. *Let $f \in L^1(0, T; X)$. The function $x \in B$ defined by*

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)\delta \\ &\quad + \int_0^t S(t-s) f\left(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\right) ds, \quad t \in [0,T] \qquad (4) \\ x_0(t) &= \phi(t), \quad -r \le t \le 0 \qquad (5) \end{aligned}$$

is called mild solution of the initial value problem (1)-(2).

we, now list the following hypotheses for our furhter discussion.

(H_1) There are constants $K \geq 1$ and $K_1 > 0$ such that

$$\|C(t)\| \leq K \quad \text{and} \quad \|S(t)\| \leq K_1,$$

for all $t \in [0, T]$.

(H_2) For every $t \in [0, T]$, $\psi \in C$ and $x \in X$, there exist a continuous function $p : [0, T] \to \mathbb{R}_+$ such that

$$\|f(t, \psi, x)\| \leq p(t)\big[\|\psi\|_C + \|x\|\big].$$

(H_3) There exist a continuous function $q : [0, T] \to \mathbb{R}_+$ such that

$$\|g(t, \psi)\| \leq q(t)\|\psi\|_C$$

for every $t \in [0, T]$ and $\psi \in C$.

(H_4) For every $t \in [0, T], \psi_1, \psi_2 \in C$ and $x_1, x_2 \in X$, there exists a constant M such that

$$\|f(t, \psi_1, x_1) - f(t, \psi_2, x_2)\| \leq M[\|\psi_1 - \psi_2\|_C + \|x_1 - x_2\|]$$

(H_5) There exists a constant N such that

$$\|g(t, \psi_1) - g(t, \psi_2)\| \leq N\|\psi_1 - \psi_2\|_C,$$

for all $t \in [0, T]$ and $\psi_1, \psi_2 \in C$.

We use modified version of Banach contraction mapping principle to prove our results.

Lemma 2.1 ([16], p. 196)**.** *Let X be a Banach space. Let D be an operator which maps the elements of X into itself for which D^r is a contraction, where r is positive integer. Then D has a unique fixed point in X.*

The following Pachpatte's inequality is the key instrument in our subsequent discussion.

Lemma 2.2 ([13], p. 758)**.** *Let $u(t), p(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on $\mathbb{R}_+$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t p(s)\Big[u(s) + \int_0^s q(\tau)u(\tau)d\tau\Big]ds,$$

holds for all $t \in \mathbb{R}_+$, where u_0 is a nonnegative constant, then

$$u(t) \leq u_0\Big[1 + \int_0^t p(s)\exp\Big(\int_0^s \big(p(\tau) + q(\tau)\big)d\tau\Big)ds\Big],$$

holds for all $t \in \mathbb{R}_+$.

With these preparations, now, we are in position to state our main results.

Theorem 2.1. *Suppose that the hypotheses* $(H_1), (H_4)$ *and* (H_5) *are satisfied. Then for each* $\phi, \delta \in C \times X$*, the initial value problem (1)-(2) has a unique mild solution* $x \in B$ *on* $[-r, T]$*. Moreover, the mapping* $(\phi, \delta) \longrightarrow x$ *is Lipschitz continuous from* $C \times X$ *into* B*.*

Theorem 2.2. *Suppose that the hypotheses* $(H_1)-(H_3)$ *hold. Then, every solution of the initial value problem (1)-(2) is bounded on* $[-r, T]$*.*

3 Proofs of the Theorem 2.1 and 2.2

Proof of Theorem 2.1. Define a mapping $F : B \to B = C([-r, T], X)$ by

$$(Fx)(t) = \begin{cases} \phi(t), \quad -r \le t \le 0, \\ \\ C(t)\phi(0) + S(t)\delta \\ \quad + \displaystyle\int_0^t S(t-s) f(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau) ds, \quad 0 \le t \le T. \end{cases} \tag{6}$$

In order to find mild solution of the initial value problem (1)-(2), it is sufficient to find the fixed point of an operator equation $x(t) = (Fx)(t)$. Since k is continuous real valued function on a compact set $[0, T] \times [0, T]$, there is a constant $L > 0$ such that

$$|k(t, s)| \le L \quad \text{for} \quad 0 \le s \le t \le T \tag{7}$$

Let $x, y \in B = C([-r, T], X)$. Then by using equation (6), condition (7) and hypotheses (H_1), (H_4) and (H_5) we have

$$\begin{aligned} \|(Fx)(t) - (Fy)(t)\| &\le \int_0^t \|S(t-s)\| \|f\left(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\right) \\ &\qquad - f\left(s, y_s, \int_0^s k(s,\tau) g(\tau, y_\tau) d\tau\right)\| ds \\ &\le K_1 M \int_0^t \left[\|x_s - y_s\|_C + LN \int_0^s \|x_\tau - y_\tau\|_C d\tau \right] ds \\ &\le K_1 M \int_0^t \left[\|x - y\|_B + LN \int_0^s \|x - y\|_B d\tau \right] ds \\ &= K_1 t [M + LMN \frac{t}{2}] \|x - y\|_B. \end{aligned} \tag{8}$$

Case 1: Suppose $t \geq r$. Then, for every $\theta \in [-r, 0]$, we have $t + \theta \geq 0$. For such $\theta' s$, from (8) we have

$$\begin{aligned}\|(Fx)_t(\theta) - (Fy)_t(\theta)\| &= \|(Fx)(t+\theta) - (Fy)(t+\theta)\| \\ &\leq K_1 M \int_0^{t+\theta} \{\|x-y\|_B + LN \int_0^s \|x-y\|_B d\tau\} ds \\ &\leq K_1 M \int_0^{t} \{\|x-y\|_B + LN \int_0^s \|x-y\|_B d\tau\} ds,\end{aligned}$$

which yields

$$\begin{aligned}\|(Fx)_t - (Fy)_t\|_C &\leq K_1 M \int_0^{t} \{\|x-y\|_B + LN \int_0^s \|x-y\|_B d\tau\} ds \\ &\leq t[K_1 M(1 + LN\frac{t}{2})]\|x-y\|_B \qquad (9)\end{aligned}$$

Case 2: Suppose $0 \leq t < r$. Then for all $\theta \in [-r, -t)$, we have $t + \theta < 0$. For such $\theta' s$, we observe, from (6), that

$$\begin{aligned}\|(Fx)_t(\theta) - (Fy)_t(\theta)\| &= \|(Fx)(t+\theta) - (Fy)(t+\theta)\| \\ &= 0,\end{aligned}$$

which implies

$$\|(Fx)_t - (Fy)_t\|_C = 0. \qquad (10)$$

For $\theta \in [-t, 0],\ t + \theta \geq 0$. Then, for such $\theta' s$ we obtain as in the case 1,

$$\|(Fx)_t - (Fy)_t\|_C \leq t[K_1 M(1 + LN\frac{t}{2})]\|x-y\|_B \qquad (11)$$

Thus, for every $\theta \in [-r, 0]$, $(0 \leq t \leq r)$, from (10) and (11), we obtain

$$\|(Fx)_t - (Fy)_t\|_C \leq t[K_1 M(1 + LN\frac{t}{2})]\|x-y\|_B \qquad (12)$$

For every $t \in [0, T]$, from inequalities (9) and (12), we set

$$\|(Fx)_t - (Fy)_t\|_C \leq t[K_1 M(1 + LN\frac{t}{2})]\ \|x-y\|_B \qquad (13)$$

By using the equation (6), the result (13), condition (7), and hypotheses (H_1), (H_4) and (H_5) we get

$$\begin{aligned}
\|(F^2x)(t)-(F^2y)(t)\| &= \|(F(Fx))(t)-(F(Fy))(t)\| \\
&= \|(Fx_1)(t)-(Fy_1)(t)\| \\
&\le \int_0^t \|S(t-s)\| \|f\Big(s, x_{1_s}, \int_0^s k(s,\tau)g(\tau, x_{1_\tau})d\tau\Big) \\
&\qquad - f\Big(s, y_{1_s}, \int_0^s k(s,\tau)g(\tau,\ y_{1_\tau})d\tau\Big)\|ds \\
&\le K_1M\int_0^t \Big[\|x_{1_s}-y_{1_s}\|_C + LN\int_0^s \|x_{1_\tau}-y_{1_\tau}\|_C d\tau\Big]ds \\
&\le K_1M\int_0^t \Big\{K_1Ms[1+LN\frac{s}{2}]\|x-y\|_B \\
&\qquad\qquad + LN\int_0^s K_1M\tau[1+LN\frac{\tau}{2}]\|x-y\|_B d\tau\Big\}ds \\
&= \frac{t^2}{2}K_1^2M^2\|x-y\|_B[1+2LN\frac{t}{3}+L^2N^2\frac{t^2}{12}] \\
&\le \frac{t^2}{2}K_1^2M^2\|x-y\|_B[1+2LN\frac{t}{2}+L^2N^2\frac{t^2}{4}] \\
&= \frac{t^2}{2}[K_1M(1+LN\frac{t}{2})]^2\|x-y\|_B
\end{aligned} \tag{14}$$

By considering Case 1 and Case 2 as before and applying same arguments we prove that:

$$\|(F^2x)_t-(F^2y)_t\|_C \le \frac{t^2}{2}[K_1M(1+LN\frac{t}{2})]^2\|x-y\|_B \tag{15}$$

for every $t\in[0,T]$.
Using equation (6), result (15), the condition (7), and hypotheses (H_1), (H_4) and

(H_5) we obtain

$$
\begin{aligned}
\|(F^3x)(t)-(F^3y)(t)\| &= \|F(F^2x)(t)-F(F^2y)(t)\| \\
&= \|(Fx_2)(t)-(Fy_2)(t)\| \\
&\le K_1M\int_0^t\left[\|x_{2_s}-y_{2_s}\|_C+LN\int_0^s\|x_{2_\tau}-y_{2_\tau}\|_C d\tau\right]ds \\
&\le K_1M\int_0^t\left\{\frac{s^2}{2}\left[K_1M(1+LN\frac{s}{2})\right]^2\|x-y\|_B\right. \\
&\qquad\left.+LN\int_0^s\frac{\tau^2}{2}\left[K_1M(1+LN\frac{\tau}{2})\right]^2\|x-y\|_B d\tau\right\}ds \\
&\le K_1^3M^3\|x-y\|_B\frac{t^3}{3!}\left[1+LNt+L^2N^2\frac{t^2}{3}+L^3N^3\frac{t^3}{40}\right] \\
&\le K_1^3M^3\|x-y\|_B\frac{t^3}{3!}\left[1+3LN\frac{t}{2}+3L^2N^2\frac{t^2}{4}+L^3N^3\frac{t^3}{8}\right] \\
&= \frac{t^3}{3!}\left[K_1M(1+LN\frac{t}{2})\right]^3\|x-y\|_B \qquad (16)
\end{aligned}
$$

By continuing the iteration process, it follows that

$$
\begin{aligned}
\|(F^nx)(t)-(F^ny)(t)\| &\le \frac{t^n}{n!}[K_1M(1+LN\frac{t}{2})]^n\|x-y\|_B \\
&\le \frac{T^n}{n!}[K_1M(1+LN\frac{T}{2})]^n\|x-y\|_B
\end{aligned}
$$

which yields

$$
\|F^nx-F^ny\|_B \le \frac{1}{n!}[K_1MT(1+LN\frac{T}{2})]^n\|x-y\|_B \qquad (17)
$$

For n large enough, $\frac{1}{n!}[K_1MT(1+LN\frac{T}{2})]^n<1$. This implies that F^n is a contraction in B for n sufficiently large and hence, by modified Banach contraction principle given in Lemma 2.1, it follows that F has a unique fixed point, say $x\in B$ which is the mild solution of initial value problem (1)–(2).

Next, we show that the mapping $(\phi,\delta)\longrightarrow x$ is Lipschitz continuous from $C\times X$ into B. Assume that y is another mild solution of the initial value problem (1) on $[-r,T]$ with $y(t)=\chi(t)$, $-r\le t\le 0$ and $y'(0)=\sigma$. By making use of definition of mild solution given by (4)-(5) and hypotheses $(H_1),(H_4)$ and (H_5), we obtain

$$\begin{aligned}\|x(t)-y(t)\| &\leq \|C(t)\|\,\|\phi(0)-\chi(0)\| + \|S(t)\|\,\|\delta-\sigma\| \\ &\qquad + \int_0^t \|S(t-s)\|\,\Big\|f\Big(s,x_s,\int_0^s k(s,\tau)g(\tau,x_\tau)d\tau\Big) \\ &\qquad - f\Big(s,y_s,\int_0^s k(s,\tau)g(\tau,y_\tau)d\tau\Big)\Big\|ds \\ &\leq K\|\phi(0)-\chi(0)\| + K_1\|\delta-\sigma\| + \int_0^t K_1M\Big[\|x_s-y_s\|_C \\ &\qquad + LN\int_0^s \|x_\tau-y_\tau\|_C d\tau\Big]ds \end{aligned} \tag{18}$$

Case1: Suppose $t \geq r$. Then, for every $\theta \in [-r,0]$, we have $t+\theta \geq 0$. For such $\theta' s$, from (18), we have

$$\begin{aligned}&\|x(t+\theta)-y(t+\theta)\| \\ &\quad\leq K\|\phi(0)-\chi(0)\| + K_1\|\delta-\sigma\| + \int_0^{t+\theta} K_1M\Big[\|x_s-y_s\|_C \\ &\qquad + LN\int_0^s \|x_\tau-y_\tau\|_C d\tau\Big]ds \\ &\quad\leq K\|\phi(0)-\chi(0)\| + K_1\|\delta-\sigma\| + \int_0^{t} K_1M\Big[\|x_s-y_s\|_C \\ &\qquad + LN\int_0^s \|x_\tau-y_\tau\|_C d\tau\Big]ds\end{aligned}$$

which yields

$$\begin{aligned}&\|x_t-y_t\|_C \\ &\quad\leq K\|\phi(0)-\chi(0)\| + K_1\|\delta-\sigma\| + \int_0^{t} K_1M\Big[\|x_s-y_s\|_C \\ &\qquad + LN\int_0^s \|x_\tau-y_\tau\|_C d\tau\Big]ds\end{aligned} \tag{19}$$

Case2: Suppose $\theta \leq t < r$. Then, for all $\theta \in [-r,-t)$, we have $t+\theta < 0$. For such $\theta' s$, we observe from (4)-(5) that

$$\begin{aligned}\|x(t+\theta)-y(t+\theta)\| &= \|x_t(\theta)-y_t(\theta)\| \\ &= \|\phi(t+\theta)-\chi(t+\theta)\|\end{aligned}$$

which yields

$$\|x_t-y_t\|_C \leq \|\phi-\chi\|_C \tag{20}$$

For $\theta \in [-t, 0],\ t+\theta \geq 0$. Then, for such $\theta' s$, we get as in Case 1,

$$\begin{aligned}\|x_t - y_t\|_C &\leq K\|\phi(0) - \chi(0)\| + K_1\|\delta - \sigma\| \\ &\quad + \int_0^t K_1 M\left[\|x_s - y_s\|_C + LN\int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \qquad (21)\end{aligned}$$

Thus, for every $\theta \in [-r, 0],\ (0 \leq t < r)$, from (20) and (21) we obtain

$$\begin{aligned}\|x_t - y_t\|_C &\leq K\|\phi - \chi\|_C + K_1\|\delta - \sigma\| \\ &\quad + \int_0^t K_1 M\left[\|x_s - y_s\|_C + LN\int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \qquad (22)\end{aligned}$$

For every $t \in [0, T]$, from inequalities (19) and (22), we get

$$\begin{aligned}\|x_t - y_t\|_C &\leq K\|\phi - \chi\|_C + K_1\|\delta - \sigma\| \\ &\quad + \int_0^t K_1 M\left[\|x_s - y_s\|_C + LN\int_0^s \|x_\tau - y_\tau\|_C d\tau\right] ds \qquad (23)\end{aligned}$$

Thanks to Pachpatte's inequality given in Lemma 2.2 and applying it to the inequality (23) with $u(t) = \|x_t - y_t\|_C$, we obtain

$$\begin{aligned}\|x_t - y_t\|_C &\leq [K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|] \\ &\qquad \left[1 + \int_0^t K_1 M \exp\left(\int_0^s (K_1 M + LN) d\tau\right) ds\right] \\ &\leq [K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|] \\ &\qquad \left[1 + K_1 M T \exp\left\{(K_1 M + LN)T\right\}\right]\end{aligned}$$

Thus, for $t \in [-r, T]$, we have

$$\begin{aligned}\|x(t) - y(t)\| &\leq [K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|] \\ &\qquad \left[1 + K_1 M T \exp\left\{(K_1 M + LN)T\right\}\right]\end{aligned}$$

and hence

$$\begin{aligned}\|x - y\|_B &\leq [K\|\phi - \chi\|_C + K_1\|\delta - \sigma\|] \\ &\qquad \left[1 + K_1 M T \exp\left\{(K_1 M + LN)T\right\}\right]\end{aligned}$$

This proves Lipschitz continuity of the mapping $(\phi, \delta) \longrightarrow x$ and hence the proof of the Theorem is complete. $\square$

Proof of Theorem 2.2. The solution of the initial value problem (1)-(2) is given by

$$x(t) = C(t)\phi(0) + S(t)\delta + \int_0^t S(t-s) f\Big(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\Big) ds \qquad t \in [0, T] \tag{24}$$

$$x_0(t) = \phi(t), \quad -r \le t \le 0 \tag{25}$$

If $t \in [0, T]$ then from (24) and using the hypotheses $(H_1) - (H_3)$, and condition (7) we have

$$\begin{aligned}\|x(t)\| &\le \|C(t)\| \, \|\phi(0)\| + \|S(t)\| \, \|\delta\| \\ &\quad + \int_0^t \|S(t-s)\| \, \|f\Big(s, x_s, \int_0^s k(s,\tau) g(\tau, x_\tau) d\tau\Big)\| ds \\ &\le K\|\phi(0)\| + K_1\|\delta\| + \int_0^t K_1 p(s) \Big[\|x_s\|_C + L\int_0^s q(\tau)\|x_\tau\|_C d\tau\Big] ds\end{aligned}$$

Since $K \ge 1$, for $-r \le t \le T$, we get

$$\|x(t)\| \le K\|\phi\|_C + K_1\|\delta\| + \int_0^t K_1 p(s) \Big[\|x_s\|_C + L\int_0^s q(\tau)\|x_\tau\|_C d\tau\Big] ds \tag{26}$$

From (26) and considering case 1 and 2 as in the proof of the Theorem 2.1, we obtain

$$\begin{aligned}\|x_t\|_C \le K\|\phi\|_C + K_1\|\delta\| &+ \int_0^t K_1 p(s)\|x_s\|_C ds \\ &+ \int_0^t K_1 p(s) \int_0^s L q(\tau)\|x_\tau\|_C d\tau ds\end{aligned} \tag{27}$$

Thanks to Pachpatte's integral inequality given in Lemma 2.2 and applying it to (27) with $u(t) = \|x_t\|_C$, we get

$$\begin{aligned}\|x_t\|_C &\le \Big[K\|\phi\|_C + K_1\|\delta\|\Big] \\ &\qquad \Big[1 + \int_0^t K_1 p(s) \exp\Big(\int_0^s (K_1 p(\tau) + L q(\tau)) d\tau\Big)\Big] ds \\ &\le \Big[K\|\phi\|_C + K_1\|\delta\|\Big]\Big[1 + \big\{K_1 P \exp(K_1 P + LQ)T\big\}T\Big]\end{aligned} \tag{28}$$

where

$$P = \max_{t\in[0,T]} p(t),\ Q = \max_{t\in[0,T]} q(t).$$

It follows that solutions $x(t)$ of initial value problem (1)-(2) are bounded on closed interval $[-r, T]$ and proof of the Theorem is complete. □

Remark 3.1. *It is to be noted that our result in above Theorem 2.2 also proves the stability of a solution $x(t)$ if $\|\phi\|_C,\ \|\delta\|$ are small enough.*

Remark 3.2. *It is important to note that cosine family $C(t)$ and sine family $S(t)$ are not bounded in $\mathbb{R}$. $C(t)$ and $S(t)$ are bounded only in finite interval and may have exponential growth on $\mathbb{R}$. Consequently, all solutions of initial value problem (1)-(2) are not bounded on $\mathbb{R}_+$.*

4 Application

In order to illustrate the applications of our results proved in previous section, we consider the following nonlinear partial integrodifferential equation of the type

$$\begin{aligned} z_{tt}(w,t) &= z_{ww}(w,t) + \frac{z(w,t-r)\sin(z(w,t-r))}{1+t^2} + \int_0^t \frac{sz(w,s-r)}{1+t}ds, \\ & \qquad\qquad t\in[0,T],\quad 0\le w\le\pi \qquad (29)\\ z(0,t) &= z(\pi,t) = 0,\ t\in[0,T], \qquad (30)\\ z(w,t) &= \phi(w,t),\ 0\le w\le\pi, -r\le t\le 0, \qquad (31)\\ z_t(w,0) &= z_0(w),\ 0\le w\le\pi, \qquad (32)\end{aligned}$$

Let $X = L^2([0,\pi])$ be Banach space with norm $\|\cdot\|_{L^2}$. Define $z(w,t) = x(t)(w)$ and $z(w,t-r) = x_t(.)w$. Now put

$$f(t, x_t, \int_0^t k(t,s)g(s,x_s)ds) = \frac{x_t\sin x_t}{1+t^2} + \int_0^t \frac{sx_s}{1+t}ds,$$
$$k(t,s) = \frac{1}{1+t} \quad \text{and} \quad g(s,x_s) = sx_s,$$

Then, we have

$$\begin{aligned} ||g(s,x_s) - g(s,y_s)|| &= |s|\ ||x_s - y_s||_C \\ &= T||x_s - y_s||_C \qquad (33)\\ |k(t,s)| = \left|\frac{1}{1+t}\right| &= \frac{1}{1+t} \le 1 \qquad (34)\end{aligned}$$

and

$$\begin{aligned}
&\left\| f(t, x_t, \int_0^t k(t,s)g(s,x_s)ds) - f(t, y_t, \int_0^t k(t,s)g(s,y_s)ds) \right\| \\
&\qquad \le \left\| \frac{x_t \sin x_t - y_t \sin y_t}{1+t^2} \right\| + \int_0^t \frac{s\|x_s - y_s\|_C}{1+t} ds \\
&\qquad = \frac{1}{1+t^2}\|x_t + y_t\|_C + \frac{t^2}{2(1+t)}\|x-y\|_B \\
&\qquad \le \frac{N_1(t)}{1+t^2}\|x_t - y_t\|_C + \frac{t^2}{2(1+t)}N_2(t)\|x-y\| \\
&\qquad \le N_3[\|x_t - y_t\|_C + \|x-y\|] \qquad (35)
\end{aligned}$$

where $N_1(t)$ and $N_2(t)$ are positive functions such that $\|x_t + y_t\|_C \le N_1(t)\|x_t - y_t\|_C$ and $\|x-y\|_B \le N_2(t)\|x-y\|$ and $N_3 = \max_{t\in[0,T]}\big(\frac{N_1(t)}{1+t^2}, \frac{t^2}{2(1+t)}N_2(t)\big)$.

Also we have

$$\|g(t,x_t)\| \le t\|x_t\|_C \qquad (36)$$

and

$$\begin{aligned}
\left\| f(t, x_t, \int_0^t k(t,s)g(s,x_s)ds) \right\| &\le \frac{\|x_t\|_C}{1+t^2} + \int_0^t \frac{s\|x_s\|_C}{1+t}ds \\
&\le \frac{\|x_t\|_C}{1+t^2} + \int_0^t \frac{s\|x\|_B}{1+t}ds \\
&\le \frac{\|x_t\|_C}{1+t^2} + \frac{t^2\|x\|_B}{2(1+t)} \\
&\le \frac{\|x_t\|_C}{1+t^2} + \frac{t^2}{2(1+t)}N_4(t)\|x\| \\
&\le N_5(t)[\|x_t\|_C + \|x\|] \qquad (37)
\end{aligned}$$

where $N_4(t)$ is positive function such that $\|x\|_B \le N_4(t)\|x\|$ and $N_5(t) = \max\big(\frac{1}{1+t^2}, \frac{t^2 N_4(t)}{2(1+t)}\big)$. Define the operator $A : X \to X$ by $Ay = y''$ with domain $D(A) = \{y \in X : y, y'$ are absolutely continuous, $y'' \in X$ and $y(0) = y(\pi) = 0\}$. Then

$$Ay = \sum_{n=1}^{\infty} -n^2 (y, y_n) y_n, \quad y \in D(A),$$

where $y_n(s) = (\sqrt{2/\pi})\sin ns, \ n = 1, 2, 3, ...$ is the orthogonal set of eigenvectors of A and it can be easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t), \ t \in \mathbb{R}$, in X and is given by

$$C(t)y = \sum_{n=1}^{\infty} \cos nt (y, y_n) y_n, \quad y \in X.$$

The associated sine family is given by

$$S(t)y = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt(y, y_n)y_n, \quad y \in X.$$

Also $S(t)$ is compact for every $t > 0$ and $\|C(t)\| \leq 1$ and $\|S(t)\| \leq 1$ for every $t \in [0, T]$. For more details, we refer the readers to [10, 18].

With these choices of A, f, g and k it is to be observed that equations (1)-(2)is an abstract formulation of (29) - (32) and hence our Theorems 2.1 and 2.2 can be applied to guarantee the existence, uniqueness and boundedness of soutions of nonlinear partial integrodifferential equations (29) - (32).

5 Conclusions

We have developed modified version of Banach contraction mapping principle and proved the existence, uniqueness and boundedness of solutions of an abstract nonlinear functional second order Volterra integrodifferential equation (1)-(2). We have also presented an example to illustrate the applications of results proved in section 3.

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