

New Characterization Of $(1, 2)S_p$ -Kernel in Bitopological Spaces

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Abstract

In this paper, the concept of $(1,2)S_p$ -Kernal in bitopological spaces is introduced. Also the closure and kernel are defined in terms of $(1,2)S_p$ -separation using $(1,2)S_p$ -open sets in bitopological spaces and some of its properties are studied.

Keywords: $(1,2)$ semi-open, $(1,2)$ pre-open, $(1,2)$ pre-closed, $(1,2)S_p$ -open sets, $(1,2)S_p$ -closed sets, $(1,2)S_p$ -kernel sets, $(1,2)S_p$ -derived sets, $(1,2)S_p$ -shell sets.

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1. Introduction

In 1963, Kelly [3] initiated the study of a triplet (X, τ_1, τ_2) , where X is a non-empty set and τ_1, τ_2 are the two topologies on X . The notion of $(1,2)\alpha$ -open set [4] in a bitopological space was introduced by Lellis Thivagar in 1991. Raja Rajeswari [6] defined and studied the concepts of ultra-kernel in bitopological spaces. In 2017, $(1,2)S_p$ -open set [2] in bitopological spaces was introduced by Hardi Ali Shareef et.al. In this paper, a new class of sets in bitopological spaces called $(1,2)S_p$ -kernel is introduced and some of its properties are derived.

2. Preliminaries

Definition 2.1. [5] A subset A of a bitopological space X is called a

- (i) $(1, 2)$ semi-open if $A \subseteq \tau_1 \tau_2 \text{Cl}(\tau_1 \text{Int}(A))$.
- (ii) $(1, 2)$ pre-open if $A \subseteq \tau_1 \text{Int}(\tau_1 \tau_2 \text{Cl}(A))$.
- (iii) $(1, 2)$ regular-open if $A = \tau_1 \text{Int}(\tau_1 \tau_2 \text{Cl}(A))$.

The collection of all $(1, 2)$ semi-open, $(1, 2)$ pre-open and $(1, 2)$ regular-open sets are denoted by $(1, 2) \text{SO}(X)$, $(1, 2) \text{PO}(X)$ and $(1, 2) \text{RO}(X)$ respectively.

Definition 2.2 [5] A subset A of a bitopological space X is called a

- (i) $(1, 2)$ α -closed if $\tau_1 \text{Cl}(\tau_1 \tau_2 \text{Int}(\tau_1 \text{Cl}(A))) \subseteq A$.
- (ii) $(1, 2)$ semi-closed if $\tau_1 \tau_2 \text{Int}(\tau_1 \text{Cl}(A)) \subseteq A$.
- (iii) $(1, 2)$ pre-closed if $\tau_1 \text{Cl}(\tau_1 \tau_2 \text{Int}(A)) \subseteq A$.
- (iv) $(1, 2)$ regular-closed if $A = \tau_1 \text{Cl}(\tau_1 \tau_2 \text{Int}(A))$.

The set of all $(1, 2)$ α -closed, $(1, 2)$ semi-closed, $(1, 2)$ pre-closed and $(1, 2)$ regular-closed sets are denoted as $(1, 2) \alpha \text{CL}(X)$, $(1, 2) \text{SCL}(X)$, $(1, 2) \text{PCL}(X)$ and $(1, 2) \text{RCL}(X)$ respectively. Also, for any subset A of X , the $(1, 2)$ α -closure, $(1, 2)$ semi-closure, $(1, 2)$ pre-closure and $(1, 2)$ regular-closure of A is denoted as $(1, 2) \alpha \text{Cl}(A)$, $(1, 2) \text{SCL}(A)$, $(1, 2) \text{PCL}(A)$ and $(1, 2) \text{RCL}(A)$ respectively.

Definition 2.3. [2] A $(1, 2)$ semi-open set A of a bitopological space X is called $(1, 2) S_p$ -open set if for each $x \in A$, there exists a $(1, 2)$ pre-closed set F such that $x \in F \subseteq A$.

Remark 2.4. If A and B are $(1, 2)$ α -open sets of X , then $A \cup B$ is also a $(1, 2)$ α -open set.

Definition 2.5. A subset A of a bitopological space X is said to be $(1, 2) S_p$ -locally closed (briefly $(1, 2) S_p \text{LC}$) if $A = C \cap D$, where C is a $(1, 2) S_p$ -open set and D is a $(1, 2) S_p$ -closed set in X . The family of $(1, 2) S_p$ -locally-closed sets is denoted by $(1, 2) S_p \text{LC}(X)$.

3. (1, 2) S_p -Kernel in Bitopological Spaces

Definition 3.1. Let A be a non-empty subset of a bitopological spaces X . Then (1, 2) S_p -kernel of A is denoted by (1, 2) S_p -Ker ($\{A\}$) and it is defined as (1, 2) S_p -Ker ($\{A\}$) = $\cap \{G \in (1, 2) S_p O(X) / A \subseteq G\}$.

Definition 3.2. Let $x \in X$. Then the (1, 2) S_p -kernel of x is defined by (1, 2) S_p -Ker ($\{x\}$) = $\cap \{G \in (1, 2) S_p O(X) / x \in G\}$.

Definition 3.3. A subset N of a bitopological space X is said to be (1, 2) S_p -neighborhood ((1, 2) S_p -nbhd) of a point $x \in X$, if there exists a (1, 2) S_p -open set U such that $x \in U \subseteq N$.

Lemma 3.4. Let X be a bitopological space. Then for any non-empty subset A of X , (1, 2) S_p -Ker ($\{A\}$) = $\{x \in X / (1, 2) S_p\text{-Cl}(\{x\}) \cap A \neq \phi\}$.

Proof: Let $x \in (1, 2)S_p\text{-Ker}(\{A\})$ and $(1, 2)S_p\text{-Cl}(\{x\}) \cap A \neq \phi$. Then $A \subseteq [X - (1, 2) S_p\text{-Cl}(\{x\})]$ and $[X - (1, 2) S_p\text{-Cl}(\{x\})]$ is a (1, 2) S_p -open set containing A but not x , which is a contradiction. Hence $(1, 2) S_p\text{-Cl}(\{x\}) \cap A \neq \phi$.

Also, let $x \notin (1, 2) S_p\text{-Ker}(\{A\})$ and $(1, 2) S_p\text{-Cl}(\{x\}) \cap A \neq \phi$. Then there exists a (1, 2) S_p -open set D containing A but not x and there exists an element $y \in (1, 2)S_p\text{-Cl}(\{x\}) \cap A$. Hence we get a (1, 2) S_p -nbhd of y , say D with $x \notin D$, which is a contradiction. Hence $x \in (1, 2) S_p\text{-Ker}(A)$.

Definition 3.5. In a bitopological space X , a subset A of X is said to be weakly (1, 2) S_p -separated from a subset B of X if there exists a (1, 2) S_p -open set G of X such that $A \subseteq G$ and $G \cap B = \phi$ or $A \cap (1, 2)S_p\text{-Cl}(B) = \phi$. It is shown in the following example.

Example 3.6. Let $X = \{a, b, c, d\}$. $\tau_1 = \{\phi, X, \{a, c\}, \{a, c, d\}\}$. $\tau_2 = \{\phi, X\}$. $\tau_1\tau_2 Cl = \{X, \phi, \{b, d\}, \{b\}\}$. (1, 2) $SO(X) = \{\phi, X, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. (1, 2) $PCL(X) = \{X, \phi, \{b, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{a, d\}, \{a, b\}, \{d\}, \{c\}, \{b\}, \{a\}\}$. (1, 2) $S_p O(X) = \{\phi, X, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. (1, 2) $S_p CL(X) = \{X, \phi, \{b, d\}, \{d\}, \{b\}\}$. Let $A = \{b\}$, $B = \{d\}$ and $G = \{a, b, c\} \in (1, 2)S_p O(X)$. Here $A \subseteq G$ and $G \cap B = \phi$ or $A \cap (1, 2)S_p\text{-Cl}(B) = \phi$. Hence a subset A of X is weakly (1, 2) S_p -separated from a subset B of X .

Theorem 3.7. Suppose X is a (1, 2) S_p -space and $A, B \in (1, 2) S_p LC(X)$. If A and B are weakly (1, 2) S_p -separated, then $A \cup B \in (1, 2) S_p LC(X)$.

Proof: Assume $A, B \in (1, 2) S_p LC(X)$ and let $A = G \cap (1, 2) S_p\text{-Cl}(A)$ and $B = E \cap (1, 2) S_p\text{-Cl}(B)$, where G and E are (1, 2) S_p -open sets of X . Put $U = G \cap [X - (1, 2) S_p\text{-Cl}(B)]$ and $V = E \cap [X - (1, 2) S_p\text{-Cl}(A)]$. Then $U \cap (1, 2)S_p\text{-Cl}(A) = G \cap [X - (1, 2) S_p\text{-Cl}(B)] \cap (1, 2) S_p\text{-Cl}(A) = [G \cap X - (1, 2) S_p\text{-Cl}(A)] \cap [X - (1, 2)S_p\text{-Cl}(B)] = A \cap [X - (1, 2) S_p\text{-Cl}(B)] = A$.

Similarly, $V \cap (1, 2) S_p\text{-Cl}(B) = [E \cap (X - (1, 2) S_p\text{-Cl}(\{A\}))] \cap (1, 2) S_p\text{-Cl}(\{A\})$

$= [E \cap [X - (1, 2) S_p\text{-Cl}(\{B\})]] \cap [(X - (1, 2) S_p\text{-Cl}(\{A\}))] = B \cap [(X - (1, 2) S_p\text{-Cl}(\{A\}))] = B$ and hence $U \cap (1, 2) S_p\text{-Cl}(B) = V \cap (1, 2) S_p\text{-Cl}(A) = \phi$. Moreover U and V are $(1, 2) S_p$ -open in X . By Remark 2.4, $(U \cup V)$ is also $(1, 2) S_p$ -open. Then $(U \cup V) \cap (1, 2) S_p\text{-Cl}(A \cup B) = [U \cap (1, 2) S_p\text{-Cl}(A)] \cup [U \cap (1, 2) S_p\text{-Cl}(B)] \cup [V \cap (1, 2) S_p\text{-Cl}(A)] \cup [V \cap (1, 2) S_p\text{-Cl}(B)] = A \cup B$. Hence $A \cup B$ is $(1, 2) S_p$ -locally closed set.

Lemma 3.8. In view of Lemma 3.4 and Definition 3.5, we have for x, y in X of a bitopological space,

- (i) $(1, 2) S_p\text{-Cl}(\{x\}) = \{y: y \text{ is not weakly } (1, 2) S_p\text{-separated from } x\}$.
- (ii) $(1, 2) S_p\text{-Ker}(\{x\}) = \{y: x \text{ is not weakly } (1, 2) S_p\text{-separated from } y\}$.

Definition 3.9. For any point x of a bitopological space X ,

- (i) The derived set of x is denoted by $(1, 2) S_p\text{-d}(\{x\})$ and is defined to be the set $(1, 2) S_p\text{-d}(\{x\}) = (1, 2) S_p\text{-Cl}(\{x\}) - \{x\} = \{y: y \neq x \text{ and } y \text{ is not weakly } (1, 2) S_p\text{-separated from } x\}$.
- (ii) the shell of a point x of X is denoted by $(1, 2) S_p\text{-shl}(\{x\})$ and is defined to be the set $(1, 2) S_p\text{-shl}(\{x\}) = (1, 2) S_p\text{-Ker}(\{x\}) - \{x\} = \{y: y \neq x \text{ and } x \text{ is not weakly } (1, 2) S_p\text{-separated from } x\}$.

Definition 3.10: Let X be a bitopological space. Then we define

- (i) $(1, 2) S_p\text{-N-D} = \{x: x \in X \text{ and } (1, 2) S_p\text{-d}(\{x\}) = \phi\}$.
- (ii) $(1, 2) S_p\text{-N-Shl} = \{x: x \in X \text{ and } (1, 2) S_p\text{-shl}(\{x\}) = \phi\}$.
- (iii) $(1, 2) S_p\text{-N} = (1, 2) S_p\text{-Cl}(\{x\}) \cap (1, 2) S_p\text{-Ker}(\{x\})$.

Theorem 3.11. Let $x, y \in X$. Then the following conditions hold good:

- (i) $y \in (1, 2) S_p\text{-Ker}(\{x\})$ if and only if $x \in (1, 2) S_p\text{-Cl}(\{y\})$.
- (ii) $y \in (1, 2) S_p\text{-shl}(\{x\})$ if and only if $x \in (1, 2) S_p\text{-d}(\{y\})$.
- (iii) $y \in (1, 2) S_p\text{-Cl}(\{x\})$ implies $(1, 2) S_p\text{-Cl}(\{y\}) \subseteq (1, 2) S_p\text{-Cl}(\{x\})$.
- (iv) $y \in (1, 2) S_p\text{-Ker}(\{x\})$ implies $(1, 2) S_p\text{-Ker}(\{y\}) \subseteq (1, 2) S_p\text{-Ker}(\{x\})$.

Proof: The proof of (i) and (ii) are obvious from the Lemma 3.8.

(iii) Let $z \in (1, 2) S_p\text{-Cl}(\{y\})$. Then z is not weakly $(1, 2) S_p$ -separated from y which implies there exists a $(1, 2) S_p$ -open set G containing x such that $G \cap \{y\} \neq \phi$. Hence $y \in G$ and by assumption $G \cap \{x\} \neq \phi$. Hence z is not weakly $(1, 2) S_p$ -separated from x which implies $z \in (1, 2) S_p\text{-Cl}(\{x\})$. Therefore $(1, 2) S_p\text{-Cl}(\{y\}) \subseteq (1, 2) S_p\text{-Cl}(\{x\})$.

(iv) Let $z \in (1, 2) S_p\text{-Ker}(\{y\})$. Then y is not weakly $(1, 2) S_p$ -separated from z which implies $y \in (1, 2) S_p\text{-Cl}(\{z\})$. Hence $(1, 2) S_p\text{-Cl}(\{y\}) \subseteq (1, 2) S_p\text{-Cl}(\{z\})$. By assumption $y \in (1, 2) S_p\text{-Ker}(\{x\})$ that implies $x \in (1, 2) S_p\text{-Cl}(\{y\})$. Then $(1, 2) S_p\text{-Cl}(\{x\}) \subseteq (1, 2) S_p\text{-Cl}(\{y\})$. Ultimately, $(1, 2) S_p\text{-Cl}(\{x\}) \subseteq (1, 2) S_p\text{-Cl}(\{z\})$ which implies $x \in (1, 2) S_p\text{-Cl}(\{z\})$, that is $z \in (1, 2) S_p\text{-Ker}(\{x\})$. Therefore $(1, 2) S_p\text{-Ker}(\{y\}) \subseteq (1, 2) S_p\text{-Ker}(\{x\})$.

Theorem 3.12. Let x, y be in X . Then

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(i) for every $x \in X$, $(1,2)S_p\text{-shl}(\{x\})$ is a degenerate set if and only if for all $x, y \in X$, $x \neq y$, $(1,2)S_p\text{-d}(\{x\}) \cap (1,2)S_p\text{-d}(\{y\}) = \emptyset$.

(ii) for every $x \in X$, $(1,2)S_p\text{-d}(\{x\})$ is a degenerate set if and only if for every $x, y \in X$, $x \neq y$, $(1,2)S_p\text{-shl}(\{x\}) \cap (1,2)S_p\text{-shl}(\{y\}) = \emptyset$.

Proof. (i) Let $(1,2)S_p\text{-d}(\{x\}) \cap (1,2)S_p\text{-d}(\{y\}) \neq \emptyset$. Then there exists a $z \in X$ such that $z \in (1,2)S_p\text{-d}(\{x\})$ and that $z \in (1,2)S_p\text{-d}(\{y\})$. Then $z \neq y \neq x$ and $z \in (1,2)S_p\text{-Cl}(\{x\})$ and $z \in (1,2)S_p\text{-Cl}(\{y\})$. That is $x, y \in (1,2)S_p\text{-Ker}(\{z\})$. Hence $(1,2)S_p\text{-Ker}(\{z\})$ implies $(1,2)S_p\text{-shl}(\{z\})$ is not a degenerate set, which is a contradiction. Hence $(1,2)S_p\text{-d}(\{x\}) \cap (1,2)S_p\text{-d}(\{y\}) = \emptyset$.

Also, let $x, y \in (1,2)S_p\text{-shl}(\{z\})$ is not a degenerate set. Then $x \neq y \neq z$ and $x, y \in (1,2)S_p\text{-Ker}(\{z\})$. Then z is an element of both $(1,2)S_p\text{-Cl}(\{x\})$ and $(1,2)S_p\text{-Cl}(\{y\})$ which implies $(1,2)S_p\text{-Cl}(\{x\}) \cap (1,2)S_p\text{-Cl}(\{y\}) \neq \emptyset$ which is a contradiction. Hence $(1,2)S_p\text{-shl}(\{z\})$ is a degenerate set.

The proof of (ii) is similar that of (i).

Theorem 3.13. If $y \in (1,2)S_p\text{-}\langle x \rangle$, then $(1,2)S_p\text{-}\langle x \rangle = (1,2)S_p\text{-}\langle y \rangle$,

Proof. If $y \in (1,2)S_p\text{-}\langle x \rangle$, then $y \in (1,2)S_p\text{-Cl}(\{x\})$ and $y \in (1,2)S_p\text{-Ker}(\{x\})$ and by Theorem 3.10, $(1,2)S_p\text{-Cl}(\{y\}) \subseteq (1,2)S_p\text{-Cl}(\{x\})$ and $(1,2)S_p\text{-Ker}(\{y\}) \subseteq (1,2)S_p\text{-Ker}(\{x\})$. which implies $(1,2)S_p\text{-Cl}(\{y\}) \cap (1,2)S_p\text{-Ker}(\{y\}) \subseteq (1,2)S_p\text{-Cl}(\{x\}) \cap (1,2)S_p\text{-Ker}(\{x\})$. Thus $(1,2)S_p\text{-}\langle y \rangle \subseteq (1,2)S_p\text{-}\langle x \rangle$. Now, $y \in (1,2)S_p\text{-Cl}(\{x\})$ implies $x \in (1,2)S_p\text{-Ker}(\{y\})$ and $y \in (1,2)S_p\text{-Ker}(\{x\})$ which implies $x \in (1,2)S_p\text{-Cl}(\{y\})$. Which implies $(1,2)S_p\text{-Cl}(\{x\}) \cap (1,2)S_p\text{-Ker}(\{x\}) \subseteq (1,2)S_p\text{-Ker}(\{y\}) \cap (1,2)S_p\text{-Cl}(\{y\})$. Thus $(1,2)S_p\text{-}\langle x \rangle \subseteq (1,2)S_p\text{-}\langle y \rangle$. Hence $(1,2)S_p\text{-}\langle x \rangle = (1,2)S_p\text{-}\langle y \rangle$.

Theorem 3.14. For all $x, y \in X$, either $(1,2)S_p\text{-}\langle x \rangle \cap (1,2)S_p\text{-}\langle y \rangle = \emptyset$ or $(1,2)S_p\text{-}\langle x \rangle = (1,2)S_p\text{-}\langle y \rangle$.

Proof. Let $(1,2)S_p\text{-}\langle x \rangle \cap (1,2)S_p\text{-}\langle y \rangle \neq \emptyset$, then there exists $z \in X$ such that $z \in (1,2)S_p\text{-}\langle x \rangle$ and $z \in (1,2)S_p\text{-}\langle y \rangle$. By theorem 3.13, $(1,2)S_p\text{-}\langle z \rangle = (1,2)S_p\text{-}\langle x \rangle = (1,2)S_p\text{-}\langle y \rangle$. Hence $(1,2)S_p\text{-}\langle x \rangle = (1,2)S_p\text{-}\langle y \rangle$.

Theorem 3.15. For any two points x, y in X , the following statements are equivalent.

(i) $(1,2)S_p\text{-Ker}(\{x\}) \neq (1,2)S_p\text{-Ker}(\{y\})$.

(ii) $(1,2)S_p\text{-Cl}(\{x\}) \neq (1,2)S_p\text{-Cl}(\{y\})$

Proof. (i) $\Rightarrow$ (ii) Let us assume that $(1,2)S_p\text{-Ker}(\{x\}) \neq (1,2)S_p\text{-Ker}(\{y\})$. Then there exists a point $z \in (1,2)S_p\text{-Ker}(\{x\})$ but $z \notin (1,2)S_p\text{-Ker}(\{y\})$. As $z \in (1,2)S_p\text{-Ker}(\{x\})$, $x \in (1,2)S_p\text{-Cl}(\{z\})$ and $(1,2)S_p\text{-Cl}(\{x\}) \subseteq (1,2)S_p\text{-Cl}(\{z\})$. Also since $z \notin (1,2)S_p\text{-Ker}(\{y\})$. By Lemma 3.4, $(1,2)S_p\text{-Cl}(\{z\}) \cap \{y\} = \emptyset$ which implies $(1,2)S_p\text{-Cl}(\{x\}) \cap \{y\} = \emptyset$ and y is weakly $(1,2)S_p$ -separated from x , that is $y \notin (1,2)S_p\text{-Cl}(\{x\})$. Hence $(1,2)S_p\text{-Cl}(\{y\}) \neq (1,2)S_p\text{-Cl}(\{x\})$.

(ii) $\Rightarrow$ (i) Suppose $(1,2)S_p\text{-Cl}(\{x\}) \neq (1,2)S_p\text{-Cl}(\{y\})$. Then there exists a point

$z \in (1, 2)S_p\text{-Cl}(\{x\})$ but $z \notin (1, 2)S_p\text{-Cl}(\{y\})$. Also a $(1, 2)S_p$ -open set containing z and x but not y implies $y \notin (1, 2)S_p\text{-Ker}(\{x\})$. Hence $(1, 2)S_p\text{-Ker}(\{y\}) \neq (1, 2)S_p\text{-Ker}(\{x\})$.

4. Conclusions

In this paper, the new characterization of $(1, 2)S_p$ -kernel was introduced and some of its properties are discussed. Later on Research be reached out with certain applications.

References

- [1] Dhanalakshmi. S and Durga Devi. N: “Some Generalization of $(1, 2)S_p$ -locally closed sets in bitopological spaces”, J. Math. Comput. Sci. 11(2021), No. 5, 5931-5936.
- [2] Hardi Ali Shareef, Durga Devi Natarajan, Raja Rajeswari Ramajeyam and Thangavelu Periannan: “ $(1, 2) S_p$ -open sets in Bitopological spaces”, Journal of Zankoy Sulaimani (2017) 19-2(Part A), 195-201.
- [3] Kelly. J.C: “Bitopological spaces”, Proc. London. Math. Soc. 1963; 13: 71-89.
- [4] Lellis Thivagar. M: “Generalization of pairwise α -continuous function”, Pure and Applied Mathematics and Sciences, Vol XXXIII, No. 1-2, 1991, 55-63.
- [5] Lellis Thivagar. M, Meera Devi. B and Navalagi. G: “ $(1, 2)$ Externally disconnectedness Via Bitopological open sets”, International Journal of General Topology, Vol.4, Nos 1-2, January – December 2011, 915.
- [6] Raja Rajeswari. R: “Bitopological concepts of some separation properties”, Ph.D., Thesis, Madurai Kamaraj University, Madurai, India, 2009.