Nano δI-closed Sets in Nano Ideal Topological Spaces

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Abstract

This article investigates the notion of N-δI-closed sets and its properties in ideal nano topological space using the closure operator $Ncl_{\delta I}(S)=\{x\in \mathbf{U} / Nint(Ncl^{*}(W))\cap S \neq \emptyset$, for each $W\in W_N(x)\}$. We establish $Ncl_{\delta I}(S)$ is the intersection of all N-δI-closed supersets of S and $Nint_{\delta I}(S)$ is the union of all N-δI-open subsets of S in this space. Apart from that N-δI-closed sets occurs between the class of Nδ-closed sets and N-closed sets. Moreover, we explore the concepts of N-δI-interior, N-δI-Exterior, N-δI-Border, and N-δI-Frontier and discuss its properties.

Keywords: nano ideal topological space; nano δI-open sets; nano δI-closed sets.

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1. Introduction and Preliminaries

A non-empty sub collection $\mathscr{J}$ of P(U) is called ideal [2] if it satisfies: (i) S $\in \mathscr{J}$ and T $\subset$ S $\Rightarrow$ T $\in \mathscr{J}$, and (ii) S $\in \mathscr{J}$ and T $\in \mathscr{J} \Rightarrow$ S $\cup$ T $\in \mathscr{J}$. In [1], Jankovic and Hamlet introduced and studied the local function and the kuratowski closure operator cl*(.) in ideal topological spaces. In [3], nano closed set, nano interior, and nano closure in nano topological spaces were defined and researched by Lellis Thivagar. In [4], Lellis Thivagar and Richard introduced the lower (resp., upper) approximation and border as $L_{(\mathscr{R})}(X) = \cup\{\mathscr{R}(x): \mathscr{R}(x) \subset X, x \in U\}$, where $\mathscr{R}(x)$ denotes the equivalence class determined by $x \in U$ (resp., $U_{(\mathscr{R})}(X) = \cup\{\mathscr{R}(x): \mathscr{R}(x) \cap X \neq \phi, x \in U\}$) and $B_{(\mathscr{R})}(X) = U_{(\mathscr{R})}(X) - L_{(\mathscr{R})}(X)$ of $X \subset U$ using the equivalence relation $\mathscr{R}$. In [10], Carmel discussed some weak nano open sets. In [5, 6, 7], Parimala et al. introduced nano ideal topological spaces or ideal nano topological spaces, N-local function which is defined as: $S_N^*(\mathscr{J}, \mathbb{N}) = \{x \in U: W \cap S \notin \mathscr{J}$, for every $W \in W_N(x)\}$ where $W_N(x) = \{W / x \in W$ and $W \in \mathscr{J}\}$ and the closure operator n-cl*(.) defined as: n-cl*(S) = $S \cup S_N^*$, for $S \subset U$ and investigated its some basic properties. Parimala et al. [8] presented the concept of Nδ-open sets. A set S is known as Nδ-open set if $\text{n-cl}_\delta(S) = S$, where $\text{n-cl}_\delta(S) = \cup\{x \in U:$ n-int(n-cl(W)) $\cap S \neq \phi$, W is n-open and $x \in W\}$. Premkumar and Rameshpandi [9] introduced R-NI-open sets. A set S is known as R-NI-open [9] (resp., nano regular open [3]) set if n-int(n-cl*(S))=S (resp., n-int(n-cl(S) = S)). The complement of an R-NI-open (resp., nano regular open) set is R-NI-closed (resp. nano regular closed). Yuksel, Acikgoz, and Noiri [11] introduced and studied the notion of δ-I-open set in an ideal topological space. By this motivation here we focus on nano δI-closed sets in nano ideal topological spaces.

Property 1.1. [3] If (U, $\mathscr{R}$) is an approximation space and W, T $\subseteq$ U. Then,

(i) $L_{(\mathscr{R})}(W) \subseteq W \subseteq U_{(\mathscr{R})}(W)$.

(ii) $L_{(\mathscr{R})}(\phi) = U_{(\mathscr{R})}(\phi) = \phi$.

(iii) $L_{(\mathscr{R})}(U) = U_{(\mathscr{R})}(U) = U$.

(iv) $U_{(\mathscr{R})}(W \cup Z) = U_{(\mathscr{R})}(X) \cup U_{(\mathscr{R})}(Z)$.

(v) $U_{(\mathscr{R})}(W \cap Z) \subseteq U_{(\mathscr{R})}(W) \cap U_{(\mathscr{R})}(Y)$.

(vi) $L_{(\mathscr{R})}(W \cup Z) \supseteq L_{(\mathscr{R})}(W) \cup L_{(\mathscr{R})}(Z)$.

(vii) $L_{(\mathscr{R})}(W \cap Z) = L_{(\mathscr{R})}(W) \cap L_{(\mathscr{R})}(Z)$.

(viii) $L_{(\mathscr{R})}(W) \subseteq L_{(\mathscr{R})}(Z)$ and $U_{(\mathscr{R})}(W) \subseteq U_{(\mathscr{R})}(W)$ if $W \subseteq Z$.

(ix) $U_{(\mathscr{R})}(W^c) = [L_{(\mathscr{R})}(W)]^c$ and $L_{(\mathscr{R})}(W^c) = [U_{(\mathscr{R})}(W)]^c$.

(x) $U_{(\mathscr{R})}[U_{(\mathscr{R})}(W)] = L_{(\mathscr{R})}[U_{(\mathscr{R})}(W)] = U_{(\mathscr{R})}(W)$.

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Definition 1.2. [3] Let $\mathbf{U}$ be the universal set and $\mathscr{R}$ be an equivalence relation defined on $\mathbf{U}$. $T_{(\mathscr{R})}(W)=\{\mathbf{U}, \phi, L_{(\mathscr{R})}(W), U_{(\mathscr{R})}(W), B_{(\mathscr{R})}(W)\}$ where $W \subseteq \mathbf{U}$, Then $T_{(\mathscr{R})}(W)$ by property 1.1 satisfies the given below axioms:

(i) $\mathbf{U}$ and $\phi \in T_{(\mathscr{R})}(W)$.
(ii) Arbitrary union of members of $T_{(\mathscr{R})}(W)$ is in $T_{(\mathscr{R})}(W)$.
(iii) Finite intersection of members of $T_{(\mathscr{R})}(W)$ is in $T_{(\mathscr{R})}(W)$.

Then, $T_{(\mathscr{R})}(W)$ is a topology on $\mathbf{U}$ called the nano topology on $\mathbf{U}$ with respect to W. The pair $(\mathbf{U}, T_{(\mathscr{R})}(W))$ is known as nano topological space (simply, NTS). The members of $T_{(\mathscr{R})}(W)$ are known as nano open sets (briefly, N-open sets) and their complements are N-closed sets. We denote, $T_{(\mathscr{R})}(W)=\mathbb{N}$ and so $(\mathbf{U}, \mathbb{N})$ is the nano topological space.

The nano topology $(\mathbf{U}, \mathbb{N})$ with ideal $\mathscr{J}$ forms the triplet $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ called nano ideal topological space or ideal nano topological space [5]. We use ideal nano topological space for convenience in understanding. We simply mention an ideal nano topological space by INTS. Also, for $S \subseteq \mathbf{U}$ we denote the nano interior (resp., nano closure, nano δ-closure, nano *-closure) of S by Nint(S) (resp., Ncl(S), $Ncl_\delta(S)$, Ncl*(S)).

2. N-δI-closed set

Definition 2.1. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be an INTS. For $S \subseteq \mathbf{U}$, $Ncl_{\delta I}(S)=\{x \in \mathbf{U} \,/\, Nint(Ncl^*(W)) \cap S \neq \emptyset$, for each $W \in W_N(x)\}$ where $W_N(x)=\{W \,/\, x \in W$ and $W \in \mathbb{N}\}$. If $Ncl_{\delta I}(S) = S$ then S is known as a nano δI-closed set and $\mathbf{U} - S$ is nano δI-open set.

Rest of the paper nano δI-closed (resp., nano δI-open) sets are simply denoted by N-δI-closed (resp., N-δI-open) and the N-δI-open sets collection is mentioned by $N\text{-}T_{\delta I}$. Generally, Nδ-open sets are N-δI-open and N-δI-open sets are N-open sets. But their reverse implications are not always hold as in the forthcoming Example. Let $\mathbf{U}$={e, f, g, h}, X={g}, $\mathbf{U}/\mathscr{R}$={{e}, {f}, {g, h}}, $\mathbb{N}=\{\mathbf{U}, \emptyset, \{g, h\}\}$ and I={∅, {g}}. Then, $n\text{-}T_{\delta I}=\{\mathbf{U}, \emptyset\}$. If I ={∅, {g}, {h}, {g, h}} then $N\delta C(\mathbf{U})=\{\mathbf{U}, \emptyset\}$ and $n\text{-}T_{\delta I}$ ={U, ∅, {g, h}}.

Theorem 2.2. Let S and T be subsets of an INTS $(\mathbf{U}, \mathbb{N}, \mathscr{J})$. Then,
(i) Nint $(Ncl^*(S))$ is R-NI-open.
(ii) If S and T are R-NI-open, then $S \cap T$ is R-NI-open.
(iii) N-Regular open sets are R-NI-open.
(iv) R-NI-open sets are N-δI-open.
(v) If S is an N-δI-open set, then $S = \cup\{W/W$ is R-NI-open$\}$.

Proof.(i)Let $W = Nint(Ncl^*(S))$, $S \subseteq \mathbf{U}$. Then, $Nint(Ncl^*(W)) = Nint(Ncl^*(Nint(Ncl^*(S))) \subseteq Nint(Ncl^*(S)) = W = Nint(W) \subseteq Nint(Ncl^*(S)))$.
(ii) The proof from the fact that $Nint(Ncl^*(S \cap T)) \subseteq Nint(Ncl^*(S) \cap Ncl^*(T))$.

(iii) The proof is from the fact that, $Ncl^*(S)) \subseteq Ncl(S)$.
(iv) Let S be an R-NI-open set. Then, for every $x \in S$, $(S^C) \cap S = \phi$. Hence, $x \notin Ncl_{\delta I}(S^C)$. Always, $S \subseteq Ncl_{\delta I}(S)$. Hence, $Ncl_{\delta I}(S^C) = S^C$.
(v) Since S is an N-δI-open set, $N\text{-}cl_{\delta I}(S^C) = S^C$. For every $x \in S$, $x \notin Ncl_{\delta I}(S^C)$. Therefore, there exists a $W \in W_N(x)$ such that $Nint(Ncl^*(W)) \cap (S^C) = \emptyset$. Therefore, $x \in Nint(Ncl^*(W)) \subseteq S$ and hence $S = \cup\{Nint(Ncl^*(W)) / x \in S\}$. Hence, by (i) proof completes.

Theorem 2.3. Let S and T be subsets of an INTS $(\mathbf{U}, \mathbb{N}, \mathscr{J})$. Then,
(i) $S \subseteq Ncl_{\delta I}(S)$.
(ii) If $S \subseteq T$, then $Ncl_{\delta I}(S) \subseteq Ncl_{\delta I}(T)$.
(iii) $Ncl_{\delta I}(S) = \cap\{H \subseteq \mathbf{U} / S \subseteq H$, H is an N-δI-closed set$\}$.
(iv) If S_j is an N-δI-closed subset of $\mathbf{U}$ for every $j \in J$, then $\cap\{S_j / j \in J\}$ is N-δI-closed.
(v) $N\text{-}cl_{\delta I}(S)$ is N-δI-closed.

Proof. (i) Let $x \in S$. Then, $W \cap S \neq \emptyset$ for all $W \in W_N(x)$. But, $W \cap S \subseteq Nint(Ncl^*(W)) \cap S$. Therefore, $x \in Ncl_{\delta I}(S)$.
(ii)Let $x \in Ncl_{\delta I}(S)$. Suppose that, $x \notin Ncl_{\delta I}(T)$. Then there is a $W \in W_N(x)$ with $Nint(Ncl^*(W) \cap T = \emptyset$. Therefore, $Nint(Ncl^*(W) \cap S = \emptyset$. A contradiction.
(iii) Let $x \in Ncl_{\delta I}(S)$. $H \subseteq S$, H is N-δI-closed. Then, $\emptyset \neq Nint(Ncl^*(W)) \cap S \subseteq Nint(Ncl(W)) \cap H$. Therefore, $x \in Ncl_{\delta I}(H)$ and hence $x \in H$. Conversely, suppose that $x \notin Ncl_{\delta I}(S)$. Then there is a $W \in W_N(x)$ for which $Nint(Ncl^*(W)) \cap S = \emptyset$. By Theorem 2.2, $\mathbf{U} - (Nint(Ncl^*(W)))$ is a N-δI-closed set contains S. $x \notin \mathbf{U} - (Nint(Ncl^*(W)))$. Therefore, $x \notin \cap\{H \subseteq \mathbf{U} / S \subseteq H$ and H is N-δI-closed$\}$.
(iv) Since $Ncl_{\delta I}(\cap_{j \in J} S_j) \subseteq Ncl_{\delta I}(S_j) = S_j$ for each j. Therefore, $Ncl_{\delta I}(\cap_{j \in J} S_j) \subseteq \cap_{j \in J} S_j$. Hence, we have, $Ncl_{\delta I}(\cap_{j \in J} S_j) = \cap_{j \in J} S_j$, by(i).
(v) The proof is from (iii) and (iv).

Definition 2.4. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be an INTS and S a subset of $X \subseteq \mathbf{U}$. A point $x \in \mathbf{U}$ is said to be an N-δI-Limit point of S if for every $W \in N\text{-}T_{\delta I}$ containing x, $W \cap (S - \{x\} \neq \emptyset$. The set of all N-δI-Limit points of S is called an N-δI-Derived set of S and is simply mentioned by $N\text{-}D_{\delta I}(S)$.

Theorem 2.5. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be an INTS and $X \subseteq \mathbf{U}$. Then for $S, T \subseteq X$,
(i) $N\text{-}D_{\delta I}(\emptyset) = \emptyset$.
(ii) If $S \subseteq T$, then $N\text{-}D_{\delta I}(S) \subseteq N\text{-}D_{\delta I}(T)$.
(iii) $N\text{-}D_{\delta I}(S) \cup N\text{-}D_{\delta I}(T) \subseteq N\text{-}D_{\delta I}(S \cup T)$.
(iv) $N\text{-}D_{\delta I}(S \cap T) \subseteq N\text{-}D_{\delta I}(S) \cap N\text{-}D_{\delta I}(T)$.
(v) $N\text{-}D_{\delta I}(N\text{-}D_{\delta I}(S)) - S \subseteq N\text{-}D_{\delta I}(S)$.
(vi) $N\text{-}D_{\delta I}(S \cup N\text{-}D_{\delta I}(S)) \subseteq S \cup N\text{-}D_{\delta I}(S)$.

Proof. (i) Let $x \in \mathbf{U}$. Then for every $W \in N\text{-}T_{\delta I}$ containing x, $W \cap (\emptyset \cap \{x\}^C) = \emptyset$. Hence, $N\text{-}D_{\delta I}(\emptyset) = \emptyset$.

(ii) Let $x \in N\text{-}D_{\delta I}(S)$. Then, $W \cap (S - \{x\}) \neq \emptyset$ for every $W \in N\text{-}T_{\delta I}$ containing x. Since $S \subseteq T$, $G \cap (T - \{x\}) \neq \emptyset$. This implies that $x \in N\text{-}D_{\delta I}(T)$.
(iii) Since S and T are the subsets of $S \cup T$, $N\text{-}D_{\delta I}(S) \subseteq N\text{-}D_{\delta I}(S \cup T)$ and $N\text{-}D_{\delta I}(T) \subseteq N\text{-}D_{\delta I}(S \cup T)$ by (ii). Hence, $N\text{-}D_{\delta I}(S) \cup N\text{-}D_{\delta I}(T) \subseteq N\text{-}D_{\delta I}(S \cup T)$.
(iv) Since $S \cap T$ is the subset of both S and T, again by (ii) the proof completes.
(v) Let $x \in N\text{-}D_{\delta I}(N\text{-}D_{\delta I}(S)) - S$ and $W \in N\text{-}T_{\delta I}$ containing x. Then, $W \cap (N\text{-}D_{\delta I}(S) - \{x\}) \neq \emptyset$. Let $u \in W \cap (N\text{-}D_{\delta I}(S) - \{x\})$. Then, $u \in W$ and $u \in N\text{-}D_{\delta I}(S)$. Then, $W \cap (S - \{u\}) \neq \emptyset$. Let $v \in W \cap (S - \{u\})$. Then $v \neq x$ for $v \in S$ and $x \notin S$. Therefore, $W \cap (S - \{x\}) \neq \emptyset$ and so $x \in N\text{-}D_{\delta I}(S)$.
(vi) Let $x \in N\text{-}D_{\delta I}(S \cup N\text{-}D_{\delta I}(S))$. If $x \in S$ then the proof is clear. If $x \in N\text{-}D_{\delta I}(S \cup N\text{-}D_{\delta I}(S)) - S$, then for $W \in N\text{-}T_{\delta I}$ containing x, $W \cap (S \cup N\text{-}D_{\delta I}(S) - \{x\}) \neq \emptyset$. Thus, $W \cap (S - \{x\}) \neq \emptyset$ or $W \cap (N\text{-}D_{\delta I}(S) - \{x\}) \neq \emptyset$. Also, by (iv) $W \cap (S - \{x\}) \neq \emptyset$. Therefore, $x \in N\text{-}D_{\delta I}(S)$. Hence, $N\text{-}D_{\delta I}(S \cup N\text{-}D_{\delta I}(S)) \subseteq S \cup N\text{-}D_{\delta I}(S)$.

Theorem 2.6. Let S be any subset of an INTS $(\mathbf{U}, \mathbb{N}, \mathscr{J})$. Then $Ncl_{\delta I}(S) = S \cup N\text{-}D_{\delta I}(S)$.

Proof. Since $N\text{-}D_{\delta I}(S) \subseteq Ncl_{\delta I}(S)$, $S \cup N\text{-}D_{\delta I}(S) \subseteq Ncl_{\delta I}(S)$. Let $x \in Ncl_{\delta I}(S)$. If $x \in S$, then the proof is clear. Suppose $x \notin S$, then $W \cap (S - \{x\}) \neq \emptyset$, for every $W \in N\text{-}T_{\delta I}$ containing x. Therefore, $x \in N\text{-}D_{\delta I}(S)$.

Definition 2.7. A set M contained in $\mathbf{U}$ is said to be a nano δI-Neighbourhood (briefly, N-δI-Nbd) of a member $x \in \mathbf{U}$ if there exists a $W \in N\text{-}T_{\delta I}$ containing x such that $x \in W \subseteq M$. The set of all N-δI-Nbd of $x \in \mathbf{U}$ is called nano δI-Neighbourhood system of x (briefly, N-δI-NbdS(x)).

Theorem 2.8. Arbitrary union of a family of N-δI-Nbd of a point $x \in \mathbf{U}$ is a N-δI-Nbd of x.

Proof. Let $\{S_\alpha : \alpha \in J\}$ be an arbitrary family of N-δI-Nbd of $x \in \mathbf{U}$. Then, S_α is N-δI-Nbd of x for every α. Therefore, there is a $W \in N\text{-}T_{\delta I}$ containing x with $x \in W \subseteq S_\alpha$. But for every $\alpha \in J$, $S_\alpha \subseteq \cup S_\alpha$. Therefore, $x \in G \subseteq \cup S_\alpha$. Hence, $\cup S_\alpha$ is a N-δI-Nbd of x.

Theorem 2.9. For any $x \in \mathbf{U}$, N-δI-NbdS(x) satisfies,
(i) N-δI-NbdS(x) $\neq \emptyset$.
(ii) If a subset $S \in$ N-δI-NbdS(x) then $x \in S$.
(iii) If $S \in$ N-δI-NbdS(x) and $S \subseteq M$ then $M \in$ N-δI-NbdS(x).

Proof. (i) For each $x \in W$ and $W \in N\text{-}T_{\delta I}$, $x \in W \subseteq W$. Hence, W is a N-δI-Nbd(x) and so $W \in$ N-δI-NbdS(x). Therefore, N-δI-NbdS(x) $\neq \emptyset$.
(ii) Since $S \in$ N-δI-NbdS(x), S is a N-δI-Nbd(x). Then there exist a $W \in N\text{-}T_{\delta I}$ such that $x \in W \subseteq S$.

(iii) Since S $\in$ N-δI-NbdS(x), there exist a W$\in$ N-$T_{\delta I}$ such that, x $\in$ W $\subseteq$ S $\subseteq$ M. Hence, M $\in$ N-δI-NbdS(x).

Theorem 2.10. Let the arbitrary union of a family of N-δI-open sets is N-δI-open and S $\subseteq$ $\mathbf{U}$. Then, S is N-δI-open iff S is N-δI-Nbd of each its points.

Proof. Let S $\in$ N-$T_{\delta I}$ containing x. Then S is N-δI-Nbd(x) for every x $\in$ S. Conversely, let S be N-δI-Nbd of each x $\in$ S. Then for every x $\in$ S, there exist a $W_x \in$ N-$T_{\delta I}$ containing x such that x $\in$ $W_x \subseteq$ S. If x $\in$ S then there is atleast one $W_x \in$ N-$T_{\delta I}$ containing x such that x $\in$ $W_x \subseteq \cup_{x\in S}W_x$. Hence, S $\subseteq \cup_{x\in S}W_x$. Again, y $\in \cup_{x\in S}W_x$ implies y $\in W_x$ for some x $\in$ S. Then y $\in$ S. Therefore, $\cup_{x\in S}W_x \subseteq$ S. Hence, S = $\cup_{x\in S}W_x$. As each $W_x \in$ N-$T_{\delta I}$, S $\in$ N-$T_{\delta I}$.

Theorem 2.11. Let the family of N-δI-open subsets of an INTS ($\mathbf{U}$, $\mathbb{N}$, $\mathscr{I}$) is closed under finite intersection. If S is N-δI-closed and x $\in$ S^C then there exists an N-δI-Nbd W of x such that W $\cap$ S = $\emptyset$.

Proof. Let S be an N-δI-closed set. Then by Theorem 2.10, S^C is N-δI-Nbd of every one of its points. Let x $\in$ S^C. Therefore, there exists a W$\in$ N-$T_{\delta I}$ such that x $\in$ W $\subseteq$ S^C. Hence, W$\cap$S = $\emptyset$.

Definition 2.12. A point x $\in$ $\mathbf{U}$ is known as N-δI-interior point of S if there exists W$\in$ N-$T_{\delta I}$ containing x such that W $\subseteq$ S. The collection of all N-δI-interior point of S is called N-δI-interior of S and is mentioned by $Nint_{\delta I}(S)$.

Theorem 2.13. For a subset S of an INTS ($\mathbf{U}$, $\mathbb{N}$, $\mathscr{I}$), $Nint_{\delta I}(S) = \cup\{W \setminus W\in$ N-$T_{\delta I}$ and $W \subseteq S\}$.

Proof. Let x $\in$ $Nint_{\delta I}(S)$. Then, x is the N-δI-interior point of S. Therefore, there is a W$\in$ N-$T_{\delta I}$ and x $\in$ W $\subseteq$ S. Therefore, x $\in \cup\{W \setminus W\in$ N-$T_{\delta I}$ and $W \subseteq S\}$. On the other hand, suppose, x $\in \cup\{W \setminus W\in$ N-$T_{\delta I}$ and $W \subseteq S\}$. Then, x $\in$ W and W $\subseteq$ S for some W$\in$ N-$T_{\delta I}$. Hence, x is the N-δI-interior point of S.

Theorem 2.14. Let S and T be subsets of an INTS ($\mathbf{U}$, $\mathbb{N}$, $\mathscr{I}$). Then,

(i) N-δI-interior of S is the biggest N-δI-open subset in S.
(ii) S is a N-δI-open iff S = $Nint_{\delta I}(S)$.
(iii) $Nint_{\delta I}(\emptyset) = \emptyset$, and $Nint_{\delta I}(\mathbf{U}) = \mathbf{U}$.
(iv) If S $\subseteq$ T then $Nint_{\delta I}(S) \subseteq Nint_{\delta I}(T)$.
(v) $Nint_{\delta I}(S) \cup Nint_{\delta I}(T) \subseteq Nint_{\delta I}(S \cup T)$.
(vi) $Nint_{\delta I}(S \cap T) \subseteq Nint_{\delta I}(S) \cap Nint_{\delta I}(T)$.
(vii) $Nint_{\delta I}(Nint_{\delta I}(S)) = Nint_{\delta I}(S)$.
(viii) $Nint_{\delta I}(S) = S - N\text{-}D_{\delta I}(\mathbf{U} - S)$.
(ix) $\mathbf{U} - Nint_{\delta I}(S) = Ncl_{\delta I}(\mathbf{U} - S)$.

Proof. (i) Let $W \subseteq S$ and W be any N-δI-open subset of S and $x \in W$. Then $x \in W \subseteq S$. Since $W \in N\text{-}T_{\delta I}$, x is the N-δI-interior point of S. Therefore, for $x \in W$ implies that $x \in Nint_{\delta I}(S)$. This implies that, every N-δI-open subset of S is in $Nint_{\delta I}(S)$. Therefore, $Nint_{\delta I}(S)$ is the biggest N-δI-open subset in S.
(ii) Necessity: Let $S \in N\text{-}T_{\delta I}$. Since $S \subseteq S$, S is the biggest N-δI-open subset of S. Hence the proof completes by (i).
Sufficiency: Let $S = Nint_{\delta I}(S)$. Then proof is clear again by (i).
(iii) Always, $\mathbf{U}, \emptyset \in N\text{-}T_{\delta I}$. Therefore, $Nint_{\delta I}(\mathbf{U}) = \mathbf{U}$ and $Nint_{\delta I}(\emptyset) = \emptyset$, by (ii).
(iv) Let $S \subseteq T$. Then for $x \in Nint_{\delta I}(S)$, there exist a $W \in N\text{-}T_{\delta I}$ containing x such that $x \in W \subseteq S$. Therefore, $x \in W \subseteq T$. Hence, $x \in Nint_{\delta I}(T)$.
(v) Since $S \cup T$ contains both S and T by (iv), $Nint_{\delta I}(S) \subseteq Nint_{\delta I}(S \cup T)$ and $Nint_{\delta I}(T) \subseteq Nint_{\delta I}(S \cup T)$. Therefore, $Nint_{\delta I}(S) \cup Nint_{\delta I}(T) \subseteq Nint_{\delta I}(S \cup T)$.
(vi) The proof is clear by (iv).
(vii) By (ii) and (i), the proof is clear.
(viii) Let $x \in S - (N\text{-}D_{\delta I}(\mathbf{U} - S))$. Then $x \in S$ and $x \notin N\text{-}D_{\delta I}(\mathbf{U} - S)$. Therefore, x is not a N-δI-Limit point of $\mathbf{U} - S$. Therefore, there exist a $W \in N\text{-}T_{\delta I}$ with $x \in W$, $W \cap (\mathbf{U} - S) = \emptyset$ and so $W \subseteq S$. Therefore, $x \in W \subseteq S$ gives $x \in Nint_{\delta I}(S)$. For other side, let $x \in Nint_{\delta I}(S)$. Then $x \in Nint_{\delta I}(S) \subseteq S$. But $Nint_{\delta I}(S)$ is the biggest N-δI-open set and $x \in Nint_{\delta I}(S)$, and not containing the points of $(\mathbf{U} - S)$. Therefore, x is not an N-δI-limit point of $(\mathbf{U} - S)$ and so $x \in S - (N\text{-}D_{\delta I}(U - S))$. Hence, $Nint_{\delta I}(S) \subseteq S - (N\text{-}D_{\delta I}(U - S))$.
(ix) $\mathbf{U} - Nint_{\delta I}(S) = \mathbf{U} - (S - N\text{-}D_{\delta I}(S^C)) = \mathbf{U} - (S \cap (\mathbf{U} - N\text{-}D_{\delta I}(S^C))) = (S^C) \cup N\text{-}D_{\delta I}(S^C) = Ncl_{\delta I}(S^C)$, by (viii) and by Theorem 2.6.

Theorem 2.15. Let S and T be subsets of an INTS $(\mathbf{U}, \mathbb{N}, \mathscr{I})$. Then,
(i) $Nint_{\delta I}(\mathbf{U} - S) \subseteq \mathbf{U} - Nint(S)$.
(ii) $Nint_{\delta I}(S - T) \subseteq Nint_{\delta I}(S) - Nint_{\delta I}(T)$.

Proof. (i) Let $x \in Nint_{\delta I}(\mathbf{U} - S)$. Then $x \notin S$ and hence $x \notin Nint_{\delta I}(S)$. Therefore, $x \in \mathbf{U} - Nint_{\delta I}(S)$.
(ii) $Nint_{\delta I}(S - T) = Nint_{\delta I}(S \cap (\mathbf{U} - T)) \subseteq Nint_{\delta I}(S) \cap Nint_{\delta I}(\mathbf{U} - T) \subseteq Nint_{\delta I}(S) \cap (\mathbf{U} - Nint_{\delta I}(T)) = Nint_{\delta I}(S) - Nint_{\delta I}(T)$, by (i).

3. Nano δI-Exterior (resp. Border and Frontier)

Definition 3.1. Let $(\mathbf{U}, \mathbb{N}, \mathscr{I})$ be an INTS and $S \subseteq \mathbf{U}$. Then,
(i) the nano δI-Exterior of S (briefly, $N\text{-}Ext_{\delta I}(S)$) is defined as $N\text{-}Ext_{\delta I}(S) = Nint_{\delta I}(\mathbf{U} - S)$.
(ii) the nano δI-Border of S (briefly, $N\text{-}Br_{\delta I}(S)$) is defined as $N\text{-}Br_{\delta I}(S) = S - Nint_{\delta I}(S)$.
(iii) the nano δI-Frontier of S (briefly, $N\text{-}Fr_{\delta I}(S)$) is defined as $N\text{-}Fr_{\delta I}(S) = Ncl_{\delta I}(S) - Nint_{\delta I}(S)$.

Theorem 3.2. Let $(\mathbf{U}, \mathbb{N}, \mathscr{I})$ be an INTS and $S, T \subseteq \mathbf{U}$. Then,
(i) $N\text{-}Ext_{\delta I}(\emptyset) = \mathbf{U}$ and $N\text{-}Ext_{\delta I}(\mathbf{U}) = \emptyset$.

(ii) $N\text{-}Ext_{\delta I}(S)$ is N-δI-open set.
(iii) $N\text{-}Ext_{\delta I}(S) = \mathbf{U} - Ncl_{\delta I}(S)$.
(iv) If $S \subseteq T$ then $N\text{-}Ext_{\delta I}(T) \subseteq N\text{-}Ext_{\delta I}(S)$.
(v) $N\text{-}Ext_{\delta I}(N\text{-}Ext_{\delta I}(S)) = Nint_{\delta I}(Ncl_{\delta I}(S))$.
(vi) $N\text{-}Ext_{\delta I}(S \cup T) \subseteq N\text{-}Ext_{\delta I}(S) \cup N\text{-}Ext_{\delta I}(T)$.
(vii) $N\text{-}Ext_{\delta I}(S) \cap N\text{-}Ext_{\delta I}(T) \subseteq N\text{-}Ext_{\delta I}(S \cap T)$.
(viii) $N\text{-}Ext_{\delta I}(S) = N\text{-}Ext_{\delta I}(\mathbf{U} - N\text{-}Ext_{\delta I}(S))$.
(ix) $Nint_{\delta I}(S) \subseteq N\text{-}Ext_{\delta I}(N\text{-}Ext_{\delta I}(S))$.
(x) $S \cap N\text{-}Ext_{\delta I}(S) = \emptyset$.
(xi) $N\text{-}Ext_{\delta I}(S) \subseteq \mathbf{U} - S$.

Proof. (i) clear by Definition.
(ii) The proof is obvious.
(iii) $N\text{-}Ext_{\delta I}(S) = Nint_{\delta I}(\mathbf{U} - S) = \mathbf{U} - Ncl_{\delta I}(S)$.
(iv) Since $S \subseteq T$, $Nint_{\delta I}(\mathbf{U} - T) \subseteq Nint_{\delta I}(\mathbf{U} - S)$. Hence, the proof is clear by Definition.
(v) $N\text{-}Ext_{\delta I}(N\text{-}Ext_{\delta I}(S)) = N\text{-}Ext_{\delta I}(Nint_{\delta I}(\mathbf{U} - S)) = N\text{-}Ext_{\delta I}(\mathbf{U} - Ncl_{\delta I}(S)) = Nint_{\delta I}(\mathbf{U} - (\mathbf{U} - Ncl_{\delta I}(S))) = Nint_{\delta I}(Ncl_{\delta I}(S))$.
(vi) by (iv) proof completes.
(vii) again, by (iv) proof is clear.
(viiii)$N\text{-}Ext_{\delta I}(\mathbf{U} - N\text{-}Ext_{\delta I}(S)) = Nint_{\delta I}(\mathbf{U} - (\mathbf{U} - N\text{-}Ext_{\delta I}(S))) = Nint_{\delta I}(\mathbf{U} - Nint_{\delta I}(S)) = Nint_{\delta I}(\mathbf{U} - S) = N\text{-}Ext_{\delta I}(S)$.
(ix) Since $S \subseteq Ncl_{\delta I}(S)$, $Nint_{\delta I}(S) \subseteq Nint_{\delta I}(Ncl_{\delta I}(S)) = Nint_{\delta I}(\mathbf{U} - Nint_{\delta I}(\mathbf{U} - S) = N\text{-}Ext_{\delta I}(Nint_{\delta I}(\mathbf{U} - S)) = N\text{-}Ext_{\delta I}(N\text{-}Ext_{\delta I}(S))$.
(x) $S \cap N\text{-}Ext_{\delta I}(S) = S \cap Nint_{\delta I}(\mathbf{U} - S) \subseteq S \cap (\mathbf{U} - S) = \emptyset$.
(xi) $N\text{-}Ext_{\delta I}(S) = Nint_{\delta I}(\mathbf{U} - S) \subseteq U - S$.

Theorem 3.3. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be a INTS and $S \subseteq \mathbf{U}$. Then S is an N-δI-open if and only if $N\text{-}Br_{\delta I}(S) = \emptyset$.

Proof. Let $S \in N\text{-}T_{\delta I}$. Then $Nint_{\delta I}(S) = S$. Hence, $N\text{-}Br_{\delta I}(S) = \emptyset$. Conversely, suppose $N\text{-}Br_{\delta I}(S) = \emptyset$. Then, $S - Nint_{\delta I}(S) = \emptyset$.

Theorem 3.4. Let $(\mathbf{U}, \mathbb{N}, \mathscr{J})$ be a INTS and $S, T \subseteq \mathbf{U}$. Then the following holds true.
(i) $N\text{-}Br_{\delta I}((Nint_{\delta I}(S)) = \emptyset$.
(ii) $Nint_{\delta I}(N\text{-}Br_{\delta I}(S)) = \emptyset$.
(iii) $N\text{-}Br_{\delta I}(N\text{-}Br_{\delta I}(S)) = N\text{-}Br_{\delta I}(S)$.
(iv) $N\text{-}int_{\delta I}(S) \cap N\text{-}Br_{\delta I}(S) = \emptyset$.

Proof. (i) $N\text{-}Br_{\delta I}(Nint_{\delta I}(S)) = Nint_{\delta I}(S) - Nint_{\delta I}(Nint_{\delta I}(S)) = Nint_{\delta I}(S) - Nint_{\delta I}(S) = \emptyset$.
(ii)$Nint_{\delta I}(N\text{-}Br_{\delta I}(S))=Nint_{\delta I}(S-Nint_{\delta I}(S))=Nint_{\delta I}(S \cap (\mathbf{U}-Nint_{\delta I}(S))) \subseteq Nint_{\delta I}(S) \cap Nint_{\delta I}(\mathbf{U} - Nint_{\delta I}(S)) = \emptyset$ since, $Nint_{\delta I}(S) \subseteq S$.
(iii) $N\text{-}Br_{\delta I}(N\text{-}Br_{\delta I}(S)) = N\text{-}Br_{\delta I}(S) - Nint_{\delta I}(N\text{-}Br_{\delta I}(S)) = N\text{-}Br_{\delta I}(S)$, by (ii).
(iv) $Nint_{\delta I}(S) \cap N\text{-}Br_{\delta I}(S) = Nint_{\delta I}(S) \cap (S - Nint_{\delta I}(S)) = Nint_{\delta I}(S) \cap (S \cap (\mathbf{U} - Nint_{\delta I}(S))) = Nint_{\delta I}(S) \cap \emptyset = \emptyset$.

Theorem 3.5. Let S and T be two subsets of a INTS ($\mathbf{U}$, $\mathbb{N}$, $\mathscr{J}$). Then,
(i) $Ncl_{\delta I}(S) = Nint_{\delta I}(S) \cup N\text{-}Fr_{\delta I}(S)$.
(ii) $Nint_{\delta I}(S) \cap N\text{-}Fr_{\delta I}(S) = \emptyset$.
(iii) $N\text{-}Fr_{\delta I}(Nint_{\delta I}(S)) \subseteq N\text{-}Fr_{\delta I}(S)$.
(iv) $N\text{-}Br_{\delta I}(S) \subseteq N\text{-}Fr_{\delta I}(S)$.

Proof. (i) $Nint_{\delta I}(S) \cup N\text{-}Fr_{\delta I}(S) = Nint_{\delta I}(S) \cup (Ncl_{\delta I}(S) - Nint_{\delta I}(S)) = Ncl_{\delta I}(S)$.
(ii) $Nint_{\delta I}(S) \cap N\text{-}Fr_{\delta I}(S) = Nint_{\delta I}(S) \cap (Ncl_{\delta I}(S) - Nint_{\delta I}(S)) = \emptyset$.
(iii) $N\text{-}Fr_{\delta I}(Nint_{\delta I}(S))=Ncl_{\delta I}(Nint_{\delta I}(S))-Nint_{\delta I}(Nint_{\delta I}(S)) \subseteq Ncl_{\delta I}(S)-Nint_{\delta I}(S) = N\text{-}Fr_{\delta I}(S)$.
(iv) $N\text{-}Br_{\delta I}(S)=S - Nint_{\delta I}(S) = S \cap (\mathbf{U} - Nint_{\delta I}(S)) \in Ncl_{\delta I}(S) \cap (\mathbf{U} - Nint_{\delta I}(S)) = Ncl_{\delta I}(S) - Nint_{\delta I}(S) = N\text{-}Fr_{\delta I}(S)$.

4 Discussion and Conclusion

The introduction of nano topology created a new space in topological research. Many of the weaker and stronger forms of open sets were studied in this space. In this article we studied N-δI-closed sets in ideal nano topological spaces. The closure operator discussed is N-δI-closed as it is the intersection of all N-δI-closed set contains $S \subseteq \mathbf{U}$. Therefore it is the smallest N-δI-closed set contains S. We proved $Ncl_{\delta I}(S)$ is the union of S and its N-δI-Derived set. It is also proved that the arbitrary union of a family of N-δI-Nbd of a point $x \in \mathbf{U}$ is a N-δI-Nbd of x. Similarly the $Nint_{\delta I}(S)$ is N-δI-open as it is the union of all N-δI-open subsets of S so that it is the largest N-δI-open subset of S. It is deduced that N-δI-closed sets occurs between Nδ-closed sets and N-closed sets and counter example have been shown to evident the reverse implication do not imply both.
Moreover, we introduced the concepts of N-δI-interior, N-δI-Exterior, N-δI-Border, and N-δI-Frontier and several necessary properties of all these are investigated. Thus this concept can be used to investigate many concepts of topological spaces and we also have views on the continuation of this work.

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