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B-Gabor type frames in separable Hilbert spaces

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Abstract

It is guaranteed that Gabor like structured frames do exist in any finite dimensional Hilbert space via an invertible map from $l^2(\mathbb{Z}_N)$ J Thomas and Nambudiri [2022]. Hence the question: whether it is possible to obtain structured class of frames in separable Hilbert spaces? is relevant. In this article we obtain a structured class of frames for separable Hilbert spaces which are generalizations of Gabor frames for $L^2(\mathbb{R})$ in their construction aspects. We call them as B-Gabor type frames and present a characterization of the frame operators associated with these frames when B is a unitary map. Some significant properties of the associated frame operators are discussed.
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1 Introduction

Elegant and fast growing theory of frames in Hilbert spaces clenches its own space in both pure and applied mathematics due to its wide applications. Time-frequency analysis of signals in $L^2(\mathbb{R})$, as suggested by D. Gabor [1946] in *Theory of Communication*, requires a special system of the form $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$, where $g \in L^2(\mathbb{R})$ and E_b, T_a $(a, b > 0)$ are the modulation and translation operators on $L^2(\mathbb{R})$ respectively. In their study of non harmonic Fourier series Duffin and Schaeffler [1952] were introduced frames in Hilbert spaces. M. Janssen [1981] designed it an independent topic of mathematical investigation by his outstanding work. Frames were brought to life by I. Daubechies and Meyer [1986] with the fundamental works and put forth the idea of combining Gabor analysis with frame theory.

Ample generalizations of the concept of frames have been proposed; frame of subspaces Cazassa and Kutyniok [2004], pseudo-frames Li and Ogawa [2004], oblique frames Christensen and Eldar [2004] and so on, where in all, the mathematical theory of Gabor frames (also known as Weyl-Heisenberg frames) plays the crucial role. These frames are very specific as they are being generated by translations and modulations of a single vector in the space concerned and hence they have a specific structure. Another class of structured frames in $L^2(\mathbb{R})$ are wavelet frames Nambudiri and Parthasarathy [2021]. Motivated from these class of structured frames, here we introduce a structured class of frames for separable Hilbert spaces $\mathcal{H}$ which are generalizations of Gabor frames for $L^2(\mathbb{R})$ in their construction aspects. The concept of Gabor type unitary systems in D. Han [2000] is generalized by considering a system of invertible operators in place of unitary systems hence we obtain a class of structured frames in $\mathcal{H}$, we call them as B-Gabor type frames.

A foremost object in frame theory is the frame operator associated with a given frame. In particular, Gabor frame operators, which are very special in their construction, are acquiring notable research attention. A complete characterization of Gabor frame operators on $L^2(\mathbb{R})$ is recently appeared in Nambudiri and Parthasarathy [2013]. In this paper we present a characterization of the frame operators associated with B-Gabor type frames when B is a unitary map.

The arrangement of this article is, Section 2 provides some basic definitions and results which are very essential for the study. B-Gabor type frames in separable Hilbert spaces and a complete characterization of frame operators associated with these frames is presented in Section 3. Our basic references for both abstract frame theory and the theory of Gabor frames are Christensen [2016] and Gröchenig [2001].

2 Preliminaries

A countable sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a Hilbert space $\mathcal{H}$ is said to be a *frame* in $\mathcal{H}$, if there are constants $\alpha, \beta > 0$, such that

$$\alpha\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq \beta\|f\|^2 \ \forall f \in \mathcal{H}.$$

These constants α, β are called *frame bounds*. $\{f_k\}_{k=1}^{\infty}$ is called a *tight frame*, if the frame bounds are equal and is called a *Parseval frame or normalized tight frame* when $\alpha = \beta = 1$. Whenever a sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ satisfies the upper frame inequality, then it is said to be a *Bessel sequence* or a *semi-frame sequence*.

If $\{f_k\}_{k=1}^{\infty}$ is a frame in a Hilbert space $\mathcal{H}$, then the map S defined using this collection by $Sf = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$ for all $f \in \mathcal{H}$ is a bounded linear operator on $\mathcal{H}$, called the *frame operator* associated with the frame $\{f_k\}_{k=1}^{\infty}$. It is easy to observe that the frame operator of a tight frame is a scalar multiple of the identity operator and that of a normalized tight frame is the identity operator, Christensen [2016]. For a Bessel sequence $\{f_k\}_{k=1}^{\infty}$, the operator defined above may not be invertible and it is called a *semi-frame operator*.

Remark 2.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S and frame bounds α, β in a Hilbert space $\mathcal{H}$. Then,*

(i) S is bounded, invertible, self-adjoint and positive.

(ii) $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a frame with frame bounds β^{-1}, α^{-1} and $\{S^{-1/2}f_k\}_{k=1}^{\infty}$ is a normalized tight frame.

(iii) If α and β are the optimal frame bounds for $\{f_k\}_{k=1}^{\infty}$, then the bounds β^{-1}, α^{-1} are the optimal frame bounds for $\{S^{-1}f_k\}_{k=1}^{\infty}$. The frame operator for $\{S^{-1}f_k\}_{k=1}^{\infty}$ is S^{-1}. Further we have $\beta^{-1}I \leq S^{-1} \leq \alpha^{-1}I$.

Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S in $\mathcal{H}$, then the frame $\{S^{-1}f_k\}_{k=1}^{\infty}$ is called the *(canonical) dual frame* of the frame $\{f_k\}_{k=1}^{\infty}$. As follows, every frame in $\mathcal{H}$ admits the *frame decomposition* in two ways.

Theorem 2.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S in a Hilbert space $\mathcal{H}$. Then for all $f \in \mathcal{H}$, $f = \sum_{k=1}^{\infty}\langle f, S^{-1}f_k\rangle f_k$ and $f = \sum_{k=1}^{\infty}\langle f, f_k\rangle S^{-1}f_k$. Both the series converge unconditionally for all $f \in \mathcal{H}$.*

Gabor frames in $L^2(\mathbb{R})$ are generated by two classes of operators on $L^2(\mathbb{R})$, namely the translation and modulation operators. For $a, b \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$, the translation operator T_a on $L^2(\mathbb{R})$ is defined by $(T_a f)(x) = f(x-a),\ x \in \mathbb{R}$ and the modulation operator E_b on $L^2(\mathbb{R})$ by $(E_b f)(x) = e^{2\pi i b x} f(x),\ x \in \mathbb{R}$.

A frame in $L^2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ for some non zero g in $L^2(\mathbb{R})$ (known as *window function or generating function*) and $a, b > 0$ is called a *Gabor frame or Weyl-Heisenberg frame*. Gabor analysis aims at representing functions $f \in L^2(\mathbb{R})$ as superposition of translated and modulated versions of a fixed function $g \in L^2(\mathbb{R})$.

3 Gabor type frames and operators in separable Hilbert spaces

We begin with a simple finding, analogous to Corollary 5.3.2 in Christensen [2016], on the interplay of bounded linear operators between separable Hilbert spaces and is given as a lemma which is very essential for our further discussion.

Lemma 3.1. *Every invertible bounded linear operator between two separable Hilbert spaces maps frames in one to frames in the other.*

Proof. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be an invertible bounded linear operator between separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and let $B = (A^*)^{-1}$. Then $B : \mathcal{H}_1 \to \mathcal{H}_2$ is also an invertible bounded linear operator and $A = (B^{-1})^*$. Let $\{g_k\}_{k\in\mathbb{N}}$ be a frame in $\mathcal{H}_1$ with frame bounds $0 < \alpha \leq \beta < \infty$. Then for all $f \in \mathcal{H}_1$,

$$\alpha\|f\|^2 \leq \sum_{k\in\mathbb{N}} |\langle f, g_k\rangle|^2 \leq \beta\|f\|^2$$

$$\Rightarrow \alpha\|B^{-1}x\|^2 \leq \sum_{k\in\mathbb{N}} |\langle B^{-1}x, g_k\rangle|^2 \leq \beta\|B^{-1}x\|^2 \text{ for all } x \in \mathcal{H}_2$$

$$\Rightarrow \|B\|^{-2}\alpha\|x\|^2 \leq \sum_{k\in\mathbb{N}} |\langle x, B^{-1*}g_k\rangle|^2 \leq \beta\|B^{-1}\|^2\|x\|^2 \text{ for all } x \in \mathcal{H}_2$$

since $\alpha\|x\|^2 \leq \alpha\|B\|^2\|B^{-1}x\|^2 \Rightarrow \|B\|^{-2}\alpha\|x\|^2 \leq \alpha\|B^{-1}x\|^2$.

Thus $\{B^{-1*}g_k\}_{k\in\mathbb{N}} = B^{-1*}(\{g_k\}_{k\in\mathbb{N}}) = A(\{g_k\}_{k\in\mathbb{N}})$ is frame in $\mathcal{H}_2$ with frame bounds $0 < \|B\|^{-2}\alpha \leq \beta\|B^{-1}\|^2 < \infty$. □

Note that, if $\{g_k\}_{k\in\mathbb{N}}$ is a frame in the Hilbert space $\mathcal{H}_1$, with frame operator S and if $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded invertible map, then $\{Ag_k\}_{k\in\mathbb{N}}$ is a frame in $\mathcal{H}_2$ and has frame operator ASA^* Nambudiri and Parthasarathy [2012].

For a couple of invertible bounded linear operators B and C on a separable Hilbert space $\mathcal{H}$, the family $(B, C, x_0) = \{B^m C^n x_0 : m, n \in \mathbb{Z}\}$ generated by a fixed x_0 in $\mathcal{H}$ is of interest of discussion here. We look at different aspects of these systems when they yield Bessel sequences or frames in $\mathcal{H}$. The adjoint and the inverse of each one of the operator acting in this system admits an interesting relation as follows.

Lemma 3.2. *Let B and C be two invertible bounded linear operators on a separable Hilbert space $\mathcal{H}$. If, for $x_0 \in \mathcal{H}$, $\{B^m C^n x_0 : m, n \in \mathbb{Z}\}$ is a Bessel sequence in $\mathcal{H}$, then $B^m S_B (B^*)^m = S_B$ and $C^n S_C (C^*)^n = S_C$ for all $m, n \in \mathbb{Z}$ where S_B and S_C are semi-frame operators of the Bessel sequences $\{B^m x_0 : m \in \mathbb{Z}\}$ and $\{C^n x_0 : n \in \mathbb{Z}\}$ respectively.*

Proof. The proof follows by observing

$$\begin{aligned} S_B B^* x &= \sum_{m\in\mathbb{Z}} \langle B^* x, B^m x_0 \rangle B^m x_0 \\ &= \sum_{m\in\mathbb{Z}} \langle x, B^{m+1} x_0 \rangle B^m x_0 \\ &= \sum_{k\in\mathbb{Z}} \langle x, B^k x_0 \rangle B^{k-1} x_0 \\ &= B^{-1} \Big(\sum_{k\in\mathbb{Z}} \langle x, B^k x_0 \rangle B^k x_0 \Big) \\ &= B^{-1} S_B x \text{ for all } x \in \mathcal{H}. \end{aligned}$$

Similarly it is true that, $S_C C^* x = C^{-1} S_C x$ for all $x \in \mathcal{H}$. Now the conclusion follows easily. □

Here is another interesting observation,

Lemma 3.3. *Let B and C be two invertible bounded linear operators on a separable Hilbert space $\mathcal{H}$. If the family $\{B^m C^n x_0 : m, n \in \mathbb{Z}\}$, $x_0 \in \mathcal{H}$, is a Bessel sequence in $\mathcal{H}$ with semi-frame operator S, then $S(B^*)^l = (B^{-1})^l S$ for all $l \in \mathbb{Z}$.*

Proof. For each $l \in \mathbb{Z}$, the image of the frame $\mathcal{B} = \{B^m C^n x_0 : m, n \in \mathbb{Z}\}$ under the operator B^l is $\mathcal{B}$ itself since $B^l(\mathcal{B}) = B^l(\{B^m C^n x_0 : m, n \in \mathbb{Z}\}) = \{B^{m+l} C^n x_0 : m, n \in \mathbb{Z}\} = \{B^k C^n x_0 : k, n \in \mathbb{Z}\} = \mathcal{B}$. Since the frame operator of this image frame is $B^l S (B^*)^l$ we obtain $B^l S (B^*)^l = S$ for all $l \in \mathbb{Z}$. □

The rest of our discussion focuses on specific operators B and C in the system $(B, C, x_0) = \{B^m C^n x_0 : m, n \in \mathbb{Z}\}$ which are generated by translation and modulation operators on $L^2(\mathbb{R})$ through invertible bounded linear operators $B : L^2(\mathbb{R}) \to \mathcal{H}$. We will use following definitions for upcoming discussions and results.

Definition 3.1. *For an invertible bounded linear operator $B : L^2(\mathbb{R}) \to \mathcal{H}$ and a parameter $\alpha \in \mathbb{R}$, we define B-translation T^B_α on $\mathcal{H}$ by $T^B_\alpha = B T_\alpha B^{-1}$ and B-modulation E^B_α on $\mathcal{H}$ by $E^B_\alpha = B E_\alpha B^{-1}$ where T_α and E_α are respectively the translation and modulation operators on $L^2(\mathbb{R})$.*

Following properties about B-translation T^B_α and B-modulation E^B_α can be easily acquired from the definition.

1. The operators T_α^B and E_α^B are invertible operators on $\mathcal{H}$. They are unitary if $B : L^2(\mathbb{R}) \to H$ is unitary.

2. For all $k \in \mathbb{Z}$, $(T_\alpha^B)^k = T_{k\alpha}^B$ and $(E_\alpha^B)^k = E_{k\alpha}^B$.
   Hence $(E_b^B)^m (T_a^B)^n = E_{mb}^B T_{na}^B$ for all $m, n \in \mathbb{Z}$.

3. Using the commutator relation of E_b and T_a, we see $E_b^B T_a^B = e^{2\pi iba} T_a^B E_b^B$.

4. Since $T_a^* = T_{-a} = (T_a)^{-1}$ we have, $(T_a^B)^* = T_{-a}^{(B^*)^{-1}}$
   if B is self adjoint operator $(T_a^B)^* = T_{-a}^{B^{-1}}$,
   if B is unitary operator $(T_a^B)^* = T_{-a}^B$.
   Similarly, since $E_b^* = E_{-b} = (E_b)^{-1}$, it follows that $(E_a^B)^* = E_{-a}^{(B^*)^{-1}}$,
   if B is self adjoint operator $(E_a^B)^* = E_{-a}^{B^{-1}}$,
   if B is unitary operator $(E_a^B)^* = E_{-a}^B$

Definition 3.2. *For $a > 0$ and $b > 0$, the family $\{E_{mb}^B T_{na}^B\, x_0 : m, n \in \mathbb{Z}\}$ generated by $x_0 \in \mathcal{H}$ is called a B-Gabor type family in $\mathcal{H}$. Such a family is called a B-Gabor type frame (B-Gabor type Bessel sequence) if it forms a frame (Bessel sequence) in $\mathcal{H}$, where $B : L^2(\mathbb{R}) \to \mathcal{H}$ is invertible.*

Proposition 3.1. *Suppose that $B : L^2(\mathbb{R}) \to \mathcal{H}$ is an invertible map. The family $\{E_{mb}^B T_{na}^B\, x_0 : m, n \in \mathbb{Z}\}$ forms a B-Gabor type frame in $\mathcal{H}$ if and only if $\{E_{mb} T_{na} B^{-1} x_0 : m, n \in \mathbb{Z}\}$ forms a Gabor frame in $L^2(\mathbb{R})$.*

Proof. Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ is invertible. Assume that the family $\{E_{mb}^B T_{na}^B\, x_0 : m, n \in \mathbb{Z}\}$ forms a B-Gabor type frame in $\mathcal{H}$. By lemma 3.1, $B^{-1}\{E_{mb}^B T_{na}^B\, x_0 : m, n \in \mathbb{Z}\}$ is a frame in $L^2(\mathbb{R})$.
But $B^{-1}\{E_{mb}^B T_{na}^B\, x_0 : m, n \in \mathbb{Z}\} = \{E_{mb} T_{na} B^{-1} x_0 : m, n \in \mathbb{Z}\}$, is a Gabor frame in $L^2(\mathbb{R})$.

Conversely assume that, $\{E_{mb} T_{na} g_0 : m, n \in \mathbb{Z}\}$ forms a Gabor frame in $L^2(\mathbb{R})$. Again by lemma 3.1, $B\{E_{mb} T_{na}\, g_0 : m, n \in \mathbb{Z}\} = \{B E_{mb} B^{-1} B T_{na} B^{-1} B g_0 : m, n \in \mathbb{Z}\} = \{E_{mb}^B T_{na}^B B g_0 : m, n \in \mathbb{Z}\}$ is a B-Gabor type frame in $\mathcal{H}$. □

Following is a useful observation about the existence of B-Gabor type Parseval frames in separable Hilbert space $\mathcal{H}$.

Proposition 3.2. *For each pair of B-translation and B-modulation parameters $a, b > 0$ satisfying $0 < ab \leq 1$, there exists B-Gabor type frame in $\mathcal{H}$ with BB^* as its frame operator.*

Proof. For each pair of translation and modulation parameters $a, b > 0$ satisfying $0 < ab \leq 1$, there exists a tight Gabor frame in $L^2(\mathbb{R})$ with identity operator as frame operator (see Nambudiri and Parthasarathy [2013]), say $\{E_{mb} T_{na} g_0 :$

$m, n \in \mathbb{Z}\}$. Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ is a bounded invertible linear map. Then image of this Gabor frame $\{E_{mb}T_{na}g_0 : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ under B is a Gabor type frame in $\mathcal{H}$ with frame operator as $BIB^* = BB^*$. □

Corolary 3.1. *In the above proposition, if B is a unitary operator then the identity operator I is the frame operator of the B-Gabor type frame in $\mathcal{H}$ and hence it is a Parseval frame in $\mathcal{H}$.*

If the family $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$, $x_0 \in \mathcal{H}$ is a B-Gabor type frame (Bessel sequence) in $\mathcal{H}$ then its frame operator S, defined by,
$S(x) = \sum\limits_{m,n\in\mathbb{Z}} \langle x, E^B_{mb}T^B_{na}x_0\rangle E^B_{mb}T^B_{na}x_0$, for all $x \in \mathcal{H}$ is called *B-Gabor type frame operator (B-Gabor type semi-frame operator)*.

We look into some properties of these operators and present characterization of such operators. In the above situation, if S is the frame operator of the Gabor frame $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ and if S' is the frame operator of the B-Gabor type frame $\{E^B_{mb}T^B_{na}Bg : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$, then

$$\begin{aligned} S'x &= \sum_{k\in\mathbb{N}} \langle x, E^B_{mb}T^B_{na}Bg\rangle E^B_{mb}T^B_{na}Bg \\ &= B(\sum_{k\in\mathbb{N}} \langle B^*x, E_{mb}T_{na}g\rangle E_{mb}T_{na}g) \\ &= BSB^*(x), \text{ for all } x \in \mathcal{H}. \end{aligned}$$

Now we give a result analogous to lemma 12.3.1 in Christensen [2016], which says that the semi-frame operator S of a Bessel sequence $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ commutes with the involved modulation and translation operators. We will prove that, if $B : L^2(\mathbb{R}) \to \mathcal{H}$ is a unitary operator then the semi-frame operator S for a B-Gabor type Bessel sequence $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$ commutes with involved B-modulation and B-translation operators.

Lemma 3.4. *Let $\mathcal{H}$ be a separable Hilbert space, $B : L^2(\mathbb{R}) \to \mathcal{H}$ is a unitary operator. For $x_0 \in \mathcal{H}$ and $a, b > 0$, if $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ is a B-Gabor type Bessel sequence in $\mathcal{H}$ with semi-frame operator S. Then the following holds:*

1. *$SE^B_{mb}T^B_{na} = E^B_{mb}T^B_{na}S$ for all $m, n \in \mathbb{Z}$.*

2. *If $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ is a B-Gabor type frame for $\mathcal{H}$, then $S^{-1}E^B_{mb}T^B_{na} = E^B_{mb}T^B_{na}S^{-1}$ for all $m, n \in \mathbb{Z}$.*

Proof. From the definition of a B-Gabor type semi-frame operator S on $\mathcal{H}$ and from the properties of B-modulation and B-translation operators, we have for all $x \in \mathcal{H}$,

$$\begin{aligned} SE^B_{mb}T^B_{na}x &= \sum_{m',n' \in \mathbb{Z}} \langle E^B_{mb}T^B_{na}x, E^B_{m'b}T^B_{n'a}x_0\rangle E^B_{m'b}T^B_{n'a}x_0 \\ &= \sum_{m',n' \in \mathbb{Z}} \langle x, T^B_{-na}E^B_{-mb}E^B_{m'b}T^B_{n'a}x_0\rangle E^B_{m'b}T^B_{n'a}x_0 \\ &= \sum_{m',n' \in \mathbb{Z}} \langle x, T^B_{-na}E^B_{(m'-m)b}T^B_{n'a}x_0\rangle E^B_{m'b}T^B_{n'a}x_0 \end{aligned}$$

using commutator relation of E_{mb} and T_{na} properly and performing the suitable change of variables, a direct computation shows that,

$$SE^B_{mb}T^B_{na}x = E^B_{mb}T^B_{na}Sx$$

This proves (1). In order to prove (2) we recall the fact that, frame operator is invertible. Hence, whenever $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ is a B-Gabor type frame for $\mathcal{H}$; the result follows by applying S^{-1} to both sides of (1). $\square$

In particular we can observe that, the frame operator S of the B-Gabor type frame $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$ commutes with all B-translations T^B_{na} and all B-modulations E^B_{mb}. Construction of frames with a given operator as their frame operator is an interesting theme having practical relevance in the context of frames. As shown in Nambudiri and Parthasarathy [2012], positive and invertible operators can become frame operators on separable Hilbert spaces. However, in view of lemma 3.4, those operators on $\mathcal{H}$ which are positive and invertible but fail to commute with certain B-translation and B-modulation operators cannot correspond to any B-Gabor type frame in $\mathcal{H}$. Here, we present a complete characterization of frame operators of B-Gabor type frame in $\mathcal{H}$.

Theorem 3.1. *A bounded linear operator on a separable Hilbert space $\mathcal{H}$ can be realized as a B-Gabor type frame operator if and only if it is positive, invertible and commutes with all the B-translations T^B_{na} and all B-modulations E^B_{mb}. Where $B : L^2(\mathbb{R}) \to \mathcal{H}$ is a unitary operator.*

Proof. Assume that $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ is a B-Gabor type frame in $\mathcal{H}$ with frame operator S, then it is clear that S is positive, invertible bounded linear operator on $\mathcal{H}$ which commutes with, from lemma 3.4, all the B-translations T^B_{na} and all B-modulations E^B_{mb}.

Conversely assume that S is positive, invertible bounded linear operator on $\mathcal{H}$ which commutes with all the B-translations T^B_{na} and all the B-modulations E^B_{mb}. Then its positive square root $S^{1/2}$ is also having these properties. Now by corollary 3.1, there is a B-Gabor type frame $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$ with identity operator I as the frame operator. Invertibility and commutativity of $S^{1/2}$ ensures that $\{S^{1/2}E^B_{mb}T^B_{na}x_0 = E^B_{mb}T^B_{na}S^{1/2}x_0 : m,n \in \mathbb{Z}\}$ is also a B-Gabor type frame.

Moreover for all $x \in \mathcal{H}$,

$$\sum_{m,n\in\mathbb{Z}} \langle x, E^B_{mb}T^B_{na}S^{1/2}x_0\rangle E^B_{mb}T^B_{na}S^{1/2}x_0 = S^{1/2}\sum_{m,n\in\mathbb{Z}} \langle S^{1/2}x, E^B_{mb}T^B_{na}x_0\rangle E^B_{mb}T^B_{na}x_0$$
$$= S^{1/2}I(S^{1/2}x) = S(x).$$

Hence the frame operator of $\{E^B_{mb}T^B_{na}S^{1/2}x_0 : m,n \in \mathbb{Z}\}$ is nothing but S. $\square$

Two more characterization of B-Gabor type frame operator in $\mathcal{H}$ with independent proofs using the idea in J. Laurence [2005] is given below. Here note that the operator B need not be unitary instead an invertible operator will work.

Theorem 3.2. *A bounded linear operator S on a separable Hilbert space $\mathcal{H}$ is a B-Gabor type frame operator if and only if it is of the form $S = TT^*$ where $T = BA$, A is a surjective operator on $L^2(\mathbb{R})$ which commutes with some translation T_a and some modulation E_b for some $a,b > 0$ with $ab \leq 1$ and $B : L^2(\mathbb{R}) \to \mathcal{H}$ is invertible.*

Proof. Assume that $S = TT^*$, where $T = BA$, $B : L^2(\mathbb{R}) \to \mathcal{H}$ is invertible and A is a surjective operator on $L^2(\mathbb{R})$ which commutes with some translation T_a and some modulation E_b for some $a,b > 0$ with $ab \leq 1$. There is always a Parseval Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$. Surjectivity and commutativity of A ensure that $A(\{E_{mb}T_{na}g\}) = \{AE_{mb}T_{na}g = E_{mb}T_{na}Ag : m,n \in \mathbb{Z}\}$ is also a Gabor frame in $L^2(\mathbb{R})$. Hence $B(\{E_{mb}T_{na}Ag : m,n \in \mathbb{Z}\})$ is a B-Gabor type frame in $\mathcal{H}$.

Let $BAg = x_0$. Then for all $x \in \mathcal{H}$,

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \langle x, E^B_{mb}T^B_{na}x_0\rangle E^B_{mb}T^B_{na}x_0 \\
&= \sum_{m,n\in\mathbb{Z}} \langle x, BE_{mb}T_{na}B^{-1}BAg\rangle BE_{mb}T_{na}B^{-1}BAg \\
&= \sum_{m,n\in\mathbb{Z}} \langle x, BE_{mb}T_{na}Ag\rangle BE_{mb}T_{na}Ag \\
&= \sum_{m,n\in\mathbb{Z}} \langle x, BAE_{mb}T_{na}g\rangle BAE_{mb}T_{na}g \\
&= BA\sum_{m,n\in\mathbb{Z}} \langle A^*B^*x, E_{mb}T_{na}g\rangle E_{mb}T_{na}g \\
&= BAA^*B^*x = TT^*(x) = S(x).
\end{aligned}$$

Hence the frame operator of $\{E^B_{mb}T^B_{na}x_0 : m,n \in \mathbb{Z}\}$ is nothing but S.

Conversely, if S is the frame operator of a B-Gabor type frame $\{E^B_{mb}T^B_{na}x_0 : m,n \in \mathbb{Z}\}$ in $\mathcal{H}$, then $B^{-1}(\{E^B_{mb}T^B_{na}x_0 : m,n \in \mathbb{Z}\})$ is a Gabor frame in $L^2(\mathbb{R})$. By Theorem 7 in Nambudiri and Parthasarathy [2021], the frame operator S' of this Gabor frame takes the form $S' = AA^*$ where A is surjective and commutes with all T_{na} and E_{mb}. Since B maps this Gabor frame to the B-Gabor type frame

$\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$, the resulting frame operator is $BS'B^* = BAA^*B^*$. Thus $S = BA(BA)^*$, as desired. □

In the next characterization of B-Gabor type frame operator, note that the map A need not be commutative with modulations and translations and in the proof we use only operator algebra.

Theorem 3.3. *An operator $S : \mathcal{H} \to \mathcal{H}$ is a B-Gabor type frame operator if and only if there is a surjective map $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and an invertible operator $B : L^2(\mathbb{R}) \to \mathcal{H}$ such that B^*B and AA^* are Gabor frame operator on $L^2(\mathbb{R})$, with the property $S = \eta\eta^*$, where $\eta = BA$.*

Proof. Suppose that S is a B-Gabor type frame operator on $\mathcal{H}$. Then there is a Gabor frame operator S' on $L^2(\mathbb{R})$ and an invertible operator $B : L^2(\mathbb{R}) \to \mathcal{H}$ so that $BS'B^* = S$. Since S' is a Gabor frame operator on $L^2(\mathbb{R})$ by Theorem 7 of Nambudiri and Parthasarathy [2021], $S' = AA^*$ for some surjective map A on $L^2(\mathbb{R})$. Thus $S = BAA^*B^* = \eta\eta^*$, where $\eta = BA$.
Conversely assume that $S = \eta\eta^*$ with the properties in hypothesis. Since AA^* is a Gabor frame operator on $L^2(\mathbb{R})$, the image of this frame under B is a B-Gabor type frame and having the frame operator $S = B(AA^*)B^*$, hence S is a B-Gabor type frame operator on $\mathcal{H}$. □

Now we make an interesting specific observation on Gabor frames in $L^2(\mathbb{R})$. Let $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ be a Gabor frame in $L^2(\mathbb{R})$. Since $ab \leq 1$, (by interchanging the parameters a and b,) there is $\tilde{g} \in L^2(\mathbb{R})$ such that $\{E_{na}T_{mb}\tilde{g} : m, n \in \mathbb{Z}\}$ is also a Gabor frame in $L^2(\mathbb{R})$. Fourier transform is one of the very important operator in harmonic analysis (see Christensen [2016]). The commutator relations of the Fourier transform $\mathcal{F}$ on $L^2(\mathbb{R})$ with translations and modulations yield

$$\begin{aligned}\mathcal{F}(\{E_{na}T_{mb}\tilde{g} : m, n \in \mathbb{Z}\}) &= \{\mathcal{F}E_{na}T_{mb}\tilde{g} : m, n \in \mathbb{Z}\}\\ &= \{\mathcal{F}E_{na}\mathcal{F}^{-1}\mathcal{F}T_{mb}\mathcal{F}^{-1}(\mathcal{F}\tilde{g}) : m, n \in \mathbb{Z}\}\\ &= \{E^{\mathcal{F}}_{na}T^{\mathcal{F}}_{mb}(\mathcal{F}\tilde{g}) : m, n \in \mathbb{Z}\}, \text{ $\mathcal{F}$-Gabor type frame}\end{aligned}$$

Again,

$$\begin{aligned}\mathcal{F}(\{E_{na}T_{mb}\tilde{g} : m, n \in \mathbb{Z}\}) &= \{\mathcal{F}E_{na}T_{mb}\tilde{g} : m, n \in \mathbb{Z}\}\\ &= \{\mathcal{F}E_{na}T_{mb}\mathcal{F}^{-1}(\mathcal{F}\tilde{g}) :\in \mathbb{Z}\}\\ &= \{T_{na}\mathcal{F}\mathcal{F}^{-1}E_{-mb}(\mathcal{F}\tilde{g}) :\in \mathbb{Z}\}\\ &= \{T_{na}E_{-mb}(\mathcal{F}\tilde{g}) :\in \mathbb{Z}\}\\ &= \{T_{na}E_{kb}h : k, n \in \mathbb{Z}\} \text{ , where } h = \mathcal{F}\tilde{g}.\end{aligned}$$

Now we state another interesting observation as one theorem.

Theorem 3.4. *Associated with every Gabor frame $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$, there is a frame in $L^2(\mathbb{R})$ of the form $\{T_{na}E_{mb}h : m, n \in \mathbb{Z}\}$ for some $h \in L^2(\mathbb{R})$ and this frame is a $\mathcal{F}$-Gabor type frame in $L^2(\mathbb{R})$.*

Consider an invertible map $B : L^2(\mathbb{R}) \to \mathcal{H}$ such that B^*B is a Gabor frame operator on $L^2(\mathbb{R})$ and $\mathcal{F}$ is the Fourier transform on $L^2(\mathbb{R})$. If S is a $B\mathcal{F}$-Gabor type frame operator on $\mathcal{H}$. Then there is Gabor frame operator S' on $L^2(\mathbb{R})$ such that $(B\mathcal{F})S'(B\mathcal{F})^* = S$.

Now,
$$\begin{aligned} SE_{mb}^{B\mathcal{F}} &= (B\mathcal{F})S'(B\mathcal{F})^*(B\mathcal{F})E_{mb}(B\mathcal{F})^{-1} \\ &= (B\mathcal{F})S'\mathcal{F}^*(B^*B)\mathcal{F}E_{mb}\mathcal{F}^{-1}B^{-1} \\ &= (B\mathcal{F})S'\mathcal{F}^*(B^*B)T_{mb}\mathcal{F}\mathcal{F}^{-1}B^{-1} \\ &= (B\mathcal{F})S'\mathcal{F}^*(B^*B)T_{mb}B^{-1} \\ &= (B\mathcal{F})S'\mathcal{F}^*T_{mb}(B^*B)B^{-1} \\ &= (B\mathcal{F})S'\mathcal{F}^{-1}T_{mb}B^* \\ &= (B\mathcal{F})S'E_{mb}\mathcal{F}^{-1}B^* \\ &= (B\mathcal{F})E_{mb}S'\mathcal{F}^*B^* \\ &= (B\mathcal{F})E_{mb}(\mathcal{F}^{-1}B^{-1})(B\mathcal{F})S'\mathcal{F}^*B^* \\ &= E_{mb}^{B\mathcal{F}}S \end{aligned}$$

Thus, $SE_{mb}^{B\mathcal{F}} = E_{mb}^{B\mathcal{F}}S$, in a similar manner we can see that, $ST_{na}^{B\mathcal{F}} = T_{na}^{B\mathcal{F}}S$.

Hence, every $B\mathcal{F}$-Gabor type frame operator on Hilbert space $\mathcal{H}$ commutes with $B\mathcal{F}$-modulations and $B\mathcal{F}$-translations. Also it can be easily verify that $(B\mathcal{F})^*(B\mathcal{F})$ is invertible, positive and self adjoint. Hence $(B\mathcal{F})^*(B\mathcal{F})$ is a Gabor frame operator and hence commutes with modulation and translation by Nambudiri and Parthasarathy [2013]. This operator $B\mathcal{F}$ has the following important property:

(1). $$\begin{aligned} (B\mathcal{F}B^{-1})E_{mb}^{B} &= B\mathcal{F}(B^{-1}B)E_{mb}B^{-1} = B\mathcal{F}E_{mb}B^{-1} \\ &= BT_{mb}\mathcal{F}B^{-1} = BT_{mb}B^{-1}B\mathcal{F}B^{-1} = T_{mb}^{B}(B\mathcal{F}B^{-1}) \end{aligned}$$

(2). $$\begin{aligned} (B\mathcal{F}B^{-1})T_{na}^{B} &= B\mathcal{F}(B^{-1}B)T_{na}B^{-1} = B\mathcal{F}T_{na}B^{-1} \\ &= BE_{-na}\mathcal{F}B^{-1} = BE_{-na}B^{-1}B\mathcal{F}B^{-1} = E_{-na}^{B}(B\mathcal{F}B^{-1}) \end{aligned}$$

If $\mathcal{H}$ is a separable Hilbert space and $B : L^2(\mathbb{R}) \to \mathcal{H}$ is an invertible operator with B^*B Gabor frame operator on $L^2(\mathbb{R})$. Then $(B\mathcal{F}B^{-1})$ can be viewed as a generalized B-Fourier transform on $\mathcal{H}$.

Proposition 3.3. *For a given $A \in \mathcal{B}(\mathcal{H}_1)$ and invertible operators $B : \mathcal{H}_1 \to \mathcal{H}_2$ and $C : \mathcal{H}_2 \to \mathcal{H}_1$, the operator $BC \in \mathcal{B}(\mathcal{H}_2)$ commutes with $BAC \in \mathcal{B}(\mathcal{H}_2)$ if and only if $CB \in \mathcal{B}(\mathcal{H}_1)$ commutes with A.*

Proof. Since both B and C are invertible,
$$\begin{aligned} (BC)(BAC) &= (BAC)(BC) \\ &\Leftrightarrow C(BAC)(C)^{-1} = B^{-1}(BAC)B \\ &\Leftrightarrow (CB)A = A(CB). \end{aligned}$$
□

A useful consequence of Proposition 3.3 is the following.

Corolary 3.2. *Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ and $C : \mathcal{H} \to L^2(\mathbb{R})$ be invertible. Then $BC \in \mathcal{B}(\mathcal{H})$ commutes with the operators $BT_{na}C$ and $BE_{mb}C$, $m, n \in \mathbb{Z}$ if and only if CB commutes with T_{na} and E_{mb}, $m, n \in \mathbb{Z}$ on $L^2(\mathbb{R})$.*

For a given pair of invertible operators $B : L^2(\mathbb{R}) \to \mathcal{H}$ and $C : \mathcal{H} \to L^2(\mathbb{R})$, assume that CB commutes with T_{na} and E_{mb} for all $m, n \in \mathbb{Z}$. Then each Gabor frame $\mathcal{G} = \{E_{mb}T_{na}g :\in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ also yields a Gabor type frame of the form $\{(BE_{mb}C)(BT_{na}C)x_0 :\in \mathbb{Z}\}$ in $\mathcal{H}$. Indeed, this family is a frame, as is seen below.

Since CB is invertible, $(CB)^2(\mathcal{G})$ is a Gabor frame in $L^2(\mathbb{R})$. Hence its image under B is also a frame in $\mathcal{H}$. This frame is given by

$$\begin{aligned} B((CB)^2)(\mathcal{G}) &= B((CB)^2)(\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}) \\ &= B(\{E_{mb}(CB)T_{na}(CB)g : m, n \in \mathbb{Z}\}) \\ &= \{(BE_{mb}C)(BT_{na}C)Bg : m, n \in \mathbb{Z}\} \end{aligned}$$

and hence the claim.

In particular, let $C = B^*$. Assume that $CB = B^*B$ commutes with T_{na} and E_{mb} for all $m, n \in \mathbb{Z}$. As above, each Gabor frame $\mathcal{G} = \{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ yields a Gabor type frame of the form $\{(BE_{mb}B^*)(BT_{na}B^*)x_0 : m, n \in \mathbb{Z}\}$ in $\mathcal{H}$.

Now, $(BT_aB^*)^2 = (BT_aB^*)(BT_aB^*) = BT_aT_aB^*BB^* = BT_{2a}B^*(BB^*)$. Hence, in general, $(BT_{na}B^*) \neq (BT_aB^*)^n$ and $(BE_{mb}B^*) \neq (BE_bB^*)^m$ even if B^*B commutes with T_{na} and E_{mb} for all $m, n \in \mathbb{Z}$. This makes B-Gabor type frames more significant in the aspects of structure of Gabor frames.

However, it is possible to get more types of Gabor frames in separable Hilbert spaces by relaxing the invertibility of the operator B by surjectivity. In addition, if $\mathcal{G}$ is a Parseval Gabor frame in $L^2(\mathbb{R})$, then the frame operator of this frame is $B(B^*B)^2B^* = B(B^*B)(B^*B)B^* = (BB^*)^3$ which commutes with the operators $BT_{na}B^*$ and $BE_{mb}B^*$, as is seen in Corollary 3.2.

More generally, there can be different Gabor type frames in $\mathcal{H}$ with same frame operator and generator. For instance, fix an invertible map $B : L^2(\mathbb{R}) \to \mathcal{H}$ and choose a Gabor frame $\mathcal{G} = \{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ with frame operator S. Then each invertible map $D : \mathcal{H} \to L^2(\mathbb{R})$ (preferably, $D \neq B^{-1}, B^*$) yields a frame in $\mathcal{H}$ of the form $B(\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}) = \{(BE_{mb}D)(D^{-1}T_{na}B^{-1})Bg : m, n \in \mathbb{Z}\}$. Obviously, Bg is the generator of this frame and BSB^* is its frame operator.

4 Conclusion

In this article we introduced a class of structured frames in separable Hilbert spaces $\mathcal{H}$ using an invertible map B from $L^2(\mathbb{R})$. Since this structure is similar to Gabor structure existing in literature, we name them as B-Gabor type frames. In light of the characterization of the Gabor frame operators on $L^2(\mathbb{R})$, recently appeared in Nambudiri and Parthasarathy [2013], we presented a characterization of frame operators associated with B-Gabor type frames in $\mathcal{H}$ when B is unitary.

Several significant properties of B-Gabor type frame operators are discussed. One can check whether such structured frames exists in separable Hilbert space $\mathcal{H}$ by replacing the invertible map with some other suitable maps.

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