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Split Legendary Domination in Graphs

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Abstract

The concept of domination arises due to the necessity of various types of facility location problem in a network which includes the one to one correspondence between them. In a network, nodes with nearly equal capacity may interact with each other in better way. Motivating by this, the equivalent property of dominating set of a graph G and its line graph G_l are studied. This make us to introduce the legendary domination. In this paper, by posting the disconnected property on the legendary dominating set, the split legendary domination are introduced and defined. Further, the graph theoretical parameters are studied in terms of elements of G and its relationship with other domination parameters are presented.

Keywords: Split domination number, line domination number, legendary domination number, split legendary domination number.
2020 AMS subject classifications: 05C69, 05C76 [1]

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1 Introduction

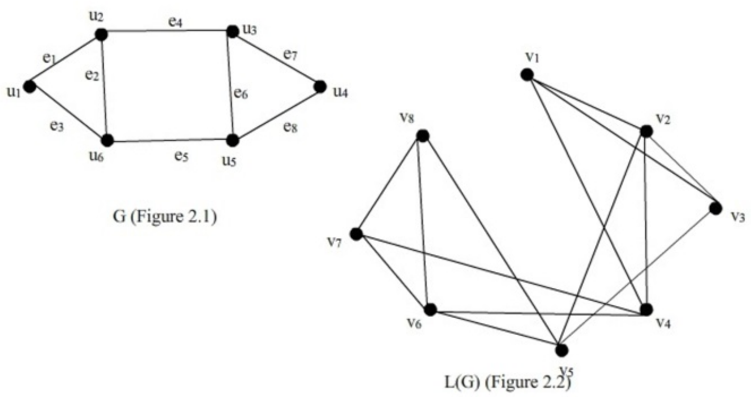
A graph $G = (V, E)$ we mean a finite, undirected, simple and connected graph with p vertices and q edges. Terms not defined here are used in the sense of Harary [1]. In mathematical discipline of graph theory, the line graph of an undirected graph G is another G_l that represents the adjacencies between edges of G. A line graph can be viewed in several ways for example, the edges of G can be considered as elements of the graph and the adjacency in G_l determined by relations between the elements. The line graph of G, denoted G_l, is the graph with vertex set $E(G)$, where vertices x and y are adjacent in G_l if and only if edges x and y share a common vertex in G. This was introduced by Harary and Norman [2]. The basic motivation for the concept of domination arises due to the necessity of various types of facility location problem in a network which includes the one to one correspondence between them.A set $D \subseteq V(G)$ of a graph $G = (V, E)$ is a dominating set of G, if every vertex in $V\backslash D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. This concept was introduced by Ore in [14]. The study of domination in line graph is named as line domination, the study of domination in edges of a graph is called as edge domination. A set $D \subseteq V(G_l)$ is said to be line dominating set of G, if every vertex not in D is adjacent to a vertex in D. The line domination number of G is denoted by $\gamma_l(G)$ is the minimum cardinality of a line dominating set of G_l. It was introduced by S. R. Jayaram in the year 1987 [3]. In the society, persons with nearly equal status, tend to be friendly. In an industry, employees with nearly equal powers form association and move closely. Motivating this, the equivalent property of the dominating set G and its line graph G_l is studied.This make us to defined the legendary domination parameters [7]. The legendary domination is relating vertex domination and edge domination of a graph G. A dominating set $D \subseteq V(G)$ of graph $G = (V, E)$ is said to be legendary dominating set of G, if D is a dominating set of G and G_l. The legendary domination number is the minimum cardinality taken over all legendary dominating sets of G and is denoted by $\gamma_{le}(G)$. A dominating set $D \subseteq V(G)$ is a split dominating set, if the induced subgraph $\langle V\backslash D\rangle$ is disconnected. This concept was introduced by V. R. Kulli and B. Janakiram in [6]. A line dominating set $D \subseteq V(L(G))$ is a split line dominating set, if the induced subgraph $\langle V(L(G))\rangle$ is disconnected. The minimum cardinality of such a set is called split line dominating number of G and is denoted by $\gamma_{sl}(G)$. This was introduced by M.H.Muddebihal, V.A.Panfarosh, Anil R.Sedamkar [13]. In this paper, the split legendary domination are introduced by posting the disconnected property on the legendary dominating set. The bounds of this parameter was studied and its exact value was found for some standard graphs. The connectivity $k = K(G_l)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph.

2 Preliminaries

Definition 2.1. *A split dominating set $X \subseteq V(G_l)$ is said to be split legendary dominating set (sled−set), if G has a split dominating set of cardinality X. The split legendary domination number is the minimum cardinality taken over all split legendary dominating set of G and is denoted by $\gamma_{sle}(G)$.*

Remark 2.1. *1. Let G, $L(G) = G_l$ be respectively denote a (p,q) graph, and its line graph. Also, take $V(G) = \{u_1, u_2, ..., u_p = u_n\}, E(G) = \{e_1, e_2, ..., e_q = e_n\}$ $V(G_l) = V_l = \{v_1, v_2, ..., v_{pl}\}, E(G_l) = \{f_1, f_2, ..., f_{ql}\}$*
2. All graphs consider in this paper are having at least one slesd-set.

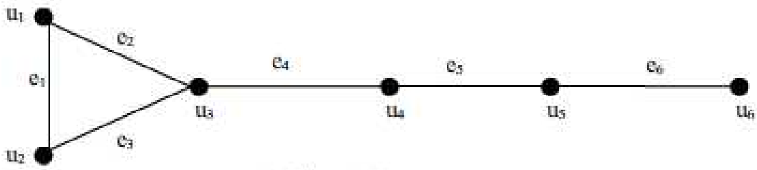
Example 2.1. *For the graph G_l in Figure 2.2, the vertex set $D = \{v_1, v_3, v_4, v_5\}$ in G_l is the minimum split dominating set of G_l and hence $\gamma_{sl} = 4$. Also note that the graph G in Figure 2.1, the vertex set $D = \{u_1, u_3, u_7\}$ is the minimum split dominating set of G and hence $\gamma_s(G) = 3$. By the definition 2.1, the vertex set $X = \{v_1, v_3, v_4, v_5\} \subseteq G_l$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 4$. Here note that $\gamma_s(G) < \gamma_{sl}(G) \leq \gamma_{sle}(G)$.*



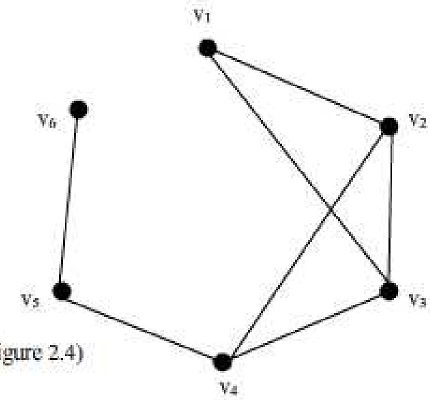
G (Figure 2.1)

L(G) (Figure 2.2)

Example 2.2. *For the graph G_l in Figure 2.4, the vertex set $D = \{v_3, v_5\}$ in G_l is the minimum split dominating set of G_l and hence $\gamma_{sl} = 2$. Also note that the graph G in Figure 2.3, the vertex set $D = \{u_3, u_5\}$ is the minimum split dominating set of G and hence $\gamma_s(G) = 2$. By the definition 2.1, the vertex set $X = \{v_3, v_5\} \subseteq G_l$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 2$. Here note that $\gamma_s(G) = \gamma_{sl}(G) = \gamma_{sle}(G)$.*

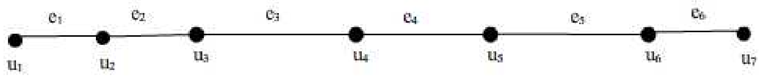


G (Figure 2.3)



L(G) (Figure 2.4)

Example 2.3. *For the graph G_l in Figure 2.6, the vertex set $D = \{v_2, v_5\}$ in G_l is the minimum split dominating set of G_l and hence $\gamma_{sl} = 2$. Also note that the graph G in Figure 2.5, the vertex set $D = \{u_2, u_5, u_7\}$ is the minimum split dominating set of G and hence $\gamma_s(G) = 3$. By the definition 2.1, the vertex set $X = \{v_1, v_2, v_5\} \subseteq G_l$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$. Here note that $\gamma_{sl}(G) < \gamma_s(G) \leq \gamma_{sle}(G)$.*



G (Figure 2.5)

L(G) (Figure 2.6)

Remark 2.2. *(i) For the trivial graph, there is no split legendary dominating set. (ii) For a connected graph, one can always find a split legendary dominating set. Therefore, for the existence of split legendary domination number, hereafter we assume that G is a connected graph with atleast two edge.*

In view of the graphs given in Example 2.1, 2.2, 2.3, we have the following result.

Theorem 2.1. *For any graph G,*

i $\gamma_s(G) \leq \gamma_{sle}(G)$.

ii $\gamma_{sl}(G) \leq \gamma_{sle}(G)$.

Proof. Since every split legendary dominating set of G is a split dominating set of G gives the result $\gamma_s(G) \leq \gamma_{sle}(G)$.

Since every split legendary dominating set of G is a split dominating set of G_l gives the result $\gamma_{sl}(G) \leq \gamma_{sle}(G)$. □

Corolary 2.1. *For any graph G,*$\gamma_{sle}(G) = max(\gamma_s(G), \gamma_{sl}(G))$.

Proof. By Theorem 2.1, we have the minimum cardinality of the split legendary domination number is atleast the minimum domination number of G or G_l.

Hence $\gamma_{sle}(G)$ is the maximum value of $\gamma_s(G)$ and $\gamma_{sl}(G)$. □

3 Characterizations, Bounds and Exact values

The following result gives split legendary dominating sets of G which are minimal.

Theorem 3.1. *A split legendary dominating set* $X \subseteq G_l$ *is minimal for each vertex* $x \epsilon X$, *one of the following three conditions holds:*

i There exists vertex $y \epsilon V_l - X$, *such that*$N(y) \cap X = \{x\}$.

ii x *is an isolated vertex in* $< X >$.

iii $< (V_l - x) \cup \{x\} >$ *is connected.*

Proof. Let G be a graph and G_l be its line graph with vertex set V_l. Let $X \subseteq V_l$ be a split legendary split dominating set of G, since X is minimal and there exists a vertex $x \epsilon X$ such that x does not hold any of the above conditions, then by condition (i) and (ii), $X_1 = X - \{x\}$ is a dominating set of G_l. Also by (iii) $< V_l - X >$ is disconnected. This implies that X_1 is a split legendary dominating set of G, which is contradiction to the fact X is minimal. □

The following theorem gives the necessary and sufficient conditions for a dominating set to be a split legendary dominating set of G.

Theorem 3.2. *A dominating set* $X \subseteq G_l$ *is a split legendary dominating set of G if and only if there exists two vertices* v_1, v_2 *in the induced subgraph* $< V_l \backslash X >$ *such that* $v_1 - v_2$ *path of* G_l *contains atleast one vertex of* X.

Proof. Let G_l be a line graph of graph G and $X \subseteq V_l$ be a split legendary dominating set of G, since cardinality of X is a split dominating set of G.

It implies that the induced subgraph $< V_l \backslash X >$ such that there is no path between v_1 and v_2. In other words, G_l being connected, the original path between v_1 and v_2 in G_l does not exists due to the removal of X from V_l.

Thus, it follows that there exists two vertices v_1, v_2 in induced subgraph $< V_l \backslash X >$ such that every $v_1 - v_2$ path of G_l contains atleast one vertex in X.

Conversely, suppose the induced subgraph $< V_l \backslash X >$ is connected. Then there exists a path between v_1 and v_2 in $< V_l \backslash X >$, which is a contradiction to the fact that $v_1 - v_2$ path of G_l contains atleast one vertex of X. Hence, $< V_l \backslash X >$ is disconnected. therefore X is a split legendary dominating set of G. □

Now we give a lower bound for split legendary dominating sets of G in terms of vertex connectivity $k(G_l)$

Theorem 3.3. *For any graph G, $k(G_l) \leq \gamma_{sle}(G)$*

Proof. Let G_l be a line graph of G. Let S be a set of minimum number of vertices of G such that whose removal makes the graph disconnected. By the definition of vertex connectivity of a graph G_l, we have

$$|S| = k(G_l)..........(1)$$

Let $X \subseteq V_l$ be a split legendary dominating set with minimum cardinality, since cardinality of X is a split dominating set of G. Therefore,

$$|X| = \gamma_{sle}(G)..........(2)$$

As X also has the property of disconnecting the graph, we have $|S| \leq |X|$. By equation (1) and (2) we have $k(G_l) \leq \gamma_{sle}(G)$. □

The following result provides an upper bound for split legendary dominating sets of G in terms of maximum degree $\Delta(G_l)$.

Theorem 3.4. *For any graph G, $\gamma_{sle}(G) \leq p_l \frac{\Delta(G_l)}{\Delta(G_l)+1}$.*

Proof. Let G_l be a line graph of G and $X \subseteq V_l$ be the split legendary dominating set of G, since X is minimal by Theorem 3.1, it follows that for each vertex $x \epsilon X$, there exists a vertex $y \epsilon V_l - X$ such that x is adjacent to y. This implies that $V_l - X$ is a split legendary dominating set of G. Thus, $\gamma_l(G) \leq |V_l - X| p_l - \gamma_{sle}(G)$.
By the Theorem, "For any graph G, $\gamma_l(G) \geq \frac{p_l}{(\Delta(G_l)+1)}$.
Proof: Let X be a γ_l-set of G. Each vertex dominates atmost itself and $\Delta(G_l)$ other vertices. Hence the result".

Hence, the upper bound follows from the fact $\gamma_l(G) \geq \frac{p_l}{(\Delta(G_l)+1)}$. □

The exact values of split legendary domination number $\gamma_{sle}(G)$ for some standard graphs are given below.

Theorem 3.5. *For complete graph K_n, $\gamma_{sle}(G) = 2n - 4, n \geq 4$.*

Proof. Let G be a complete graph K_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, ..., v_{\frac{n(n-1)}{2}}\}$. The vertex set $X = \{v_1, v_2, ..., v_{n-2}, v_n, v_{n+1}, v_{n+2}, ..., v_{2n-3}\}$ is a sled $-$ set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

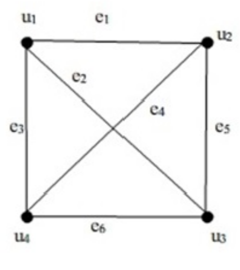
$$\gamma_{sle}(G) \leq |X| = 2n - 4..........(1)$$

Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a complete graph, X has atleast $2n-4$ vertices.
Hence,

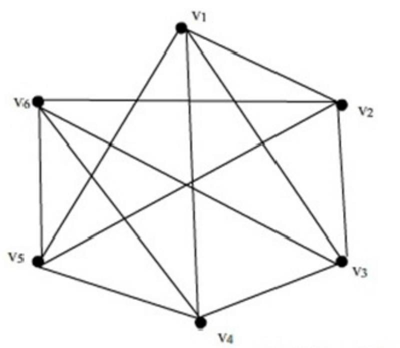
$$\gamma_{sle}(G) = |X| \geq 2n - 4..........(2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = 2n - 4, n \geq 4$. □

Example 3.1. *For the graph G_l in Figure 2.8, the vertex set $X = \{v_1, v_2, v_4, v_5\} \subseteq V(G_l)$ is the minimum legendary split dominating set of G and hence $\gamma_{sle}(G) = 4$.*



G (Figure 2.7)



L(G) (Figure 2.8)

Theorem 3.6. *For fan graph $F_{1,n}$, $\gamma_{sle}(G) = n, n \geq 3$.*

Proof. Let G be a fan graph $F_{1,n}$ with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, ..., v_{2n-1}\}$. The vertex set $X = \{v_2, v_3, v_4, v_5, ...v_{n+1}\}$ is a sled $-$ set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{les}(G) \leq |X| = n..........(1)$$

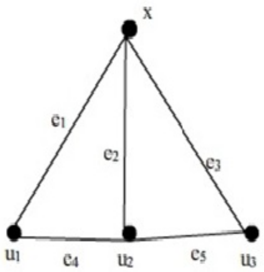
Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a fan graph, X has atleast n vertices.

Hence,

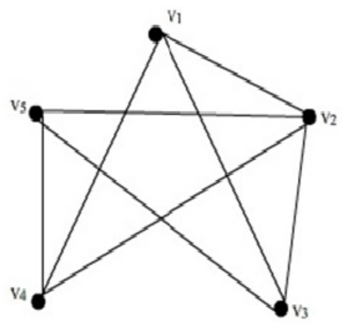
$$\gamma_{sled}(G) = |X| \leq n..........(2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = n, n \geq 2$. □

Example 3.2. *For the graph G_l in Figure 2.10, the vertex set $X = \{v_2, v_3, v_4\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.*



G (Figure 2.9)



L(G) (Figure 2.10)

Theorem 3.7. *For friendship graph F_n, $\gamma_{sle}(G) = n + 1, n \geq 2$.*

Proof. Let G be a friendship graph F_n with atleast five vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, ..., v_{3n}\}$. The vertex set $X = \{v_1, v_{3i}/i = 1, 2, ...n\}$ is a sled $-$ set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = n + 1..........(1)$$

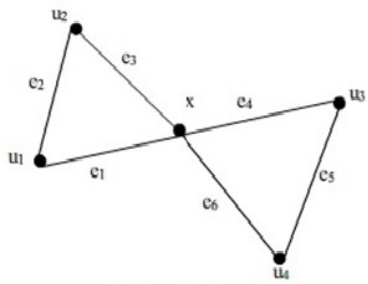
Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a friendship graph, X has atleast $n + 1$ vertices.

Hence,

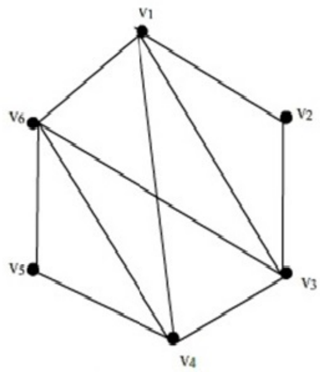
$$\gamma_{sle}(G) = |X| \leq n + 1..........(2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = n + 1, n \geq 2$. □

Example 3.3. *For the graph G_l in Figure 2.12, the vertex set $X = \{v_1, v_3, v_6\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.*



G (Figure 2.11)



L(G) (Figure 2.12)

Theorem 3.8. *For ladder graph L_n,*

$$\gamma_{sle}(G) = \begin{cases} 2, & n = 2 \\ n, & n \geq 3 \end{cases}$$

Proof. Let G be a ladder graph L_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, ..., v_{3(n-2)}\}$.

Case (i) n = 2

In this case, the vertex set $X = \{v_2, v_4\}$ is a minimum split Legendary dominating set of G, since induced subgraph $< V_l \backslash X >$ is disconnected and hence $\gamma_{sle}(G) = 2$

Case (ii) $n \geq 3$

In this case, the vertex set $X = \{v_1, v_{n+1}\} \cup \{v_4, v_7, v_{10}, ..., v_{3n-5}\}$ is a minimum split Legendary dominating set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = n..........(1)$$

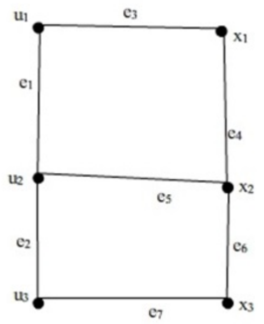
Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a ladder graph, X has atleast n vertices.

Hence,

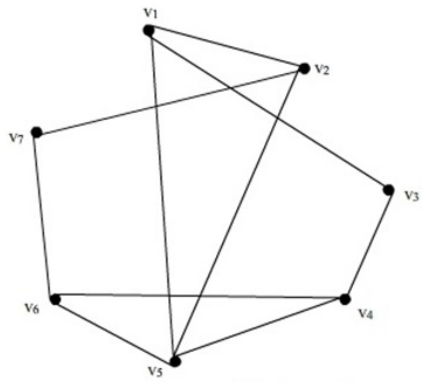
$$\gamma_{sle}(G) = |X| \leq n..........(2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = n, n \geq 3$. □

Example 3.4. *For the graph G_l in Figure 2.14, the vertex set $X = \{v_1, v_4, v_7\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.*



G (Figure 2.13)



L(G) (Figure 2.14)

Theorem 3.9. *For wheel graph W_n,*

$$\gamma_{sle}(G) = \begin{cases} 4, & n = 3 \\ \lceil \frac{n+3}{2} \rceil, & n \geq 4 \end{cases}$$

Proof. Let G be a wheel graph W_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, ..., v_{2n}\}$.

Case (i) n = 3

In this case, the vertex set $X = \{v_2, v_3, v_4, v_5\}$ is a minimum split legendary dominating set of G, since induced subgraph $< V_l \backslash X >$ is disconnected and hence $\gamma_{sle}(G) = 4$

Case (ii) $n \equiv 0(mod 2)$

In this case, the vertex set $X = \{v_2, v_n, v_{n+1}, v_{n+2}\} \cup \{v_{2i}/i = 2, 3, ..., n-4\}$ is a minimum split Legendary dominating set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n+3}{2} = \lceil \frac{n+3}{2} \rceil..........(1)$$

Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a wheel graph, X has atleast $\frac{n+3}{2}$ vertices.

Hence,

$$\gamma_{sle}(G) = |X| \leq \frac{n+3}{2} = \lceil \frac{n+3}{2} \rceil..........(2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = \lceil \frac{n+3}{2} \rceil$.

Case (iii) $n \equiv 1(mod 2)$

In this case, the vertex set $X = \{v_2, v_n, v_{n+1}, v_{n+2}\} \cup \{v_{2i}/i = 2, 3, ..., n-5\}$ is a minimum split legendary dominating set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n+4}{2} = \lceil \frac{n+3}{2} \rceil..........(3)$$

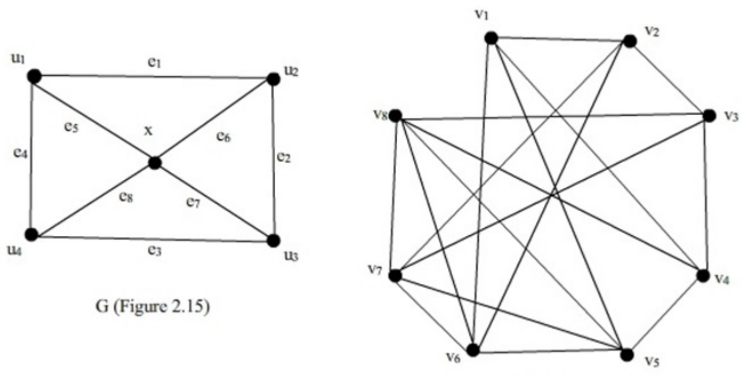
Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a wheel graph, X has atleast $\frac{n+4}{2}$ vertices.

Hence,

$$\gamma_{sle}(G) = |X| \leq \frac{n+4}{2} = \lceil \frac{n+3}{2} \rceil..........(4)$$

Then from equation (3) and (4) we have $\gamma_{sle}(G) = \lceil \frac{n+3}{2} \rceil$. □

Example 3.5. *For the graph G_l in Figure 2.16, the vertex set $X = \{v_2, v_4, v_5, v_6\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 4$.*



G (Figure 2.15)

L(G) (Figure 2.16)

Theorem 3.10. *For path P_n,*

$$\gamma_{sle}(G) = \begin{cases} 2, & n = 4,5,6 \\ \lceil \frac{n}{3} \rceil, & n \geq 7 \end{cases}$$

Proof. Let G be a path P_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, ..., v_{n-1}\}$.

Case (i) n = 4,5,6

In this case, the vertex set $X = \{v_2, v_4\}$ is a minimum split Legendary dominating set of G, since induced subgraph $< V_l \backslash X >$ is disconnected and hence $\gamma_{sle}(G) = 2$

Case (ii) $n \equiv 0(mod 3)$

In this case, the vertex set $X = \{v_{3i-1}/i = 1, 2, 3, ..., n-7\} \cup \{v_{n-1}\}$ is a minimum split legendary dominating set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n}{3} = \lceil \frac{n}{3} \rceil..........(1)$$

Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a path, X has atleast $\frac{n}{3}$ vertices.

Hence,

$$\gamma_{les}(G) = |X| \leq \frac{n}{3} = \lceil \frac{n}{3} \rceil ....(2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = \lceil \frac{n}{3} \rceil$.

Case (iii) $n \equiv 1(mod 3)$

In this case, the vertex set $X = \{v_{3i-1}/i = 2, 3, ..., n-5\} \cup \{v_{n-1}\}$ is a minimum split legendary dominating set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n+2}{3} = \lceil \frac{n}{3} \rceil ....(3)$$

Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a path, X has atleast $\frac{n+2}{3}$ vertices.

Hence,

$$\gamma_{sle}(G) = |X| \leq \frac{n+2}{3} = \lceil \frac{n}{3} \rceil ....(4)$$

Then from equation (3) and (4) we have $\gamma_{sle}(G) = \lceil \frac{n}{3} \rceil$.

Case (iv) $n \equiv 2(mod 3)$

In this case, the vertex set $X = \{v_{3i-1}/i = 2, 3, ..., n-6\} \cup \{v_{n-1}\}$ is a minimum split legendary dominating set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n+1}{3} = \lceil \frac{n}{3} \rceil ....(5)$$

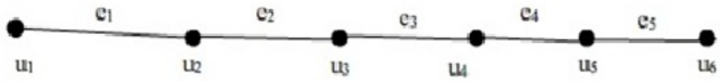
Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a path, X has atleast $\frac{n+1}{3}$ vertices.

Hence,

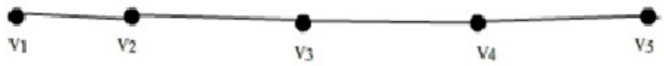
$$\gamma_{sle}(G) = |X| \leq \frac{n+1}{3} = \lceil \frac{n}{3} \rceil ....(6)$$

Then from equation (5) and (6) we have $\gamma_{sle}(G) = \lceil \frac{n}{3} \rceil$. □

Example 3.6. *For the graph G_l in Figure 2.18, the vertex set $X = \{v_2, v_5\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 2$.*



G (Figure 2.17)



L(G) (Figure 2.18)

Theorem 3.11. *For triangular snake graph T_n, $\gamma_{sle}(G) = n + 1, n \geq 2$.*

Proof. Let G be a triangular snake graph T_n with atleast five vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, ..., v_{\frac{n(n-1)}{2}}\}$. The vertex set $X = \{v_2, v_{3i}/i = 1, 2, ...n\}$ is a lesd $-$ set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = n + 1..........(1)$$

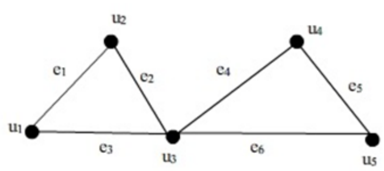
Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a triangular snake graph, X has atleast $n + 1$ vertices.

Hence,

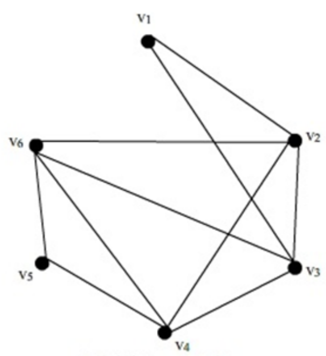
$$\gamma_{sle}(G) = |X| \leq n + 1..........(2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = n + 1, n \geq 2$. □

Example 3.7. *For the graph G_l in Figure 2.20, the vertex set $X = \{v_2, v_3, v_6\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.*



G (Figure 2.19)



L(G) (Figure 2.20)

Theorem 3.12. *For book graph B_n,*

$$\gamma_{sle}(G) = \begin{cases} 2, & n = 1 \\ n+1, & n \geq 2 \end{cases}$$

Proof. Let G be a book graph B_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, ..., v_{3n}\}$.

Case (i) n = 1

In this case, the vertex set $X = \{v_1, v_3\}$ is a minimum split legendary dominating set of G, since induced subgraph $< V_l \backslash X >$ is disconnected and hence $\gamma_{sle}(G) = 2$

Case (ii) $n \geq 2$

In this case, the vertex set $X = \{v_2, v_4\} \cup \{v_{3i}/i = 2, 3, ..., n\}$ is a minimum split legendary dominating set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = n + 1..........(1)$$

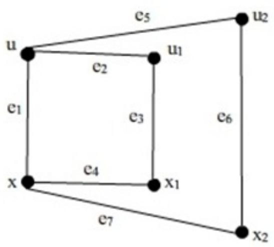
Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a book graph, X has atleast $n + 1$ vertices.

Hence,

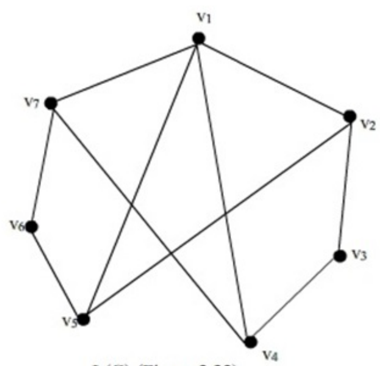
$$\gamma_{sle}(G) = |X| \leq n + 1..........(2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = n + 1, n \geq 3$. □

Example 3.8. *For the graph G_l in Figure 2.22, the vertex set $X = \{v_2, v_4, v_6\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.*



G (Figure 2.21)



L(G) (Figure 2.22)

Theorem 3.13. *For cycle graph C_n,*

$$\gamma_{sle}(G) = \lfloor \frac{n}{2} \rfloor, \quad n \geq 4$$

Proof. Let G be a cycle C_n with atleast four vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, ..., v_n\}$.
Case (i) $n \equiv 0(mod 2)$

In this case, the vertex set $X = \{v_2, v_n\} \cup \{v_4, v_6, v_8, ..., v_{n-2}\}$ is a minimum split legendary dominating set of G, since induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n}{2} = \lfloor \frac{n}{2} \rfloor..........(1)$$

Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a cycle graph, X has atleast $\frac{n}{2}$ vertices.

Hence,

$$\gamma_{sle}(G) = |X| \leq \frac{n}{2} = \lfloor \frac{n}{2} \rfloor..........(2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = \lfloor \frac{n}{2} \rfloor$.
Case (ii) $n \equiv 1(mod 2)$

In this case, the vertex set $X = \{v_2, v_n\} \cup \{v_4, v_6, v_8, ..., v_{n-3}\}$ is a minimum split legendary dominating set of G, since induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor..........(3)$$

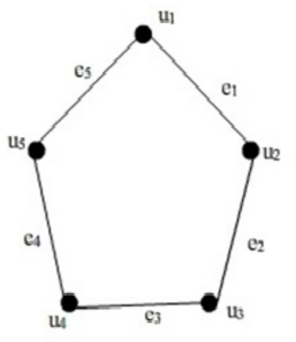
Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a cycle graph, X has atleast $\frac{n-1}{2}$ vertices.

Hence,

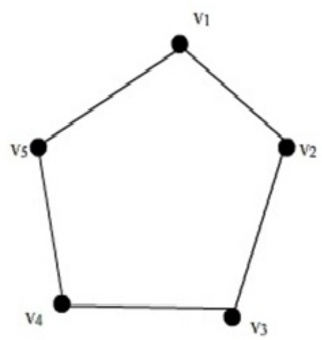
$$\gamma_{sle}(G) = |X| \leq \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor..........(4)$$

Then from equation (3) and (4) we have $\gamma_{sle}(G) = \lfloor \frac{n}{2} \rfloor$. □

Example 3.9. *For the graph G_l in Figure 2.24, the vertex set $X = \{v_2, v_5\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 2$.*



G (Figure 2.23)



L(G) (Figure 2.24)

Theorem 3.14. *For crown graph C_n^+, $\gamma_{sle}(G) = n, n \geq 3$.*

Proof. Let G be a crown graph C_n^+ with atleast six vertices. Let G_l be a line graph of G with vertex set $V_l = \{v_1, v_2, ..., v_{2n}\}$. The vertex set $X = \{v_2, v_3, v_4, v_5, ..., v_{n+1}\}$ is a sled $-$ set of G, since the induced subgraph $< V_l \backslash X >$ is disconnected. Therefore

$$\gamma_{sle}(G) \leq |X| = n..........(1)$$

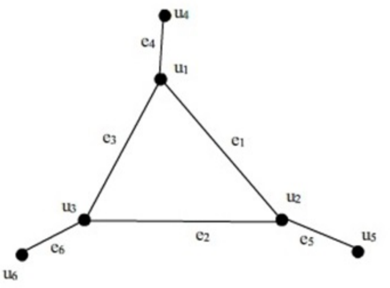
Let X be a $\gamma_{sle}-$ set of G. Then by Theorem 3.2, X must contains atleast one vertex in every $v_1 - v_2$ path of G_l, where v_1, v_2 are the vertices in the disconnected subgraph $< V_l \backslash X >$. As G being a crown graph, X has atleast n vertices.

Hence,

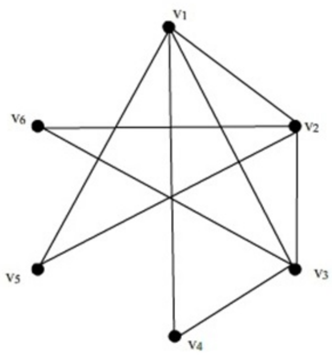
$$\gamma_{sle}(G) = |X| \leq n..........(2)$$

Then from equation (1) and (2) we have $\gamma_{sle}(G) = n, n \geq 3$. $\square$

Example 3.10. *For the graph G_l in Figure 2.26, the vertex set $X = \{v_2, v_3, v_4\} \subseteq V(G_l)$ is the minimum split legendary dominating set of G and hence $\gamma_{sle}(G) = 3$.*



G (Figure 2.25)



L(G) (Figure 2.26)

4 Discussion

The applications of domination in graphs lies in various fields in solving real life problems. It includes social network theory, land surveying, radio stations, computer communication networks and school bus routing. The online social network has been developed significantly in the recent years as medium of communication, sharing the information and spreading the influence. Recent advances in technology have provided portable computers with wireless interfaces that allowed network communication among mobile users. The resulting computing environment, often referred to as wireless mobile computing, no longer requires users to maintain a fixed and universally known position in the network and enables almost unrestricted mobility. A network with clustering is divided into several clusters. Within each cluster, one of the nodes is elected as a dominating nodes and with the rest being adjacent dominating nodes. The dominating node will collect data from its adjacent dominating nodes directly.

The present work, the split legendary dominating set is introduced and the corresponding split legendary domination number is defined. Further, the relationship with other domination parameter are obtained.The characterization of the split legendary dominating set is also given and the exact values of the parameter is found for complete graph, cycle graph, path, wheel graph, crown graph, ladder graph, book graph, triangular snake graph, fan graph and friendship graph.By using the split legendary domination one can optimize, the parameters involved in network theory.

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